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SEMICIRCULAR-LIKE AND SEMICIRCULAR LAWS ON BANACH $*$-PROBABILITY SPACES INDUCED BY DYNAMICAL SYSTEMS OF THE FINITE ADELE RING $A_{\mathbb{Q}}$

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Abstract. Starting from the finite adele ring $A_{\mathbb{Q}}$, we construct semigroup dynamical systems of $A_{\mathbb{Q}}$, acting on certain C^*-probability spaces. From such dynamical-systematic C^*-probability spaces, we construct Banach-space operators acting on the C^*-probability spaces and corresponding Banach $*$-probability spaces. In particular, we are interested in Banach-space operators whose free distributions are the (weighted-)semicircular law(s).

1. Introduction

The main purpose of this paper is to construct-and-study *semicircular-like* and *semicircular* elements induced by *crossed product algebras* of a *semigroup dynamical system* of the *finite adele ring* $A_{\mathbb{Q}}$.

To do that, we study (i) *functional analysis* on the $*$-*algebra* $\mathcal{M}_{\mathcal{P}}$, consisting of *measurable functions* on the finite adele ring $A_{\mathbb{Q}}$, in terms of "nontraditional" senses of *free probability theory* and its *Hilbert-space representation* and the corresponding C^*-*algebra* $M_{\mathcal{P}}$, (ii) a system of C^*-*probability spaces* $M_{\mathcal{P}}^{p,j}$ of $M_{\mathcal{P}}$, for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, where $\mathcal{P}$ is the set of all *primes* in the set $\mathbb{N}$ of all *natural numbers* and $\mathbb{Z}$ is the set of all *integers*, (iii) *Banach-space operators* acting on the C^*-subalgebras $\mathfrak{S}_{\mathcal{P}}$ of $M_{\mathcal{P}}$ generated by certain *projections* for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, (iv) certain semigroup dynamical system of the σ-algebra $\sigma(A_{\mathbb{Q}})$ acting on

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arbitrarily fixed *unital* C^**-probability spaces* (A, ψ) and the corresponding *crossed product algebras* of the systems, (v) functional analysis on the structures of (iv) and (vi) establish-and-study *Banach* $*$*-probability spaces* $\mathfrak{LS}_A(p, j)$ for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then, our semicircular-like laws and the semicircular law are obtained "locally" for a prime p. Such "local" semicircular-like laws and the semicircular law will be globalized over primes under Adelic analysis.

Our main results not only illustrate relations between primes and Banach-space operators, but also provide connections among *number theory*, *representation theory*, *operator theory*, *operator algebra theory*, and *dynamical system theory*, via *free probability theory*.

1.1. **Remark: NonTraditional vs. Traditional.** In the beginning of this section, we mentioned about "nontraditional" senses of *free probability theory*. Note that the (traditional) free probability theory is a *noncommutative* operator-algebraic version of *measure theory* and *statistics* (e.g., [3, 4, 5, 9, 11, 29] through [18, 33] through [37]). However, such noncommutative free probability well-covers the cases where given algebras are commutative, even though the freeness on such algebras is trivial. In other words, the techniques and concepts in (noncommutative) free probability are applicable to commutative algebras (if we are not interested in freeness on them).

Recall that A is a noncommutative (topological $*$-)algebra, and φ is a (bounded, or unbounded) linear functional on A; then the pair (A, φ) is said to be a (*noncommutative*) *free probability space*. In the following text, even though a given algebra B is commutative, if ψ is a well-defined linear functional on B, then we will say the pair (B, ψ) is a free probability space "nontraditionally," and use notations, techniques, and concepts of free probability theory for studying statistical data of operators of B in terms of ψ, as in the earlier works of [7, 8, 10, 11].

With help of such (nontraditional) free-probability-theoretic approaches, we consider (traditional) free-probability-theoretic structures of Banach $*$-algebras under *crossed products for dynamical systems*.

1.2. **Background and Motivation.** The relations between primes and operators have been studied in various different approaches (e.g., [2, 12, 6, 13, 15, 14, 19, 20, 21, 22, 23, 25, 26, 32]). For instance, in [9], we considered free-probabilistic structures on a *Hecke algebra* $\mathcal{H}\left(GL_2(\mathbb{Q}_p)\right)$ for *primes* p, where $GL_2(X)$ are the *general linear groups* in the *matricial sets* $M_2(X)$ over X.

Independently, in [11], by using number-theoretic information from a certain nontraditional C^**-probability space* induced by a p*-adic number field* $\mathbb{Q}_p$, for arbitrarily fixed $p \in \mathcal{P}$, we established and studied *weighted-semicircular elements* in a certain *Banach* $*$*-probability space* (implying p-adic number-theoretic data). Such weighted-semicircular elements naturally generate *semicircular elements*.

In [7], we extended the (weighted-)semicircularity of [11] in a *free product Banach* $*$*-probability space* over primes. The main results of [7] demonstrate that indeed the (weighted-)semicircularity of [11] are well-determined as traditional free-probabilistic objects.

By globalizing the main results of [7], we could construct weighted-semicircular and semicircular elements from the finite adele ring $A_{\mathbb{Q}}$ in [10], by applying nontraditional free-probability-theoretic approaches of [8]. In this paper, we generalize the main results of [10] in the traditional sense of free probability theory under dynamical systems.

1.3. **Overview and Main Results.** In sections 2, we briefly introduce backgrounds and motivations of our proceeding works.

Our nontraditional free-probabilistic model on the $*$-algebra $\mathcal{M}_{\mathcal{P}}$ is established from *Adelic analysis*, and the statistical data on $\mathcal{M}_{\mathcal{P}}$ are considered in section 3. Then, a suitable Hilbert-space representation of our free-probabilistic model of $\mathcal{M}_{\mathcal{P}}$, preserving the statistical data implying number-theoretic information, is constructed in section 4. Under representation, the corresponding C^*-algebra $M_{\mathcal{P}}$ is defined.

In sections 5 and 6, functional analysis on the C^*-algebra $M_{\mathcal{P}}$ is considered by putting a system of linear functionals dictated by the Adelic integration under free-probability-theoretic language. In particular, distributions of generating operators of $M_{\mathcal{P}}$ are studied by computing moments of them. We in particular focus on certain projections of $M_{\mathcal{P}}$. C^*-subalgebras $\mathfrak{S}_p$ and the corresponding nontraditional C^*-probability spaces generated by the projections are observed.

In sections 7, 8, and 9, we construct semigroup dynamical systems of the σ-algebra $\sigma\left(A_{\mathbb{Q}}\right)$, regarding $\sigma\left(A_{\mathbb{Q}}\right)$ as a semigroup with set-intersection and the corresponding crossed product C^*-algebras of the dynamical systems. The traditional free-probabilistic structures are constructed, and free-distributional data on them is studied for our main purpose. Especially, Theorem 9.2(and Corollaries 9.3 and 9.4 illustrates how our dynamical systems affects the original free probability determined by (weighted-)semicircular law(s).

In section 10, we consider weighted-semicircular elements and semicircular elements induced by certain Banach-space operators in Banach $*$-probability spaces of section 9, locally for fixed primes. See Theorem 10.2 and Corollary 10.3.

Finally, in sections 11 and 12, we globalize the weighted-semicircularity and the semicircularity of section 10. See Theorems 11.1 and 11.2 and Theorem 12.2.

2. Preliminaries

In this section, we briefly mention about backgrounds for our proceeding works. See also [8, 9, 17, 16, 33] for motivations from number theory.

2.1. **Free Probability.** Readers can review analytic-and-combinatorial *free probability theory* from [31, 37] (also see, e.g., [30, 34, 35, 36]). *Free probability* is understood as the *noncommutative* operator-algebraic version of classical *measure theory* and *statistics*. The classical *independence* is replaced to the *freeness*, by replacing *measures* on sets to *linear functionals* on algebras. It has various applications not only in pure mathematics (e.g., [29, 27, 28, 24, 18]), but also in applied fields (for example, see [1] through [11]). In particular, we will use combinatorial approach of *Speicher* (e.g., [31]).

In the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed.

2.2. **p-Adic Analysis on $\mathbb{Q}_p$.** In this section, we briefly review *p-adic calculus* on the $*$-algebras $\mathcal{M}_p$ of measurable functions on p-adic number fields $\mathbb{Q}_p$ for $p \in \mathcal{P}$. For more about p-adic analysis, see [33]. Also, for applications of p-adic and Adelic analysis, see [15, 14, 25, 26].

For a fixed prime $p \in \mathcal{P}$, one can define the *p-norm* $|.|_p$ on the set $\mathbb{Q}$ of all *rational numbers* by

$$|x|_p = \left|ap^k\right|_p = \tfrac{1}{p^k},$$

whenever x is factorized by ap^k for some $a \in \mathbb{Q}$, $k \in \mathbb{K}$. For instance,

$$\left|\tfrac{4}{3}\right|_2 = \left|\tfrac{1}{3} \cdot 2^2\right|_2 = \tfrac{1}{2^2} = \tfrac{1}{4},$$
$$\left|\tfrac{4}{3}\right|_3 = \left|4 \cdot 3^{-1}\right|_3 = \tfrac{1}{3^{-1}} = 3,$$

and

$$\left|\tfrac{4}{3}\right|_q = \left|\tfrac{4}{3} \cdot q^0\right| = \tfrac{1}{q^0} = 1 \qquad \text{for all } q \in \mathcal{P} \setminus \{2, 3\}.$$

The p-adic number field $\mathbb{Q}_p$ is defined to be the maximal $|.|_p$-norm completion in $\mathbb{Q}$. So, $\mathbb{Q}_p$ forms a *Banach space* in $\mathbb{Q}$ under $|.|_p$.

Remark that all elements x of $\mathbb{Q}_p$ are uniquely expressed by

$$x = \sum_{k=-N}^{\infty} x_k p^k, \qquad \text{with } x_k \in \{0, 1, \ldots, p-1\},$$

for some $N \in \mathbb{N}$, decomposed by

$$x = \sum_{k=-N}^{-1} x_k p^k + \sum_{l=0}^{\infty} x_l p^k.$$

If $x = \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$, then x is said to be a *p-adic integer*. Note that any p-adic integer x satisfies $|x|_p \leq 1$. The subset

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$$

consisting of all p-adic integers is called the *unit disk of* $\mathbb{Q}_p$.

Under the p-adic addition and the p-adic multiplication of [35], $\mathbb{Q}_p$ forms a well-defined *ring*, algebraically.

Let us understand this *Banach ring* $\mathbb{Q}_p$ as a *measure space*,

$$\mathbb{Q}_p = (\mathbb{Q}_p,\ \sigma(\mathbb{Q}_p),\ \mu_p),$$

where $\sigma(\mathbb{Q}_p)$ is the *σ-algebra of* $\mathbb{Q}_p$, consisting of all *μ_p-measurable subsets*, where μ_p is a left-and-right additive-invariant *Haar measure on* $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1.$$

If we define

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{2.1}$$

for all $k \in \mathbb{Z}$, satisfying $U_0 = \mathbb{Z}_p$, then these μ_p-measurable subsets U_k satisfy

$$\mathbb{Q}_p = \underset{k \in \mathbb{Z}}{\cup} U_k,$$

$$\mu_p\left(U_k\right) = \frac{1}{p^k}, \qquad \text{for all } k \in \mathbb{Z}, \tag{2.2}$$

and

$$\cdots \subset U_2 \subset U_1 \subset U_0 \subset U_1 \subset U_2 \subset \cdots$$

(e.g., see [33]).

Define now subsets ∂_k of $\mathbb{Q}_p$ by

$$\partial_k = U_k \setminus U_{k+1}, \qquad \text{for all } k \in \mathbb{Z}, \tag{2.3}$$

where U_k are in the sense of (2.1).

We call such μ_p-measurable subsets ∂_k of (2.3), the *k-th boundaries* (*of* U_k) in $\mathbb{Q}_p$ for all $k \in \mathbb{Z}$. By (2.2) and (2.3), one obtains that

$$\mathbb{Q}_p = \underset{k\in\mathbb{Z}}{\sqcup} \partial_k,$$

where $\sqcup$ means the *disjoint union* and

$$\mu_p\left(\partial_k\right) = \mu_p\left(U_k\right) - \mu_p\left(U_{k+1}\right) = \frac{1}{p^k} - \frac{1}{p^{k+1}} \qquad \text{for all } k \in \mathbb{Z}. \tag{2.4}$$

Now, let $\mathcal{M}_p$ be an *algebra*,

$$\mathcal{M}_p = \mathbb{C}\left[\left\{\chi_S : S \in \sigma\left(\mathbb{Q}_p\right)\right\}\right], \tag{2.5}$$

where χ_S are the usual *characteristic functions* of S. So, $f \in \mathcal{M}_p$, if and only if

$$f = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S \chi_S \qquad \text{with } t_S \in \mathbb{C},$$

where $\sum$ is the *finite sum.*

Then this algebra $\mathcal{M}_p$ of (2.5) forms a $*$-*algebra over* $\mathbb{C}$, equipped with the *adjoint*

$$\left(\sum_{S\in\sigma(G_p)} t_S \chi_S\right)^* \overset{def}{=} \sum_{S\in\sigma(G_p)} \overline{t_S} \chi_S,$$

where $\overline{t_S}$ are the *conjugates of* t_S in $\mathbb{C}$.

If $f \in \mathcal{M}_p$, then one can define the *p-adic integral* φ_p *of* f by

$$\varphi_p(f) \overset{def}{=} \int_{\mathbb{Q}_p} f \; d\mu_p = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S \mu_p(S). \tag{2.6}$$

Note that, by (2.4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{2.7}$$

satisfying

$$\varphi_p\left(\chi_S\right) = \int_{\mathbb{Q}_p} \chi_S \; d\mu_p = \int_{\mathbb{Q}_p} \sum_{j\in\Lambda_S} \chi_{S\cap\partial_j} \; d\mu_p$$

by (2.4)

$$= \sum_{j\in\Lambda_S} \mu_p\left(S \cap \partial_j\right)$$

by (2.6)

$$\leq \sum_{j\in\Lambda_S} \mu_p\left(\partial_j\right) = \sum_{j\in\Lambda_S} \left(\tfrac{1}{p^j} - \tfrac{1}{p^{j+1}}\right),$$

by (2.4); that is,

$$\int_{\mathbb{Q}_p} \chi_S \, d\mu_p \leq \sum_{j \in \Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right),$$

for all $S \in \sigma(\mathbb{Q}_p)$, where Λ_S is in the sense of (2.7).

More precisely, one can get the following proposition.

Proposition 2.1 (See [8]). *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that*

$$\begin{gathered} 0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \qquad \text{for all } j \in \Lambda_S, \\ \int_{\mathbb{Q}_p} \chi_S \, d\mu_p = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right), \end{gathered} \tag{2.8}$$

where Λ_S is in the sense of (2.7).

2.3. **The adele Ring and the Finite adele Ring.** In this section, we introduce the *adele ring* $\mathbb{A}_{\mathbb{Q}}$ and the *finite adele ring* $A_{\mathbb{Q}}$. For more about the adele ring $\mathbb{A}_{\mathbb{Q}}$ and the corresponding *Adelic analysis*, see [33].

Definition 2.2. Let $\mathcal{P}_\infty = \mathcal{P} \cup \{\infty\}$, and identify $\mathbb{Q}_\infty$ with the Banach field $\mathbb{R}$ equipped with the usual-(distance-)metric topology. Let $\mathbb{A}_{\mathbb{Q}}$ be a set

$$\mathbb{A}_{\mathbb{Q}} = \left\{ (x_p)_{p \in \mathcal{P}_\infty} \;\middle|\; \begin{array}{c} x_p \in \mathbb{Q}_p, \qquad \text{for all} p \in \mathcal{P}_\infty, \\ \text{where only finitely many } x_p\text{'s are in } \mathbb{Q}_p \setminus \mathbb{Z}_p, \\ \text{but all other } x_p\text{'s are contained in } \mathbb{Z}_p \text{ of } \mathbb{Q}_p \end{array} \right\}, \tag{2.9}$$

equipped with the addition $(+)$

$$(x_p)_{p \in \mathcal{P}_\infty} + (y_p)_{p \in \mathcal{P}_\infty} = (x_p + y_p)_{p \in \mathcal{P}_\infty}, \tag{2.10}$$

and the multiplication $(\cdot)$

$$(x_p)_{p \in \mathcal{P}_\infty} (y_p)_{p \in \mathcal{P}_\infty} = (x_p y_p)_{p \in \mathcal{P}_\infty}, \tag{2.11}$$

where the entries $x_p + y_p$ of (2.10) and the entries $x_p y_p$ of (2.11) , respectively, are the p-adic addition and the p-adic multiplication on $\mathbb{Q}_p$ (e.g., [33]) for all $p \in \mathcal{P}$, and where $x_\infty + y_\infty$, and $x_\infty y_\infty$ are the usual $\mathbb{R}$-addition and the usual $\mathbb{R}$-multiplication, respectively.

The adele ring $\mathbb{A}_{\mathbb{Q}}$ is equipped with the product topology of the p-adic-norm topologies for $\mathbb{Q}_p$'s, for all $p \in \mathcal{P}$, and the usual-metric topology of $\mathbb{Q}_\infty = \mathbb{R}$, providing the $\mathbb{A}_{\mathbb{Q}}$-norm $|.|_{\mathbb{Q}}$,

$$\left| (x_p)_{p \in \mathcal{P}_\infty} \right|_{\mathbb{Q}} = \prod_{p \in \mathcal{P}_\infty} |x_p|_p, \tag{2.12}$$

where $|.|_p$ are the p-adic norms on $\mathbb{Q}_p$, for all $p \in \mathcal{P}$, and $|.|_\infty$ is the usual absolute value $|.|$ on $\mathbb{R} = \mathbb{Q}_\infty$.

From the above definition, the set $\mathbb{A}_{\mathbb{Q}}$ of (2.9) forms a *ring* algebraically, equipped with the binary operations (2.10) and (2.11), and this ring $\mathbb{A}_{\mathbb{Q}}$ is a *Banach space* under its $|.|_{\mathbb{Q}}$-norm of (2.12). Thus, the set $\mathbb{A}_{\mathbb{Q}}$ of (2.9) forms a *Banach ring* induced by the family

$$\mathcal{Q} = \{\mathbb{Q}_p\}_{p\in\mathcal{P}} \cup \{\mathbb{Q}_\infty = \mathbb{R}\}.$$

Suppose that $X = (x_p)_{p\in\mathcal{P}_\infty} \in \mathbb{A}_{\mathbb{Q}}$, and assume that there are $p_1, \ldots, p_N \in \mathcal{P}_\infty$, for some $N \in \mathbb{N}$, such that

$$x_{p_l} \in \mathbb{Q}_p \setminus \mathbb{Z}_p,$$

for $l = 1, \ldots, N$, and

$$x_q \in \mathbb{Z}_q$$

for $q \in \mathcal{P}_\infty \setminus \{p_1, \ldots, p_N\}$. Then, by (2.9) and (2.12),

$$\begin{aligned} |X|_{\mathbb{Q}} &= \left(\prod_{l=1}^{N} |x_{p_l}|_{p_l}\right)\left(\prod_{q\in\mathcal{P}_\infty\setminus\{p_1,\ldots,p_N\}} |x_q|_q\right) \\ &\leq \left(\prod_{l=1}^{N} |x_{p_l}|_{p_l}\right)\cdot 1 = \left(\prod_{l=1}^{N} |x_{p_l}|_{p_l}\right) < \infty. \end{aligned}$$

Equivalent to the definition (2.9), the adele ring $\mathbb{A}_{\mathbb{Q}}$ is in fact the *weak-direct product of* $\mathcal{Q}$, expressed by

$$\mathbb{A}_{\mathbb{Q}} = \prod_{p\in\mathcal{P}_\infty}{}' \mathbb{Q}_p \tag{2.13}$$

(e.g., [33]), where Π' means the *weak-direct product of topological rings.*

Definition 2.3. Let $\mathbb{A}_{\mathbb{Q}}$ be the adele ring (2.9) or (2.13). Define a set $A_{\mathbb{Q}}$ by

$$A_{\mathbb{Q}} = \left\{ (x_p)_{p\in\mathcal{P}} \;\middle|\; \begin{array}{c} x_p \in \mathbb{Q}_p, \text{ for all } p \in \mathcal{P}, \\ \text{and } \left(0,\ (x_p)_{p\in\mathcal{P}}\right) \in \mathbb{A}_{\mathbb{Q}} \end{array} \right\}, \tag{2.14}$$

equipped with the inherited binary operations (2.10) and (2.11) of $\mathbb{A}_{\mathbb{Q}}$, under subspace topology of the norm topology (2.12). Then this topological ring $A_{\mathbb{Q}}$ is said to be the finite adele ring.

By (2.13) and (2.14), one can conclude that

$$A_{\mathbb{Q}} = \prod_{p\in\mathcal{P}}{}' \mathbb{Q}_p, \tag{2.15}$$

where Π' is the weak-direct product of topological rings.

By (2.15) and [35], the finite adele ring $A_{\mathbb{Q}}$ of (2.14) can be regarded as a measure space equipped with its measure,

$$\mu = \underset{p\in\mathcal{P}}{\times} \mu_p, \tag{2.16}$$

on the *σ-algebra* $\sigma(A_{\mathbb{Q}})$ *of* $A_{\mathbb{Q}}$, which is the *product σ-algebra of* $\{\sigma(\mathbb{Q}_p)\}_{p\in\mathcal{P}}$, where (2.16) means the *product measure.*

So, one can define the $*$-algebra $\mathcal{M}_{\mathcal{P}}$ by

$$\mathcal{M}_{\mathcal{P}} = \mathbb{C}\left[\{\chi_Y : Y \in \sigma\left(A_{\mathbb{Q}}\right)\}\right]. \tag{2.17}$$

Remark here that $Y \in \sigma\left(A_{\mathbb{Q}}\right)$ if and only if

$$Y = \underset{p \in \mathcal{P}}{\Pi} S_p, \qquad \text{with } S_p \in \sigma\left(\mathbb{Q}_p\right),$$

by (2.15) (under additional conditions; See (2.21) below for details).
By (2.17), $f \in \mathcal{M}_{\mathcal{P}}$ if and only if

$$f = \sum_{Y \in \sigma(A_{\mathbb{Q}})} s_Y \chi_Y, \qquad \text{with } s_Y \in \mathbb{C}, \tag{2.18}$$

where $\sum$ means the *finite sum*.
Thus, one obtains the *(finite-)Adelic integration* φ *of* $f \in \mathcal{M}_{\mathcal{P}}$ by

$$\varphi(f) \overset{def}{=} \int_{A_{\mathbb{Q}}} f \, d\mu = \sum_{Y \in \sigma(A_{\mathbb{Q}})} t_Y \mu\left(Y\right), \tag{2.19}$$

whenever f is in the sense of (2.18) in $\mathcal{M}_{\mathcal{P}}$, where μ is the product measure (2.16).

Definition 2.4. Let $\mathcal{M}_{\mathcal{P}}$ be in the sense of (2.17), and let φ be the linear functional (2.19) on $\mathcal{M}_{\mathcal{P}}$. Then the pair

$$\left(\mathcal{M}_{\mathcal{P}},\ \varphi\right) \tag{2.20}$$

is called the finite-Adelic ($*$-)probability space (under the nontraditional sense in section 1.1).

Recall that our finite adele ring $A_{\mathbb{Q}}$ is a weak-direct product of $\{\mathbb{Q}_p\}_{p \in \mathcal{P}}$ by (2.15), and hence, $Y \in \sigma\left(A_{\mathbb{Q}}\right)$ if and only if there exist $N \in \mathbb{N}$ and $p_1, \ldots, p_N \in \mathcal{P}$ such that

$$Y = \underset{p \in \mathcal{P}}{\Pi} S_p, \text{ where } S_p \in \sigma\left(\mathbb{Q}_p\right), \text{ with } S_p = \begin{cases} S_p \subset \mathbb{Q}_p & \text{if } p \in \{p_1, \ldots, p_N\}, \\ \mathbb{Z}_p & \text{otherwise,} \end{cases} \tag{2.21}$$

for all $p \in \mathcal{P}$.
Thus, if $Y \in \sigma\left(A_{\mathbb{Q}}\right)$ satisfying (2.21), one has

$$\varphi\left(\chi_Y\right) = \int_{A_{\mathbb{Q}}} \chi_Y \, d\mu = \int_{A_{\mathbb{Q}}} \chi_{\underset{p \in \mathcal{P}}{\Pi} S_p} \, d\mu$$

$$= \underset{p \in \mathcal{P}}{\Pi} \left(\int_{\mathbb{Q}_p} \chi_{S_p} d\mu_p\right)$$

by (2.16)

$$= \left(\prod_{l=1}^{N} \mu_{p_l}\left(S_p\right)\right) \left(\underset{q \in \mathcal{P} \setminus \{p_1, \ldots, p_N\}}{\Pi} \mu_q\left(\mathbb{Z}_q\right)\right)$$

by (2.21)

$$= \prod_{l=1}^{N} \mu_{p_l}(S_p) = \prod_{l=1}^{N} \left(\varphi_{p_l}\left(\chi_{S_{p_l}}\right)\right), \tag{2.22}$$

since $\mu_q(\mathbb{Z}_q) = 1$, for all $q \in \mathcal{P}$, where φ_p are the p-*adic integrations* (2.6) for all $p \in \mathcal{P}$.

Proposition 2.5. *Let $Y \in \sigma(A_{\mathbb{Q}})$, satisfying (2.21), and let $\chi_Y \in (\mathcal{M}_{\mathcal{P}}, \varphi)$. Then*

$$\varphi(\chi_Y^n) = \prod_{l=1}^{N} \left(\sum_{j \in \Lambda_{S_{p_l}}} r_j^{S_{p_l}} \left(\frac{1}{p_l^j} - \frac{1}{p_l^{j+1}} \right) \right), \tag{2.23}$$

for all $n \in \mathbb{N}$, where $r_j^{S_{p_l}}$ are in the sense of (2.8) for all $j \in \Lambda_{S_{p_l}}$ and for all $l = 1, \ldots, N$.

Proof. The formula (2.23) is obtained by (2.8), (2.21), and (2.22). □

Notice that, by the construction (2.17) of $\mathcal{M}_{\mathcal{P}}$, one can conclude that

$$\mathcal{M}_{\mathcal{P}} \overset{\text{Alg}}{=} \prod_{p \in \mathcal{P}}' \mathcal{M}_p, \tag{2.24}$$

where Π' means the *weak-direct product of $*$-algebras*, where "$\overset{\text{Alg}}{=}$" means "being pure-algebraic $*$-isomorphic." The isomorphism (2.24) holds because of (2.15) (and (2.21)).

Proposition 2.6. *Let $(\mathcal{M}_{\mathcal{P}}, \varphi)$ be the finite-Adelic probability space. Then*

$$\mathcal{M}_{\mathcal{P}} = \prod_{p \in \mathcal{P}}' \mathcal{M}_p \text{ and } \varphi = \prod_{p \in \mathcal{P}} \varphi_p, \tag{2.25}$$

where $\mathcal{M}_p$ and φ_p, respectively, are in the sense of (2.5) and (2.6) for all $p \in \mathcal{P}$.

Proof. The $*$-isomorphism theorem of $\mathcal{M}_{\mathcal{P}}$ in (2.25) is obtained by (2.24). The equivalence (2.25) for φ is guaranteed by (2.16) and (2.23). □

3. Analysis on $\mathcal{M}_{\mathcal{P}}$

Let $(\mathcal{M}_{\mathcal{P}}, \varphi)$ be the finite-Adelic probability space. By abusing notation, one may / can re-write the relation (2.25) by

$$(\mathcal{M}_{\mathcal{P}}, \varphi) = \prod_{p \in \mathcal{P}}' (\mathcal{M}_p, \varphi_p). \tag{3.1}$$

Recall that, in [7, 11], we call the pairs $(\mathcal{M}_p, \varphi_p)$, the p-*adic probability spaces* for all $p \in \mathcal{P}$.

Proposition 3.1. *Let* $Y_1, \ldots, Y_n \in \sigma(A_{\mathbb{Q}})$, *and let* $\chi_{Y_l} \in (\mathcal{M}_{\mathcal{P}}, \varphi)$ *for* $l = 1, \ldots, n$ *for some* $n \in \mathbb{N}$. *Then there exist a unique "finite" subset* P_o *of* $\mathcal{P}$ *and* $X_p \in \sigma(\mathbb{Q}_p)$, *for all* $p \in P_o$, *such that*

$$\varphi\left(\prod_{l=1}^{n} \chi_{Y_l}\right) = \prod_{p \in P_o}\left(\sum_{j \in \Lambda_{X_p}} r_j^{X_p}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\right), \tag{3.2}$$

where $r_j^{X_p}$ *are in the sense of* (2.8) *and* Λ_{X_p} *are in the sense of* (2.7).

Proof. The formula (3.2) is obtained by (2.23) and (3.1). See [8] for more details. □

The above formula (3.2) characterizes the (free) distributions of generating elements of our finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$. As a corollary of (3.2), one obtains the following result.

Corollary 3.2. *Let* $Y_l = \prod_{p \in \mathcal{P}} S_p^l \in \sigma(A_{\mathbb{Q}})$, *for* $l = 1, \ldots, n$ *for some* $n \in \mathbb{N}$, *where*

$$S_p^l = \begin{cases} \partial_{k_{p_t,l}}^{p_t} & \text{if } p_t \in \{p_{l,1}, \ldots, p_{l,N_l}\}, \\ \mathbb{Z}_p & \text{otherwise}, \end{cases} \tag{3.3}$$

where $\partial_{k_p}^p$ *are the* k_p*-th boundaries for* $k_p \in \mathbb{Z}$ *in* $\mathbb{Q}_p$, *for* $p \in \mathcal{P}$, *and where* $k_{p_t,1}, \ldots, k_{p_t,N_l} \in \mathbb{Z}$, *for all* $l = 1, \ldots, n$, *all* $p \in \mathcal{P}$. *Now, let*

$$P_o = \bigcup_{l=1}^{n} \{p_{l,1}, \ldots, p_{l,N_l}\} \text{ in } \mathcal{P}.$$

Then one obtains that

$$\varphi\left(\prod_{l=1}^{n} \chi_{Y_l}\right) = \prod_{p \in P_o} \beta_{k_p}\left(\frac{1}{p^{k_p}} - \frac{1}{p^{k_p+1}}\right), \tag{3.4}$$

where p^{k_p} *are in the sense of* (3.3), *where*

$$\beta_{k_p} = \begin{cases} 1 & \text{if } \bigcap_{l=1}^{n} S_p^l \neq \varnothing, \\ 0 & \text{otherwise}, \end{cases}$$

for all $p \in P_o$.

Proof. The formula (3.4) is shown by (3.2), under the condition (3.4). See [8] for details. □

Let $Y_l \in \sigma(A_{\mathbb{Q}})$ be in the sense of (3.3), for $l = 1, \ldots, n$, and let

$$X = \prod_{l=1}^{n} \chi_{Y_l} \in (\mathcal{M}_{\mathcal{P}}, \varphi). \tag{3.5}$$

Definition 3.3. Such elements X of (3.5) are called *boundary-product elements* of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$. Let X be a boundary-product element (3.5) of $(\mathcal{M}_{\mathcal{P}}, \varphi)$. Assume that Y_l are in the sense of (3.3) and that P_o is in the sense of the above corollary. Assume further that, for all $p \in P_o$, the corresponding integers k_p are "non-negative"; that is,

$$k_p \geq 0 \qquad \text{for all } p \in P_o. \tag{3.6}$$

Then this boundary-product element X is said to be a (+)-boundary element of $(\mathcal{M}_{\mathcal{P}},\ \varphi)$.

Let $\phi : \mathbb{N} \to \mathbb{C}$ be the *Euler totient function* defined by an *arithmetic function*,

$$\phi(n) = \left| \left\{ k \in \mathbb{N} \,\middle|\, \begin{array}{c} 1 \leq k \leq n \\ \gcd(n,\ k) = 1 \end{array} \right\} \right|, \tag{3.7}$$

for all $n \in \mathbb{N}$, where $|S|$ mean the *cardinalities of sets* S, and gcd means the *greatest common divisor*. It is well-known that

$$\phi(n) = n \left(\prod_{p \in \mathcal{P},\ p \mid n} \left(1 - \frac{1}{p}\right)\right), \qquad \text{for all } n \in \mathbb{N}, \tag{3.8}$$

where "$p \mid n$" means "p divides n" or "n is divisible by p." For instance,

$$\phi(p) = p - 1 = p\left(1 - \tfrac{1}{p}\right),$$

for all $p \in \mathcal{P}$, by (3.7) and (3.8).

Remark that the Euler totient function ϕ is a *multiplicative arithmetic function* in the sense that

$$\phi(n_1 n_2) = \phi(n_1)\phi(n_2), \qquad \text{whenever } \gcd(n_1, n_2) = 1 \text{ for all } n_1, n_2 \in \mathbb{N}. \tag{3.9}$$

Theorem 3.4. *Let X be a (+)-boundary element (3.5) of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ satisfying (3.6). Then there exist a finite subset P_o of $\mathcal{P}$ and*

$$K_o = \{k_p \in \mathbb{N}_0 : p \in P_o\} \text{ of } \mathbb{Z},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, such that

$$n_X = \prod_{p \in P_o} p^{k_p} \in \mathbb{N},$$

$$0 < r_X = \prod_{p \in P_o} \frac{1}{p^{k_p+1}} = \left(\frac{1}{\prod\limits_{p \in P_o} p}\right)\left(\frac{1}{n_X}\right) \leq 1 \text{ in } \mathbb{Q}, \tag{3.10}$$

and

$$\varphi(X) = r_X\ \phi(n_X).$$

Proof. Let X be a (+)-boundary element (3.5) in $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ satisfying (3.6). Then, by (3.4), there exist the subsets P_o of $\mathcal{P}$, and K_o of $\mathbb{Z}$, such that

$$\varphi(X) = \prod_{p \in P_o} \left(\tfrac{1}{p^{k_p}} - \tfrac{1}{p^{k_p+1}}\right),$$

with $k_p \in K_o$, with $k_p \geq 0$ in $\mathbb{Z}$. Observe that

$$\varphi(X) = \prod_{p \in P_o} \tfrac{1}{p^{k_p}}\left(1 - \tfrac{1}{p}\right) = \left(\prod_{p \in P_o^+} \tfrac{1}{p^{k_p+1}}\right) \phi(n_X),$$

where

$$n_X = \left(\underset{p \in P_o}{\Pi} p^{k_p} \right) \in \mathbb{N},$$

by (3.7), (3.8), and (3.9). □

The relations in (3.10) characterize the free distributions of (+)-boundary elements of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}},\ \varphi)$, in terms of the Euler-totient-functional values.

Theorem 3.5. *Let $n \in \mathbb{N}$ be prime-factorized by*

$$n = p_1^{k_{p_1}} p_2^{k_{p_2}} \dots p_N^{k_{p_N}} \text{ in } \mathbb{N}, \tag{3.11}$$

where $k_{p_1}, \dots,\ k_{p_N} \in \mathbb{N}$ for some $N \in \mathbb{N}$. Then there exists a (+)-boundary element X of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}},\ \varphi)$, such that

$$X = \underset{p \in \mathcal{P}}{\Pi} \chi_{Y_p} \in (\mathcal{M}_{\mathcal{P}},\ \varphi), \qquad \text{with } P_o = \{p_1, \dots, p_N\} \subset \mathcal{P}, \tag{3.12}$$

$$K_o = \{k_{p_1}, \dots,\ k_{p_N}\} \subset \mathbb{N}_0,$$

and

$$Y_p = \begin{cases} \partial_{k_p}^p & \text{if } p \in P_o, \\ \mathbb{Z}_p & \text{otherwise,} \end{cases}$$

for all $p \in \mathcal{P}$, where $k_p \in K_o$, satisfying that

$$\phi(n) = n_o n \varphi(X), \qquad \text{with } n_o = \underset{p \in P_o}{\Pi} p \in \mathbb{N}. \tag{3.13}$$

Proof. Let X be a (+)-boundary element (3.12) in $(\mathcal{M}_{\mathcal{P}},\ \varphi)$. Then

$$\varphi(X) = \left(\frac{1}{\underset{p \in P_o}{\Pi} p^{k_p}} \right) \left(\underset{p \in P_o}{\Pi} \frac{1}{p} \right) (\phi(n)) = \left(\tfrac{1}{n}\right) \left(\frac{1}{\underset{p \in P_o^+}{\Pi} p} \right) (\phi(n)),$$

where n is in the sense of (3.11). So, we obtain (3.13). □

The above two theorems illustrate connections between our analysis and number theory.

4. Representation of $(\mathcal{M}_{\mathcal{P}},\ \varphi)$

Let $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ be the finite-Adelic probability space,

$$\begin{aligned} (\mathcal{M}_{\mathcal{P}},\ \varphi) &= \underset{p \in \mathcal{P}}{\Pi'} (\mathcal{M}_p,\ \varphi_p) \\ &= \left(\underset{p \in \mathcal{P}}{\Pi'} \mathcal{M}_p,\ \underset{p \in \mathcal{P}}{\Pi} \varphi_p \right). \end{aligned} \tag{4.1}$$

In [7, 11], we established-and-studied Hilbert-space representations $(\mathfrak{H}_p,\ \alpha^p)$ of the p-adic probability spaces $(\mathcal{M}_p,\ \varphi_p)$ for $p \in \mathcal{P}$. By (4.1), one can construct a Hilbert-space representation of $\mathcal{M}_{\mathcal{P}}$ from the representations,

$$\{(\mathfrak{H}_p,\ \alpha^p) : p \in \mathcal{P}\}$$

of [7, 11]. However, instead of using them, we provide the following equivalent construction.

Define a form

$$[,] : \mathcal{M}_{\mathcal{P}} \times \mathcal{M}_{\mathcal{P}} \to \mathbb{C}$$
$$[f_1, f_2] \overset{def}{=} \int_{A_{\mathbb{Q}}} f_1 f_2^* d\mu = \varphi(f_1 f_2^*) \qquad \text{for all } f_1, f_2 \in \mathcal{M}_{\mathcal{P}}. \tag{4.2}$$

Proposition 4.1. *The form $[,]$ of (4.2) on the finite-Adelic $*$-algebra $\mathcal{M}_{\mathcal{P}}$ is an inner product. Equivalently, the pair $(\mathcal{M}_{\mathcal{P}},\ [,])$ forms an inner product space.*

Proof. The form $[,]$ of (4.2) is an inner product on $\mathcal{M}_{\mathcal{P}}$. See [9] for details. □

From the inner product $[,]$ of (4.2), one can construct the norm $\|.\|$ and the metric $d(,)$, canonically.

Definition 4.2. Let d be the metric induced by the inner product $[,]$ of (4.2). Then the maximal d-metric-topology closure $H_{\mathcal{P}}$ in $\mathcal{M}_{\mathcal{P}}$ is called the finite-Adelic Hilbert space.

By the definition of finite-Adelic Hilbert space $H_{\mathcal{P}}$, our $*$-algebra $\mathcal{M}_{\mathcal{P}}$ is acting on $H_{\mathcal{P}}$ via a linear morphism α from $\mathcal{M}_{\mathcal{P}}$ into the operator algebra $B\,(H_{\mathcal{P}})$ (consisting of all bounded operators on $H_{\mathcal{P}}$ under the operator-norm);

$$\alpha(f)(h) \overset{denote}{=} \alpha_f(h) = fh, \qquad \text{for all } h \in H_{\mathcal{P}}, \tag{4.3}$$

for all $f \in \mathcal{M}_{\mathcal{P}}$. That is, the algebra-action α of (4.3) assigns each element f of $\mathcal{M}_{\mathcal{P}}$ to the *multiplication operator* $\alpha(f) = \alpha_f$ *with its symbol* f in the *operator algebra* $B\,(H_{\mathcal{P}})$ consisting of all bounded linear operators on $H_{\mathcal{P}}$.

Notation 4.1 For convenience, we denote the multiplication operators $\alpha\,(\chi_Y) = \alpha_{\chi_Y}$ simply by α_Y, for all $Y \in \sigma\,(A_{\mathbb{Q}})$ from below.

Observe that, for any $f_1,\ f_2 \in \mathcal{M}_{\mathcal{P}}$,

$$\alpha_{f_1 f_2} = \alpha_{f_1}\alpha_{f_2} \text{ on } H_{\mathcal{P}}, \tag{4.4}$$

and, for any $f \in \mathcal{M}_{\mathcal{P}}$,

$$\alpha_f^* = \alpha_{f^*} \text{ on } H_{\mathcal{P}} \tag{4.5}$$

(e.g., [8, 10]).

Proposition 4.3. *Let $H_{\mathcal{P}}$ be the finite-Adelic Hilbert space, and let α be in the sense of (4.3). Then the pair $(H_{\mathcal{P}},\ \alpha)$ is a Hilbert-space representation of the finite-Adelic $*$-algebra $\mathcal{M}_{\mathcal{P}}$.*

Proof. The linear morphism α of (4.3) is a $*$-homomorphism from $\mathcal{M}_{\mathcal{P}}$ to $B(H_{\mathcal{P}})$, by (4.4) and (4.5). $\square$

By the above proposition, one can understand all elements f of $\mathcal{M}_{\mathcal{P}}$ as a Hilbert-space operator α_f acting on $H_{\mathcal{P}}$.

Definition 4.4. Let $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ be the finite-Adelic probability space, and let $(H_{\mathcal{P}},\ \alpha)$ be the Hilbert-space representation of $\mathcal{M}_{\mathcal{P}}$. Then we call $(H_{\mathcal{P}},\ \alpha)$, the finite-Adelic representation of $(\mathcal{M}_{\mathcal{P}},\ \varphi)$. Define now the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ by a C^*-subalgebra of $B(H_{\mathcal{P}})$,

$$M_{\mathcal{P}} = C^*(\mathcal{M}_{\mathcal{P}}) \overset{def}{=} \overline{\mathbb{C}\left[\alpha\left(\mathcal{M}_{\mathcal{P}}\right)\right]} \tag{4.6}$$

where $\overline{X}$ means the operator-norm-topology closures of subsets X of $B(H_{\mathcal{P}})$.

5. Functional Analysis on $M_{\mathcal{P}}$

Let $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ be the finite-Adelic probability space, and let $M_{\mathcal{P}}$ be the finite-Adelic C^*-algebra (4.6) of $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ under the finite-Adelic representation $(H_{\mathcal{P}},\ \alpha)$. In this section, we will consider functional analysis on the C^*-algebra $M_{\mathcal{P}}$ by constructing a system $\{\varphi_{p,j}\}_{p\in\mathcal{P},\ j\in\mathbb{Z}}$ of linear functionals $\varphi_{p,j}$'s (implying number-theoretic information) on $M_{\mathcal{P}}$.

Define a linear functional $\varphi_{p,j}$ on $M_{\mathcal{P}}$ by

$$\varphi_{p,j}(T) = \left[T\left(\chi_{B_j^p}\right),\ \chi_{B_j^p}\right], \qquad \text{for all } T \in M_{\mathcal{P}}, \tag{5.1}$$

for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, where $[,]$ is the inner product (4.2) on the finite-Adelic Hilbert space $H_{\mathcal{P}}$, and where

$$B_j^p = \underset{q\in\mathcal{P}}{\Pi} Y_q \text{ in } \sigma(A_{\mathbb{Q}})$$

with

$$Y_q = \begin{cases} \partial_j^p & \text{if } q = p, \\ \mathbb{Z}_q & \text{otherwise,} \end{cases}$$

for all $q \in \mathcal{P}$; that is,

$$\chi_{B_j^p} = \chi_{\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\cdots\times \underset{p\text{-th position}}{\partial_j^p} \times \cdots\cdots} \in H_{\mathcal{P}}.$$

All vectors h of the finite-Adelic Hilbert space $H_{\mathcal{P}}$ have their expressions,

$$h = \sum_{Y\in\sigma(A_{\mathbb{Q}})} t_Y\ \chi_Y, \text{ with } t_Y \in \mathbb{C},$$

where $\sum$ is a finite, or an infinite (a limit of finite) sum(s) under the Hilbert-space topology, while every operator T of $M_{\mathcal{P}}$ has its expression,

$$T = \sum_{Y\in\sigma(A_{\mathbb{Q}})} s_Y\ \alpha_Y, \text{ with } s_Y \in \mathbb{C},$$

where $\sum$ is a finite, or an infinite (limit of finite) sum(s) under the C^*-topology for $M_{\mathcal{P}}$, and where α_Y are in the sense of **Notation 4.1**.

So, the linear functionals $\varphi_{p,j}$ of (5.1) are well-defined on $M_{\mathcal{P}}$, and hence, one can get the mathematical pairs,

$$M_{\mathcal{P}}^{p,j} \overset{denote}{=} (M_{\mathcal{P}},\ \varphi_{p,j}) \qquad \text{for all } p \in \mathcal{P} \text{ and } \ j \in \mathbb{Z}. \tag{5.2}$$

Definition 5.1. Let $M_{\mathcal{P}}^{p,j} = (M_{\mathcal{P}},\ \varphi_{p,j})$ be a mathematical pair (5.2), where $\varphi_{p,j}$ is a linear functional (5.1), for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then we call this pair $M_{\mathcal{P}}^{p,j}$, the $(p,\ j)$(-finite)-Adelic C^*-probability space of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, nontraditionally.

In the rest of this section, let us fix $p \in \mathcal{P}$ and $j \in \mathbb{Z}$ and the corresponding (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$ of (5.2).

Consider that, if $\alpha_Y = \alpha_{\chi_Y} \in M_{\mathcal{P}}^{p,j}$, for $Y \in \sigma(A_{\mathbb{Q}})$, satisfying

$$Y = \underset{p\in\mathcal{P}}{\Pi} S_p, \qquad \text{with } S_p \in \sigma(\mathbb{Q}_p),\ \text{where } S_p = \begin{cases} S_p \neq \mathbb{Z}_p & \text{if } p \in P_Y, \\ \mathbb{Z}_p & \text{if } p \notin P_Y, \end{cases} \tag{5.3}$$

for all $p \in \mathcal{P}$, where

$$P_Y = \{p \in \mathcal{P} : S_p \neq \mathbb{Z}_p\}$$

is a finite subset in $\mathcal{P}$.

Then we have

$$\varphi_{p,j}(\alpha_Y) = \left[\alpha_Y(\chi_{B_j^p}),\ \chi_{B_j^p}\right] = \int_{A_{\mathbb{Q}}} \chi_{Y\cap B_j^p}\, d\mu = \mu\left(Y \cap B_j^p\right)$$

$$\begin{cases} \left(\underset{q\in P_Y}{\Pi} \mu_q(S_q \cap \mathbb{Z}_q)\right)\left(\mu_p\left(\mathbb{Z}_p \cap \partial_j^p\right)\right) & \text{if } p \notin P_Y, \\ \\ \left(\underset{q\in P_Y\setminus\{p\}}{\Pi} \mu_q(S_q \cap \mathbb{Z}_q)\right)\left(\mu_p\left(S_p \cap \partial_j^p\right)\right) & \text{if } p \in P_Y. \end{cases} \tag{5.4}$$

Thus, one obtains the following result.

Theorem 5.2. *Let α_Y be an element of the (p, j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$, where $Y \in \sigma(A_{\mathbb{Q}})$ is in the sense of (5.3). Then*

$$\varphi(\alpha_Y^n) = \left(\underset{q\in(P_Y\cup\{p\})\setminus\{p\}}{\Pi} \mu_q(S_q \cap \mathbb{Z}_q)\right)\left(\mu_p\left(S_p \cap \partial_j^p\right)\right) \qquad \textit{for all } n \in \mathbb{N}. \tag{5.5}$$

Proof. Since α_Y is a projection in $M_{\mathcal{P}}$, one has $\alpha_Y^n = \alpha_Y$ for all $n \in \mathbb{N}$. So, the formula (5.5) is obtained by (5.4). Remark that the formula (5.5) is nothing but a re-expression of conditional formulas in (5.4). □

Now, let Y be in the sense of (5.3), with specific condition as follows:

$$Y = \underset{p\in\mathcal{P}}{\Pi} S_p, \qquad \text{with } S_p \in \sigma(\mathbb{Q}_p),\ \text{where } S_p = \begin{cases} \partial_{k_p}^p & \text{if } p \in P_Y \\ \mathbb{Z}_p & \text{if } p \notin P_Y, \end{cases} \tag{5.6}$$

for all $p \in \mathcal{P}$, where $k_p \in \mathbb{Z}$ for $p \in P_Y$, and $P_Y = \{p_1, \ldots, p_N\}$ in $\mathcal{P}$ for some $N \in \mathbb{N}$.

If Y is in the sense of (5.6), then the corresponding element α_Y of the (p,j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$ satisfies that

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \left(\underset{q\in(P_Y\cup\{p\})\setminus\{p\}}{\Pi}\mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(S_p \cap \partial_j^p\right)\right)$$

$$\begin{cases} \left(\underset{q\in P_Y}{\Pi}\mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right)\left(\mu_p\left(\mathbb{Z}_p \cap \partial_j^p\right)\right) & \text{if } p \notin P_Y, \\ \delta_{j,k_p}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\left(\underset{q\in P_Y\setminus\{p\}}{\Pi}\mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right) & \text{if } p \in P_Y, \end{cases} \tag{5.7}$$

by (5.4) and (5.5), for all $n \in \mathbb{N}$, where δ is the *Kronecker delta*. Therefore, one obtains the following special case of (5.5) with help of (5.7).

Corollary 5.3. *Let Y be in the sense of (5.6) in $\sigma(A_{\mathbb{Q}})$, and let α_Y be the corresponding element of the (p,j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$. Then*

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \delta_{j,Y}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\left(\underset{q\in P_Y\setminus\{p\}}{\Pi}\mu_q\left(\partial_{k_q}^q \cap \mathbb{Z}_q\right)\right), \tag{5.8}$$

for all $n \in \mathbb{N}$, where

$$\delta_{j,Y} = \begin{cases} \delta_{j,k_p} & \text{if } p \in P_Y, \\ 0 & \text{if } p \notin P_Y \text{ and } j < 0, \\ 1 & \text{otherwise,} \end{cases}$$

where P_Y is in the sense of (5.6).

Proof. The formula (5.8) is proven by (5.5) and (5.7). □

Note that the operator α_Y of the above corollary is an operator $\alpha(\chi_Y)$ induced by a *boundary-product element* χ_Y of the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}}, \varphi)$, and hence, they provide building blocks for computing (free) distributions of all operators in $M_{\mathcal{P}}$. So, as in section 3, we focus on studying (free-)distributional data of these operators α_Y for investigating statistical data on $M_{\mathcal{P}}$.

Definition 5.4. Let α_Y be the operator of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, generated by the μ-measurable subset Y of (5.6). Then we call such an operator α_Y, a boundary-product operator of $M_{\mathcal{P}}$.

Now, let Y and $P_Y \subset \mathcal{P}$ be in the sense of (5.6). Then P_Y is partitioned by

$$P_Y = P_Y^+ \sqcup P_Y^- \text{ in } \mathcal{P}, \tag{5.9}$$

where

$$P_Y^+ = \{q \in P_Y : k_q \geq 0 \text{ in } \mathbb{Z}\},$$

and

$$P_Y^- = \{q \in P_Y : k_q < 0 \text{ in } \mathbb{Z}\}.$$

Then the formula (5.8) can be refined as follows with help of (5.9).

Theorem 5.5. *Let Y be in the sense of (5.6), inducing a finite subset $P_Y = P_Y^+ \sqcup P_Y^-$ of $\mathcal{P}$, as in (5.9). If $\alpha_Y \in M_{\mathcal{P}}^{p,j}$, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, then*

$$\varphi_{p,j}\left(\alpha_Y^n\right) = \begin{cases} \delta_{j,k_p}\left(\prod_{q \in P_Y}\left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right) & \text{if } P_Y^- = \varnothing, \\ 0 & \text{if } P_Y^- \neq \varnothing, \end{cases} \tag{5.10}$$

for all $n \in \mathbb{N}$, where $\varnothing$ means the empty set.

Proof. The proof of (5.10) is done by (5.8) and (5.9). Indeed, if $P_Y^- \neq \varnothing$ in P_Y, and if $q \in P_Y^-$ in P_Y, and hence, $k_q < 0$ in $\mathbb{Z}$, then

$$\partial_{k_q}^q \cap \mathbb{Z}_q = \varnothing \text{ in } \mathbb{Q}_q,$$

implying

$$\mu\left(Y \cap B_j^p\right) = 0,$$

for "any" $p \in \mathcal{P}$, $j \in \mathbb{Z}$. (Also, see the definition $\delta_{j,Y}$ of (5.8), implying the above discussion.)

So, whenever $P_Y^- \neq \varnothing$, the free moments $\varphi_{p,j}\left(\alpha_Y^n\right)$ vanish. $\square$

The above distributional data (5.10) allow us to have the following result.

Theorem 5.6. *Let α_Y be a boundary operator of the (p,j)-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$ for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. If the subset P_Y of (5.9) satisfies $P_Y^- = \varnothing$, equivalently, if $P_Y = P_Y^+$ in $\mathcal{P}$, then there exist*

$$n_Y = \prod_{q \in P_Y^+ \cup \{p\}} q^{k_q} \in \mathbb{N}_0, \text{ such that } \varphi_{p,j}\left(\alpha_Y^n\right) = \frac{\delta_{j,k_p}}{n_Y n_{p,j}}\phi(n_{p,j}), \qquad \text{for all } n \in \mathbb{N}, \tag{5.11}$$

where

$$n_{p,j} = \prod_{q \in P_Y^+ \cup \{p\}} q \text{ in } \mathbb{N},$$

where ϕ is the Euler totient function.

Proof. Recall that, for a fixed $p \in \mathcal{P}$, $j \in \mathbb{Z}$, if Y is a μ-measurable set (5.6) of $A_{\mathbb{Q}}$, satisfying $P_Y^- = \varnothing$, then the corresponding boundary-product operator α_Y in the finite-Adelic C^*-probability space $M_{\mathcal{P}}^{p,j}$ satisfies

$$\varphi_{p,j}\left(\alpha_Y\right) = \delta_{j,k_p}\left(\prod_{q \in P_Y \cup \{p\}}\left(\frac{1}{q^{k_q}} - \frac{1}{q^{k_q+1}}\right)\right)$$

by (5.10)

$$= \delta_{j,k_p}\left(\prod_{q \in P_Y \cup \{p\}} \frac{q^{k_q}}{q^{k_q+1}}\left(1 - \frac{1}{q}\right)\right)$$

$$= \delta_{j,k_p}\left(\frac{1}{n_Y}\right)\left(\frac{1}{n_{p,j}}\right)\phi(n_{p,j})$$

where

$$n_Y = \underset{q \in P_{Y,p}}{\Pi} q^{k_q}, \qquad n_{p,j} = \underset{q \in P_{Y,p}}{\Pi} q,$$

in $\mathbb{N}$, and hence, it goes to

$$= \delta_{j,k_p} \left(\frac{1}{n_Y n_{p,j}}\right) \phi(n_{p,j})$$

for all $n \in \mathbb{N}$. $\square$

The above results (5.10) and (5.11) illustrate connections between our C^*-probabilistic structures and number-theoretic information. Also, they show a relation between the $*$-distributional data of section 3 and our C^*-probabilistic data, whenever $P_Y = P_Y^+$ in $\mathcal{P}$.

6. Projections in $M_{\mathcal{P}}^{p,j}$

Let $M_{\mathcal{P}}^{p,j} = (M_{\mathcal{P}},\ \varphi_{p,j})$ be the (p,j)-Adelic C^*-probability space of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, and a linear functional $\varphi_{p,j}$ of (5.1) for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

For $q \in \mathcal{P}$ and $k \in \mathbb{Z}$, let $B_k^q \in \sigma(A_{\mathbb{Q}})$ be in the sense of (5.1); that is,

$$B_k^q = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \cdots \times \underset{q\text{-th position in } \mathcal{P}}{\partial_k^q} \times \cdots, \tag{6.1}$$

in $A_{\mathbb{Q}}$.

For $B_k^q \in \sigma(A_{\mathbb{Q}})$ of (6.1), let

$$\alpha_{q,k} \overset{denote}{=} \alpha_{B_k^q} = \alpha\left(\chi_{B_k^q}\right) \in M_{\mathcal{P}}, \tag{6.2}$$

as a boundary-product operator, for all $q \in \mathcal{P}$, $k \in \mathbb{Z}$.

Then, by (6.1) and (6.2), we obtain the following special result of (5.10).

Corollary 6.1. *Let $\alpha_{q,k} \in M_{\mathcal{P}}^{p,j}$ be in the sense of (6.2) for $p, q \in \mathcal{P}$ and $j, k \in \mathbb{Z}$. Then*

$$\varphi_{p,j}\left(\alpha_{q,k}^n\right) = \begin{cases} \frac{\delta_{j,\{p,q\}}}{n_{\{p,q\}} n_{p,j}}\ \phi(n_{p,j}) & \text{if } k \geq 0, \\ 0 & \text{if } p \neq q \text{ and } k < 0, \end{cases} \tag{6.3}$$

with

$$\delta_{j,\{p,q\}} = \begin{cases} \delta_{j,k} & \text{if } p \neq q, \\ 1 & \text{if } p = q, \end{cases}$$

$$n_{\{p,q\}} = \underset{s \in \{p,q\}}{\Pi} s^{k_s}, \qquad \text{where } k_s = \begin{cases} j & \text{if } s = p, \\ k & \text{if } s = q, \end{cases}$$

and

$$n_{p,j} = \underset{s \in \{p,q\}}{\Pi} s \text{ in } \mathbb{N}$$

for all $n \in \mathbb{N}$.

Proof. The proof of (6.3) is straightforward by (5.10). $\square$

The above free distribution (6.3) of the projection $\alpha_{q,k}$ in $M_{\mathcal{P}}^{p,j}$ is refined by the following four formulas (6.4), (6.5), (6.6), and (6.7): if $p = q$ and $j = k$, then

$$\varphi_{p,j}\left(\alpha_{p,j}^{n}\right) = \frac{1}{p^{j+1}}\phi(p) = \frac{(p-1)}{p^{j+1}} = \frac{1}{p^{j}} - \frac{1}{p^{j+1}}; \tag{6.4}$$

if $p = q$ and $j \neq k$, then

$$\varphi_{p,j}\left(\alpha_{p,k}^{n}\right) = \frac{\delta_{j,k}}{n_{\{p\}}n_{p,j}}\phi(n_{p,j}) = 0; \tag{6.5}$$

if $p \neq q$ and $j = k$, then

$$\begin{aligned} \varphi_{p,j}\left(\alpha_{q,j}^{n}\right) &= \tfrac{\delta_{k\geq 0}}{(p^{j}q^{j})(pq)}\phi(pq) \\ &= \delta_{k\geq 0}\left(\tfrac{1}{p^{j}} - \tfrac{1}{p^{j+1}}\right)\left(\tfrac{1}{q^{j}} - \tfrac{1}{q^{j+1}}\right), \end{aligned} \tag{6.6}$$

where

$$\delta_{k\geq 0} = \begin{cases} 1 & \text{if } k \geq 0, \\ 0 & \text{if } k < 0, \end{cases}$$

and if $p \neq q$ and $j \neq k$, then, because $\delta_{j,k} = 0$,

$$\varphi_{p,j}\left(\alpha_{q,k}^{n}\right) = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{6.7}$$

6.1. **The C^*-Subalgebra $\mathfrak{S}_p$ of $M_{\mathcal{P}}$.** Let $M_{\mathcal{P}}^{p,j} = (M_{\mathcal{P}},\ \varphi_{p,j})$ be (p,j)-Adelic C^*-probability spaces, and let $\alpha_{q,k} = \alpha_{B_k^q}$ be projections (6.2) in the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ for all $p,\ q \in \mathcal{P}$ and $j,\ k \in \mathbb{Z}$.

Definition 6.2. Let $M_{\mathcal{P}}$ be the finite-Adelic C^*-algebra. Define a C^*-subalgebra $\mathfrak{S}_p$ of $M_{\mathcal{P}}$ by the C^*-algebra generated by the family

$$\Omega_p = \{\alpha_{p,k} \in M_{\mathcal{P}} : k \in \mathbb{Z}\} \tag{6.8}$$

of projections $\alpha_{p,k}$'s of (6.2) for an arbitrarily fixed $p \in \mathcal{P}$. That is,

$$\mathfrak{S}_p = C^*\left(\Omega_p\right) = \overline{\mathbb{C}\left[\Omega_p\right]} \text{ in } M_{\mathcal{P}}, \tag{6.9}$$

for all $p \in \mathcal{P}$, where $\overline{X}$ means the C^*-topology closures of subsets X in $M_{\mathcal{P}}$. We call $\mathfrak{S}_p$ of (6.9), the p-adic projection (C^*-)subalgebra of $M_{\mathcal{P}}$ for all $p \in \mathcal{P}$.

Let $p \in \mathcal{P}$, and let $\alpha_{p,k}$ and $\alpha_{p,j}$ be generating projections of the p-adic projection subalgebra $\mathfrak{S}_p$ of $M_{\mathcal{P}}$ for $k,\ j \in \mathbb{Z}$. Then

$$\alpha_{p,k}\alpha_{p,j} = \delta_{k,j}\alpha_{p,j} \text{ in } \mathfrak{S}_p, \tag{6.10}$$

by (2.4).

Proposition 6.3. *Let $\mathfrak{S}_p$ be the p-adic projection subalgebra (6.9) of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$. Then*

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\oplus}\left(\mathbb{C}\cdot\alpha_{p,j}\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \tag{6.11}$$

in $M_{\mathcal{P}}$*, where* "$\overset{*\text{-iso}}{=}$" *means "being* $*$*-isomorphic."*

Proof. Since $\mathfrak{S}_p$ is generated by the family Ω_p of (6.8), and the generators $\alpha_{p,j}$'s satisfy the orthogonality (6.10)), the $*$-isomorphic relations (6.11) hold in $M_{\mathcal{P}}$. $\square$

Let $\mathfrak{S}_p$ be the p-adic projection subalgebra of the finite-Adelic C^*-algebra of $M_{\mathcal{P}}$. Then, by determining the restrictions $\varphi_{p,j}\mid_{\mathfrak{S}_p}$ of the linear functionals $\varphi_{p,j}$ of (5.1), also denoted by $\varphi_{p,j}$, one can define C^*-probability spaces,

$$\mathfrak{S}_{p,j}=\left(\mathfrak{S}_p,\ \varphi_{p,j}\right), \tag{6.12}$$

for all $j\in\mathbb{Z}$, for any fixed $p\in\mathcal{P}$.

Definition 6.4. Let $\mathfrak{S}_p$ be the p-adic projection subalgebra (6.9) of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, and let $\mathfrak{S}_{p,j}$ be C^*-probability spaces (6.12) for all $j\in\mathbb{Z}$. Then we call $\mathfrak{S}_{p,j}$, the $(p,\ j)$-projection (C^*-)probability spaces.

The free distributions of generating operators $\alpha_{p,k}$'s in the (p,j)-projection probability spaces $\mathfrak{S}_{\mathcal{P}}^{p,j}$ are characterized by (6.4), refined by (6.5) and (6.6).

Proposition 6.5. *Let* $\mathfrak{S}_{p,j}=\left(\mathfrak{S}_p,\ \varphi_{p,j}\right)$ *be the* (p,j)*-projection probability space for* $p\in\mathcal{P}$ *and* $j\in\mathbb{Z}$*, and let* $\alpha_{p,k}$ *be generating operators of* $\mathfrak{S}_p$ *for all* $k\in\mathbb{Z}$*. Then*

$$\varphi_{p,j}\left(\alpha_{p,k}^{n}\right)=\delta_{j,k}\left(\frac{\phi(p)}{p^{j+1}}\right)=\delta_{j,k}\left(\frac{1}{p^{j}}-\frac{1}{p^{j+1}}\right) \tag{6.13}$$

for all $n\in\mathbb{N}$.

Proof. The formula (6.13) is obtained from (6.4) (or, (6.5) and (6.6)) and (6.11). $\square$

6.2. **On C^*-Probability Spaces $\mathfrak{S}(p)=\left(\mathfrak{S}_p,\ \varphi_p\right)$.** Let $p\in\mathcal{P}$ be fixed, and let $\mathfrak{S}_{p,j}=\left(\mathfrak{S}_p,\ \varphi_{p,j}\right)$ be $(p,\ j)$-projection probability spaces (6.12) for all $j\in\mathbb{Z}$. Recall that the C^*-algebra $\mathfrak{S}_p$ satisfies the structure theorem (6.11). Thus, one can define the following linear functional φ_p on $\mathfrak{S}_p$ by

$$\varphi_p=\sum_{k\in\mathbb{Z}}^{\oplus}\varphi_{p,j}\ \text{on}\ \underset{k\in\mathbb{Z}}{\oplus}\left(\mathbb{C}\cdot\alpha_{p,k}\right)=\mathfrak{S}_p;$$

by a linear morphism,

$$\varphi_p\left(\underset{j\in\mathbb{Z}}{\oplus}t_j\alpha_{p,j}\right)=\sum_{j\in\mathbb{Z}}t_j\varphi_{p,j}\left(\alpha_{p,j}\right). \tag{6.14}$$

By the definition (6.14) of φ_p, one has

$$\varphi_p\left(\underset{j\in\mathbb{Z}}{\oplus}t_j\alpha_{p,j}\right)=\sum_{j\in\mathbb{Z}}t_j\left(\frac{1}{p^j}-\frac{1}{p^{j+1}}\right)=\sum_{j\in\mathbb{Z}}\frac{t_j\phi(p)}{p^{j+1}} \tag{6.15}$$

and

$$\varphi_p\left(\alpha_{p,k}\right)=\varphi_{p,k}\left(\alpha_{p,k}\right)=\frac{1}{p^k}-\frac{1}{p^{k+1}}=\frac{\phi(p)}{p^{k+1}}, \tag{6.16}$$

for all $k \in \mathbb{Z}$, by (6.13).

Note that the above linear functional φ_p of (6.14) is well-defined by (6.11).

Definition 6.6. Let $\mathfrak{S}_p$ be the p-adic projection subalgebra of $M_{\mathcal{P}}$ for $p \in \mathcal{P}$, and let φ_p be the linear functional (6.14) on $\mathfrak{S}_p$. Then the pair

$$\mathfrak{S}(p) = (\mathfrak{S}_p,\ \varphi_p) \tag{6.17}$$

is called the p-projection (free-)probability space (generated by (p, j)-projection probability spaces $\{\mathfrak{S}_{p,j}\}_{j\in\mathbb{Z}}$) for all $p \in \mathcal{P}$.

The free-distributional data on the p-projection probability spaces $\mathfrak{S}(p)$ of (6.17) are characterized by (6.16) for all $p \in \mathcal{P}$.

Proposition 6.7. *Let $\mathfrak{S}(p) = (\mathfrak{S}_p,\ \varphi_p)$ be a p-projection probability space for $p \in \mathcal{P}$, and let $\{\alpha_{p,k}\}_{k\in\mathbb{Z}}$ be the generators of $\mathfrak{S}(p)$. Then*

$$\varphi_p\left(\alpha_{p,k}^n\right) = \frac{\phi(p)}{p^{k+1}} = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \qquad \text{for all } k \in \mathbb{Z}, \tag{6.18}$$

for all $n \in \mathbb{N}$.

Proof. Note that the generating operators $\alpha_{p,k}$'s of $\mathfrak{S}(p)$ are projections satisfying $\alpha_{p,k}^n = \alpha_{p,k}$ for all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$. So, the formula (6.18) holds by (6.16). □

Note that the formula (6.18) gives the full characterization of free distributions on $\mathfrak{S}(p)$ by (6.11) and (6.15).

6.3. **On a C^*-Probability Space over $\mathfrak{S}$.** Now, let $\mathfrak{S}(p) = (\mathfrak{S}_p,\ \varphi_p)$ be p-projection probability spaces (6.17) for all $p \in \mathcal{P}$, where φ_p are the linear functionals (6.16) on the p-projection C^*-algebra $\mathfrak{S}_p$ in the finite-Adelic C^*-algebra $M_{\mathcal{P}}$. By (4.1), one can construct the C^*-probability space

$$\mathfrak{S} \overset{denote}{=} (\mathfrak{S},\ \varphi) \tag{6.19}$$

from $\{\mathfrak{S}(p)\}_{p\in\mathcal{P}}$ as the pair of the weak direct product algebra $\mathfrak{S}$,

$$\mathfrak{S} = \underset{p\in\mathcal{P}}{\Pi'}\mathfrak{S}_p, \tag{6.20}$$

where Π' means the *weak direct product of C^*-algebras*, and

$$\varphi = \underset{p\in\mathcal{P}}{\Pi}\varphi_p, \tag{6.21}$$

satisfying that

$$\varphi\left((a_p)_{p\in\mathcal{P}}\right) = \underset{p\in\mathcal{P}}{\Pi}\varphi_p\left(a_p\right) \qquad \text{for all } (a_p)_{p\in\mathcal{P}} = \underset{p\in\mathcal{P}}{\Pi} a_p \in \mathfrak{S}.$$

That is, like in (4.1), by abusing notation, one has

$$\mathfrak{S} = (\mathfrak{S},\ \varphi) = \underset{p\in\mathcal{P}}{\Pi'}\left(\mathfrak{S}_p, \varphi_p\right) = \underset{p\in\mathcal{P}}{\Pi'}\mathfrak{S}(p), \tag{6.22}$$

by (6.19), (6.20), and (6.21).

By the very definition (6.19) and its characterization (6.22), the C^*-probability space $\mathfrak{S}$ is a well-defined C^*-probabilistic sub-structures of $M_{\mathcal{P}} = (M_{\mathcal{P}},\ \varphi)$.

Definition 6.8. Let $\mathfrak{S} = (\mathfrak{S},\ \varphi)$ be in the sense of (6.22). Then this pair $\mathfrak{S}$ is called "the" projection (C^*-)probability space (in $M_{\mathcal{P}}$).

Note that, by (6.20) or (6.22), if T is an element of the projection probability space $\mathfrak{S}$ of (6.22), then it is generated by the generating operators $\{\alpha^{p,k}\}_{p\in\mathcal{P},k\in\mathbb{Z}}$, formed by

$$\alpha^{p,k} = \left(\beta_{q,k_q}\right)_{q\in\mathcal{P},\ k_q\in\mathbb{Z}} \in \mathfrak{S},\ \text{with}\ \beta_{q,k_q} = \begin{cases} \alpha_{[p,k]} & \text{if } q = p,\ \text{with } k \in \mathbb{Z} \\ \alpha_{\mathbb{Z}_q} & \text{if } q \neq p, \end{cases} \tag{6.23}$$

for all $q \in \mathcal{P}$, for $k \in \mathbb{Z}$, where

$$\alpha_{[p,k]} = \alpha\left(\chi_{\partial_p^k}\right),\ \text{for } k \in \mathbb{Z},\ \text{and } \alpha_{\mathbb{Z}_q} = \alpha\left(\chi_{\mathbb{Z}_q}\right),\ \text{for all } q \neq p \text{ in } \mathcal{P}, \tag{6.24}$$

in $\mathfrak{S}$. Note that

$$\alpha^{p,k} = \alpha_{p,k} \text{ in } M_{\mathcal{P}},$$

by (6.24) where $\alpha_{p,k}$ are in the sense of (6.2).

But, the notation $\alpha_{p,k}$ is from the definition of $M_{\mathcal{P}}$ naturally, and $\alpha^{p,k}$ are obtained from the definition (6.22) of $\mathfrak{S}$. So, whenever we want to distinguish the origins of them, or to focus on structures where they belong, the different notations will be used below.

Theorem 6.9. *Let $\alpha^{p,j}$ be a generating operator (6.23) of the projection probability space $\mathfrak{S}$ of (6.22) for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then*

$$\varphi\left((\alpha^{p,j})^n\right) = \frac{\phi(p)}{p^{j+1}} = \frac{1}{p^j} - \frac{1}{p^{j+1}} \tag{6.25}$$

for all $n \in \mathbb{N}$.

Proof. Let $\alpha^{p,j}$ be a generating operator (6.23) of $\mathfrak{S}$ for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then it is a projection in $\mathfrak{S}$, satisfying $\left(\alpha^{p,j}\right)^n = \alpha^{p,j}$, for all $n \in \mathbb{N}$, by (6.24). So, it suffices to consider $\varphi\left(\alpha^{p,j}\right)$ to obtain the free-distributional data (6.25). Observe that

$$\varphi\left(\alpha^{p,j}\right) = \left(\underset{q\in\mathcal{P}}{\Pi}\varphi_q\right)\left(\alpha^{p,j}\right)$$

by (6.21), (6.22), (6.23), and (6.24).

$$= \varphi_p\left(\alpha_{p,j}\right) = \varphi_{p,j}\left(\alpha_{p,j}\right)$$

by (6.14)

$$= \tfrac{\phi(p)}{p^{j+1}} = \tfrac{1}{p^j} - \tfrac{1}{p^{j+1}},$$

by (6.18).

Therefore, the free-distributional data (6.25) holds. □

Note that, by (6.22), every operator of $\mathfrak{S}$ is a limit of linear combinations of operators T formed by

$$T=\left(T^{q,k_q}\right)_{q\in\mathcal{P}}=\underset{q\in\mathcal{P}}{\Pi}T^{q,k_q}, \tag{6.26}$$

for some $k_q\in\mathbb{Z}$, with a unique finite subset P_T of $\mathcal{P}$, such that

$$T^{q,k_q}=\begin{cases}\alpha^{q,k_q} & \text{if } q\in P_T,\\ \alpha_{B_q^0} & \text{if } q\in\mathcal{P}\setminus P_T,\end{cases} \tag{6.27}$$

for all $q\in\mathcal{P}$.

Notation 6.1 Let T be an element (6.26) of the projection probability space $\mathfrak{S}$ of (6.22), satisfying (6.27). Then, by abusing notation, we write T simply by

$$T=\underset{p\in P_T}{\Pi'}\alpha^{p,k_p}, \tag{6.28}$$

where the set P_T in (6.28) is a finite subset of $\mathcal{P}$ satisfying (6.26) and (6.27) for the operator T.

Corollary 6.10. *Let $T=\underset{p\in P_T}{\Pi'}\alpha^{p,k_p}\in\mathfrak{S}$ be in the sense of **Notation 6.1**. Then*

$$\varphi(T^n)=\underset{p\in P_T}{\Pi}\frac{\phi(p)}{p^{k_p+1}}=\frac{1}{N_T}\phi(n_T), \tag{6.29}$$

for all $n\in\mathbb{N}$, where

$$N_T=\underset{p\in P_T}{\Pi}p^{k_p+1}\in\mathbb{Q},\ \textit{and}\ n_T=\underset{p\in P_T}{\Pi}p\in\mathbb{N}.$$

Proof. Let $T=\underset{p\in P_T}{\Pi'}\alpha^{p,k_p}\in\mathfrak{S}$ be in the sense of (6.28). Then

$$T^n=\left(\underset{p\in P_T}{\Pi'}\alpha^{p,k_p}\right)^n=\underset{p\in P_T}{\Pi'}\left(\alpha^{p,k_p}\right)^n=\underset{p\in P_T}{\Pi'}\alpha^{p,k_p}=T,$$

in $\mathfrak{S}$ for all $n\in\mathbb{N}$. So,

$$\varphi(T^n)=\varphi(T)=\varphi\left(\underset{p\in P_T}{\Pi'}\alpha^{p,k_p}\right)$$

$$=\underset{p\in P_T}{\Pi}\varphi_p\left(\alpha^{p,k_p}\right)=\underset{p\in P_T}{\Pi}\varphi_{p,k_p}\left(\alpha_{p,k_p}\right)$$

$$=\underset{p\in P_T}{\Pi}\left(\frac{1}{p^{k_p}}-\frac{1}{p^{k_p+1}}\right)=\underset{p\in P_T}{\Pi}\frac{\phi(p)}{p^{k_p+1}}$$

by (6.25)

$$=\frac{\underset{p\in P_T}{\Pi}\phi(p)}{\underset{p\in P_T}{\Pi}p^{k_p+1}}=\frac{\phi\left(\underset{p\in P_T}{\Pi}p\right)}{\underset{p\in P_T}{\Pi}p^{k_p+1}},$$

by (3.10), for all $n\in\mathbb{N}$. $\square$

7. Semigroup C^*-Dynamical Systems Induced by $A_{\mathbb{Q}}$

Now, independent from the above analytic results (but to generalize those results), let us consider semigroup dynamical systems of the σ-algebra $\sigma(A_{\mathbb{Q}})$ of the finite adele ring $A_{\mathbb{Q}}$. In this section, we regard $\sigma(A_{\mathbb{Q}})$ as a semigroup,

$$\sigma(A_{\mathbb{Q}}) = (\sigma(A_{\mathbb{Q}}),\ \cap), \tag{7.1}$$

where $\cap$ is the usual *intersection of sets*.

Indeed, under finite intersection, $\sigma(A_{\mathbb{Q}})$ is closed, and definitely the set-operation $\cap$ is associative on $\sigma(A_{\mathbb{Q}})$, and hence, the pair $\sigma(A_{\mathbb{Q}})$ of (7.1) forms a well-defined *semigroup*.

Now, let $(A,\ \psi)$ be an arbitrarily fixed *unital C^*-probability space*, where A is a C^*-algebra in the operator algebra $B(H)$ (consisting of all bounded operators on a Hilbert space H), satisfying

$$\psi(1_A) = 1,$$

where 1_A is the *unit* (or the operator-multiplication-identity) in A.

Let $\mathcal{M}_{\mathcal{P}}$ be the finite-Adelic algebra (2.24) generated by $\sigma(A_{\mathbb{Q}})$, and let $H_{\mathcal{P}}$ be the finite-Adelic Hilbert space of section 4, where the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ induced by the finite-Adelic probability space $(\mathcal{M}_{\mathcal{P}},\ \varphi)$ act.

Now, consider a new Hilbert space $\mathfrak{H}$,

$$\mathfrak{H} = H \otimes H_{\mathcal{P}}, \tag{7.2}$$

where H is the Hilbert space where a given C^*-algebra A acts, and $H_{\mathcal{P}}$ is the finite-Adelic Hilbert space, and let

$$A_{\mathcal{P}} = A \otimes_{\mathbb{C}} M_{\mathcal{P}} \tag{7.3}$$

be the tensor product C^*-algebra of A and $M_{\mathcal{P}}$ acting on the Hilbert space $\mathfrak{H}$ of (7.2).

Define now a *semigroup-action π of* the semigroup $\sigma(A_{\mathbb{Q}})$ of (7.1) acting on the C^*-algebra $A_{\mathcal{P}}$ of (7.3) by a morphism π,

$$S \in \sigma(A_{\mathbb{Q}}) \longmapsto \pi(S) \overset{denote}{=} \pi_S \in End(A_{\mathcal{P}}),$$

satisfying

$$\pi_S(a \otimes \alpha_Y) = a \otimes \alpha_Y \alpha_S, \tag{7.4}$$

for all $S,\ Y \in \sigma(A_{\mathbb{Q}})$ and $a \in A$, where

$$End(A_{\mathcal{P}}) = \left\{ E : A_{\mathcal{P}} \to A_{\mathcal{P}} \,\middle|\, \begin{array}{c} E \text{ is a} \\ *\text{-endomorphism} \\ \text{on } A_{\mathcal{P}} \end{array} \right\}.$$

Recall that $*$-endomorphisms are surjective (bounded) $*$-homomorphisms from a (topological) $*$-algebra onto itself.

By the definition (7.4) of the morphism π, it is not difficult to check that

$$\pi_{S_1 \cap S_2} = \pi_{S_1}\pi_{S_2} \text{ on } A_{\mathcal{P}}, \tag{7.5}$$

for all $S_1,\ S_2 \in \sigma(A_{\mathbb{Q}})$.

Moreover, one can check that

$$\pi_S\left((a \otimes \alpha_Y)^*\right) \begin{array}{l} = a^* \otimes \alpha_Y^* \alpha_S = a^* \otimes \alpha_Y^* \alpha_S^* \\ = a^* \otimes (\alpha_S \alpha_Y)^* = a^* \otimes (\alpha_Y \alpha_S)^* \\ = (a \otimes \alpha_Y \alpha_S)^* = (\pi_S(a \otimes \alpha_Y))^* \end{array} \tag{7.6}$$

for all $a \in A$ and $S,\, Y \in \sigma(A_{\mathbb{Q}})$.

By (7.6), we have

$$\pi_S\left(T^*\right) = (\pi_S(T))^*, \qquad \text{for all } T \in A_{\mathcal{P}} \tag{7.7}$$

for all $S \in \sigma(A_{\mathbb{Q}})$.

So, indeed, the images π_S of our morphism π of (7.2) are well-determined $*$-endomorphisms on the C^*-algebra $A_{\mathcal{P}}$, for all $S \in \sigma(A_{\mathbb{Q}})$, by (7.5) and (7.7); that is,

$$\pi_S \in End\,(A_{\mathcal{P}}) \qquad \text{for all } S \in \sigma(A_{\mathbb{Q}}). \tag{7.8}$$

Notation 7.1 In the rest of this paper, for convenience, we denote $\pi_S(T)$ simply by T^S for all $T \in A_{\mathcal{P}}$ and for all $S \in \sigma(A_{\mathbb{Q}})$.

Definition 7.1. Let B be an arbitrary topological $*$-algebra (C^*-algebra, or W^*-algebra, or Banach $*$-algebra, etc.), and let K be a semigroup, and let

$$\theta : K \to End(B)$$

be a morphism whose images $\theta(k) \overset{denote}{=} \theta_k$ are $*$-endomorphisms on B, for all $k \in K$, satisfying

$$\theta_{k_1 k_2} = \theta_{k_1}\theta_{k_2} \qquad \text{for all } k_1,\, k_2 \in K.$$

Then the triple $(B,\ K,\ \theta)$ is called the semigroup (topological-$*$-)dynamical system of K acting on B via a semigroup-action θ.

By the above definition, we obtain the following result.

Proposition 7.2. *Let $(A,\ \psi)$ be the fixed unital C^*-probability space, and let $\sigma(A_{\mathbb{Q}})$ be the σ-algebra of the finite adele ring $A_{\mathbb{Q}}$, regarded as a semigroup (7.1). Then the triple $(A_{\mathcal{P}},\ \sigma(A_{\mathbb{Q}}),\ \pi)$ is the well-defined semigroup C^*-dynamical system of the semigroup $\sigma(A_{\mathbb{Q}})$ acting on the tensor product C^*-algebra $A_{\mathcal{P}} = A \otimes_{\mathbb{C}} M_{\mathcal{P}}$ of (7.3) via a semigroup-action π of (7.4).*

Proof. The proof is trivial by (7.4), (7.5), and (7.8). □

Definition 7.3. Let $(A_{\mathcal{P}},\ \sigma(A_{\mathbb{Q}}),\ \pi)$ be the semigroup C^*-dynamical system of the semigroup $\sigma(A_{\mathbb{Q}})$ of (7.1) acting on the C^*-algebra $A_{\mathcal{P}}$ of (7.3) via a semigroup-action π of (7.4). Then we call it the finite-Adelic A-dynamical system.

Now, let $(B,\ K,\ \theta)$ be the semigroup C^*-dynamical system of a semigroup K acting on a C^*-algebra B via a semigroup-action θ. Then it induces a new C^*-algebra $\mathbb{B}_K$ generated by both B and $\theta(K)$,

$$\mathbb{B}_K \overset{denote}{=} B \times_\theta K, \tag{7.9}$$

dictated by the θ-relation:

$$(b_1, k_1)(b_2, k_2) = (b_1\theta_{k_1}(b_2),\ k_1k_2) \text{ and } (b,\ k)^* = (\theta_k(b^*),\ k)\,, \tag{7.10}$$

for all $b,\ b_1,\ b_2 \in B$ and $k,\ k_1,\ k_2 \in K$.

Definition 7.4. A C^*-algebra $\mathbb{B}_K$ of (7.9) is called the crossed product C^*-algebra of $(B,\ K,\ \theta)$, and the relations in (7.10) are called the θ-relation on $\mathbb{B}_K$.

For the finite-Adelic A-dynamical system $(A_{\mathcal{P}},\ \sigma(A_{\mathbb{Q}}),\ \pi)$, define the corresponding crossed product C^*-algebra,

$$\mathfrak{A}_{\mathcal{P}} \overset{denote}{=} A_{\mathcal{P}} \times_{\pi} \sigma(A_{\mathbb{Q}}), \tag{7.11}$$

satisfying the π-relation:

$$\begin{array}{rl} (a_1 \otimes T_1,\ S_1)\,(a_2 \otimes T_2, S_2)7 & = \Big((a_1 \otimes T_1)\,(a_2 \otimes T_2)^{S_1},\ S_1 \cap S_2\Big) \\ \text{and } (a \otimes T, S)^* & = \big((a^* \otimes T^*)^S,\ S\big)\,, \end{array} \tag{7.12}$$

for all $a,\ a_1,\ a_2 \in A$ and $T, T_1, T_2 \in M_{\mathcal{P}}$ and $S, S_1, S_2 \in \sigma(A_{\mathbb{Q}})$, under **Notation 7.1**.

The definitions (7.11) and (7.12) are from (7.9) and (7.10), respectively.

Definition 7.5. Let $(A_{\mathcal{P}},\ \sigma(A_{\mathbb{Q}}),\ \pi)$ be the finite-Adelic A-dynamical system, and let $\mathfrak{A}_{\mathcal{P}}$ be the corresponding crossed product C^*-algebra (7.11) with the π-relation (7.12). Then we call $\mathfrak{A}_{\mathcal{P}}$, the finite-Adelic A-dynamical-(crossed-product-C^*-)algebra (of $(A_{\mathcal{P}},\ \sigma(A_{\mathbb{Q}}),\ \pi)$).

By (7.4) and (7.6), the π-relation (7.12) on the finite-Adelic A-dynamical algebra $\mathfrak{A}_{\mathcal{P}}$ of (7.11) can be re-written as follows:

$$\begin{array}{rl} (a_1 \otimes T_1, S_1)(a_2 \otimes T_2, S_2) & = ((a_1 \otimes T_1)(a_2 \otimes T_2\alpha_{S_1}),\ S_1 \cap S_2) \\ & = (a_1a_2 \otimes T_1T_2\alpha_{S_1},\ S_1 \cap S_2)\,, \\ \text{and } (a \otimes T, S)^* & = (a^* \otimes T^*\alpha_S,\ S)\,, \end{array} \tag{7.13}$$

for all $a,\ a_1,\ a_2 \in A$ and $T,\ T_1,\ T_2 \in M_{\mathcal{P}}$ and $S,\ S_1,\ S_2 \in \sigma(A_{\mathbb{Q}})$.

Suppose that $\mathbb{B}_K$ be the crossed product C^*-algebra (7.9) of a semigroup C^*-dynamical system $(B,\ K,\ \theta)$, satisfying the θ-relation (7.10). Now, consider the C^*-algebra $\mathcal{K} = C^*(K)$ generated by the semigroup K,

$$k \in K \longmapsto e_k \in \mathcal{K} \subset B(H_K),$$

such that

$$e_{k_1}e_{k_2} = e_{k_1k_2} \text{ in } \mathcal{K}, \qquad \text{for all } k_1,\ k_2 \in K,$$

where H_K is a Hilbert space where $\mathcal{K}$ acts (under a suitable representation).

Define the *conditional tensor product C^*-algebra*,

$$\mathbb{B}^K = B \otimes_{\theta} \mathcal{K}, \tag{7.14}$$

by the C^*-subalgebra of the usual tensor product C^*-algebra $B \otimes_{\mathbb{C}} \mathcal{K}$ generated by B and $\mathcal{K}$, satisfying the θ-*condition*:

$$\begin{array}{rl} (b_1 \otimes e_{k_1})(b_2 \otimes e_{k_2}) & = b_1\theta_{k_1}(b_2) \otimes e_{k_1}e_{k_2} \\ & = b_1\theta_{k_1}(b_2) \otimes e_{k_1k_2}, \\ \text{and} \quad (b \otimes e_k)^* & = \theta_k(b^*) \otimes e_k \end{array} \tag{7.15}$$

for all b, b_1, $b_2 \in B$ and k, k_1, $k_2 \in K$.

Proposition 7.6. *Let $\mathbb{B}_K$ be the crossed product C^*-algebra (7.9) of a semigroup C^*-dynamical system $(B,\ K,\ \theta)$, and let $\mathbb{B}^K$ be the conditional tensor product C^*-algebra (7.14). Then*

$$\mathbb{B}_K \overset{*\text{-}iso}{=} \mathbb{B}^K, \tag{7.16}$$

where "$\overset{*\text{-}iso}{=}$" in (7.16) means "being C^*-isomorphic."

Proof. Let $\mathbb{B}_K$ and $\mathbb{B}^K$ be in the sense of (7.9) and (7.14), respectively. Define a linear morphism $\Phi : \mathbb{B}_K \to \mathbb{B}^K$ by a linear transformation satisfying that

$$\Phi\left((b,\ k)\right) = b \otimes e_k \in \mathbb{B}^K \qquad \text{for all } (b,\ k) \in \mathbb{B}_K.$$

By the above definition, the linear transformation Φ preserves the generators of $\mathbb{B}_K$ to the generators of $\mathbb{B}^K$. So, it is not only bounded but also bijective.

Also, by the θ-relation (7.10) on $\mathbb{B}_K$ and the θ-condition (7.15) on $\mathbb{B}^K$, one has that

$$\begin{array}{rl} \Phi\left((b_1, k_1)(b_2, k_2)\right) & = \Phi\left((b_1\theta_{k_1}(b_2)),\ k_1k_2\right) \\ & = b_1\theta_{k_1}(b_2) \otimes e_{k_1k_2} = b_1\theta_{k_1}(b_2) \otimes e_{k_1}e_{k_2} \\ & = \left(b_1 \otimes e_{k_1}\right)\left(b_2 \otimes e_{k_2}\right) \end{array}$$

in $\mathbb{B}^K$ for all $(b_1,\ k_1)$, $(b_2,\ k_2) \in \mathbb{B}_K$. Thus, this bounded linear transformation Φ is multiplicative.

Moreover,

$$\begin{array}{rl} \Phi\left((b, k)^*\right) & = \Phi\left((\theta_k(b^*),\ k)\right) \\ & = \theta_k(b^*) \otimes e_k = \left(b \otimes e_k\right)^*, \end{array}$$

in $\mathbb{B}^K$, for all $(b,\ k) \in \mathbb{B}_K$, by (7.10) and (7.15).

Thus, this multiplicative bijective bounded linear transformation Φ is a $*$-homomorphism from $\mathbb{B}_K$ onto $\mathbb{B}^K$; equivalently, it is a $*$-isomorphisms. Therefore, two C^*-algebras $\mathbb{B}_K$ and $\mathbb{B}^K$ are $*$-isomorphic. □

Now, let $\mathfrak{A}_{\mathcal{P}} = A_{\mathcal{P}} \times_{\pi} \sigma(A_{\mathbb{Q}})$ be our finite-Adelic A-dynamical algebra, and let $M_{\mathcal{P}}$ be the finite-Adelic C^*-algebra. Define a conditional tensor product C^*-algebra

$$\mathfrak{A}^{o}_{\mathcal{P}} = A_{\mathcal{P}} \otimes_{\pi} M_{\mathcal{P}} = (A \otimes_{\mathbb{C}} M_{\mathcal{P}}) \otimes_{\pi} M_{\mathcal{P}}, \tag{7.17}$$

by the C^*-subalgebra of the usual tensor product C^*-algebra $A_{\mathcal{P}} \otimes_{\mathbb{C}} M_{\mathcal{P}}$, satisfying the π-condition:

$$\begin{array}{rl} \left((a_1 \otimes T_1) \otimes \alpha_{S_1}\right)\left((a_2 \otimes T_2) \otimes \alpha_{S_2}\right) & = \left((a_1 \otimes T_1)(a_2 \otimes T_2)^{S_1}\right) \otimes \alpha_{S_1 \cap S_2} \\ & = \left(a_1a_2 \otimes T_1T_2\alpha_{S_1}\right) \otimes \alpha_{S_1 \cap S_2} \end{array}$$

and

$$\begin{array}{ll} ((a \otimes T) \otimes \alpha_S)^* & = (a^* \otimes T^*)^S \otimes \alpha_S \\ & = (a^* \otimes T^* \alpha_S) \otimes \alpha_S, \end{array} \tag{7.18}$$

for all a_1, a_2, $a \in A$ and T_1, T_2, $T \in M_{\mathcal{P}}$ and S_1, S_2, $S \in \sigma(A_{\mathbb{Q}})$.

Corollary 7.7. *Let $\mathfrak{A}_{\mathcal{P}}$ be the finite-Adelic A-dynamical algebra, and let $\mathfrak{A}_{\mathcal{P}}^o$ be the conditional tensor product C^*-algebra (7.17) satisfying the π-condition (7.18). Then*

$$\mathfrak{A}_{\mathcal{P}} \overset{*\text{-}iso}{=} \mathfrak{A}_{\mathcal{P}}^o. \tag{7.19}$$

Proof. The isomorphism theorem (7.19) is a special case of (7.16). □

Assumption and Notation 7.2 (in short, **AN 7.2** from below) By the above isomorphism theorem (7.19), in the following text, we use $\mathfrak{A}_{\mathcal{P}}$ and $\mathfrak{A}_{\mathcal{P}}^o$ alternatively, case-by-case. Also, we use the notation $\mathfrak{A}_{\mathcal{P}}$ for both $\mathfrak{A}_{\mathcal{P}}$ and $\mathfrak{A}_{\mathcal{P}}^o$, and we call them, the finite-Adelic A-dynamical algebra. Also, we use π-relation of (7.12) (or (7.13)) and the π-condition (7.18) alternatively, under the identified term, the π-relation.

Let $\mathfrak{A}_{\mathcal{P}}$ be the finite-Adelic A-dynamical algebra. Then, by (7.19), one can understand it as a C^*-subalgebra of the usual tensor product C^*-algebra $A_{\mathcal{P}} \otimes_{\mathbb{C}} M_{\mathcal{P}}$; more precisely, it is a C^*-subalgebra of

$$(A \otimes_{\mathbb{C}} M_{\mathcal{P}}) \otimes_{\mathbb{C}} M_{\mathcal{P}},$$

by (7.3) and (7.17).

Define now a $*$-endomorphism

$$\Psi : A \otimes_{\mathbb{C}} M_{\mathcal{P}} \otimes_{\mathbb{C}} M_{\mathcal{P}} \to A \otimes_{\mathbb{C}} M_{\mathcal{P}} = A_{\mathcal{P}}$$

by a multiplicative surjective bounded linear transformation satisfying

$$\Psi\,(a \otimes T_1 \otimes T_2) = a \otimes T_1 T_2 \tag{7.20}$$

for all $a \in A$ and T_1, $T_2 \in M_{\mathcal{P}}$.

Then, by the commutativity on $M_{\mathcal{P}}$, this morphism Ψ of (7.20) is indeed a well-defined $*$-endomorphism. Since our finite-Adelic A-dynamical algebra $\mathfrak{A}_{\mathcal{P}}$ is a C^*-subalgebra,

$$(A \otimes_{\mathbb{C}} M_{\mathcal{P}}) \otimes_{\pi} M_{\mathcal{P}} \text{ in } A_{\mathcal{P}} \otimes_{\mathbb{C}} M_{\mathcal{P}},$$

the $*$-endomorphism Ψ is naturally inherited to

$$\Psi = \Psi \mid_{\mathfrak{A}_{\mathcal{P}}} : \mathfrak{A}_{\mathcal{P}} \to A_{\mathcal{P}}, \tag{7.21}$$

under π-relation.

Now, since a C^*-algebra A is from our fixed unital C^*-probability space (A, ψ) and the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ have a system of linear functionals $\{\varphi_{p,j}\}_{p \in \mathcal{P},\, j \in \mathbb{Z}}$ of (5.1), one can determine linear functionals $\{\psi_{p,j}\}_{p \in \mathcal{P},\, j \in \mathbb{Z}}$ on the C^*-algebra $A_{\mathcal{P}} = A \otimes_{\mathbb{C}} M_{\mathcal{P}}$ by linear morphisms satisfying that

$$\psi_{p,j}\,(a \otimes T) = \varphi_{p,j}\,(\psi(a)T) = \varphi_{p,j}(T)\psi(a), \tag{7.22}$$

for all $a \in A$ and $T \in M_{\mathcal{P}}$, for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Let $\mathfrak{A}_{\mathcal{P}}$ be the finite-Adelic A-dynamical algebra. Define linear functionals $\varphi_{p,j}^{A}$ on $\mathfrak{A}_{\mathcal{P}}$ by the linear morphisms,

$$\varphi_{p,j}^{A} \overset{def}{=} \psi_{p,j} \circ \Psi \text{ on } \mathfrak{A}_{\mathcal{P}}, \tag{7.23}$$

for all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, where $\psi_{p,j}$ are in sense of (7.22), and Ψ is in the sense of (7.21). By the linearity of $\psi_{p,j}$'s and Ψ, the morphisms $\varphi_{p,j}^{A}$ of (7.23) are indeed well-defined linear functionals on $\mathfrak{A}_{\mathcal{P}}$.

Note that, by the definition (7.23), one can get that

$$\begin{aligned} \varphi_{p,j}^{A}\left((a \otimes \alpha_{Y}) \otimes \alpha_{S}\right) &= \psi_{p,j}\left(a \otimes \alpha_{Y \cap S}\right) \\ &= \varphi_{p,j}\left(\alpha_{Y \cap S}\right)\psi(a) \end{aligned} \tag{7.24}$$

for all $a \in A$ and Y, $S \in \sigma(A_{\mathbb{Q}})$.

Definition 7.8. Let $\left(\mathfrak{A}_{\mathcal{P}},\ \varphi_{p,j}^{A}\right)$ be the (traditional) C^{*}-probability spaces of the finite-Adelic A-dynamical C^{*}-algebra $\mathfrak{A}_{\mathcal{P}}$ and the linear functionals $\varphi_{p,j}^{A}$ of (7.23) for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then they are said to be the A-dynamical (p, j)-(finite-Adelic-)C^{*}-probability spaces over $M_{\mathcal{P}}$ for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

As we have seen in sections 5 and 6, one can construct free-probabilistic sub-structures of the A-dynamical (p, j)-C^{*}-probability spaces $\left(\mathfrak{A}_{\mathcal{P}},\ \varphi_{p,j}^{A}\right)$, because of the structure theorem (7.19).

Let $\mathfrak{S}_{A,p}$ be C^{*}-subalgebras of $\mathfrak{A}_{\mathcal{P}}$ defined by

$$\mathfrak{S}_{A,p} = A \otimes_{\pi} \mathfrak{S}_{p}, \qquad \text{for all } p \in \mathcal{P}, \tag{7.25}$$

where $\otimes_{\pi}$ is the conditional tensor product in the sense of (7.25) satisfying the π-relation (7.18), where $\mathfrak{S}_{p}$ is the p-adic projection subalgebra (6.9) of the finite-Adelic C^{*}-algebra $M_{\mathcal{P}}$. Note that the above sub-structures (7.25) of $\mathfrak{A}_{\mathcal{P}}$ are well-determined by (7.19).

By understanding $\mathfrak{S}_{p}$ as the p-projection probability spaces $\mathfrak{S}(p)$ of (6.17), one can construct new C^{*}-probability spaces,

$$\mathfrak{S}_{A,p} \overset{denote}{=} \left(\mathfrak{S}_{A,p},\ \varphi_{A,p}\right), \qquad \text{for all } p \in \mathcal{P}, \tag{7.26}$$

where $\varphi_{A,p}$ are the linear functionals satisfying that

$$\begin{aligned} \varphi_{A,p}\left((a \otimes T) \otimes \alpha_{S}\right) &= \varphi_{A,p}\left(a \otimes T\alpha_{S}\right) \\ &= \varphi_{p}\left(T\alpha_{S}\right)\psi(a), \end{aligned} \tag{7.27}$$

for all $S \in \sigma(A_{\mathbb{Q}})$ and $a \in (A,\ \psi)$, where φ_{p} are the linear functionals (6.14) on $\mathfrak{S}_{p}$, for all $p \in \mathcal{P}$.

These linear functionals $\varphi_{A,p}$ of (7.27) are well-determined in $\mathfrak{S}_{A,p}$ of (7.25) because of the well-definedness of the linear functionals φ_{p} of (6.14), by (7.24).

Remark that the projection probability space $\mathfrak{S}$ of (6.19) is constructed by

$$\mathfrak{S} = (\mathfrak{S},\ \varphi) = \underset{p \in \mathcal{P}}{\Pi'}\, \mathfrak{S}(p).$$

Hence, motivated by (6.19) and (7.26), one can construct a new C^{*}-probability space,

$$\mathfrak{S}_{A}=A_{\mathcal{P}}\otimes_{\pi}\mathfrak{S}\subset A_{\mathcal{P}}\otimes_{\pi}M_{\mathcal{P}}, \tag{7.28}$$

equipped with the linear functional φ_{A} defined by the linear morphism satisfying that

$$\varphi_{A}\left((a\otimes T_{1})\otimes T_{2}\right)=\varphi(T_{1}T_{2})\psi(a), \tag{7.29}$$

as in (7.27), for all $a\in(A,\,\psi)$ and $T_{1}\in M_{\mathcal{P}}$ and $T_{2}\in\mathfrak{S}=\underset{p\in\mathcal{P}}{\Pi'}\mathfrak{S}_{p}$, where φ is in the sense of (6.19).

Definition 7.9. Let $\mathfrak{S}_{A}=(\mathfrak{S}_{A},\ \varphi_{A})$ be the C^{*}-probability space (7.28). We call it the A-dynamical-projection (C^{*}-)probability space. We call the C^{*}-algebra $\mathfrak{S}_{A}$, the A-dynamical-projection C^{*}-algebra.

Let $\mathfrak{S}_{A}$ be the A-dynamical-projection probability space (7.28). If

$$T_{a}^{p,k}=\left(a\otimes\alpha^{p,k}\right)\otimes\alpha^{p,k}\in\mathfrak{S}_{A}, \tag{7.30}$$

then

$$\left(T_{a}^{p,k}\right)^{n}=\left((a\otimes\alpha^{p,k})\otimes\alpha^{p,k}\right)\cdots\left((a\otimes\alpha^{p,k})\otimes\alpha^{p,k}\right)$$

$$=\left(a_{p,k}a_{p,k}^{B_{p}^{k}}a_{p,k}^{B_{p}^{k}\cap B_{p}^{k}}\cdots a_{p,k}^{\overbrace{B_{p}^{k}\cap\cdots\cap B_{p}^{k}}^{(n-1)\text{-times}}}\right)\otimes\left(\alpha^{p,k}\right)^{n};$$

by (7.28) and by the *induction on* (7.18), where B_{p}^{k} are in the sense of (6.1) and

$$a_{p,k}\overset{denote}{=}a\otimes\alpha^{p,k}\text{ in }A_{\mathcal{P}},$$

then it goes to

$$=\left(a_{p,k}a_{p,k}^{B_{p}^{k}}a_{p,k}^{B_{p}^{k}}\cdots a_{p,k}^{B_{p}^{k}}\right)\otimes\alpha^{p,k}$$

because $B_{p}^{k}\cap...\cap B_{p}^{k}=B_{p}^{k}$ in $\sigma(A_{\mathbb{Q}})$, for all $p\in\mathcal{P}$ and $k\in\mathbb{Z}$, by (6.23)

$$=a_{p,k}\left(a_{p,k}^{B_{p}^{k}}\right)^{n-1}\otimes\alpha^{p,k} \tag{7.31}$$

with axiomatization

$$\left(a_{p,k}^{B_p^k}\right)^0 = 1_A \otimes \alpha^{p,k} \qquad \text{for all } n \in \mathbb{N}.$$

But, in this case where $T_a^{p,k}$ is in the sense of (7.30), one can verify that

$$\left(T_a^{p,k}\right)^n = a_{p,k}\left(a_{p,k}^{B_p^k}\right)^{n-1} \otimes \alpha^{p,k}$$

by (7.31)

$$= \left(\left(a \otimes \alpha^{p,k}\right)\left(a \otimes \alpha^{p,k}\alpha^{p,k}\right)^{n-1}\right) \otimes \alpha^{p,k}$$

by (7.24), (7.27), and (7.29)

$$= \left(\left(a \otimes \alpha^{p,k}\right)\left(a \otimes \alpha^{p,k}\right)^{n-1}\right) \otimes \alpha^{p,k}$$
$$= \left(a \otimes \alpha^{p,k}\right)^n \otimes \alpha^{p,k}$$
$$= \left(a^n \otimes \alpha^{p,k}\right) \otimes \alpha^{p,k};$$

that is,

$$\left(T_a^{p,k}\right)^n = \left(a^n \otimes \alpha^{p,k}\right) \otimes \alpha^{p,k} = T_{a^n}^{p,k}, \tag{7.32}$$

in the sense of (7.30).

Thus, one can have that

$$\varphi_A\left(\left(T_a^{p,k}\right)^n\right) = \varphi\left(\alpha^{p,k}\alpha^{p,k}\right)\,\psi\left(a^n\right)$$

by (7.27), (7.28), and (7.32)

$$= \varphi_p\left(\alpha_{p,k}\right)\,\psi\left(a^n\right) = \left(\frac{\phi(p)}{p^{k+1}}\right)\,\psi\left(a^n\right)$$

$$= \left(\frac{1}{p^k} - \frac{1}{p^{k+1}}\right)\psi\left(a^n\right) \tag{7.33}$$

for all $n \in \mathbb{N}$.

Proposition 7.10. *Let $T_a^{p,j} = (a \otimes \alpha^{p,j}) \otimes \alpha^{p,j}$ be a free random variable (7.30) in the A-dynamical-projection probability space $\mathfrak{S}_A$ of (7.28). Then*

$$\begin{aligned} \varphi_A\left(\left(T_a^{p,j}\right)^n\right) &= \tfrac{\phi(p)}{p^{j+1}}\psi\left(a^n\right) \\ &= \left(\tfrac{1}{p^j} - \tfrac{1}{p^{j+1}}\right)\,\psi\left(a^n\right) \end{aligned} \tag{7.34}$$

for all $n \in \mathbb{N}$.

Proof. The free-distributional data (7.34) is obtained by (7.33). □

More general to (7.30), let

$$T_{a,q,l}^{p,k} = \left(a \otimes \alpha^{q,l}\right) \otimes \alpha^{p,k} \in \mathfrak{S}_A, \tag{7.35}$$

for $a \in (A,\,\psi)$, $p,\,q \in \mathcal{P}$ and $k, l \in \mathbb{Z}$.

Assumption In the rest of the paper, whenever operators $T_{a,q,l}^{p,k}$ of (7.35) are considered in $\mathfrak{S}_A$, we automatically assume that

$$a \neq 0_A \text{ and that } \psi(a) \neq 0 \text{ in } \mathbb{C},$$

where 0_A is the zero element of (A, ψ). Remark that the operators $T_a^{p,k}$ of (7.30) are regarded as $T_{a,p,k}^{p,k}$ of (7.35) in $\mathfrak{S}_A$, and hence, the above conditions will be automatically assumed for $T_a^{p,k}$, from below.

Then, similar to (7.33) and (7.34), one can get the following free-distributional data.

Theorem 7.11. *Let $T_{a,q,l}^{p,j}$ be a free random variable (7.35) of the A-dynamical-projection probability space $\mathfrak{S}_A$. Then*

$$\varphi_A\left((T_{a,q,l}^{p,j})^n\right) = \begin{cases} \psi(a^n)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) & \text{if } (p,j) = (q,l), \\ \\ \psi(a^n)\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right)\left(\frac{1}{q^l} - \frac{1}{q^{l+1}}\right) & \text{if } (p,j) \neq (q,l) \end{cases} \tag{7.36}$$

for all $n \in \mathbb{N}$.

Proof. Suppose that $(p, j) = (q, l)$ in $\mathcal{P} \times \mathbb{Z}$. Then the operator $T_{a,p,j}^{p,j}$ of (7.35) is identical to the operator $T_a^{p,j}$ of (7.30). Therefore, the first formula in (7.36) holds.

Assume now that $(p, j) \neq (q, l)$ in $\mathcal{P} \times \mathbb{Z}$. Then, one can get the second formula of (7.36) by (6.13), (6.18), (6.25), (6.29), (7.27), and (7.28).

Therefore, the free-distributional data (7.36) on the A-dynamical-projection probability space $\mathfrak{S}_A$. □

8. On the A-Dynamical-Projection Probability Space $(\mathfrak{S}_A, \varphi_A)$

In this section, we use same concepts and notations from previous sections.

Let $\mathfrak{S}_A = (\mathfrak{S}_A, \varphi_A)$ be the A-dynamical-projection probability space (7.28), and let

$$T_{a,q,l}^{p,j} = \left(a \otimes \alpha^{q,l}\right) \otimes \alpha^{p,j} \text{ and} T_a^{p,j} = T_{a,p,j}^{p,j} \tag{8.1}$$

respectively, be the simplest generating operators (7.35) and (7.30) of the A-dynamical-projection C^*-algebra $\mathfrak{S}_A$ for all $a \in (A, \psi)$, $p, q \in \mathcal{P}$, and $j, l \in \mathbb{Z}$.

Recall that, if $T_{a,q,l}^{p,j}$ is in the sense of (8.1) in $\mathfrak{S}_A$, then

$$\begin{aligned} \left(T_{a,q,l}^{p,j}\right)^n &= \left(a^n \otimes \alpha^{q,l}\alpha^{p,j}\right) \otimes \alpha^{p,j}, \text{ in } \mathfrak{S}_A, \\ \text{and } \varphi_A\left(\left(T_{a,q,l}^{p,j}\right)^n\right) &= \left(\frac{\phi(pq)}{p^{j+1}q^{l+1}}\right)\psi\left(a^n\right), \end{aligned} \tag{8.2}$$

for all $n \in \mathbb{N}$, by (7.36).

As special cases of (8.2),

$$\begin{aligned} \left(T_a^{p,j}\right)^n &= \left(a^n \otimes \alpha^{p,j}\alpha^{p,j}\right) \otimes \alpha^{p,j} = T_{a^n}^{p,j}, \text{ in } \mathfrak{S}_A, \\ \text{and } \varphi_A\left(\left(T_a^{p,j}\right)^n\right) &= \left(\frac{\phi(p)}{p^{j+1}}\right)\psi(a^n), \end{aligned} \tag{8.3}$$

for all $n \in \mathbb{N}$, by (7.34).

Now, we focus on the A-dynamical-projection C^*-algebra $\mathfrak{S}_A$. Define linear morphisms c_A and a_A acting on $\mathfrak{S}_A$ by the bounded linear transformations satisfying

$$c_A\left(T_{a,q,l}^{p,j}\right) = T_{a,q,l}^{p,j+1} \text{ and } a_A\left(T_{a,q,l}^{p,j}\right) = T_{a,q,l}^{p,j-1}, \tag{8.4}$$

in $\mathfrak{S}_A$, for all $a \in (A,\, \psi)$, $p,\, q \in \mathcal{P}$, and $j, l \in \mathbb{Z}$, where $T_{a,q,l}^{p,j}$ are generating operators (8.1) of $\mathfrak{S}_A$.

Thus, by (8.4), one has

$$c_A\left(T_a^{p,j}\right) = T_a^{p,j+1} \text{ and } a_A\left(T_a^{p,j}\right) = T_a^{p,j-1}, \tag{8.5}$$

in $\mathfrak{S}_A$, where $T_a^{p,j} = T_{a,p,j}^{p,j}$ are in the sense of (8.1).

By definition, one can understand the linear transformations c_A and a_A of (8.4) as elements of the *operator space* $B\left(\mathfrak{S}_A\right)$ (in the sense of [12]), consisting of all bounded linear transformations "on $\mathfrak{S}_A$." That is, by regarding our A-dynamical-projection algebra $\mathfrak{S}_A$ as a Banach space, the morphisms c_A and a_A of (8.4) are well-determined Banach-space operators on $\mathfrak{S}_A$.

Definition 8.1. Let c_A and a_A be in the sense of (8.4) in the operator space $B\left(\mathfrak{S}_A\right)$. Then we call c_A and a_A, the A-dynamical (Adelic-)creation and the A-dynamical (Adelic-)annihilation on the A-dynamical-projection C^*-algebra $\mathfrak{S}_A$, respectively. Define a new element $l_A \in B\left(\mathfrak{S}_A\right)$ by

$$l_A = c_A + a_A. \tag{8.6}$$

Then we call l_A, the A-dynamical (Adelic-)radial operator on $\mathfrak{S}_A$.

For any generating operator $T_{a,q,l}^{p,j} \in \mathfrak{S}_A$ of (8.1), one obtains that

$$\begin{aligned} c_A a_A\left(T_{a,q,l}^{p,j}\right) &= c_A\left(T_{a,q,l}^{p,j-1}\right) = T_{a,q,l}^{p,j} \\ \text{and } a_A c_A\left(T_{a,q,l}^{p,j}\right) &= a_A\left(T_{a,q,l}^{p,j+1}\right) = T_{a,q,l}^{p,j}, \end{aligned} \tag{8.7}$$

and hence,

$$c_A a_A = 1_{\mathfrak{S}_A} = a_A c_A \text{ on } \mathfrak{S}_A, \tag{8.8}$$

by (8.7), where $1_{\mathfrak{S}_A} \in B\left(\mathfrak{S}_A\right)$ is the identity operator,

$$1_{\mathfrak{S}_A}(T) = T \qquad \text{for all } T \in \mathfrak{S}_A.$$

By the relation (8.8), the following result is obtained.

Lemma 8.2. *Let c_A and a_A, respectively, be the A-dynamical creation and the A-dynamical annihilation* (8.4) *on $\mathfrak{S}_A$. Then*

$$c_A^{n_1} a_A^{n_2} = a_A^{n_2} c_A^{n_1} \text{ on } \mathfrak{S}_A, \tag{8.9}$$

for all $n_1,\, n_2 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with axiomatization

$$c_A^0 = 1_{\mathfrak{S}_A} = a_A^0 \text{ on } \mathfrak{S}_A.$$

Proof. The equality (8.9) holds for all $n_1, n_2 \in \mathbb{N}$, by induction on (8.8). Moreover, under the above axiomatization, the relation (8.9) holds for all $n_1,\ n_2 \in \mathbb{N}_0$, too. $\square$

Let l_A be the A-dynamical radial operator (8.6) on $\mathfrak{S}_A$. Then, by (8.9), one has that

$$l_A^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) c_A^k a_A^{n-k} \text{ on } \mathfrak{S}_A, \qquad \text{for all } n \in \mathbb{N}, \tag{8.10}$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} \qquad \text{for all } k \leq n \in \mathbb{N}_0.$$

Define now a cyclic Banach algebra $\mathfrak{L}_A$ by

$$\mathfrak{L}_A = \overline{\mathbb{C}[\{l_A\}]} \text{ in } B\left(\mathfrak{S}_A\right), \tag{8.11}$$

where $\overline{Y}$ means the operator-norm-topology closures of subsets Y of the operator space $B\left(\mathfrak{S}_A\right)$, where the operator-norm $\|.\|$ on $B\left(\mathfrak{S}_A\right)$ is defined to be

$$\|T\| = \sup \left\{ \|Tx\|_{\mathfrak{S}_A} \left| \begin{array}{c} x \in \mathfrak{S}_A, \text{ where} \\ \|x\|_{\mathfrak{S}_A} = 1 \end{array} \right. \right\},$$

where $\|.\|_{\mathfrak{S}_A}$ is the C^*-norm on $\mathfrak{S}_A$ (e.g., [13]).

From the construction (8.11), it is a well-defined Banach "algebra," embedded in the Banach space $B\left(\mathfrak{S}_A\right)$. Moreover, by the cyclicity of $\mathfrak{L}_A$, one can define the adjoint $(*)$ on it by

$$\left(\sum_{k=0}^{n} t_k l_A^{n_k}\right)^* = \sum_{k=0}^{n} \overline{t_k}\ l_A^{n_k},$$

where $t_k \in \mathbb{C}$ with their conjugates $\overline{t_k}$ in $\mathbb{C}$. Then it is a well-defined *adjoint on* $\mathfrak{L}_A$ (i.e., all elements are *adjointable* under $(*)$ in $B(\mathfrak{S}_A)$, in the sense of [12]), and hence, $\mathfrak{L}_A$ forms a *Banach $*$-algebra*. We call the Banach $*$-algebra $\mathfrak{L}_A$ of (8.11), the *A-dynamical radial (Banach-$*$-)algebra*.

Now, define the *tensor product Banach $*$-algebra* $\mathfrak{L}\mathfrak{S}_A$ by

$$\begin{aligned} \mathfrak{L}\mathfrak{S}_A &\overset{def}{=} \mathfrak{L}_A \otimes_{\mathbb{C}} \mathfrak{S}_A \\ &= \mathfrak{L}_A \otimes_{\mathbb{C}} \left((A \otimes_{\mathbb{C}} M_{\mathcal{P}}) \otimes_{\pi} \mathfrak{S} \right), \end{aligned} \tag{8.12}$$

where the first tensor product $\otimes_{\mathbb{C}}$ in the second equality of (8.12) means the (usual) *tensor product of Banach $*$-algebras*, and the second tensor product $\otimes_{\mathbb{C}}$ is the *tensor product of C^*-algebras*, where $\otimes_{\pi}$ is the *conditional tensor product* under the π-relation.

Definition 8.3. Let $\mathfrak{L}\mathfrak{S}_A$ be the tensor product Banach $*$-algebra (8.12) of the A-dynamical radial algebra $\mathfrak{L}_A$ of (8.11) and the A-dynamical projection algebra $\mathfrak{S}_A$. We call it the A-dynamical radial-projection (Banach-$*$-)algebra.

9. On the A-Dynamical Radial-Projection Algebra $\mathfrak{LS}_A$

Let $\mathfrak{LS}_A = \mathfrak{L}_A \otimes_{\mathbb{C}} \mathfrak{S}_A$ be the A-dynamical radial-projection algebra (8.12), where (A, ψ) is a fixed unital C^*-probability space and $\mathfrak{S}_A$ is the A-dynamical projection algebra (7.30) induced by our finite-Adelic C^*-algebra $M_{\mathcal{P}}$, and where $\mathfrak{L}_A$ is in the sense of (8.11).

Define a linear morphism $E_A : \mathfrak{LS}_A \to \mathfrak{S}_A$ by a surjective bounded linear transformation satisfying that

$$E_A \left(l_A^n l \otimes \left(\underset{p \in P, q \in Q}{\Pi'} T_{a,q,k_q}^{p,k_p} \right) \right)$$
$$\overset{def}{=} \begin{cases} \dfrac{\left(\underset{p \in P}{\Pi} \left(p^{k_p+1}\right)^{n+1}\right)\left(\underset{q \in Q}{\Pi} q^{k_q+1}\right)^{n+1}}{\left(\underset{p \in P}{\Pi} \phi(p)\right)\left(\underset{q \in P}{\Pi} \phi(q)\right)\left([\frac{n}{2}]+1\right)} l_A^n \left(T_{a,q,k_q}^{p,k_p}\right) & \text{if } (p, k_p) \neq (q, k_q), \\ \\ \dfrac{\underset{p \in P}{\Pi} \left(p^{k_p+1}\right)^{n+1}}{\left(\underset{p \in P}{\Pi} \phi(p)\right)\left([\frac{n}{2}]+1\right)} l_A^n \left(T_a^{p,k_p}\right) & \text{if } (p, k_p) = (q, k_q), \end{cases} \tag{9.1}$$

for all generating operators $l_A \otimes \left(T_{a,q,k_q}^{p,k_p}\right)$, satisfying that

$$\begin{aligned} \left(l_A \otimes \left(T_{a,q,k_q}^{p,k_p}\right)\right)^n &= l_A^n \otimes \left(\left(a^n \otimes \alpha^{q,k_q} \alpha^{p,k_p}\right) \otimes \alpha^{p,k_p}\right) \\ \text{and } \left(l_A \otimes \left(T_a^{p,k_p}\right)\right)^n &= l_A^n \otimes \left(T_{a^n}^{p,k_p}\right), \end{aligned} \tag{9.2}$$

for all $n \in \mathbb{N}_0$, $a \in (A, \psi)$, and for all finite subsets P and Q of $\mathcal{P}$, for $k_r \in \mathbb{Z}$ and $r \in P$, where $[\frac{n}{2}]$ mean the *minimal integers greater than or equal to* $\frac{n}{2}$ for all $n \in \mathbb{N}$; for instance,

$$[\tfrac{3}{2}] = 2 = [\tfrac{4}{2}].$$

Recall that the tensor factors T_{a,q,k_q}^{p,k_p} and $T_a^{p,k_p} = T_{a,p,k_p}^{p,k_p}$ in (9.2) are the generating operators (8.1) of $\mathfrak{S}_A$.

This morphism E_A of (9.1) is indeed a well-defined bounded linear transformation from $\mathfrak{LS}_A$ "onto" $\mathfrak{S}_A$ because of (8.11), (8.12), (7.30), and (6.20).

Now, on $\mathfrak{LS}_A$, define linear functionals $\tau_{p,j}^A$ by the bounded linear morphism satisfying

$$\tau_{p,j}^A = \varphi_{p,j}^A \circ E_A \text{ on } \mathfrak{LS}_A, \tag{9.3}$$

where $\varphi_{p,j}^A$ are in the sense of (7.23) satisfying (7.27) for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Note that, by the well-definedness of the linear functional φ_A of (7.29), these linear functionals (9.3) are well-defined.

Definition 9.1. The well-defined Banach $*$-probability spaces

$$\mathfrak{LS}_A(p,j) \overset{denote}{=} \left(\mathfrak{LS}_A,\ \tau^A_{p,j}\right), \tag{9.4}$$

are called the A-dynamical (radial-projection-)(p,j)-filterization of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Let $\mathfrak{LS}_A(p,j)$ be the A-dynamical (p,j)-filterization (9.4) of $M_{\mathcal{P}}$. Then one can get the following free-distributional data.

Theorem 9.2. *Let $\mathfrak{LS}_A(p,j) = \left(\mathfrak{LS}_A,\ \tau^A_{p,j}\right)$ be the A-dynamical (p,j)-filterization of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$ for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Let $a \in (A,\ \psi)$, and let*

$$U_a^{p,j} = l_A \otimes \left(T_a^{p,j}\right) \in \mathfrak{LS}_A(p,j), \tag{9.5}$$

where $T_a^{p,j} = T_{a,p,j}^{p,j} \in \mathfrak{S}_A$ is in the sense of (8.1). Then

$$\begin{aligned} \tau^A_{p,j}\left(\left(T_a^{p,j}\right)^n\right) &= \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)\left(\psi\left(a^n\right)\right), \\ \text{where } \omega_n &= \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \tag{9.6}$$

for all $n \in \mathbb{N}$, and where

$$c_m = \tfrac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix} = \tfrac{(2m)!}{m!(m+1)!}$$

are the m-th Catalan numbers for all $m \in \mathbb{N}_0$.

Proof. Let $U_a^{p,j}$ be in the sense of (9.5) in the A-dynamical (p,j)-filterization $\mathfrak{LS}_A(p,j)$, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Then

$$\left(U_a^{p,j}\right)^n = \left(l_A \otimes \left(T_a^{p,j}\right)\right)^n = l_A^n \otimes \left(T_{a^n}^{p,j}\right), \tag{9.7}$$

by (9.2) with identity $a^0 = 1_A$ (for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$).
So, one can get that:

$$\begin{aligned} \tau^A_{p,j}\left(\left(U_a^{p,j}\right)^n\right) &= \tau^A_{p,j}\left(l_A^n \otimes \left(T_{a^n}^{p,j}\right)\right) \quad \text{by (9.7)} \\ &= \varphi^A_{p,j}\left(E_A\left(l_A^n \otimes T_{a^n}^{p,j}\right)\right) \\ &= \left(\varphi^A_{p,j}\right)\left(\frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1}\ l_A^n\left(T_{a^n}^{p,j}\right)\right) \end{aligned}$$

$$= \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1}\varphi^A_{p,j}\left(l_A^n\left(\left(a^n \otimes \alpha^{p,j}\right) \otimes\ \alpha^{p,j}\right)\right) \tag{9.8}$$

for all $n \in \mathbb{N}$.

Observe now that, for any $n \in \mathbb{N}$,

$$\begin{aligned} l_A^{2n-1} &= \sum_{k=0}^{2n-1} \binom{2n-1}{k} c_A^k a_A^{2n-k-1} \\ \text{and } l_A^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} c_A^k a_A^{2n-k}, \end{aligned} \tag{9.9}$$

by (8.10), where c_A and a_A are the A-dynamical creation and A-dynamical annihilation on $\mathfrak{S}_A$, respectively.

Thus, by (9.9), one can realize that l_A^{2n-1} does not contain $1_{\mathfrak{S}_A}$-terms, and l_A^{2n} contains its $1_{\mathfrak{S}_A}$-term,

$$\binom{2n}{n} c_A^n a_A^n = \binom{2n}{n} (c_A a_A)^n = \binom{2n}{n} 1_{\mathfrak{S}_A},$$

for all $n \in \mathbb{N}$, with help of (8.8) and (8.9).

So, the formula (9.8) goes to

$$\tau_{p,j}^A \left((T_a^{p,j})^n \right) = \frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)\left([\frac{n}{2}]+1\right)} \, \varphi_{p,j}^A \left(l_A^n \left(T_{a^n}^{p,j} \right) \right)$$

$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)\left(\frac{n}{2}+1\right)} \right) \varphi_{p,j}^A \left(l_A^n \left(T_{a^n}^{p,j} \right) \right)$$

where

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{9.10}$$

by (9.9), and hence,

$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)\left(\frac{n}{2}+1\right)} \right) \varphi_{p,j}^A \left(\left(\binom{n}{\frac{n}{2}} \right) T_{a^n}^{p,j} + [\text{Rest terms}] \right)$$

by (8.10)

$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)\left(\frac{n}{2}+1\right)} \right) \varphi_{p,j}^A \left(\left(\binom{n}{\frac{n}{2}} \right) T_{a^n}^{p,j} \right)$$

by (9.3)

$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)\left(\frac{n}{2}+1\right)} \right) \binom{n}{\frac{n}{2}} \varphi_{p,j}\left(\alpha^{p,j}\right) \psi\left(a^n\right)$$

by (9.2)

$$= \omega_n \left(\frac{\left(p^{j+1}\right)^{n+1}}{\phi(p)\left(\frac{n}{2}+1\right)} \right) \left(\frac{\frac{n}{2}+1}{\frac{n}{2}+1} \binom{n}{\frac{n}{2}} \right) \left(\frac{\phi(p)}{p^{j+1}} \right) \psi\left(a^n\right)$$

by (6.13)

$$= \left(\omega_n \left(p^{j+1}\right)^n c_{\frac{n}{2}}\right) \psi\left(a^n\right)$$

for all $n \in \mathbb{N}$, where

$$c_m = \frac{1}{m+1} \left(\begin{array}{c} 2m \\ m \end{array} \right) = \frac{(2m)!}{m!(m+1)!}$$

are the m-th Catalan numbers for all $m \in \mathbb{N}_0$.

Therefore, if $U_a^{p,j}$ is a free random variable (9.5) in the A-dynamical (p,j)-filterization $\mathfrak{L}\mathfrak{S}_A(p,j)$ of (9.4), then

$$\tau_{p,j}^A \left((U_a^{p,j})^n \right) = \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}} \right) \left(\psi(a^n)\right),$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (9.10). Therefore, the free-distributional data (9.6) holds. $\square$

In the above theorem, if $a \in (A, \psi)$ is self-adjoint, then a generating operator $U_a^{p,j}$ of (9.5) is self-adjoint in $\mathfrak{L}\mathfrak{S}_A$, too. Indeed,

$$\begin{aligned} \left(U_a^{p,j}\right)^* &= \left(l_A \otimes T_a^{p,j}\right)^* \\ &= l_A \otimes \left(\left(a^* \otimes \alpha^{p,j}\right) \otimes \alpha^{p,j}\right) = U_a^{p,j}, \end{aligned}$$

in $\mathfrak{L}\mathfrak{S}_A$, since $a^* = a$ in A, under the π-relation on tensor-factor $\mathfrak{S}_A$ of $\mathfrak{L}\mathfrak{S}_A$.

Therefore, if a is self-adjoint in (A, ψ), then, by the self-adjointness of $U_a^{p,j}$, the above formula (9.6) fully characterizes the free distribution of $U_a^{p,j}$ in the A-dynamical (p,j)-filterization $\mathfrak{L}\mathfrak{S}_A(p,j)$ for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Corollary 9.3. *Let $T_{1_A}^{p,j} = l_A \otimes \left(T_{1_A}^{p,j}\right)$ be in the sense of (9.5) in $\mathfrak{L}\mathfrak{S}_A(p,j)$, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, where 1_A is the unit of A. Then*

$$\tau_{p,j}^A \left(\left(U_{1_A}^{p,j}\right)^n \right) = \omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{9.11}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (9.6) and $c_{\frac{n}{2}}$ are the $\left(\frac{n}{2}\right)$-th Catalan numbers.

Proof. Since (A, ψ) is assumed to be a unital C^*-probability space, we have

$$\psi(1_A^n) = \psi(1_A) = 1 \qquad \text{for all } n \in \mathbb{N}.$$

Thus, one can get that

$$\tau_{p,j}^A \left(\left(U_{1_A}^{p,j}\right)^n \right) = \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}} \right) \cdot \psi(1_A^n), \tag{9.12}$$

by (9.6), for all $n \in \mathbb{N}$. So, the free distribution (9.11) holds for $U_{1_A}^{p,j}$ in $\mathfrak{L}\mathfrak{S}_A(p,j)$ by (9.12). $\square$

Also, one can get the following corollary, too.

Corollary 9.4. *Let $U_a^{p,j} = l_A \otimes T_a^{p,j}$ be in the sense of (9.5) in the A-dynamical (p,j)-filterization $\mathfrak{LS}_A(p,j)$ of $M_{\mathcal{P}}$ for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. Assume that a is self-adjoint and that*

$$\psi(a^n) = \psi(a)^n \qquad \text{for all } n \in \mathbb{N}. \tag{9.13}$$

Then

$$\tau_{p,j}^{A}\left(\left(U_a^{p,j}\right)^n\right) = \omega_n \left(p^{2(j+1)}\psi(a)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{9.14}$$

for all $n \in \mathbb{N}$.

Proof. Let $U_a^{p,j}$ be in the sense of (9.5) in $\mathfrak{LS}_A(p,j)$, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, where a given self-adjoint operator a satisfies the additional conditions (9.13) in (A, ψ). Then

$$\tau_{p,j}^{A}\left((U_a^{p,j})^n\right) = \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)\left(\psi\left(a^n\right)\right)$$

by (9.6)

$$= \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)\left(\psi(a)\right)^n$$

by (9.13)

$$= \omega_n \left(p^{2(j+1)}\psi(a)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$. So, the free-distributional data (9.14) holds for $U_a^{p,j}$ in $\mathfrak{LS}_A(p,j)$, under (9.13). ☐

The above free-probabilistic results, expressed by (9.6), (9.11), and (9.14), not only generalize the main results of [10], but also universalize the main results of [7, 11].

10. Weighted-Semicircular Elements in $\mathfrak{LS}_A$

In this section, we use same concepts and notations in previous sections. Let $\mathfrak{LS}_A$ be the A-dynamical radial-projection algebra, and let

$$\mathfrak{LS}_A(p,j) = \left(\mathfrak{LS}_A,\ \tau_{p,j}^{A}\right)$$

be the A-dynamical (p,j)-filterizations, for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

For fixed $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, and a self-adjoint $a \in (A, \psi)$, the operator $U_a^{p,j}$,

$$U_a^{p,j} = l_A \otimes \left(T_a^{p,j}\right) \in \mathfrak{LS}_A \tag{10.1}$$

has its free distribution determined by

$$\tau_{p,j}^{A}\left(\left(U_a^{p,j}\right)^n\right) = \left(\omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right)\left(\psi(a^n)\right), \tag{10.2}$$

for all $n \in \mathbb{N}$, by (9.6).

Moreover, if the fixed self-adjoint operator $a \in A$ satisfies

$$\psi(a^n) = \left(\psi(a)\right)^n \qquad \text{for all } n \in \mathbb{N}; \tag{10.3}$$

then

$$\tau_{p,j}^{A}\left(\left(U_a^{p,j}\right)^n\right) = \omega_n \left(p^{2(j+1)}\psi(a)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{10.4}$$

for all $n \in \mathbb{N}$, by (9.14).

10.1. **Weighted-Semicircular and Semicircular Elements.** Let (B, φ_B) be an arbitrary *topological $*$-probability space*, where B is a *topological $*$-algebra* and φ_B is a (bounded, or unbounded) *linear functional on* B.

Definition 10.1. A self-adjoint free random variable b is said to be weighted-semicircular in (B, φ_B) with weight $t_0 \in \mathbb{C}$, (or in short, t_0-semicircular in (B, φ_B)), if b satisfies the free-cumulant computation,

$$k_n^B(b, \ldots, b) = \begin{cases} k_2^B(b,\ b) = t_0 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{10.5}$$

for all $n \in \mathbb{N}$, where $k_n^B(\ldots)$ is the free cumulant on B (in the sense of [31]) with respect to φ_B under the Möbius inversion of [31].

A self-adjoint free random variable b is semicircular in (B, φ), if b is 1-semicircular in the sense of (10.5); that is,

$$k_n^B(b, \ldots, b) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{10.6}$$

for all $n \in \mathbb{N}$.

By the *Möbius inversion of* [31], one can characterize the weighted-semicircularity (10.5) as follows: a self-adjoint element b is t_0-semicircular in (B, φ_B) if and only if

$$\varphi_B(b^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}}\right), \tag{10.7}$$

where ω_n are in the sense of (9.6) for all $n \in \mathbb{N}$, and c_k are the k-th Catalan numbers for all $k \in \mathbb{N}_0$.

Similarly,a free random variable b is semicircular in (B, φ_B) if and only if b is 1-semicircular in (B, φ_B), if and only if

$$\varphi_B(b^n) = \omega_n c_{\frac{n}{2}}, \tag{10.8}$$

by (10.6) for all $n \in \mathbb{N}$.

So, we use the t_0-semicircularity (10.5) (resp., the semicircularity (10.6)) and its characterization (10.7) (resp., (10.8)) alternatively.

10.2. **Weighted-Semicircular Elements in $\mathfrak{LS}_A(p, j)$.** For $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, let $\mathfrak{LS}_A(p, j)$ be the A-dynamical (p, j)-filterization, and let

$$U_a^{p,j} = l_A \otimes T_a^{p,j} = l_A \otimes ((a \otimes \alpha^{p,j}) \otimes \alpha^{p,j})$$

be in the sense of (10.1) in $\mathfrak{LS}_A(p, j)$, where a is self-adjoint in (A, ψ), and hence, having its free distribution (10.2).

By (10.5) and (10.7), one can obtain the following weighted-semicircular elements in $\mathfrak{LS}_A(p, j)$ for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Theorem 10.2. *Let $U_a^{p,j}$ be in the sense of (10.1) in the A-dynamical (p, j)-filterization $\mathfrak{LS}_A(p, j)$ of $M_{\mathcal{P}}$, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, where a self-adjoint operator a satisfies the following additional condition (10.3) in (A, ψ); that is,*

$$\psi(a^n) = (\psi(a))^n \qquad \textit{for all } n \in \mathbb{N}.$$

Then $U_a^{p,j}$ is $(p^{j+1}\psi(a))^2$-semicircular in $\mathfrak{LS}_A(p, j)$. That is,

$$a \textit{ satisfies } (10.3) \;\Rightarrow U_a^{p,j} \textit{ is } \left(p^{j+1}\psi(a)\right)^2\textit{-semicircular}, \tag{10.9}$$

for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Proof. Suppose a self-adjoint free random variable a satisfies the additional condition (10.3) in a fixed unital C^*-probability space (A, ψ). Then, by (10.2) and (10.4), one obtains the free distribution of the operator $U_a^{p,j}$ of (10.1), determined by the free moments,

$$\tau_{p,j}^A\left((U_a^{p,j})^n\right) = \omega_n \left(\left(p^{j+1}\psi(a)\right)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}}$$

for all $n \in \mathbb{N}$.

So, by (10.7), this operator $U_a^{p,j}$ is $(p^{j+1}\psi(a))^2$-semicircular in $\mathfrak{LS}_A(p, j)$, since it is self-adjoint in the A-dynamical radial-projection algebra $\mathfrak{LS}_A$. Therefore, the statement (10.9) holds true. □

As a corollary of the weighted-semicircularity (10.9) on $\mathfrak{LS}_A(p, j)$, we have the following result.

Corollary 10.3. *Let $U_{1_A}^{p,j}$ be in the sense of (10.1) in the A-dynamical (p, j)-filterization $\mathfrak{LS}_A(p, j)$, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$, where 1_A is the unit of A. Then this operator $U_{1_A}^{p,j}$ is $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_A(p, j)$.*

$$U_{1_A}^{p,j} \textit{ is } p^{2(j+1)}\textit{-semicircular in } \mathfrak{LS}_A(p, j), \tag{10.10}$$

for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Proof. First of all, this operator $U_{1_A}^{p,j}$ is self-adjoint in $\mathfrak{LS}_A$,, and the free distribution of it is determined by the free-moments,

$$\tau_{p,j}^A\left(\left(T_{1_A}^{p,j}\right)^n\right) = \omega_n \left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \qquad \text{for all } n \in \mathbb{N},$$

by (9.11). Therefore, one can conclude that $U_a^{p,j}$ is $p^{2(j+1)}$-semicircular in $\mathfrak{LS}_A(p, j)$.

Alternatively, the weighted-semicircularity (10.9) directly allows the $p^{2(j+1)}$-semicircularity of $U_{1_A}^{p,j}$, because 1_A is a self-adjoint element of (A, ψ) satisfying the condition (10.3). □

The main results (10.9) and (10.10) of this section show that, starting from the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, and the semigroup-dynamical systems of $\sigma(A_{\mathbb{Q}})$, one can construct weighted-semicircular elements. Therefore, they generalize (operator-theoretically) and globalize (number-theoretically) the weighted-semicircularity of [7, 10, 11].

11. Semicircular Elements in $\mathfrak{LS}_A(p, j)$

Let $\mathfrak{LS}_A(p, j)$ be the A-dynamical (p, j)-filterization, for $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. In section 10.2, we considered weighted-semicircular elements in $\mathfrak{LS}_A(p, j)$. In particular, if an operator $U_a^{p,j}$ of (10.1) satisfies the condition (10.3), then it is $\left(p^{2(j+1)}\psi(a)^2\right)$-semicircular in $\mathfrak{LS}_A(p, j)$, by (10.9).

In this section, we fix $p \in \mathcal{P}$, $j \in \mathbb{Z}$, and the corresponding A-dynamical (p, j)-filterization $\mathfrak{LS}_A(p, j)$. Also, define an operator,

$$S_a^{p,j} = \frac{1}{p^{j+1}} U_a^{p,j} \in \mathfrak{LS}_A(p, j), \tag{11.1}$$

where $U_a^{p,j}$ is in the sense of (10.1), and where a is self-adjoint in (A, ψ).

Theorem 11.1. *Let $S_a^{p,j} = \frac{1}{p^{j+1}} U_a^{p,j}$ be in the sense of (11.1) in the A-dynamical (p, j)-filterization $\mathfrak{LS}_A(p, j)$, where a is self-adjoint in (A, ψ). Assume further that a satisfies the condition (10.3); that is,*

$$\psi\left(a^n\right) = \left(\psi(a)\right)^n \qquad \text{for all } n \in \mathbb{N}.$$

Then the operator $S_a^{p,j}$ is $\psi(a)^2$-semicircular in $\mathfrak{LS}_A(p, j)$. That is,

$$a \text{ satisfies (10.3)} \ \Rightarrow S_a^{p,j} \text{ is } \psi(a)^2\text{-semicircular.} \tag{11.2}$$

Proof. Let $a \in (A, \psi)$ be a self-adjoint free random variable satisfying the condition (10.3). Now, let $k_n^{A,p,j}(\dots)$ be the free cumulant (in the sense of [31]) on the A-dynamical radial-projection algebra $\mathfrak{LS}_A$ in terms of the linear functional $\tau_{p,j}^A$. Then

$$k_n^{A,p,j}\left(\underbrace{S_a^{p,j},\ S_a^{p,j}, \dots,\ S_a^{p,j}}_{n\text{-times}}\right) = k_n^{A,p,j}\left(\tfrac{1}{p^{j+1}}U_a^{p,j}, \dots,\ \tfrac{1}{p^{j+1}}U_a^{p,j}\right)$$

by (11.1)

$$= \left(\tfrac{1}{p^{j+1}}\right)^n k_n^{A,p,j}\left(U_a^{p,j},\ U_a^{p,j}, \dots,\ U_a^{p,j}\right)$$

by the bimodule map property of free cumulants (e.g., [31])

$$= \begin{cases} \left(\tfrac{1}{p^{j+1}}\right)^2 k_2^{A,p,j}\left(U_a^{p,j},\ U_a^{p,j}\right) & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

by the weighted-semicircularity (10.9) of $U_a^{p,j}$

$$= \begin{cases} \left(\frac{1}{p^{j+1}}\right)^2 \left(p^{2(j+1)}\psi(a)^2\right) & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

by (10.9)

$$= \begin{cases} \psi(a)^2 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Therefore, by the weighted-semicircularity (10.5), this operator $S_a^{p,j}$ is $\psi(a)^2$-semicircular in $\mathfrak{LS}_A(p, j)$. □

By the weighted-semicircularity (11.2), we have the following semicircularity on $\mathfrak{LS}_A(p, j)$. Recall first that we automatically assume $\psi(a) \neq 0$ from section 8.

Theorem 11.2. *Let $S_a^{p,j} \in \mathfrak{LS}_A(p, j)$ be in the sense of (11.1). Now, suppose a self-adjoint $a \in (A, \psi)$ satisfies (10.3), and let*

$$X_a^{p,j} = \frac{1}{\psi(a)} S_a^{p,j} = \frac{1}{p^{j+1}\psi(a)} U_a^{p,j} \in \mathfrak{LS}_A(p, j).$$

Then this free random variable $X_a^{p,j}$ is semicircular in $\mathfrak{LS}_A(p, j)$; that is,

$$a \text{ satisfies (10.3)} \Rightarrow \frac{1}{p^{j+1}\psi(a)} U_a^{p,j} \text{ is semicircular in } \mathfrak{LS}_A(p, j). \tag{11.3}$$

Proof. The proof of (11.3) is similar to that of (11.2). But, here, we provide a different type of proofs. Let $X_a^{p,j}$ be as above, where a satisfies the condition (10.3) in (A, ψ). Then

$$\tau_{p,j}^A\left((X_a^{p,j})^n\right) = \tau_{p,j}^A\left(\left(\frac{1}{p^{j+1}\psi(a)} U_a^{p,j}\right)^n\right)$$

$$= \left(\frac{1}{p^{j+1}\psi(a)}\right)^n \tau_{p,j}^A\left((U_a^{p,j})^n\right)$$

$$= \left(\frac{1}{p^{j+1}\psi(a)}\right)^n \left(\omega_n \left(p^{j+1}\psi(a)\right)^n c_{\frac{n}{2}}\right)$$

by the $\left(p^{j+1}\psi(a)\right)^2$-semicircularity of $U_a^{p,j}$ under (10.3)

$$= \omega_n c_{\frac{n}{2}}$$

for all $n \in \mathbb{N}$. Therefore, by (10.8), this free random variable $X_a^{p,j}$ is semicircular in $\mathfrak{LS}_A(p, j)$. So, the statement (11.3) holds. □

The main result (11.3) of this section generalize and globalize the semicircularity of [7, 10, 11].

By the weighted-semicircularity (11.2), one also obtains the following semicircularity on $\mathfrak{LS}_A(p, j)$ independent from (11.3).

Corollary 11.3. *Let $S_{1_A}^{p,j} = \frac{1}{p^{j+1}} U_{1_A}^{p,j}$ be in the sense of (11.1) in $\mathfrak{LS}_A(p, j)$. Then it is semicircular in $\mathfrak{LS}_A(p, j)$. That is,*

$$S_{1_A}^{p,j} \text{ is semicircular in } \mathfrak{LS}_A(p,j). \tag{11.4}$$

Proof. Since the unit 1_A of $(A,\ \psi)$ satisfies the self-adjointness and

$$\psi(1_A^n) = \psi(1_A) = 1 = \psi(1_A)^n,$$

for all $n \in \mathbb{N}$, the operator $S_{1_A}^{p,j}$ is semicircular in $\mathfrak{LS}_A(p,j)$, by (11.3). Therefore, the statement (11.4) holds. □

12. Weighted-Semicircularity and Semicircularity on $\mathfrak{LS}_A$

In this section, we globalize the main results of sections 9 and 10. Let $(A,\ \psi)$ be a fixed unital C^*-probability space as above, and let

$$\begin{aligned}\mathfrak{LS}_A &= \mathfrak{L}_A \otimes_{\mathbb{C}} \mathfrak{S}_A \\ &= \mathfrak{L}_A \otimes_{\mathbb{C}} ((A \otimes_{\mathbb{C}} M_{\mathcal{P}}) \otimes_\pi \mathfrak{S})\end{aligned}$$

be the A-dynamical radial-projection algebra.

By defining linear functionals $\tau_{p,j}^A = \varphi_{p,j}^A \circ E_A$ of (9.3) on $\mathfrak{LS}_A$, one obtains the corresponding A-dynamical (p,j)-filterizations (9.4),

$$\mathfrak{LS}_A(p,j) = \left(\mathfrak{LS}_A,\ \tau_{p,j}^A\right) \tag{12.1}$$

for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Define now a new linear functional τ_A on $\mathfrak{LS}_A$ by a linear transformation,

$$\tau_A = \varphi_A \circ E_A = \left(\prod_{p\in\mathcal{P}} \left(\sum_{j\in\mathbb{Z}}^{\oplus} \varphi_{p,j}^A\right)\right) \circ E_A, \tag{12.2}$$

where φ_A is in the sense of (7.28) and (7.29) and E_A is in the sense of (9.1). Then this linear functional τ_A of (12.2) is well-defined, and it globalize our linear functionals $\{\tau_{p,j}^A\}_{p\in\mathcal{P},\ j\in\mathbb{Z}}$ on $\mathfrak{LS}_A$. So, the pair $(\mathfrak{LS}_A,\ \tau_A)$ forms a well-defined Banach $*$-probability space.

Definition 12.1. The Banach $*$-probability space

$$\mathbb{LS}_A \overset{denote}{=} (\mathfrak{LS}_A,\ \tau_A) \tag{12.3}$$

is called the A-dynamical filterization of the finite-Adelic C^*-algebra $M_{\mathcal{P}}$, where τ_A is the linear functional (12.2) on the A-dynamical radial-projection algebra $\mathfrak{LS}_A$.

On the A-dynamical filterization $\mathbb{LS}_A$ of (12.3), we obtain the following weighted-semicircularity.

Theorem 12.2. *Let $\mathbb{LS}_A = (\mathfrak{LS}_A,\ \tau_A)$ be the A-dynamical filterization (12.3). Suppose that $a \in (A,\ \psi)$ is a self-adjoint free random variable satisfying that*

$$\psi(a^n) = \psi(a)^n \text{ in } \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \tag{12.4}$$

for all $n \in \mathbb{N}$. Then the operator $U_a^{p,j}$ of (10.1) is $\left(p^{j+1}\psi(a)\right)^2$-semicircular in $\mathfrak{LS}_A$ for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$. That is,

$$a \text{ satisfies } (12.4) \ \Rightarrow U_a^{p,j} \text{ is } \left(p^{j+1}\psi(a)\right)^2\text{-semicircular}, \tag{12.5}$$

in $\mathfrak{LS}_A^0$ for all $p \in \mathcal{P}$ and $j \in \mathbb{Z}$.

Proof. Let $U_a^{p,j}$ be in the sense of (10.1) in the A-dynamical radial-projection algebra $\mathfrak{LS}_A$, where a self-adjoint free random variable $a \in (A,\ \psi)$ satisfies the additional condition (12.4). Then $U_a^{p,j}$ is self-adjoint in $\mathfrak{LS}_A$, and it satisfies that

$$\tau_A\left(\left(U_{p,j}^a\right)^n\right) = \tau_{p,j}^A\left(\left(U_{p,j}^a\right)^n\right)$$

by (12.2)

$$= \omega_n \left(p^{2(j+1)}\psi(a)^2\right)^n c_{\frac{n}{2}},$$

by (10.9), for all $n \in \mathbb{N}$.

Therefore, this operator $U_a^{p,j} \in \mathfrak{LS}_A$ is $\left(p^{j+1}\psi(a)\right)^2$-semicircular in the A-dynamical filterization $\mathfrak{LS}_A$. It proves the statement (12.5) holds under condition (12.4). □

By the above weighted-semicircularity (12.5), one obtains the following corollary.

Corollary 12.3. *Let $U_a^{p,j}$ be in the sense of (10.1) in the A-dynamical filterization $\mathfrak{LS}_A$, where a self-adjoint free random variable $a \in (A,\ \psi)$ satisfies (12.4).*

$$\frac{1}{p^{j+1}} U_a^{p,j} \text{ is } \psi(a)^2\text{-semicircular in } \mathfrak{LS}_A. \tag{12.6}$$

$$\frac{1}{p^{j+1}\psi(a)} U_a^{p,j} \text{ is semicircular in } \mathfrak{LS}_A. \tag{12.7}$$

$$U_{1_A}^{p,j} \text{ is } p^{2(j+1)}\text{-semicircular in } \mathfrak{LS}_A. \tag{12.8}$$

$$\frac{1}{p^{j+1}} U_{1_A}^{p,j} \text{ is semicircular in } \mathfrak{LS}_A. \tag{12.9}$$

Proof. The proofs of (12.6), (12.7), (12.8), and (12.9) are done by (12.5), with help of the main results of sections 10.2 and 10.3. □

The weighted-semicircularity (12.4) and its special cases (12.6), (12.7), (12.8), and (12.9) not only generalize the main results of [7, 10, 11] (operator-theoretically), but also globalize those (number-theoretically).

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