Adv. Oper. Theory 4 (2019), no. 1, 113–139
https://doi.org/10.15352/aot.1801-1293
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

NORM ESTIMATES FOR RESOLVENTS OF LINEAR OPERATORS IN A BANACH SPACE AND SPECTRAL VARIATIONS

MICHAEL GIL'

Communicated by C. Lizama

ABSTRACT. Let P_t $(a \leq t \leq b)$ be a function whose values are projections in a Banach space. The paper is devoted to bounded linear operators A admitting the representation

$$A = \int_a^b \phi(t) dP_t + V,$$

where $\phi(t)$ is a scalar function and V is a compact quasi-nilpotent operator such that $P_t V P_t = V P_t$ $(a \leq t \leq b)$. We obtain norm estimates for the resolvent of A and a bound for the spectral variation of A. In addition, the representation for the resolvents of the considered operators is established via multiplicative operator integrals. That representation can be considered as a generalization of the representation for the resolvent of a normal operator in a Hilbert space. It is also shown that the considered operators are Kreiss-bounded. Applications to integral operators in L^p are also discussed. In particular, bounds for the upper and lower spectral radius of integral operators are derived.

1. INTRODUCTION

Throughout this paper $\mathcal{X}$ is a Banach space with the approximation property; that is, any compact operator in $\mathcal{X}$ is a limit in the operator norm of finite rank operators, see [22]; $\mathcal{B}(\mathcal{X})$ is the set of all bounded operators in $\mathcal{X}$, and I is the unit operator in $\mathcal{X}$. For an $A \in \mathcal{B}(\mathcal{X})$, A^* is the operator adjoint to A, $\sigma(A)$ denotes

Date: Received: Jan. 11, 2018; Accepted: Apr. 13, 2018.
2010 *Mathematics Subject Classification.* Primary 47A10; Secondary 47A55, 47G10, 47A11, 47A30.
Key words and phrases. Banach space, resolvent, spectral variation, integral operator, invariant chain of projections.

the spectrum, $R_\lambda(A) = (A-\lambda I)^{-1}$ is the resolvent, and $\rho(A,\lambda) := \inf_{t\in\sigma(A)} |\lambda - t|$ is the distance between $\lambda \in \mathbb{C}$ and $\sigma(A)$. In addition,

$$\mathrm{sv}_A(\tilde{A}) := \sup_{\mu\in\sigma(\tilde{A})} \inf_{\lambda\in\sigma(A)} |\mu - \lambda|$$

is *the spectral variation of an operator* $\tilde{A} \in \mathcal{B}(\mathcal{X})$ *with respect to* A.

In the present paper we consider a class of operators from $\mathcal{B}(\mathcal{X})$ having rather rich sets of invariant subspaces and admitting the triangular representation.

The deep theory of triangular representations of nonselfadjoint operators in a Hilbert space $\mathcal{H}$ via integrals along maximal chains has been developed in the works of M.S. Brodskii, I. C. Gohberg, M.G. Krein, L.A. Sakhnovich, and other mathematicians; see [2, 3, 15, 17, 24] and references therein. We particularly extend some of the representations investigated in the mentioned works.

In the 1930, Carleman established a norm estimate for the resolvent of a Schatten–von Neumann operator via the regularized determinant; see [6]. In [12] that estimate has been slightly refined. In [11] the author has obtained a sharp norm estimate for the resolvent of a Schatten–von Neumann operator A in terms of $\rho(A,\lambda)$. The results of paper [11] have been extended to various nonselfadjoint operators in a Hilbert space [13, 14]. In the present paper for the resolvents of the considered operators in $\mathcal{X}$, we obtain a norm estimate in terms of $\rho(A,\lambda)$. It gives us a bound for $sv_A(\tilde{A})$. It should be noted that the spectral variations mainly investigated in the cases of finite rank operators and operators in a Hilbert space; see [1, 25] (see also [14] and references therein).

We discuss applications of the above pointed results to integral operators. In particular, inequalities for the upper and lower spectral radii of the Hille–Tamarkin integral operators are suggested.

An operator $T \in \mathcal{B}(\mathcal{X})$ is said to be *Kreiss bounded*, if

$$\|(\lambda I - T)^{-1}\| \le \frac{c_0}{|\lambda| - 1} \qquad (|\lambda| > 1, c_0 = const > 0)$$

see papers [21, 26]. In particular, in these papers it was shown that the operator of the indefinite integration is Kreiss-bounded. Below we show that the operators considered in this paper are Kreiss-bounded. In addition, we suggest the representation for resolvents via the multiplicative operator integrals. That representation can be considered as a generalization of the representation for the resolvents of normal operators in a Hilbert space.

A few words about the contents. The paper consists of 10 sections.

In Section 2 we introduce the notion of the maximal chain of projections in $\mathcal{X}$ and consider some properties of operators with invariant maximal chains.

Section 3 is devoted to projection functions whose values form continuous maximal chains and to operators commuting with these projection functions. In addition, the notion of the triangular representation is introduced for operators having invariant continuous projection functions.

In Section 4, norm estimates are derived for the resolvents of the operators admitting the triangular representations. In Section 5 these estimates are applied to the spectral variations of the considered operators. Sections 6–8 are devoted to applications of the above pointed results to integral operators in L^p.

The representation for the resolvents of the considered operators via multiplicative operator integrals is presented in Section 9.

Section 10 deals with operators in a Hilbert space having Shatten–von Neumann Hermitian components. Note that in paper [10] the representation of the resolvent of such operators via the spectral measure has been suggested without the proof. In [12, Chapter 10] and [14, Section 9.9] short proofs of the main result from [10] are given. In Section 10 below we considerably refine the just mentioned results from [12] and [14].

2. Maximal chains of projections

For two projections P_1 and P_2 in $\mathcal{X}$ ($P_1 \neq P_2$), we write $P_1 < P_2$ if $P_1P_2 = P_2P_1 = P_1$ (and thus $P_1\mathcal{X} \subset P_2\mathcal{X}$). A set $\mathcal{P}$ of projections in $\mathcal{X}$, containing at least two projections, is called *a chain (of projections)*, if from $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ it follows that either $P_1 < P_2$ or $P_1 > P_2$.

Let $P^-, P^+ \in \mathcal{P}$, and let $P^- < P^+$. If for every $P \in \mathcal{P}$ we have either $P < P^-$ or $P > P^+$, then the pair (P^+, P^-) is called a gap of $\mathcal{P}$. Besides, dim $(P_+ - P_-)\mathcal{X}$ is the dimension of the gap. A chain which does not have gaps is called *a continuous chain*.

A projection P in $\mathcal{X}$ is called a limit projection of a chain $\mathcal{P}$ if there exists a sequence $P_k \in \mathcal{P}$ $(k = 1, 2, \ldots)$ which strongly converges to P. A chain is said to be closed if it contains all its limit projections.

Definition 2.1. A chain $\mathcal{P}$ is said to be maximal if it is closed, contains 0 and I, all its gaps (if they exist) are one dimensional, and $\sup_{P\in\mathcal{P}} \|P\| < \infty$.

We will say that *a maximal chain $\mathcal{P}$ is invariant for $A \in \mathcal{B}(\mathcal{X})$, or A has a maximal invariant chain $\mathcal{P}$*, if $PAP = AP$ for any $P \in \mathcal{P}$.

Lemma 2.2. *Let P_1 and P_2 be two invariant projections of A, and let $P_1 < P_2$. Then $P_2^* - P_1^*$ is an invariant projection of $P_2^*A^*$.*

Proof. Since $P_2P_1 = P_1P_2 = P_1$, we have

$$\begin{aligned}(P_2 - P_1)A(P_2 - P_1) &= P_2AP_2 - P_2AP_1 - P_1AP_2 + P_1AP_1\\ &= AP_2 - P_2P_1AP_1 - P_1AP_2 + AP_1\\ &= AP_2 - P_1AP_1 - P_1AP_2 + AP_1\\ &= (P_2 - P_1)AP_2.\end{aligned}$$

As claimed. □

Let us prove the following result.

Lemma 2.3. *Let a compact operator $V \in \mathcal{B}(\mathcal{X})$ have a maximal invariant chain $\mathcal{P}$. If, in addition,*

$$(P^+ - P^-)V(P^+ - P^-) = 0, \tag{2.1}$$

for every gap (P^+, P^-) of $\mathcal{P}$ (if it exists), then V is a quasi-nilpotent operator; that is, $\sigma(V) = \{0\}$.

Proof. Indeed, since $(P^+ - P^-)V(P^+ - P^-)$ is one dimensional, by the previous lemma we have $(P^+ - P^-)^* V^* (P^+ - P^-)^* h = \mu (P^+ - P^-)^* h = V^* (P^+ - P^-)^* h$ $(\mu \in \mathbb{C}, h \in \mathcal{X}^*)$. So μ is an eigenvalue of V and $(P^+ - P^-)h$ is the corresponding eigenvector. By (2.1) $\mu = 0$. Moreover, at the points of the continuity of $\mathcal{P}$ operator V does not have eigenvectors. So V does not have nonzero eigenvalues; but V is compact. So it is quasi-nilpotent. $\square$

In particular, if a compact operator has a continuous invariant chain, then it is quasi-nilpotent.

We need also the following lemma.

Lemma 2.4. *Let V be a compact quasi-nilpotent operator having a maximal invariant chain $\mathcal{P}$. Then equality (2.1) holds for every gap (P^+, P^-) of $\mathcal{P}$ (if it exists).*

Proof. As it is shown in the proof of the previous lemma, $(P^+ - P^-)^* V^* (P^+ - P^-)^* h = \mu (P_2 - P_1)^* h$, where μ is an eigenvalue of V^*. But V^* is quasi-nilpotent. So $\mu = 0$. This proves the lemma. $\square$

In the what follows the expression $(P^+ - P^-)T(P^+ - P^-)$ for a $T \in \mathcal{B}(\mathcal{X})$ will be called *the block of the gap (P^+, P^-) of $\mathcal{P}$ on T.*

Lemma 2.5. *Let V_1 and V_2 be compact quasi-nilpotent operators having a joint maximal invariant chain $\mathcal{P}$. Then $V_1 + V_2$ is a quasi-nilpotent operator having the same maximal invariant chain.*

Proof. Since the blocks of the gaps of $\mathcal{P}$ on both V_1 and V_2, if they exist, are zero (due to Lemma 2.4), the blocks of the gaps of $\mathcal{P}$ on $V_1 + V_2$ are also zero. Now the required result is due to Lemma 2.3. $\square$

Lemma 2.6. *Let V and B be bounded linear operators in $\mathcal{X}$ having a joint maximal invariant chain $\mathcal{P}$. In addition, let V be a compact quasi-nilpotent operator. Then VB and BV are quasi-nilpotent and $\mathcal{P}$ is their maximal invariant chain.*

Proof. It is obvious that

$$PVBP = VPBP = VBP \qquad (P \in \mathcal{P}).$$

Now let $Q = P^+ - P^-$ for a gap (P^+, P^-). Then according to Lemma 2.4 equality (2.1) holds. Further, we have $QVP^- = QBP^- = 0$ and

$$\begin{aligned} QVBQ &= QVB(P^+ - P^-) = QV(P^+ BP^+ - P^- BP^-) \\ &= QV[(P^- + Q)B(P^- + Q) - P^- BP^-] = QVQBQ = 0. \end{aligned}$$

Due to Lemma 2.3, this relation implies that VB is a quasi-nilpotent operator. Similarly we can prove that BV is quasi-nilpotent. $\square$

Lemma 2.7. *Let V and B be bounded linear operators in $\mathcal{X}$ having a joint maximal invariant chain $\mathcal{P}$. In addition, let V be a compact quasi-nilpotent operator and the regular set of B be simply connected. Then $\sigma(B + V) = \sigma(B)$.*

Proof. We have

$$PR_\lambda(B)P = -\sum_{k=0}^{\infty} P\frac{B^k}{\lambda^{k+1}}P = R_\lambda(B)P \qquad (|\lambda| > \|B\|, P \in \mathcal{P}).$$

Since the set of regular points of B is simply connected, by the resolvent identity one can extend the equality $PR_\lambda(B)P = R_\lambda(B)P$ to all regular λ of B (see also [23, pp. 32–33]).

Put $T = B + V$. For any $\lambda \notin \sigma(B)$, an operator $VR_\lambda(B)$ is quasi-nilpotent due to Lemma 2.6. So $I + VR_\lambda(B)$ is boundedly invertible, and therefore,

$$R_\lambda(T) = (B + V - \lambda I)^{-1} = R_\lambda(B)(I + VR_\lambda(B))^{-1} \qquad (\lambda \notin \sigma(B)).$$

Hence it follows that λ is a regular point for T. Consequently,

$$\sigma(T) \subseteq \sigma(B). \tag{2.2}$$

So the regular set of T is also simply connected.

Now let $\lambda \notin \sigma(T)$. Since $\mathcal{P}$ is invariant for T, as above we can show that $\mathcal{P}$ is invariant for $R_\lambda(T)$. Then operator $VR_\lambda(T)$ is quasi-nilpotent due to Lemma 2.6. So $I - VR_\lambda(T)$ is boundedly invertible. Furthermore, according to the equality $B = T - V$, we get

$$R_\lambda(B) = (T - V - \lambda I)^{-1} = R_\lambda(T)(I - VR_\lambda(T))^{-1}.$$

Hence, it follows that λ is a regular point also for B, and therefore $\sigma(B) \subseteq \sigma(T)$. This proves the result. □

3. Operators having continuous maximal chains

Definition 3.1. Let P_t $(t \in [a, b])$ be a function defined on a finite segment $[a, b]$, whose values form a maximal continuous chain $\mathcal{P}$ of projections such that $P_{t_2}P_{t_1} = P_s$, $(s = \min\{t_2, t_1\})$ $P_a = 0$, and $P_b = I$. Then we will call P_t a continuous maximal projection function (CMPF).

So P_t is a particular case of a resolution of the identity.

It is assumed that there is a constant m_P dependent on P_t only such that

$$\|\sum_{k=1}^{n} a_k \Delta P_k\| \le m_P \max_j |a_j| \qquad \begin{pmatrix} n < \infty, \ \Delta P_k = P_{t_k} - P_{k-1} \\ a = t_0 < t_1 < \cdots < t_n = b \end{pmatrix} \tag{3.1}$$

for arbitrary numbers a_k and an arbitrary partitioning of $[a, b]$.

Let $\psi(t)$ be a bounded scalar function defined on $[a, b]$, and there exists a limit S of the operator sums

$$S_n = \sum_{k=1}^{n} \psi(t_k)\Delta P_k$$

in the operator norm. Then we write

$$S = \int_a^b \psi(t) dP_t; \tag{3.2}$$

$\psi(t)$ and S will be called a *P_t-integrable function* and *a P_t-scalar operator*, respectively. We write $S = \psi(T_0)$, where

$$T_0 = \int_a^b t dP_t$$

is a scalar type spectral operator [7]. So a P_t-scalar operator is a function of a scalar type spectral operator.

Due to (3.1), $\|S_n\| \le m_p \sup_t |\psi(t)|$; by the Banach–Steinhaus theorem, S is bounded and $\|S\| \le \sup_n \|S_n\| \le m_P \sup_t |\psi(t)|$.

For example, let as usually $L^p = L^p(0,1)$ $(1 \le p < \infty)$ be the space of scalar-valued functions h defined on $[0,1]$ and equipped with the norm

$$|h|_{L^p} = [\int_0^1 |h(x)|^p dx]^{1/p}.$$

Let $\hat{P}_t$ $(0 \le t \le 1)$ be the truncation projection function, defined by the relations $\hat{P}_0 = 0, \hat{P}_1 = I$, and

$$(\hat{P}_t f)(x) = \begin{cases} f(x) & \text{if } 0 \le x < t, \\ 0 & \text{if } t < x \le 1 \end{cases} \quad (t \in (0,1),\ f \in L^p). \tag{3.3}$$

It is simple to check that the values of $\hat{P}_t$ form a continuous maximal chain and the operator $\hat{S}$ defined by

$$(\hat{S}f)(x) = \hat{\psi}(x) f(x) \qquad (0 \le x \le 1, f \in L^p(0,1))$$

with an integrable function $\hat{\psi}$ can be written in the form (3.2).

Furthermore, if $\inf_{a \le t \le b} |\psi(t)| > 0$, then $1/\psi(t)$ is also P_t-integrable and according to (3.2),

$$S^{-1} = \int_a^b \frac{1}{\psi(t)} dP_t. \tag{3.4}$$

Indeed, put

$$B_n = \sum_{k=1}^n \frac{1}{\psi(t_k)} \Delta P_k.$$

Then

$$B_n S_n = S_n B_n = \sum_{k=1}^n \Delta P_k = I.$$

So $B_n = S_n^{-1}$, and by (3.1), $\|S_n^{-1}\| \le \frac{m_P}{\inf_t |\psi(t)|} < \infty$. Since

$$S_m^{-1} - S_n^{-1} = -S_m^{-1}(S_m - S_n) S_n^{-1} \to 0 \qquad (m, n \to \infty)$$

in the operator norm, we have $S_n^{-1} \to S^{-1}$. So (3.4) is valid and $\|S^{-1}\| \le \frac{m_P}{\inf_t |\psi(t)|} < \infty$.

It is simple to check that

$$\sigma(S) = \{z \in \mathbb{C} : z = \psi(t), t \in [a,b]\}.$$

Let $\lambda \neq \psi(t), t \in [a,b]$. Then according to (3.2) and (3.1),

$$(S-\lambda I)^{-1} = \int_a^b \frac{1}{\psi(t)-\lambda} dP_t \tag{3.5}$$

and

$$\|(S-\lambda I)^{-1}\| \leq \frac{m_P}{\rho(S,\lambda)} \qquad (\lambda \notin \sigma(S)). \tag{3.6}$$

Besides, $\rho(S,\lambda) = \inf_t |\psi(t)-\lambda|$.

Definition 3.2. Let $A \in \mathcal{B}(\mathcal{X})$, and let P_t be a CMPF. If $P_t A P_t = A P_t$ ($a \leq t \leq b$), then P_t is said to be an invariant CMPF of A or A has a CMPF P_t.

Definition 3.3. Let $A \in \mathcal{B}(\mathcal{X})$ have a CMPF P_t defined on $[a,b]$, and let there be a bounded P_t-integrable function ϕ such that

$$A = D + V, \tag{3.7}$$

where

$$D = \int_a^b \phi(t) dP_t,$$

and V is a compact quasi-nilpotent operator in $\mathcal{X}$. In addition, let the regular set of A be simply connected. Then we will say that A is a P_t-triangular operator, if equality (3.7) is its triangular representation, D and V are the diagonal and nilpotent parts of A, respectively, and $\phi(.)$ is a *P_t-diagonal function of A*.

Note that $P_t V P_t = P_t(A-D)P_t = VP_t$ ($a \leq t \leq b$).
According to (3.5) we have

$$(D-\lambda I)^{-1} = \int_a^b \frac{1}{\phi(t)-\lambda} dP_t \qquad (\lambda \notin \sigma(D)). \tag{3.8}$$

Corollary 3.4. *Let A be P_t-triangular, and let D and V be its diagonal part and nilpotent one, respectively. Then for any regular point λ of D, the operators $VR_\lambda(D)$ and $R_\lambda(D)V$ are quasi-nilpotent ones. Besides P_t is invariant for $VR_\lambda(D)$ and $R_\lambda(D)V$.*

Indeed, due to (3.8), P_t is invariant for $R_\lambda(D)$. Now Lemma 2.6 ensures the required result.
From Lemma 2.7, the following corollary is obtained.

Corollary 3.5. *Let A be P_t-triangular. Then $\sigma(A) = \sigma(D)$, where D is the diagonal part of A.*

Moreover, from (3.1), we have

$$R_\lambda(A) = (D+V-\lambda I)^{-1} = R_\lambda(D)(I+VR_\lambda(D))^{-1} \qquad (\lambda \notin \sigma(A)). \tag{3.9}$$

Similarly one can check that

$$R_\lambda(A) = (I+R_\lambda(D)V)^{-1} R_\lambda(D) \qquad (\lambda \notin \sigma(A)).$$

Note that

$$\rho(A,\lambda) = \rho(D,\lambda) = \inf_t |\phi(t)-\lambda|.$$

4. Norm estimates for resolvents

Definition 4.1. Let P_t be a CMPF in $\mathcal{X}$, and let E be a linear subspace of the set of compact operators in $\mathcal{X}$ endowed with a norm $N_E(.)$ having the following property: for arbitrary P_t-scalar operators $S, S_1 \in \mathcal{B}(\mathcal{X})$ the inequality

$$N_E(SBS_1) \le \|S_1\|\|S\|N_E(B) \qquad (B \in E) \tag{4.1}$$

is valid. Then E will be called a P_t-subset of compact operators.

For example, let E be the set of operators B in $L^p = L^p(0,1)$ $(1 \le p < \infty)$ defined by

$$(Bh)(x) = \int_0^1 k(x,s)h(s)ds \qquad (h \in L^p, x \in [0,1]),$$

where $k(x,s)$ is a scalar kernel defined on $[0,1]^2$ and satisfying the condition

$$M_p(B) := [\int_0^1 [\int_0^1 |k(x,s)|^{p'} ds]^{p/p'} dx]^{1/p} < \infty \qquad (1 < p < \infty, 1/p + 1/p' = 1) \tag{4.2}$$

or

$$M_1(B) := ess \sup_x \int_0^1 |k(x,s)|dx < \infty.$$

Operators satisfying condition (4.2) are called (p,p')-Hille–Tamarkin operators [22, p.245].

It is not hard to check that $N_E(.) = M_p(.)$ is a norm. Take $P_t = \hat{P}_t$ as in (3.3). Then arbitrary P_t-scalar type operators S and S_1 are the operators of the multiplication by some scalar bounded measurable functions ψ and ψ_1, respectively. In this case we have

$$\begin{aligned} M_p(SBS_1) &= [\int_0^1 [\int_0^1 |\psi(x)k(x,s)\psi_1(s)|^{p'} ds]^{p/p'} dx]^{1/p} \\ &\le \sup_x |\psi(x)| \sup_x |\psi_1(x)| [\int_0^1 [\int_0^1 |k(x,s)|^{p'} ds]^{p/p'} dx]^{1/p} \\ &= \|S_1\|\|S\|M_p(B). \end{aligned}$$

So condition (4.1) is satisfied.

Furthermore, let us suppose that, for any quasi-nilpotent operator $W \in E$, there are positive numbers θ_k $(k = 1, 2, \ldots)$ independent of W (but dependent on E) such that

$$\|W^k\| \le \theta_k N_E^k(W) \qquad (k = 1, 2, \ldots) \tag{4.3}$$

and

$$\lim_{k\to\infty} \sqrt[k]{\theta_k} = 0. \tag{4.4}$$

Then

$$\|(I - W)^{-1}\| = \|\sum_{k=0}^{\infty} W^k\| \le \sum_{k=0}^{\infty} \theta_k N_E^k(W) < \infty. \tag{4.5}$$

Now we are in a position to formulate and prove the main result of this section.

Theorem 4.2. *Let A be a P_t-triangular operator, whose nilpotent part V belongs to a P_t-subset of compact operators E such that conditions (4.1), (4.3), and (4.4) hold. Then*

$$\|R_\lambda(A)\| \leq \sum_{k=0}^{\infty} \frac{m_P^{k+1}\theta_k N_E^k(V)}{\rho^{k+1}(A,\lambda)} \qquad (\lambda \notin \sigma(A)). \tag{4.6}$$

Proof. Let D be the diagonal part of A. Due to (3.6)

$$\|(D-\lambda I)^{-1}\| \leq \frac{m_P}{\rho(D,\lambda)} \qquad (\lambda \notin \sigma(D)).$$

By Corollary 3.4, $VR_\lambda(D)$ $(\lambda \notin \sigma(D))$ is quasi-nilpotent, and according to (4.1), (4.3) and (4.4),

$$\|(VR_\lambda(D))^k\| \leq \theta_k N_E^k(VR_\lambda(D))) \leq \theta_k \|R_\lambda(D)\|^k N_E^k(V).$$

This implies

$$\|(VR_\lambda(D))^k\| \leq \frac{m_P^k\theta_k N_E^k(V)}{\rho^k(D,\lambda)}$$

and therefore by (4.5),

$$\|(I+VR_\lambda(D))^{-1}\| = \|\sum_{k=0}^{\infty}(-VR_\lambda(D))^k\| \leq \sum_{k=0}^{\infty} \frac{m_P^k\theta_k N_E^k(V)}{\rho^k(D,\lambda)}.$$

Hence (3.9) yields

$$\|R_\lambda(A)\| \leq \sum_{k=0}^{\infty} \frac{m_P^{k+1}\theta_k N_E^k(V)}{\rho^{k+1}(D,\lambda)} \qquad (\lambda \notin \sigma(D)).$$

Taking into account that, by Corollary 3.4, $\rho(D,\lambda) = \rho(A,\lambda)$, we arrive at the required result. □

Observe that (4.6) implies

$$\|R_\lambda(A)\| \leq \sum_{k=0}^{\infty} \frac{m_P^{k+1}\theta_k N_E^k(V)}{(|\lambda| - r_s(A))^{k+1}} \qquad (|\lambda| > r_s(A)),$$

where $r_s(A)$ is the (upper) spectral radius. Assume that $r_s(A) < 1$. Then

$$\|R_\lambda(A)\| \leq \frac{c_A}{|\lambda| - 1} \qquad (|\lambda| > 1)$$

with

$$c_A = \sum_{k=0}^{\infty} \frac{m_P^{k+1}\theta_k N_E^k(V)}{(1 - r_s(A))^k}.$$

We thus arrive at the following corollary.

Corollary 4.3. *Under the hypothesis of Theorem 4.2, let $r_s(A) < 1$. Then A is Kreiss-bounded.*

5. Perturbations of triangularizable operators

Let $A, \tilde{A} \in \mathcal{B}(\mathcal{X})$, and let $q := \|A - \tilde{A}\|$. Recall that the spectral variation of $\tilde{A}$ with respect to A is defined in Section 1.

Due to the Hilbert identity $R_\lambda(\tilde{A}) - R_\lambda(A) = R_\lambda(A)(A - \tilde{A})R_\lambda(\tilde{A})$, we have

$$\|R_\lambda(\tilde{A})\| \leq \|R_\lambda(A)\| + q\|R_\lambda(A)\|\|R_\lambda(\tilde{A})\|.$$

So if a $\lambda \in \mathbb{C}$ is regular for A and

$$q\|R_\lambda(A)\| < 1, \tag{5.1}$$

then λ is also regular for $\tilde{A}$. Moreover,

$$\|R_\lambda(\tilde{A})\| \leq \frac{\|R_\lambda(A)\|}{1 - q\|R_\lambda(A)\|}.$$

Assume that

$$\|R_\lambda(A)\| \leq F\left(\frac{1}{\rho(A, \lambda)}\right) \qquad (\lambda \notin \sigma(A)), \tag{5.2}$$

where $F(t)$ is a monotonically increasing non-negative continuous function of a non-negative variable such that $F(0) = 0$ and $F(\infty) = \infty$. We need the following technical lemma.

Lemma 5.1. *Let $A, \tilde{A} \in \mathcal{B}(\mathcal{X})$, and let condition (5.2) hold. Then $\mathrm{sv}_A(\tilde{A}) \leq z(F, q)$, where $z(F, q)$ is the unique positive root of the equation*

$$qF(1/z) = 1.$$

For the proof see [14, Lemma 1.10]. Now Theorem 4.2 implies the following corollary.

Corollary 5.2. *Let $A \in \mathcal{B}(\mathcal{X})$ satisfy the hypothesis of Theorem 4.2. Then, for any $\tilde{A} \in \mathcal{B}(\mathcal{X})$, we have $\mathrm{sv}_A(\tilde{A}) \leq z_E(A, q)$, where $z_E(A, q)$ is the unique positive root of the equation*

$$q\sum_{k=0}^{\infty} \frac{m_P^{k+1}\theta_k N_E^k(V)}{z^{k+1}} = 1.$$

6. Powers of Volterra operators in L^p

The results of this section have been particularly published in [12, Chapters 16 and 17], but for the convenience of the reader we present here the brief proofs. Throughout this section W is a Volterra operator in $L^p \equiv L^p(0,1)$ $(1 \leq p \leq \infty)$ defined by

$$(Wh)(x) = \int_0^x K(x, s)h(s)ds \; (h \in L^p, x \in [0, 1]), \tag{6.1}$$

where $K(x, s)$ is a scalar kernel defined on $0 \leq s \leq x \leq 1$ and satisfying the inequalities pointed below.

6.1. **Hille–Tamarkin Volterra operators.** Let $1 < p < \infty$, and let

$$M_p(W) := [\int_0^1 [\int_0^x |K(x,s)|^{p'} ds]^{p/p'} dx]^{1/p} < \infty \qquad (1/p + 1/p' = 1). \tag{6.2}$$

That is, W is a (p,p')-Hille–Tamarkin Volterra operator.

Lemma 6.1. *Under condition (6.2), the operator W defined by (6.1) satisfies the inequality*

$$|W^k|_{L^p} \le \frac{M_p^k(W)}{\sqrt[p]{k!}} \qquad (k = 1, 2, \ldots).$$

Proof. Put

$$w(x) = [\int_0^x |K(x,s)|^{p'} ds]^{p/p'}. \tag{6.3}$$

Employing Hölder's inequality, we have

$$|Wh|_{L^p}^p \le \int_0^1 w(s_1) \int_0^{s_1} |h(s_2)|^p ds_2\, ds_1 \qquad (h \in L^p).$$

Using this inequality, we obtain

$$|W^k h|_{L^p}^p \le \int_0^1 w(s_1) \int_0^{s_1} |W^{k-1} h(s_2)|^p ds_2\, ds_1.$$

Once more apply Hölder's inequality,

$$|W^k h|_{L^p}^p \le \int_0^1 w(s_1) \int_0^{s_1} w(s_2) \int_0^{s_2} |W^{k-2} h(s_3)|^p ds_3\, ds_2\, ds_1.$$

Repeating these arguments, we arrive at the relation

$$|W^k h|_{L^p}^p \le \int_0^1 w(s_1) \int_0^{s_1} w(s_2) \ldots \int_0^{s_k} |h(s_{k+1})|^p ds_{k+1} \ldots ds_2\, ds_1.$$

Hence,

$$|W^k|_{L^p}^p \le \int_0^1 w(s_1) \int_0^{s_1} w(s_2) \ldots \int_0^{s_{k-1}} w(s_k) ds_k \ldots ds_2\, ds_1. \tag{6.4}$$

It is simple to see that with $y(x) = \int_0^x w(s)ds$ we have

$$\begin{aligned}\int_0^1 w(s_1) \ldots \int_0^{s_{k-1}} w(s_k) ds_k \ldots ds_1 &= \int_0^{y(1)} \ldots \int_0^{y_{k-1}} dy_k \ldots dy_1 \\ &= \frac{y^k(1)}{k!} = \frac{(\int_0^1 w(s)ds)^k}{k!}.\end{aligned}$$

Thus (6.4) and (6.3) give us the inequality

$$|W^k|_{L^p}^p \le \frac{(\int_0^1 w(s)ds)^k}{k!} = M_p^{pk}(W).$$

As claimed. □

6.2. **Volterra operators in L^1 and L^∞.**

Lemma 6.2. *Assume that*

$$M_1(W) := ess - \sup_{s\in[0,1]} \int_s^1 |K(x,s)|dx < \infty.$$

Then the operator defined by (5.1) *satisfies the inequality*

$$|W^k|_{L^1} \le \frac{M_1^k(W)}{k!} \qquad (k = 1, 2, \dots).$$

Proof. We have

$$\int_0^x |(Wh)(s)|ds \le \int_0^x \int_0^s |K(s,s_1)h(s_1)|ds_1 ds$$
$$\le \int_0^x v(s) \int_0^s |h(s_1)|ds_1\, ds \qquad (x \in [0,1], h \in L^1),$$

where $v(s) = ess\ \sup_{s_1 \le s} |K(s,s_1)|$. Hence

$$\int_0^1 |(W^2h)(s)|ds \le \int_0^1 v(s) \int_0^s v(s_1) \int_0^{s_1} h(s_2)ds_2ds_1ds.$$

Consequently,

$$\|W^2\|_{L^1} \le \int_0^1 v(s) \int_0^s v(s_1)ds_1.$$

Continuing this process we have

$$|W^k|_{L^1} \le \int_0^1 v(s_1) \int_0^{s_1} v(s_2) \dots \int_0^{s_{k-1}} v(s_k)ds_k \dots ds_2ds_1.$$

It is simple to see that

$$\int_0^1 v(s_1) \dots \int_0^{s_{k-1}} v(s_k)ds_k \dots ds_1 = \frac{(\int_0^1 v(s)ds)^k}{k!}.$$

Thus we get the inequality

$$|W^k|_{L^1} \le \frac{(\int_0^1 v(s)ds)^k}{k!} = \frac{M_1^k(W)}{k!}.$$

As claimed. □

Recall that the space $L^\infty(0,1) = L^\infty$ is the space of scalar functions h defined on $[0,1]$ with the finite norm $|h|_{L^\infty} = ess\ \sup_{x\in[0,1]} |h(x)|$.

Repeating the arguments of the previous lemma we arrive at the following lemma.

Lemma 6.3. *Assume that*

$$M_\infty(W) := ess\ \sup_{x\in[0,1]} \int_0^x |K(x,s)|ds < \infty.$$

Then the operator defined by (5.1) *satisfies the inequality*

$$|W|_{L^\infty} \le \frac{M_\infty^k(W)}{k!} \qquad (k = 1, 2, \dots).$$

Various aspects of powers of Volterra operators have been considered in papers [8, 9, 18, 19, 20, 21, 27], but mainly the convolution operators, in particular the operators of the indefinite integration have been considered.

7. Triangularizable operators in L^p

Consider in $L^p[0,1]$ $(1 \le p < \infty)$ the operator A defined by

$$(Ah)(x) = \phi(x)h(x) + \int_x^1 k(x,s)h(s)ds \qquad (h \in L^p, x \in [0,1]), \tag{7.1}$$

where $k(x,s)$ is a scalar kernel defined on $0 \le x \le s \le 1$ and having the properties pointed below, and $\phi(x)$ is a scalar bounded Riemann-integrable function, whose values lie on an unclosed Jordan curve. The Volterra operator in (7.1) is assumed to be compact.

Let $\hat{P}_t$ $(0 \le t \le 1)$ be the truncation projection function, defined by (3.3). It is simple to check that $\hat{P}_t A \hat{P}_t = A\hat{P}_t$. Define the operators $\hat{D}$ and $\hat{V}$ by

$$(\hat{D}h)(x) = \phi(x)h(x) \text{ and } (\hat{V}h)(x) = \int_x^1 k(x,s)h(s)ds \qquad (h \in L^p, x \in [0,1]).$$

Then $\hat{P}_t\hat{V}\hat{P}_t = \hat{V}\hat{P}_t$ and

$$\hat{D} = \int_0^1 \phi(s)d\hat{P}_s.$$

Omitting the obvious calculations we arrive at the following result.

Lemma 7.1. *Let A be defined by (7.1). Then it is a $\hat{P}_s$-triangular operator, its diagonal part is $\hat{D}$, and its nilpotent part is $\hat{V}$.*

Assume that either

$$M_p(\hat{V}) := [\int_0^1 [\int_x^1 |k(x,s)|^{p'} ds]^{p/p'} dx]^{1/p} < \infty \qquad (1 < p < \infty, 1/p + 1/p' = 1), \tag{7.2}$$

or

$$M_1(\hat{V}) := \int_0^1 ess - \sup_{s\in[x,1]} |k(x,s)|dx < \infty \tag{7.3}$$

Recall that a compact operator K in $\mathcal{X}$ is said to be p-summing $(1 \le p < \infty)$, if there is a constant ν such that regardless of a natural number m and regardless of the choice $x_1, \dots, x_m \in \mathcal{X}$ we have

$$[\sum_{k=1}^m \|Kx_k\|^p]^{1/p} \le \nu \, \sup\{[\sum_{k=1}^m |(x^*, x_k)|^p]^{1/p} : x^* \in \mathcal{X}^*, \|x^*\| = 1\};$$

see [5]. Here $\mathcal{X}^*$ means the space adjoint to $\mathcal{X}$. The least ν, for which this inequality holds, is a norm and is denoted by $\pi_p(K)$. The set of p-summing operators in $\mathcal{X}$ with the finite norm π_p is a two-sided normed ideal in the set of bounded linear operators, which is denoted by Π_p; see [22]. As is well known, [22, Proposition 7.2.7] and [5, p. 43], any (p,p')-Hille–Tamarkin operator K is a p-summing operator with $\pi_p(K) \le M_p(K)$ $(1 \le p < \infty)$.

Theorem 4.2, and Lemmas 6.1 and 6.2 imply

$$|R_\lambda(A)|_{L^p} \le \sum_{k=0}^{\infty} \frac{M_p^k(\hat V)}{\sqrt[p]{k!}\rho^{k+1}(A,\lambda)} \qquad (1 \le p < \infty, \lambda \notin \sigma(A)), \tag{7.4}$$

if condition (7.2) or (7.3) holds. Besides $\rho(A,\lambda) = \rho(\hat D,\lambda) = \inf_{0\le x\le 1}|\phi(x)-\lambda|$. So

$$|R_\lambda(A)|_{L^1} \le \frac{1}{\rho(A,\lambda)} \exp[\frac{M_1(\tilde V)}{\rho(A,\lambda)}]$$

if condition (7.3) holds.

Let $p>1$. Then by the Hölder inequality for any $c>1$ we get

$$\sum_{k=0}^{\infty} \frac{c^k M_p^k(\tilde V)x^k}{c^k\sqrt[p]{k!}} \le (\sum_{k=0}^{\infty}\frac{1}{c^{kp'}})^{1/p'}(\sum_{k=0}^{\infty}\frac{c^{kp}M_p^{pk}(\tilde V)x^{kp}}{k!})^{1/p}$$
$$= \frac{c}{(c^{p'}-1)^{1/p'}}\exp[c^p M_p^p(\tilde V)x^p/p] \qquad (x>0).$$

By virtue of (7.4), we can write

$$|R_\lambda(A)|_{L^p} \le \frac{1}{(1-c^{-p'})^{1/p'}\rho(A,\lambda)} \exp\left[\frac{c^p M_p^p(\hat V)}{\rho^p(A,\lambda)p}\right] \qquad (1/p+1/p', c>1, \lambda\notin\sigma(A)).$$

Take $c = p^{1/p}$. Then we obtain

$$|R_\lambda(A)|_{L^p} \le \frac{b_p}{\rho(A,\lambda)} \exp\left[\frac{M_p^p(\hat V)}{\rho^p(A,\lambda)}\right] \qquad (1<p<\infty, \lambda\notin\sigma(A)),$$

where

$$b_p := \frac{1}{(1-p^{-p'/p})^{1/p'}}.$$

Let $q_p = |A-\tilde A|_{L^p}$, and let $z_p(\hat V, q_p)$ be the unique positive root of the equation

$$q_p F_p(\hat V, 1/z) = 1, \tag{7.5}$$

where

$$F_p(\hat V, x) = b_p x \exp[M_p^p(\hat V)x^p] \qquad (1<p<\infty)$$

and

$$F_1(\hat V, x) = x\exp[xM_1(\hat V)] \qquad (x\ge 0).$$

Note that one can take $b_1 = 1$. Now Corollary 5.2 implies the following result.

Lemma 7.2. *Let A be defined by (7.1) and satisfy one of the conditions (7.2) or (7.3). Then for any $\tilde A \in \mathcal{B}(\mathcal{X})$ we have* $\mathrm{sv}_A(\tilde A) \le z_p(\hat V, q_p)$ $(1\le p<\infty)$.

To estimate $z_p(\hat V, q_p)$ we can apply the following lemma.

Lemma 7.3. *The unique positive root z_a of the equation*

$$y\, e^y = a \qquad (a = const > 0) \tag{7.6}$$

satisfies the inequality $z_a \geq \delta_p(a)$, *where*

$$\delta_p(a) := \begin{cases} ae^{-1} & \text{if } a \leq e, \\ \frac{1}{2}\ln(ae) & \text{if } a \geq e. \end{cases}$$

Proof. Let $a \geq e$. Then $z_a \geq 1$. By the usual calculations, the function $f(y) = \frac{e^{y-1}}{y}$ has, for $y \geq 1$, a unique extremum-minimum at $y = 1$, and $f(y) \geq 1$ for $y \geq 1$. We obtain $1 \leq z_a \leq e^{z_a - 1}$, and $a = z_a e^{z_a} \leq e^{2z_a - 1}$ and therefore $z_a \geq \frac{1}{2}\ln(ea)$.

Now let $a \leq e$. Then $z_a \leq 1$. Thus $e^{z_a} \leq e$ and therefore, $a = z_a e^{z_a} \leq e z_a$, as claimed. □

Rewrite equation (7.5) with $p = 1$ and $z = 1/x$ as

$$q_1 M_1(\hat{V}) x \exp[x M_1(\hat{V})] = M_1(\hat{V}).$$

Then we obtain equation (7.6) with $y = xM_1(\hat{V})$ and $a = M_1(\hat{V})/q_1$. So

$$z_1(\hat{V}, q_1) = \frac{M_1(\hat{V})}{z_a}.$$

Now Lemma 7.3 implies $z_1(A, q) \leq \delta_1(\hat{V}, q_1)$, where

$$\delta_1(\hat{V}, q_1) := \begin{cases} q_1 e & \text{if } M_1(\hat{V}) \leq q_1 e, \\ \frac{2M_1(\hat{V})}{\ln\left(\frac{M_1(\hat{V})e}{q_1}\right)} & \text{if } M_1(\hat{V}) > q_1 e. \end{cases}$$

Now let $1 < p < \infty$. Then equation (7.5) with $z = 1/x$ takes the form

$$q_p b_p x \exp[x^p M_p^p(\hat{V})] = 1 \text{ or } b_p^p q_p^p x^p \exp[p x^p M_p^p(\hat{V})] = 1.$$

Therefore

$$(q_p b_p)^p p M_p^p(\hat{V}) x^p \exp[p x^p M_p^p(\hat{V})] = p M_p^p(\hat{V}).$$

Hence we obtain the equation (7.6) with $y = p x^p M_p^p(\hat{V})$ and

$$a = \frac{p M_p^p(\hat{V})}{(q_p b_p)^p}.$$

So

$$z_p(\hat{V}, q_p) = \frac{p^{1/p} M_p(\hat{V})}{z_a^{1/p}}.$$

From Lemma 7.3 it follows that $z_p(\hat{V}, q_p) \leq \delta_p(\hat{V}, q_p)$, where

$$\delta_p(\hat{V}, q_p) := \begin{cases} q_p b_p e^{1/p} & \text{if } M_p(\hat{V}) \leq b_p q_p (e/p)^{1/p}, \\ \frac{2^{1/p} M_p(\hat{V})}{\ln^{1/p}\left(\frac{(pe)^{1/p} M_p(\hat{V}) e}{q_p b_p}\right)} & \text{if } M_p(\hat{V}) > b_p q_p (e/p)^{1/p}. \end{cases}$$

Now Lemma 7.2 implies the following corollary.

Corollary 7.4. *Let A be defined by (7.1) and satisfy one of the conditions (7.2) or (7.3). Then, for any $\tilde{A} \in \mathcal{B}(\mathcal{X})$, we have* $\mathrm{sv}_A(\tilde{A}) \leq \delta_p(\tilde{V}, q_p)$ $(1 \leq p < \infty)$.

8. Integral operators in L^p

Throughout this section $\tilde{A}$ is a linear operator in $L^p = L^p(0,1)$ $(1 \le p < \infty)$ defined by

$$(\tilde{A}h)(x) = \phi(x)h(x) + \int_0^1 k(x,s)h(s)ds \ (h \in L^p, x \in [0,1]), \tag{8.1}$$

where ϕ is the same as in the previous section and $k(x,s)$ is a scalar kernel defined on $[0,1]^2$ and having the properties pointed below.

8.1. **The case $1 < p < \infty$.** Let

$$[\int_0^1 [\int_0^1 |k(x,s)|^{p'} ds]^{p/p'} dx]^{1/p} < \infty \qquad (1 < p < \infty,\ 1/p + 1/p' = 1). \tag{8.2}$$

So k is a Hille–Tamarkin kernel. Take A as in (7.1) namely, $A = \hat{D} + \hat{V}$, where $\hat{D}$ and $\hat{V}$ are the same as in the previous section. By (8.2)

$$\tau_p := [\int_0^1 [\int_0^x |k(x,s)|^{p'} ds]^{p/p'} dx]^{1/p} < \infty.$$

So $q_p = |A - \tilde{A}|_{L^p} \le \tau_p$. Now Corollary 7.4 and Lemma 7.2 imply the following theorem.

Theorem 8.1. *Let $\tilde{A}$ be defined by (8.1), and let condition (8.2) hold. Then*

$$\sigma(\tilde{A}) \subseteq \{z \in \mathbb{C} : |\phi(x) - z| \le z_p(A, \tau_p) \le \delta_p(\hat{V}, \tau_p), x \in [0,1]\},$$

where $z(A, \tau_p)$ is the unique positive root of the equation

$$\frac{b_p \tau_p}{z} \exp[\frac{M_p^p(\hat{V})}{z^p}] = 1$$

and

$$\delta_p(\hat{V}, \tau_p) := \begin{cases} \tau_p b_p e^{1/p} & \text{if } M_p(\hat{V}) \le b_p \tau_p (e/p)^{1/p}, \\ \frac{2^{1/p} M_p(\hat{V})}{\ln^{1/p}\left(\frac{(pe)^{1/p} M_p(\hat{V}) e}{\tau_p b_p}\right)} & \text{if } M_p(\hat{V}) > b_p \tau_p (e/p)^{1/p}. \end{cases}$$

This result is sharp; if $\tau_p = 0$, then we have $\sigma(\tilde{A}) = \{z \in \mathbb{C} : z = \phi(x), x \in [0,1]\}$. From Theorem 8.1 it follows the following corollary.

Corollary 8.2. *Under condition (8.2), the (upper) spectral radius $r_s(\tilde{A})$ of the operator $\tilde{A}$ defined by (8.1) satisfies the inequalities*

$$r_s(\tilde{A}) \le \sup_x |\phi(x)| + z_p(A, \tau_p) \le \sup_x |\phi(x)| + \delta_p(\hat{V}, \tau_p).$$

If, in addition,

$$\inf_x |\phi(x)| > z_p(A, \tau_p),$$

then the lower spectral radius $r_{low}(\tilde{A}) := \inf |\sigma(\tilde{A})|$ satisfies the inequality

$$r_{low}(\tilde{A}) \ge \inf_x |\phi(x)| - z_p(A, \tau_p).$$

Moreover, if

$$\inf_x |\phi(x)| > \delta_p(\hat{V}, \tau_p),$$

then $r_{low}(\tilde{A}) \geq \inf_x |\phi(x)| - \delta_p(\hat{V}, \tau_p)$.

8.2. **The case $p = 1$.** Now suppose that

$$\int_0^1 \sup_{s \in [0,1]} |k(x,s)| dx < \infty. \tag{8.3}$$

Then according to (7.1)

$$|A - \tilde{A}|_{L^1} \leq \tau_1 := \int_0^1 \sup_{s \in [0,x]} |k(x,s)| dx.$$

Now Lemma 7.2 and Corollary 7.4 imply the following theorem.

Theorem 8.3. *Let $\tilde{A}$ be defined by (8.1), and let condition (8.3) hold. Then*

$$\sigma(\tilde{A}) \subseteq \{z \in \mathbb{C} : |\phi(x) - z| \leq z_1(A, \tau_1) \leq \delta_1(\hat{V}, \tau_1), x \in [0,1]\},$$

where $z(A, \tau_1)$ is the unique positive root of the equation

$$\frac{\tau_1}{z} \exp[\frac{M_1(\hat{V})}{z}] = 1$$

and

$$\delta_1(\hat{V}, \tau_1) := \begin{cases} \tau_1 e & \text{if } M_1(\hat{V}) \leq \tau_1 e, \\ \frac{2M_1(\hat{V})}{\ln\left(\frac{M_1(\hat{V})e}{\tau_1}\right)} & \text{if } M_1(\hat{V}) > \tau_1 e. \end{cases}$$

From this theorem we obtain the following result.

Corollary 8.4. *Let condition (8.3) hold. Then*

$$r_s(\tilde{A}) \leq \sup_x |\phi(x)| + z_1(A, \tau_1) \leq \sup_x |\phi(x)| + \delta_1(\hat{V}, \tau_1).$$

If, in addition,

$$\inf_x |\phi(x)| > z_1(A, \tau_1),$$

then $r_{low}(\tilde{A}) \geq \inf_x |\phi(x)| - z_1(A, \tau_1)$. *Moreover, if*

$$\inf_x |\phi(x)| > \delta_1(\hat{V}, \tau_1),$$

then $r_{low}(\tilde{A}) \geq \inf_x |\phi(x)| - \delta_1(A, \tau_1)$.

9. Multiplicative representations for resolvents of operators in a Banach space

In this section we suggest a representation for the resolvent of a P_t-triangular operator. We begin with the following lemma.

Lemma 9.1. *Let a sequence of compact quasi-nilpotent operators $V_n \in \mathcal{B}(\mathcal{X})$ $(n = 1, 2, \dots)$ in the operator norm converge to an operator V. Then V is compact and quasi-nilpotent.*

Proof. From the approximation property $\mathcal{X}$ it follows that the uniform limit of compact operators is compact. So V is compact. Assume that V has an eigenvalue $\lambda_0 \neq 0$. Since V is compact, λ_0 is an isolate point of $\sigma(V)$. Consequently, there is a circle L which contains λ_0 and does not contain zero and other points of $\sigma(V)$. We have

$$\|R_z(V_n)\| - \|R_z(V)\| \leq \|R_z(V_n) - R_z(V)\| \leq \|V - V_n\| \|R_z(V_n)\| \|R_z(V)\|.$$

Hence, for sufficiently large n,

$$\|R_z(V_n)\| \leq \frac{\|R_z(V)\|}{1 - \|V - V_n\| \|R_z(V_n)\| \|R_z(V)\|}.$$

Therefore $\|R_z(V_n)\|$ are uniformly bounded on L. Since V_n $(n = 1, 2, \ldots)$ are quasi-nilpotent operators, we have

$$\int_L R_z(V_n) dz = 0$$

and

$$\int_L R_z(V) dz = \int_L (R_z(V) - R_z(V_n)) dz = \int_L R_z(V)(V - V_n) R_z(V_n) dz \to 0.$$

So $\int_L R_z(V) dz = 0$, but this is impossible, since that integral represents the eigen-projection corresponding to λ_0. This contradiction proves the lemma. $\square$

This lemma is well-known for operators in a Hilbert space [3, Lemma 17.1].

Let $\psi(.)$ be a scalar function defined and bounded on a finite real segment $[a, b]$, and let Q_t be a resolution of the identity defined on $[a, b]$, $B \in \mathcal{B}(\mathcal{X})$, and let

$$M_n = \overrightarrow{\prod_{1 \leq k \leq n}} (I + \psi(t_k) B \Delta Q_k) \qquad \begin{pmatrix} \Delta Q_k = Q_{t_k} - Q_{t_{k-1}} \\ a = t_0 < t_1 < \cdots < t_n = b \end{pmatrix}$$
$$:= (I + \psi(t_1) B \Delta Q_1)(I + \psi(t_2) B \Delta Q_2) \cdots (I + \psi(t_n) B \Delta Q_n)$$

If the sequence of operators M_n converges in the operator norm to some $M \in \mathcal{B}(\mathcal{X})$, then M is called *the right multiplicative integral.* We write

$$M = \int_{[a,b]}^{\rightarrow} (I + \psi(t) B dQ_t).$$

Lemma 9.2. *Let V be a compact quasi-nilpotent operator in $\mathcal{X}$ having an invariant CMPF P_t $(a \leq t \leq b)$, and let*

$$\sum_{k=1}^{n} \Delta P_k V \Delta P_k \to 0 \text{ as } n \to \infty \qquad \begin{pmatrix} P_k = P_{t_k} - P_{t_{k-1}} \\ a = t_0 < t_1 < \cdots < t_n = b \end{pmatrix} \tag{9.1}$$

in the operator norm. Then

$$(I - V)^{-1} = \int_{[a,b]}^{\rightarrow} (I + V dP_t).$$

Proof. Put
$$V_n = \sum_{k=1}^{n} P_{t_{k-1}} V \Delta P_k.$$
Since
$$V = \sum_{j=1}^{n} \Delta P_j V \sum_{k=1}^{n} \Delta P_k = \sum_{k=1}^{n} \sum_{j=1}^{k} \Delta P_j V \Delta P_k,$$
we have
$$V - V_n = \sum_{k=1}^{n} \Delta P_k V \Delta P_k.$$
Due to (9.1), the sequence of the operators V_n tends to V in the operator norm since V is compact. Besides, V_n is nilpotent, since with the notation $P_k = P_{t_k}$, we have
$$V_n^n = V_n^n P_n = V_n^{n-1} P_{n-1} V_n = V_n^{n-2} P_{n-2} V_n P_{n-1} V_n = \cdots = V_n P_1 \cdots V_n P_{n-1} V_n = 0.$$
Due to [14, Lemma 3.14],
$$(I - V_n)^{-1} = \overrightarrow{\prod_{2 \le k \le n}} (I + V_n \Delta P_k).$$
That lemma is proved in a Hilbert space, but in $\mathcal{X}$ the proof is similar. In addition, $(I - V_n)^{-1} \to (I - V)^{-1}$ in the operator norm; see [7, p. 585, Lemma VII.6.3]. Hence the required result follows. □

Theorem 9.3. *Let A be a P_t-triangular operator. Then*
$$(A - \lambda I)^{-1} = \int_{[a,b]} \frac{dP_\tau}{\phi(\tau) - \lambda} \; \overrightarrow{\int_{[a,b]}} \left(I + \frac{V dP_t}{\phi(t) - \lambda} \right) \qquad (\lambda \notin \sigma(A)),$$
where V is the nilpotent part of A and $\phi(.)$ is its P_t-diagonal function.

Proof. Due to Corollary 3.4, $V(D - \lambda I)^{-1}$ is quasi-nilpotent. By the previous lemma
$$(I + V(D - \lambda I)^{-1})^{-1} = \overrightarrow{\int_{[a,b]}} (I + V(D - \lambda I)^{-1} dP_t).$$
According to (3.9) we have
$$(A - \lambda I)^{-1} = (D - \lambda I)^{-1} \overrightarrow{\int_{[a,b]}} (I + V(D - \lambda I)^{-1} dP_t) \qquad (\lambda \notin \sigma(A)). \tag{9.2}$$
But
$$(D - \lambda I)^{-1} = \int_{[a,b]} \frac{dP_\tau}{\phi(\tau) - \lambda} \text{ and therefore } (D - \lambda I)^{-1} dP_t = \frac{1}{\phi(t) - \lambda} dP_t.$$
Now (9.2) yields the required result. □

10. Resolvents of nonselfadjoint operators in a Hilbert space

In this section in the case of a Hilbert space we obtain the multiplicative representation for the resolvents of nonselfadjoint operators having maximal chains and Schatten–von Neumann Hermitian components. Besides we do not assume that the chain is continuous. For operators with continuous maximal chains and real spectra, the results of the present section are similar to Theorem 9.3.

10.1. **Auxiliary results.** Let $\mathcal{H}$ be a separable Hilbert space with a scalar product $(.,.)$ and the unit operator $I = I_{\mathcal{H}}$; $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators in $\mathcal{H}$ and SN_p ($p \in [1,\infty)$) is the Schatten–von Neumann ideal of compact operators K in $\mathcal{H}$ with the finite norm $N_p(K) = (\text{trace}\,(KK^*)^{p/2})^{1/p}$. Let $A \in \mathcal{B}(\mathcal{H})$, and let

$$\Im A := (A - A^*)/2i \in SN_p \qquad (1 < p < \infty). \tag{10.1}$$

The notion of the chain of projections in $\mathcal{H}$ is defined as above. It is only required that *the projections are orthogonal*. Namely, for two orthogonal projections P_1 and P_2 in $\mathcal{H}$, we write $P_1 < P_2$ if $P_1\mathcal{H} \subset P_2\mathcal{H}$. A set $\mathcal{P}$ of orthogonal projections in $\mathcal{H}$ containing at least two orthogonal projections is called a chain, if from $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ it follows that either $P_1 < P_2$ or $P_1 > P_2$. For two chains $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 < \mathcal{P}_2$ if from $P \in \mathcal{P}_1$ it follows that $P \in \mathcal{P}_2$. In this case we say that $\mathcal{P}_1$ precedes $\mathcal{P}_2$. The chain that precedes only itself is called *a maximal chain*. Let $P^-, P^+ \in \mathcal{P}$, and $P^- < P^+$. A gap (P_+, P_-) is defined as in the case of a Banach space (see Section 2 of the present paper). Besides, $\dim\,(P_+\mathcal{H}) \ominus (P_-\mathcal{H})$ is the dimension of the gap. An orthogonal projection P in $\mathcal{H}$ is called a limit projection of a chain $\mathcal{P}$ if exists a sequence $P_k \in \mathcal{P}$ $(k = 1, 2, \dots)$ which strongly converges to P. A chain is said to be closed if it contains all its limit projections.

We need the following result proved in [15, Proposition XX.4.1, p. 478] and [3, Theorem II.14.1].

Theorem 10.1. *A chain in $\mathcal{H}$ is maximal if and only if it is closed, contains 0 and I, and all its gaps (if they exist) are one dimensional.*

So in the Hilbert space the definition of a maximal chain coincides with Definition 2.1, provided the projections in $\mathcal{H}$ are orthogonal. Any compact operator in $\mathcal{H}$ has a maximal invariant chain [17, Theorem I.3.1].

Let $\psi(P)$ be a scalar valued function of $P \in \mathcal{P}$. If for some operator T and any $\epsilon > 0$, there is a partitioning $\mathcal{P}_n$ $(n = 2, 3, \dots)$ of $\mathcal{P}$ of the form

$$0 = P_0 < P_1 < P_2 < \cdots < P_n = I \qquad (P_1, \dots, P_n \in \mathcal{P})$$

such that

$$\|T - \sum_{k=1}^{n} \psi(P_k)\Delta P_k\| < \epsilon \qquad (P_k \in \mathcal{P}_n, \Delta P_k = P_k - P_{k-1}),$$

then T is called *the integral in the Shatunovsky sense*. We write

$$T = \int_{\mathcal{P}} \psi(P)dP.$$

Besides ψ is called $\mathcal{P}$-integrable. For more details about such types integrals see [4], [15, Chapters XX and XI], and references therein.

Furthermore, let F be a function defined on a maximal chain $\mathcal{P}$ with values in $\mathcal{B}(\mathcal{H})$, and let $\mathcal{P}_n$ be a partitioning of $\mathcal{P}$. Put

$$M_n(\mathcal{P}) := \prod_{1\le k\le n}^{\rightarrow} (1+\Delta F(P_k)) := (1+\Delta F(P_1))(I+\Delta F(P_2))\cdots(I+\Delta F(P_n)).$$

Here $\Delta F(P_k) = F(P_k) - F(P_{k-1})$. If, for some $M \in \mathcal{B}(\mathcal{H})$ and any $\epsilon > 0$, there is a partitioning $\mathcal{P}_n$ of $\mathcal{P}$ such that $\|M - M_n(\mathcal{P})\| < \epsilon$, then M is called *the right multiplicative integral along chain* $\mathcal{P}$. We write

$$M = \int_{\mathcal{P}}^{\rightarrow} (I + dF(P)).$$

For more details about the multiplicative integrals and their applications see [15, p. 493]. In particular, if ψ is a scalar function defined and bounded on $\mathcal{P}$ and $B \in \mathcal{B}(\mathcal{H})$, then the integral

$$\int_{\mathcal{P}}^{\rightarrow} (I + \psi(P)BdP)$$

means the limit (if it exists) in the operator norm of the products

$$\prod_{1\le k\le n}^{\rightarrow} (I + \psi(P_k)B\Delta P_k).$$

Assume that for a given $A \in \mathcal{B}(\mathcal{H})$ there is a bounded scalar $\mathcal{P}$-integrable function $\phi(P)$ of $P \in \mathcal{P}$ such that

$$A = \int_{\mathcal{P}} \phi(P)dP + V, \tag{10.2}$$

where V is a quasi-nilpotent operator with invariant maximal chain $\mathcal{P}$.

Theorem 10.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be represented by (10.2), and let the regular set of A be simply connected. Then*

$$(A-\lambda I)^{-1} = \int_{\mathcal{P}} \frac{dQ}{\phi(Q)-\lambda} \int_{\mathcal{P}}^{\rightarrow} \left(I + \frac{VdP}{\phi(P)-\lambda}\right) \qquad (\lambda \notin \sigma(A)). \tag{10.3}$$

For the proof see [14, Theorem 9.9].

Remark 10.3. In [14] instead of the condition on the regular set it is assumed that the spectrum lies on an unclosed Jordan curve, but the proof is the same.

Furthermore, let $\phi(P)$ be real valued; then it is said to be nondecreasing, if $\phi(P) \le \phi(P_1)$ for $P < P_1$.

Under condition (10.1) representation (10.2) is valid with a nondecreasing function ϕ and $V \in SN_p$, provided $\sigma(A)$ is real, see [14, Corollary 8.4]. Thus from Theorem 10.2 we get the following result.

Corollary 10.4. *Assume that condition (10.1) holds and that $\sigma(A)$ is real. Then there are a maximal chain $\mathcal{P}$ and a nondecreasing function $\phi(P)$ defined on $\mathcal{P}$ such that (10.3) holds. Moreover, $V \in SN_p$.*

Take into account that $VdP = PVdP$ and $PV^* = PV^*P$. Let us check that $PV^*dP = 0$. Indeed, for any $P_1 > P$, we have $PV^*(P_1 - P) = PV^*P(P_1 - P) = PV^*(P - P) = 0$. If $P_1 - P$ is a gap, then $dP = P_1 - P$ and $PV^*dP = 0$. If P is a point of continuity of the chain, then letting $P_1 \to P$ $(P_1 > P)$, we get the required result.

Therefore, $VP = P(V - V^*)dP = 2iP(\Im V)dP$ $(\Im V = (V - V^*)/2i)$. If $\sigma(A)$ is real, then $\Im V = \Im A$. Now Corollary 10.4 implies the following result.

Corollary 10.5. *Let condition* (10.1) *hold, and let $\sigma(A)$ be real. Then there are a maximal chain $\mathcal{P}$ and a nondecreasing scalar valued function $\phi(P)$ defined on $\mathcal{P}$ such that*

$$(A - \lambda I)^{-1} = \int_{\mathcal{P}} \frac{dQ}{\phi(Q) - \lambda} \int_{\mathcal{P}}^{\rightarrow} \left(I + \frac{2iP(\Im A)dP}{\phi(P) - \lambda} \right) \qquad (\lambda \notin \sigma(A)). \qquad (10.4)$$

Let $A = A^*$. Then from (10.4) we have

$$(A - \lambda I)^{-1} = \int_{\mathcal{P}} \frac{dQ}{\phi(Q) - \lambda}.$$

Thus, Corollary 10.5 generalizes the representation for the resolvent of a selfadjoint operator.

Furthermore, since $\Im A$ is compact, according to Theorem I.5.2 from [16], the nonreal spectrum of A consists of no more than countable set of points which are normal; that is, they are the isolated eigenvalues having finite multiplicities.

Denote by $\mathcal{E}$ the closed linear hull of all the root vectors of A corresponding to nonreal eigenvalues $\hat{\lambda}_k(A)$. Choose in each root subspace a Jordan basis. Then we obtain vectors h_k for each of which either $Ah_k = \hat{\lambda}_k(A)h_k$, or $Ah_k = \hat{\lambda}_k(A)h_k + h_{k+1}$. Orthogonalizing the system $\{h_k\}$, we obtain the (orthonormal) Schur basis $\{e_k\}$ of the triangular representation,

$$Ae_k = a_{1k}e_1 + a_{2k}e_2 + \cdots + a_{kk}e_k \qquad (k = 1, 2, \ldots)$$

with $a_{kk} = \hat{\lambda}_k(A)$ (see [16, Section II.6]). Besides, $\mathcal{E}$ is an invariant subspace of A. Let $Z_{\mathcal{E}}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{E}$, and let $C = AZ_{\mathcal{E}} = Z_{\mathcal{E}}AZ_{\mathcal{E}}$. So $\sigma(C)$ consists of the nonreal spectrum of A. Denote $M = (I - Z_{\mathcal{E}})A(I - Z_{\mathcal{E}})$ and $W = Z_{\mathcal{E}}A(I - Z_{\mathcal{E}})$. Since $(I - Z_{\mathcal{E}})AZ_{\mathcal{E}} = 0$, we have

$$A = (Z_{\mathcal{E}} + (I - Z_{\mathcal{E}}))A(Z_{\mathcal{E}} + (I - Z_{\mathcal{E}})) = C + M + W,$$

So on $Z_{\mathcal{E}}\mathcal{H} \oplus (I - Z_{\mathcal{E}})\mathcal{H}$, A is represented by the matrix

$$A = \begin{pmatrix} C & W \\ 0 & M \end{pmatrix}.$$

Besides $\sigma(A) = \sigma(M) \cup \sigma(C)$ and $\sigma(M)$ is real. Take into account that $Ce_k = Ae_k$. So

$$Ce_k = a_{1k}e_1 + a_{2k}e_2 + \cdots + a_{kk}e_k = (D_C + V_C)e_k, \qquad (10.5)$$

where $D_C e_k = \lambda_k(C)e_k = \hat{\lambda}_k(A)e_k \qquad (k = 1, 2, \ldots)$,

$$V_C e_k = a_{1k}e_1 + a_{2k}e_2 + \cdots + a_{k-1,k}e_k \qquad (k = 2, 3, \ldots), \qquad \text{and } Ve_1 = 0.$$

Note that $M - M^* = (I - Z_{\mathcal{E}})(A - A^*)(I - Z_{\mathcal{E}})$ and $C - C^* = Z_{\mathcal{E}}(A - A^*)Z_{\mathcal{E}}$. So if condition (10.1) holds, then $N_p(M - M^*) \le N_p(A - A^*)$ and $N_p(C - C^*) \le N_p(A - A^*)$. Hence,

$$N_p(W - W^*) \le N_p(A - A^*) + N_p(M - M^*) + N_p(C - C^*) < \infty.$$

10.2. **Resolvents of infinite triangular matrices.** Let $C \in \mathcal{B}(\mathcal{H})$ be an operator represented in the orthonormal basis $\{e_k\}$ by an upper triangular matrix (a_{jk}) as in (10.5). Put

$$\tilde{P}_j = \sum_{k=1}^{j} (., e_k)e_k \ (j = 1, 2, \ldots), \ \tilde{P}_0 = 0, \ \Delta\tilde{P}_j = (., e_j)e_j, \ D_C = \sum_{k=1}^{\infty} \lambda_k(C)\Delta\tilde{P}_k$$

and $\tilde{P}_k$ $(k = 1, 2, \ldots)$ is the maximal chain of the invariant projections of operator C. We have

$$\Delta\tilde{P}_j V_C \Delta\tilde{P}_k = 0 \qquad \text{if } j \ge k.$$

Let X_k $(k = 1, 2, \ldots)$ be a sequence of bonded operators in $\mathcal{H}$. Put

$$\Pi_m := \prod_{1\le k\le m}^{\rightarrow} (I + X_k) := (I + X_1)(I + X_2)\ldots(I + X_m).$$

If there exists a limit Π_∞ of Π_m as $m \to \infty$ in the operator norm, we write

$$\Pi_\infty := \prod_{1\le k\le\infty}^{\rightarrow} X_k.$$

Lemma 10.6. *Let C be represented by* (10.5). *Then*

$$R_\lambda(C) = R_\lambda(D_C) \prod_{2\le k\le\infty}^{\rightarrow} \left(I + \frac{V_C\Delta\tilde{P}_k}{\lambda - \lambda_k(C)}\right) \qquad (\lambda \notin \sigma(C)). \tag{10.6}$$

Besides,

$$R_\lambda(D_C) = \sum_{j=1}^{\infty} \frac{\Delta\tilde{P}_k}{\lambda_k(C) - \lambda}.$$

Proof. We have

$$R_\lambda(C) = (D_C + V_C - \lambda I)^{-1} = R_\lambda(D_C)(I + V_C R_\lambda(D_C))^{-1}. \tag{10.7}$$

With $C_n = C\tilde{P}_n$, $D_n = D_C\tilde{P}_n$, and $V_n = V_C\tilde{P}_n$ we obtain

$$V_C R_\lambda(D_C) - V_n R_\lambda(D_n) = V_C R_\lambda(D_C)(I - \tilde{P}_n) \to 0$$

in the operator norm. But due to Theorem 2.10.2 of [12],

$$(I + V_n R_\lambda(D_n))^{-1} = \prod_{2\le k\le n}^{\rightarrow} \left(I + \frac{V_n\Delta\tilde{P}_k}{\lambda - \lambda_k(C)}\right) \qquad (\lambda \notin \sigma(C)).$$

Consequently,

$$(I + V_C R_\lambda(D_C))^{-1} = \prod_{2\le k\le\infty}^{\rightarrow} \left(I + \frac{V_C\Delta\tilde{P}_k}{\lambda - \lambda_k(C)}\right) \qquad (\lambda \notin \sigma(C)).$$

Hence (10.7) implies (10.6). □

Making use of the relations $V_C \Delta \tilde{P}_k = \tilde{P}_{k-1} V_C \Delta \tilde{P}_k$, $\tilde{P}_{k-1} V_C^* \tilde{P}_k = 0$, and

$$V_C \Delta \tilde{P}_k = 2i \tilde{P}_{k-1} \Im V_C \Delta \tilde{P}_k = 2i \tilde{P}_{k-1} (\Im C - \Im \lambda_k(C)) \Delta \tilde{P}_k$$

from the preceding lemma, we have

$$R_\lambda(C) = R_\lambda(D_C) \prod_{2 \le k \le \infty}^{\rightarrow} \left(I + \frac{2i \tilde{P}_{k-1} (\Im C - \Im \lambda_k(C)) \Delta \tilde{P}_k}{\lambda - \lambda_k(C)} \right) \qquad (\lambda \notin \sigma(C)).$$

10.3. **Resolvents of operators with non-real spectra.** We need the following lemma.

Lemma 10.7. *Let $A \in \mathcal{B}(\mathcal{H})$ have a nontrivial invariant projection P and a simply connected regular set. Then, for all its regular points λ, with the notation $\overline{P} = I - P$ one has*

$$P R_\lambda(A) P = R_\lambda(A) P = (AP - \lambda P)^{-1} \tag{10.8}$$

and

$$\overline{P} R_\lambda(A) \overline{P} = \overline{P} R_\lambda(A) = (\overline{P} A - \lambda \overline{P})^{-1}, \tag{10.9}$$

where $(AP - \lambda P)^{-1}$ is the inverse to $AP - \lambda P$ in subspace $P\mathcal{H}$ and $(\overline{P} A - \lambda \overline{P})^{-1}$ is the inverse to $\overline{P} A - \lambda \overline{P}$ in subspace $\overline{P}\mathcal{H}$.

Proof. Since

$$A^k P = A^{k-1} PAP = \cdots = A^{k-2} PAPAP = \cdots = PAP \ldots APAP$$

we can write $A^k P = (AP)^k P$ and $A^k P = P A^k P$. For a sufficiently large λ we have

$$R_\lambda(A) P = - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} A^k P = - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} P A^k P = P R_\lambda(A) P$$

and

$$R_\lambda(A) P = - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} A^k P = - \sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} (AP)^k P = (AP - \lambda P)^{-1}.$$

Extending this relation to all regular values, we obtain (10.8).

Similarly we can prove relations (10.9). □

Lemma 10.8. *Let $A \in \mathcal{B}(\mathcal{H})$ have a nontrivial invariant projection P and a simply connected regular set. Then*

$$R_\lambda(A) = (AP - \lambda P)^{-1} + (\overline{P} A - \lambda \overline{P})^{-1} - (AP - \lambda P)^{-1} A (\overline{P} A - \lambda \overline{P})^{-1}.$$
$$(\overline{P} = I - P, \lambda \notin \sigma(A))$$

Proof. Since $\overline{P} A \overline{P} = \overline{P} A$ and $\overline{P} A P = 0$, we have

$$A = PAP + PA\overline{P} + \overline{P} A \overline{P} = AP + PA\overline{P} + \overline{P} A. \tag{10.10}$$

Now, we check the equality

$$R_\lambda(A) = \Psi(\lambda) \qquad (\lambda \notin \sigma(A)), \tag{10.11}$$

where

$$\Psi(\lambda) := R_\lambda(A)P + \overline{P}R_\lambda(A) - R_\lambda(A)PA\overline{P}R_\lambda(A).$$

Indeed, by the previous lemma, we have $(A-\lambda I)P(A-\lambda I)^{-1}P = (A-I\lambda)(A-I\lambda)^{-1}P = P$ and

$$\overline{P}(A-I\lambda)\overline{P}R_\lambda(A)\overline{P} = \overline{P}(A-\lambda I)R_\lambda(A)\overline{P} = \overline{P}.$$

Taking into account (10.10), we obtain

$$\begin{aligned}(A-\lambda I)\Psi(\lambda) &= [(A-\lambda I)P + PA\overline{P} + \overline{P}(A-I\lambda)]\Psi(\lambda)\\ &= P - (A-\lambda I)PR_\lambda(A)PA\overline{P}R_\lambda(A) + \overline{P} + PA\overline{P}R_\lambda(A)\\ &= I - PA\overline{P}R_\lambda(A) + PA\overline{P}R_\lambda(A) = I.\end{aligned}$$

Similarly, $\Psi(\lambda)(A-I\lambda) = I$. So (10.11) is true. This proves the lemma. □

Since

$$\begin{aligned}(I - A(AP-\lambda P)^{-1})(I - A(\overline{P}A-\lambda\overline{P})^{-1}) =& I - A(AP-\lambda P)^{-1} - A(\overline{P}A-\lambda\overline{P})^{-1}\\ &+ A(AP-\lambda P)^{-1}A(\overline{P}A-\lambda\overline{P})^{-1},\end{aligned}$$

applying the previous lemma, we can write

$$I - AR_\lambda(A) = I - A(AP-\lambda P)^{-1} - A(\overline{P}A-\lambda\overline{P})^{-1} + (AP-\lambda P)^{-1}A(\overline{P}A-\lambda\overline{P})^{-1}.$$

We thus arrive at the following corollary.

Corollary 10.9. *Under the hypothesis of Lemma 10.8, one has*

$$I - AR_\lambda(A) = (I - A(AP-\lambda P)^{-1})(I - A(\overline{P}A-\lambda\overline{P})^{-1}) \qquad (\lambda \notin \sigma(A)).$$

Let us apply this corollary to relation (10.2) with $P = Z_\mathcal{E}$ and $\hat{Z}_\mathcal{E} = I - Z_\mathcal{E}$. Recall that $Z_\mathcal{E}$ is the projection of A into the invariant subspace corresponding to the nonreal spectrum, $C = AZ_\mathcal{E} = Z_\mathcal{E}AZ_\mathcal{E}$, and $M = \overline{Z}_\mathcal{E}A = \overline{Z}_\mathcal{E}A\overline{Z}_\mathcal{E}$. Then we have

$$\begin{aligned}I - AR_\lambda(A) &= (I - A(AZ_\mathcal{E} - \lambda Z_\mathcal{E})^{-1})(I - A(\hat{Z}_\mathcal{E}A - \lambda\hat{Z}_\mathcal{E})^{-1})\\ &= (I - A(C-\lambda Z_\mathcal{E})^{-1})(I - A(M - \lambda\hat{Z}_\mathcal{E}))^{-1}).\end{aligned}$$

But due to Lemma 10.6,

$$(C-\lambda Z_\mathcal{E})^{-1} = (D_C - \lambda Z_\mathcal{E})^{-1}\prod_{2\le k\le\infty}^{\rightarrow}\left(I + \frac{V_C\Delta\tilde{P}_k}{\lambda - \hat{\lambda}_k(A)}\right) \qquad (\lambda\notin\sigma(A)).$$

Recall that $\hat{\lambda}_k(A) = \lambda_k(C)$ are nonreal eigenvalues of A with their multiplicities. Due to Corollary 10.4,

$$(M-\lambda\hat{Z}_\mathcal{E})^{-1} = (D_M - \lambda\hat{Z}_\mathcal{E})^{-1}\int_{\mathcal{P}_M}^{\rightarrow}\left(I + \frac{V_M dP}{\phi(P)-\lambda}\right) \qquad (\lambda\notin\sigma(M)),$$

where $\mathcal{P}_M$ is the maximal chain of M, $\phi(P)$ is a no-decreasing scalar function, and $V_M \in SN_p$ is the nilpotent part of M. Now Corollary 10.9 implies the following theorem.

Theorem 10.10. *Let condition (10.1) hold. Then for all $\lambda \notin \sigma(A)$ one has*

$$I - AR_\lambda(A) = \left(I - A(D_C - \lambda Z_{\mathcal{E}})^{-1} \prod_{2\le k\le\infty}^{\rightarrow} \left(I + \frac{V_C \Delta \tilde{P}_k}{\lambda - \hat{\lambda}_k(A)}\right)\right)$$
$$\times \left(I - A(D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1} \int_{\mathcal{P}_M}^{\rightarrow} \left(I + \frac{V_M dP}{\phi(P) - \lambda}\right)\right).$$

Besides,

$$(D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1} = \int_{\mathcal{P}_M} \frac{dQ}{\phi(Q) - \lambda}$$

and

$$(D_C - \lambda Z_{\mathcal{E}})^{-1} = \sum_{j=1}^{\infty} \frac{\Delta \tilde{P}_k}{\hat{\lambda}_k(A) - \lambda}.$$

Note that

$$I - AR_\lambda(A) = -\lambda R_\lambda(A)$$

If A is normal, then $A = D_C + D_M$ and

$$\begin{aligned} -\lambda R_\lambda(A) &= (I - D_C(D_C - \lambda Z_{\mathcal{E}})^{-1}(I - D_M(D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1} \\ &= I - D_C(D_C - \lambda Z_{\mathcal{E}})^{-1} - D_M(D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1} \\ &\quad + D_C(D_C - \lambda Z_{\mathcal{E}})^{-1} D_M(D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1} \\ &= I - D_C(D_C - \lambda Z_{\mathcal{E}})^{-1} - D_M(D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1} \\ &= -\lambda[(D_C - \lambda Z_{\mathcal{E}})^{-1} + (D_M - \lambda \hat{Z}_{\mathcal{E}})^{-1}] \end{aligned}$$

For $\lambda \neq 0$ we have the standard representation

$$R_\lambda(A) = \int_{\mathcal{P}_M} \frac{dQ}{\phi(Q) - \lambda} + \sum_{j=1}^{\infty} \frac{\Delta \tilde{P}_k}{\hat{\lambda}_k(A) - \lambda}.$$

For regular $\lambda = 0$ we obtain this result by small perturbation.

References

1. R. Bhatia, *Perturbation bounds for matrix eigenvalues*, Reprint of the 1987 original. Classics in Applied Mathematics, 53. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.
2. L. de Branges, *Some Hilbert spaces of analytic functions II*, J. Math. Anal. Appl. **11** (1965), 44–72.
3. M. S. Brodskii, *Triangular and Jordan representations of linear operators*, Translated from the Russian by J. M. Danskin. Translations of Mathematical Monographs, Vol. 32. American Mathematical Society, Providence, R.I., 1971.
4. V. M. Brodskii, I. C. Gohberg, and M. G. Krein, *General theorems on triangular representations of linear operators and multiplicative representations of their characteristic functions*, (Russian) Funkcional. Anal. i Priložen. **3** (1969), no. 4, 1–27.
5. J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995.
6. N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, With the assistance of William G. Bade and Robert G. Bartle Interscience Publishers John Wiley & Sons New York-London 1963

7. N. Dunford and J. T. Schwartz, *Linear operators. Part III: Spectral operators*, With the assistance of William G. Bade and Robert G. Bartle. Pure and Applied Mathematics, Vol. VII. Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1971.
8. S. P. Eveson, *Norms of iterates of Volterra operators on* L^2. J. Operator Theory **50** (2003), no. 2, 369–386.
9. S. P. Eveson, *Asymptotic behaviour of iterates of Volterra operators on* $L^p(0,1)$. Integral Equations Operator Theory **53** (2005), 331--341.
10. M. I. Gil', *On the representation of the resolvent of a nonselfadjoint operator by the integral with respect to a spectral function*, Soviet Math. Dokl. **14** (1973), 1214–1217.
11. M. I. Gil', *One estimate for resolvents of nonselfadjoint operators which are "near" to selfadjoint and to unitary ones*, Math. Notes **33** (1983), 81–84.
12. M. I. Gil', *Operator functions and localization of spectra*, Lecture Notes in Mathematics, 1830. Springer-Verlag, Berlin, 2003.
13. M. I. Gil', *Kronecker's products and Kronecker's sums of operators*, Contributions in mathematics and engineering, 205–253, Springer, [Cham], 2016.
14. M. I. Gil, *Operator functions and operator equations*, World Scientific, New Jersey, 2017.
15. I. C. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of linear operators. Vol. II*, Operator Theory: Advances and Applications, 63. Birkhäuser Verlag, Basel, 1993.
16. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I. 1969
17. I. C. Gohberg and M. G. Krein, *Theory and applications of Volterra operators in Hilbert space*, Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 24 American Mathematical Society, Providence, R.I. 1970.
18. D. Kershaw, *Operator norms of powers of the Volterra operator*, J. Integral Equations Appl. **11** (1999), no. 3, 351–362.
19. N. Lao and R. Whitley, *Norms of powers of the Volterra operator*, Integral Equations Operator Theory **27** (1997), no. 4, 419–425
20. G. Little and J. B. Reade, *Estimates for the norm of the nth indefinite integral*, Bull. London Math. Soc. **30** (1998), no. 5, 539–542.
21. A. Montes-Rodriguez, J. Sanchez-Alvarez, and J. Zemanek, *Uniform Abel–Kreiss boundedness and the extremal behaviour of the Volterra operator*, Proc. London Math. Soc. (3)*91* (2005), no. 3, 761–788.
22. A. Pietsch, *Eigenvalues and s-numbers*, Cambridge Studies in Advanced Mathematics, 13. Cambridge University Press, Cambridge, 1987.
23. H. Radjavi and P. Rosenthal, *Invariant subspaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 77. Springer-Verlag, New York-Heidelberg, 1973.
24. L. Sakhnovich, $(S+N)$*-triangular operators: spectral properties and important examples*, Math. Nachr. **289** (2016), no. 13, 1680–1691.
25. G. W. Stewart and J. G. Sun, *Matrix perturbation theory*, Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA, 1990.
26. J.C. Strikwerda and B. A. Wade, *A survey of the Kreiss matrix theorem for power bounded families of matrices and its extensions*, Linear operators (Warsaw, 1994), 339–360, Banach Center Publ., 38, Polish Acad. Sci. Inst. Math., Warsaw, 1997.
27. B. Thorpe, *The norm of powers of the indefinite integral operator on* $(0,1)$, Bull. London Math. Soc. **30** (1998), no. 5, 543–548.

Department of Mathematics, Ben Gurion University of the Negev, P.0. Box 653, Beer-Sheva 84105, Israel.
E-mail address: gilmi@bezeqint.net