Adv. Oper. Theory 4 (2019), no. 1, 1–23
https://doi.org/10.15352/aot.1712-1280
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

THE BISHOP-PHELPS-BOLLOBÁS MODULUS FOR FUNCTIONALS ON CLASSICAL BANACH SPACES

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Communicated by P. Aiena

Abstract. In this manuscript, we compute the Bishop-Phelps-Bollobás modulus for functionals in classical Banach spaces, such as Hilbert spaces, spaces of continuous functions c_0 and ℓ_1.

1. Introduction

Very recently, the authors of [4] introduced a modulus that measures the accuracy of the Bishop-Phelps-Bollobás theorem (BPBt) in every Banach space. In [1, Notation 1.3] the authors introduce a function that trivially characterizes whether a pair of Banach spaces enjoy the Bishop-Phelps-Bollobás property for operators (BPBpo). However, this function does not measure the accuracy of the BPBpo in a pair of Banach spaces enjoying the BPBpo. In [2] the authors find an equivalent reformulation of the previous function for pairs of Banach spaces enjoying the BPBpo which also measures the accuracy of the BPBpo in such pairs. This equivalent reformulation is known as the BPBmo. In this manuscript we compute the BPBmo for pairs of Banach spaces in which the first space is a classical Banach space and the second space is the real field.

Let us introduce a bit of notation.

Let $\mathcal{L}(X,Y)$ be the Banach space of all linear and continuous operators from the Banach space X into the Banach space Y. We define the sets $\mathrm{suppv}(T) := \{x \in \mathsf{S}_X : \|T(x)\| = \|T\|\}$ and $\mathcal{P}(X,Y) := \{T \in \mathcal{L}(X,Y) : \mathrm{suppv}(T) \neq \varnothing\}$. We

Date: Received: Dec. 24, 2017; Accepted: Mar. 11, 2018.
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2000 *Mathematics Subject Classification.* Primary 47A05; Secondary 46B20.
Key words and phrases. Bishop-Phelps-Bollobás, modulus, classical Banach spaces.

will consider only real Banach spaces, although most of the results can also be adapted to the complex case.

For Banach spaces X and Y and for $\varepsilon \in (0, 1/2)$ and $\delta > 0$, we define the following sets:

- $\Pi_{X,Y} := \{(x, T) \in \mathsf{S}_X \times \mathsf{S}_{\mathcal{P}(X,Y)} : x \in \operatorname{suppv}(T)\}$, which we will consider endowed with the metric inherited from $X \oplus_\infty \mathcal{L}(X, Y)$.
- $A_{X,Y}(\delta) := \left\{(x, T) \in \mathsf{S}_X \times \mathsf{S}_{\mathcal{L}(X,Y)} : \|T(x)\| > 1 - \delta\right\}$.
- $P_{X,Y}(\varepsilon) := \left\{(x, T) \in \mathsf{S}_X \times \mathsf{S}_{\mathcal{L}(X,Y)} : d((x, T), \Pi_{X,Y}) \geq \varepsilon\right\}$.
- $A_{X,Y}(\varepsilon, \delta) := A_{X,Y}(\delta) \setminus P_{X,Y}(\varepsilon)$.
- $a_{X,Y}(\varepsilon) := \{\delta > 0 : A_{X,Y}(\varepsilon, \delta) = A_{X,Y}(\delta)\}$.

With all these we can trivially state that (X, Y) has the BPBpo if and only if $a_{X,Y}(\varepsilon) \neq \varnothing$ for every $\varepsilon \in (0, 1/2)$.

To end this introduction we will recall the definition of the BPBmo, introduced in [2, Definition 2.1].

Definition 1.1 (BPBmo [2])**.** *Let X and Y be Banach spaces such that (X, Y) has the BPBpo. The BPBmo of (X, Y) is the function*

$$\begin{array}{rccl} C_{X,Y}: & (0, 1/2) & \to & (0, +\infty] \\ & \varepsilon & \mapsto & C_{X,Y}(\varepsilon) := \sup a_{X,Y}(\varepsilon). \end{array}$$

2. The BPBmo for $(X, \mathbb{R})$ with X a classical Banach space

We will compute the BPBmo for $(X, \mathbb{R})$ with X a classical Banach space. For simplicity and to ease the calculations, the BPBmo is computed here in a slightly different way to the one defined in Definition 1.1. This difference relies on the fact that the sets used to define the BPBmo are now the following, where $\varepsilon \in (0, 1/2)$ and $\delta > 0$:

- $\Pi_X := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*(x) = 1\}$.
- $A_X(\delta) := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : x^*(x) \geq 1 - \delta\}$.
- $P_X(\varepsilon) := \{(x, x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} : d((x, x^*), \Pi_X) \geq \varepsilon\}$.
- $A_X(\varepsilon, \delta) := A_X(\delta) \setminus P_X(\varepsilon)$.
- $a_X(\varepsilon) := \{\delta > 0 : A_X(\varepsilon, \delta) = A_X(\delta)\}$.

Now

$$\begin{array}{rccl} C_X: & (0, 1/2) & \to & (0, +\infty) \\ & \varepsilon & \mapsto & C_X(\varepsilon) := \sup a_X(\varepsilon) \end{array}$$

Lemma 2.1. *Let X be a Banach space with $\dim(X) \geq 2$. For every $x \in \mathsf{S}_X$ and every $x^* \in \mathsf{S}_{X^*}$ with $x^*(x) < 1$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$ converging to x such that $x^*(x) < x^*(x_n)$ for all $n \in \mathbb{N}$.*

Proof. Fix $x_0 \in \mathsf{S}_X$ such that $x^*(x) < x^*(x_0) \leq 1$. For every $n \in \mathbb{N} \setminus \{2\}$ define

$$x_n := \frac{\frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)x}{\left\|\frac{1}{n}x_0 + \left(1 - \frac{1}{n}\right)x\right\|}.$$

First, let us show that x_n is well defined. If $\frac{1}{n}x_0 + \left(1-\frac{1}{n}\right)x = 0$ for some $n \in \mathbb{N} \setminus \{2\}$, then

$$\frac{1}{n} = \left\|\frac{1}{n}x_0\right\| = \left\|-\left(1-\frac{1}{n}\right)x\right\| = 1 - \frac{1}{n},$$

which implies the contradiction that $n = 2$. Finally observe that

$$x^*(x_n) \geq x^*\left(\frac{1}{n}x_0 + \left(1-\frac{1}{n}\right)x\right) > x^*(x).$$

□

Theorem 2.2. *Let X be a Banach space with $\dim(X) \geq 2$. Fix $\varepsilon \in (0, 1/2)$.*

1. $a_X(\varepsilon) \subseteq a_{X,\mathbb{R}}(\varepsilon)$.
2. *For all $\delta \in a_{X,\mathbb{R}}(\varepsilon)$ and all $(x, x^*) \in A_X(\delta)$, $d((x, x^*), \Pi_X) \leq \varepsilon$.*
3. $a_{X,\mathbb{R}}(\varepsilon) \setminus \{C_{X,\mathbb{R}}(\varepsilon)\} \subseteq a_X(\varepsilon)$.

Proof.

1. Let $\delta \in a_X(\varepsilon)$ and $(x, x^*) \in A_{X,\mathbb{R}}(\delta)$. Now we will distinguish between two cases:
   - $x^*(x) \geq 0$. Then $(x, x^*) \in A_X(\delta) = A_X(\varepsilon, \delta)$, therefore
   $$d((x, x^*), \Pi_{X,\mathbb{R}}) \leq d((x, x^*), \Pi_X) < \varepsilon.$$
   Thus $(x, x^*) \in A_{X,\mathbb{R}}(\varepsilon, \delta)$.
   - $x^*(x) < 0$. Then $(-x, x^*) \in A_X(\delta) = A_X(\varepsilon, \delta)$, therefore there exists $(y, y^*) \in \Pi_X$ such that $\|(-x, x^*) - (y, y^*)\|_\infty < \varepsilon$. Now observe that $(-y, y^*) \in \Pi_{X,\mathbb{R}}$ and
   $$\|(x, x^*) - (-y, y^*)\|_\infty = \|(-x, x^*) - (y, y^*)\|_\infty < \varepsilon,$$
   which implies that $(x, x^*) \in A_{X,\mathbb{R}}(\varepsilon, \delta)$.

   This shows that $\delta \in a_{X,\mathbb{R}}(\varepsilon)$.
2. Let $\delta \in a_{X,\mathbb{R}}(\varepsilon)$ and $(x, x^*) \in A_X(\delta)$. Observe that $|x^*(x)| \geq x^*(x) \geq 1 - \delta$. In accordance with [2, Lemma 2.2(4,5)], we have that $\delta \leq 2\varepsilon < 1$. Therefore, $|x^*(x)| = x^*(x) \geq 1 - \delta > 0$. Now we will distinguish between two cases:
   - $x^*(x) > 1 - \delta$. Then $(x, x^*) \in A_{X,\mathbb{R}}(\delta) = A_{X,\mathbb{R}}(\varepsilon, \delta)$, so there exists $(y, y^*) \in \Pi_{X,\mathbb{R}}$ such that $\|(x, x^*) - (y, y^*)\|_\infty < \varepsilon$. Note that if $y^*(y) = -1$, then
   $$x^*(x) + 1 = |x^*(x) - y^*(y)| \leq 2\|(x, x^*) - (y, y^*)\|_\infty < 2\varepsilon < 1,$$
   which implies the contradiction that $x^*(x) < 0$. Thus $y^*(y) = 1$ and hence $(y, y^*) \in \Pi_X$ and $(x, x^*) \in A_X(\varepsilon, \delta)$.
   - $x^*(x) = 1 - \delta$. By applying Lemma 2.1, we can find a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathsf{S}_X$ converging to x such that $x^*(x_n) > x^*(x)$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ arbitrarily fixed, $(x_n, x^*) \in A_{X,\mathbb{R}}(\delta) = A_{X,\mathbb{R}}(\varepsilon, \delta)$, therefore there exists $(y_n, y_n^*) \in \Pi_{X,\mathbb{R}}$ such that $\|(x_n, x^*) - (y_n, y_n^*)\|_\infty < \varepsilon$. Since $x^*(x_n) > 0$ and $\varepsilon < \frac{1}{2}$, by using a similar

reasoning as above, we conclude that $y_n^*(y_n) = 1$, that is, $(y_n, y_n^*) \in \Pi_X$. Since

$$\begin{aligned} \|(x,x^*) - (y_n, y_n^*)\|_\infty &\leq \|(x,x^*) - (x_n,x^*)\|_\infty + \|(x_n,x^*) - (y_n,y_n^*)\|_\infty \\ &= \|x - x_n\| + \|(x_n,x^*) - (y_n,y_n^*)\|_\infty \\ &< \|x - x_n\| + \varepsilon, \end{aligned}$$

which implies that $d\left((x,x^*), \Pi_X\right) \leq \varepsilon$.

(3) Recall (see [2, Lemma 2.2]) that

$$a_{X,\mathbb{R}}(\varepsilon) \setminus \{C_{X,\mathbb{R}}(\varepsilon)\} = (0, C_{X,\mathbb{R}}(\varepsilon)) \subseteq (0, 2\varepsilon) \subseteq (0,1).$$

Let $\delta \in (0, C_{X,\mathbb{R}}(\varepsilon))$ and $(x,x^*) \in A_X(\delta)$. Fix an arbitrary $\delta' \in (\delta, C_{X,\mathbb{R}}(\varepsilon))$. Then $|x^*(x)| \geq x^*(x) \geq 1 - \delta > 1 - \delta' > 0$. Then $(x,x^*) \in A_{X,\mathbb{R}}(\delta') = A_{X,\mathbb{R}}(\varepsilon, \delta')$, so there exists $(y,y^*) \in \Pi_{X,\mathbb{R}}$ such that $\|(x,x^*)-(y,y^*)\|_\infty < \varepsilon$. Note that $y^*(y) = 1$ and hence $(y,y^*) \in \Pi_X$ and $(x,x^*) \in A_X(\varepsilon,\delta)$. This shows that $\delta \in a_X(\varepsilon)$.

□

An immediate consequence of Theorem 2.2 is the following corollary.

Corollary 2.3. *Let X be a Banach space with $\dim(X) \geq 2$. For every $\varepsilon \in (0,1/2)$, $C_X(\varepsilon) = C_{X,\mathbb{R}}(\varepsilon)$.*

Finally, notice that, for $\dim(X) \geq 2$ and $\varepsilon \in (0,1/2)$, $a_X(\varepsilon)$ is also an interval of extremes 0 and $C_{X,\mathbb{R}}(\varepsilon)$, open at 0. As a consequence, if $\delta \notin a_X(\varepsilon)$, then $C_X(\varepsilon) \leq \delta$. In the further subsections we will show examples where the sup involved in $C_X(\varepsilon)$ is not attained, that is, if we let $c := C_X(\varepsilon) = C_{X,\mathbb{R}}(\varepsilon)$, then $a_X(\varepsilon) = (0,c)$ whereas $a_{X,\mathbb{R}}(\varepsilon) = (0,c]$ according to [2, Lemma 2.2].

In order to ease the calculations when computing the BPBmo for classical spaces, we will rely on the following result.

Theorem 2.4. *Let X be a Banach space. For every $\varepsilon \in (0,1/2)$, if X is reflexive, $C_X(\varepsilon) = C_{X^*}(\varepsilon)$.*

Proof. We can assume without loss of generality that $\dim(X) \geq 2$. Simply observe that:

$$\begin{aligned} C_{X^*}(\varepsilon) &= \sup\{\delta > 0 : \text{ if } (x^*,x) \in \mathsf{S}_{X^*} \times \mathsf{S}_X \text{ with } x(x^*) \geq 1 - \delta, \text{ there exists} \\ &\qquad (y^*,y) \in \Pi_{X^*} \text{ such that } \|x - y\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon\} \\ &= \sup\{\delta > 0 : \text{ if } (x,x^*) \in \mathsf{S}_X \times \mathsf{S}_{X^*} \text{ with } x^*(x) \geq 1 - \delta, \text{ there exists} \\ &\qquad (y,y^*) \in \Pi_X \text{ such that } \|x - y\| < \varepsilon \text{ and } \|x^* - y^*\| < \varepsilon\} \\ &= C_X(\varepsilon). \end{aligned}$$

□

2.1. $C_{\mathbb{R}}$.

Theorem 2.5. *For every $\varepsilon \in (0,1/2)$, $C_{\mathbb{R}}(\varepsilon) = 2$.*

Proof. Consider $\varepsilon \in (0,1/2)$. Observe that in this case, $\Pi_{\mathbb{R}} = \{(1,1),(-1,-1)\}$.

$\leq$ Let $\delta \geq 2$ and we will prove that $\delta \notin a_{\mathbb{R}}(\varepsilon)$, which will show that $\sup a_{\mathbb{R}}(\varepsilon) \leq 2$. Let $x = 1$, $x^* = -1$. We have $x^*(x) = -1 \geq 1 - \delta$, since $\delta \geq 2$. If $(y, y^*) = (1, 1)$, we have $|x - y| = 0 < \varepsilon$, but $|x^* - y^*| = 2 > \varepsilon$. If $(y, y^*) = (-1, -1)$, $|x - y| = 2 > \varepsilon$. Then $\delta \notin a_{\mathbb{R}}(\varepsilon)$ and $\sup a_{\mathbb{R}}(\varepsilon) \leq 2$.

$\geq$ Let $0 < \delta < 2$ be and we will show that $\delta \in a_{\mathbb{R}}(\varepsilon)$, so $\sup a_{\mathbb{R}}(\varepsilon) \geq 2$. If $(x, x^*) \in \mathsf{S}_{\mathbb{R}} \times \mathsf{S}_{\mathbb{R}^*}$ is such that $x^*(x) \geq 1-\delta$, necessarily it is $x^* = 1, x = 1$ or $x^* = -1, x = -1$, since $\delta < 2$. This means that $(x, x^*) \in \Pi_{\mathbb{R}}$. This shows that $\sup a_{\mathbb{R}}(\varepsilon) \geq 2$.

We finally conclude that $C_{\mathbb{R}}(\varepsilon) = 2$. $\square$

2.2. $\boldsymbol{C_{\mathcal{H}}}$**,** $\boldsymbol{\mathcal{H}}$ **Hilbert space,** $\mathbf{dim}(\boldsymbol{\mathcal{H}}) \geq 2$**.** In [3, Corollary 5.6] the authors compute $d\left((h, k), \Pi_{\mathcal{H}}\right)$ for $\mathcal{H}$ a Hilbert space when $\Pi_{\mathcal{H}}$ is endowed with the metric inherited from $\mathcal{H} \oplus_2 \mathcal{H}$.

Theorem 2.6. *Let $\mathcal{H}$ be a Hilbert space with dim$(\mathcal{H}) \geq 2$. Then*

$$C_{\mathcal{H}}(\varepsilon) = 2\varepsilon^2 - \frac{\varepsilon^4}{2}, \text{ for every } \varepsilon \in (0, 1/2).$$

Proof. Let $\varepsilon \in (0, 1/2)$. Consider $\{e_i\}_{i \in I}$ an orthonormal basis of $\mathcal{H}$ and $\{e_i^*\}_{i \in I}$ its dual orthonormal basis of $\mathcal{H}^*$. We will assume that $i = 1, 2 \in I$. Put $\delta_0 := 2\varepsilon^2 - \dfrac{\varepsilon^4}{2}$ and $\mu := 1 - \dfrac{\varepsilon^2}{2}$.

$\leq$ First, we will show that $\delta_0 \notin a_{\mathcal{H}}(\varepsilon)$, which will prove that $C_{\mathcal{H}}(\varepsilon) \leq 2\varepsilon^2 - \frac{\varepsilon^4}{2}$. Let

$$\begin{aligned} x &= e_1, \\ x^* &= x_1^* e_1^* + x_2^* e_2^*, \end{aligned}$$

where

$$\begin{aligned} x_1^* &= 2\mu^2 - 1, \\ x_2^* &= \sqrt{1 - x_1^{*2}}. \end{aligned}$$

We have $x^*(x) = 2\mu^2 - 1 = 1 - \delta_0$. If $(y, y^*) \in \Pi_{\mathcal{H}}$ is such that $\|x - y\| \geq \varepsilon$, then we are done. If $\|x - y\| < \varepsilon$, then

$$\begin{aligned} (1 - y_1)^2 + \sum_{\substack{i \in I \\ i \neq 1}} y_i^2 &< \varepsilon^2 \Leftrightarrow \\ 2 - 2y_1 &< \varepsilon^2 \Leftrightarrow \\ y_1 &> 1 - \frac{\varepsilon^2}{2} = \mu > 0, \end{aligned}$$

and we will show that $\|x^* - y^*\| > \varepsilon$. Since $(y, y^*) \in \Pi_{\mathcal{H}}$,

$$\sum_{i \in I} (y_i^* - y_i)^2 = \sum_{i \in I} y_i^{*2} + \sum_{i \in I} y_i^2 - 2 \sum_{i \in I} y_i^* y_i = 0,$$

so $y_i^* = y_i$ for every $i \in I$. Notice that

$$\begin{aligned}\|x^* - y^*\|^2 &= \sum_{i\in I}(x_i^* - y_i)^2 = (x_1^* - y_1)^2 + (x_2^* - y_2)^2 + \sum_{\substack{i\in I\\ i\neq 1,2}} y_i^2\\ &= 2 - 2y_1x_1^* - 2y_2x_2^*.\end{aligned}$$

and so

$$\begin{gathered}\|x^* - y^*\|^2 > \varepsilon^2 \Leftrightarrow\\ x_1^*y_1 + x_2^*y_2 < 1 - \frac{\varepsilon^2}{2}.\end{gathered}$$

Observe that

$$x_1^*y_1 + x_2^*y_2 \le x_1^*y_1 + x_2^*\sqrt{1 - y_1^2},$$

so it is sufficient to prove that

$$x_1^*y_1 + x_2^*\sqrt{1 - y_1^2} < 1 - \frac{\varepsilon^2}{2} = \mu.$$

Now, we have a series of equivalent inequalities:

$$\begin{aligned}y_1x_1^* + \sqrt{1 - y_1^2}x_2^* < \mu \Leftrightarrow\\ y_1x_1^* + \sqrt{1 - y_1^2}\sqrt{1 - x_1^{*2}} < \mu \Leftrightarrow\\ y_1^2 - 2\delta x_1^*y_1 + x_1^{*2} + \mu^2 - 1 > 0 \Leftrightarrow\\ y_1 < 4\mu^3 - 3\mu \text{ or } y_1 > \mu.\end{aligned}$$

Since $y_1 > \mu$, all the inequalities are true and we have $\|x^* - y^*\| > \varepsilon$ so we conclude that $C_{\mathcal{H}}(\varepsilon) \le 2\varepsilon^2 - \frac{\varepsilon^4}{2}$.

$\geq$ Suppose now that $0 < \delta < \delta_0$ and we will prove that $\delta \in a_{\mathcal{H}}(\varepsilon)$ which will show that $C_{\mathcal{H}}(\varepsilon) \ge 2\varepsilon^2 - \frac{\varepsilon^4}{2}$. First, take $x = e_1$ and $x^* \in \mathsf{S}_{\mathcal{H}^*}$ such that $x^*(x) > 1 - \delta$. This leads to $x_1^* > 1 - \delta_0$. Take

$$y = \frac{(x_1^*+1)e_1 + \sum_{\substack{i\in I\\ i\neq 1}} x_i^*e_i}{\left\|(x_1^*+1)e_1 + \sum_{\substack{i\in I\\ i\neq 1}} x_i^*e_i\right\|}, \qquad y^* = \frac{(x_1^*+1)e_1^* + \sum_{\substack{i\in I\\ i\neq 1}} x_i^*e_i^*}{\left\|(x_1^*+1)e_1^* + \sum_{\substack{i\in I\\ i\neq 1}} x_i^*e_i^*\right\|}$$

Notice that $y \in \mathsf{S}_{\mathcal{H}}$, $y^* \in \mathsf{S}_{\mathcal{H}^*}$ and

$$\sum_{i\in I}(y_i - y_i^*)^2 = 0,$$

so $y^*(y) = \sum_{i\in I} y_i^*y_i = 1$ and $(y, y^*) \in \Pi_{\mathcal{H}}$.

Observe that

$$\left\|(x_1^*+1)e_1+\sum_{\substack{i\in I\\ i\neq 1}} x_i^* e_i\right\|^2=(x_1^*+1)^2+\sum_{\substack{i\in I\\ i\neq 1}} x_i^{*2}=2+2x_1^*.$$

$$\left\|(x_1^*+1)e_1^*+\sum_{\substack{i\in I\\ i\neq 1}} x_i^* e_i^*\right\|^2=2+2x_1^*.$$

Consequently,

$$\begin{aligned}\|x-y\|^2&=\sum_{i\in I}(x_i-y_i)^2=(1-y_1)^2+\sum_{\substack{i\in I\\ i\neq 1}} y_i^2=2-2y_1\\ &=2-2\frac{x_1^*+1}{\sqrt{2+2x_1^*}}.\end{aligned}$$

And we have the following equivalent inequalities:

$$\begin{aligned}\|x-y\|^2<\varepsilon^2 &\Leftrightarrow\\ 1-\frac{x_1^*+1}{\sqrt{2+2x_1^*}}<\frac{\varepsilon^2}{2} &\Leftrightarrow\\ \frac{x_1^*+1}{\sqrt{2+2x_1^*}}>1-\frac{\varepsilon^2}{2}=\mu>0 &\Leftrightarrow\\ (x_1^*+1)^2>\mu^2(2+2x_1^*) &\Leftrightarrow\\ x_1^*+1>2\mu^2 &\Leftrightarrow\\ x_1^*>2\mu^2-1.&\end{aligned}$$

Since $x_1^*>1-\delta_0=2\mu^2-1$, we have $\|x-y\|<\varepsilon$.
On the other hand,

$$\begin{aligned}\|x^*-y^*\|^2&=\sum_{i\in I}(x_i^*-y_i^*)^2=\sum_{i\in I}x_i^{*2}+\sum_{i\in I}y_i^{*2}-2\sum_{i\in I}x_i^*y_i^*\\ &=2-2\frac{x_1^*+1}{\sqrt{2+2x_1^*}}x_1^*-2\frac{\sum_{\substack{i\in I\\ i\neq 1}}x_i^{*2}}{\sqrt{2+2x_1^*}}=2-2\frac{x_1^*+1}{\sqrt{2+2x_1^*}},\end{aligned}$$

and for the same reason, $\|x^*-y^*\|<\varepsilon$.
Consider now any $(x,x^*)\in \mathsf{S}_{\mathcal{H}}\times\mathsf{S}_{\mathcal{H}^*}$ with $x^*(x)>1-\delta$. There exists a linear and surjective isometry $T:\mathcal{H}\to\mathcal{H}$ such that $T(e_1)=x$. Let $x_0^*=x^*\circ T$. We have $x_0^*\in\mathsf{S}_{\mathcal{H}^*}$ and,

$$x_0^*(e_1)=x^*(T(e_1))=x^*(x)>1-\delta,$$

so because of the previous procedure, there exists $(y_0,y_0^*)\in\Pi_{\mathcal{H}}$ such that $\|e_1-y_0\|<\varepsilon$ and $\|x_0^*-y_0^*\|<\varepsilon$. Take $y=T(y_0)\in\mathsf{S}_{\mathcal{H}}$ and

$y^* = T^{-1} \circ y_0^* \in \mathsf{S}_{\mathcal{H}}^*$. We have:

$$\begin{aligned}\|x-y\| &= \|T(e_1)-T(y_0)\| = \|T(e_1-y_0)\| = \|e_1-y_0\| < \varepsilon,\\ \|x^*-y^*\| &= \|T^{-1}\circ x_0^* - T^{-1}\circ y_0^*\| = \|T^{-1}\|\|x_0^*-y_0^*\| = \|x_0^*-y_0^*\| < \varepsilon,\\ y^*(y) &= y_0^*\circ T^{-1}(T(y_0)) = y_0^*(y_0) = 1.\end{aligned}$$

We conclude that $C_{\mathcal{H}}(\varepsilon) = 2\varepsilon^2 - \frac{\varepsilon^4}{2}$, for every $\varepsilon \in (0, 1/2)$.

□

2.3. **$C_{\mathcal{C}(K)}$, K compact and Hausdorff, card$(K) \geq 2$.** We will need some previous lemmas to calculate $C_{\mathcal{C}(K)}$.

Recall that a positive measure $\mu : K \to \mathbb{R}^+$ is said to be *inner regular* if for every Borel set $E \subseteq K$,

$$\mu(E) = \sup\{\mu(L) : L \subseteq E,\ L \text{ compact}\}.$$

Lemma 2.7. *Let K be a compact set, $\mu : K \to \mathbb{R}^+$ a finite and positive inner regular measure and $f \in \mathcal{C}(K)$, $f \geq 0$. If $\int_K f \mathrm{d}\mu \leq 0$, then $f = 0$ for almost every $t \in K$.*

Proof.

Consider $A = \{t \in K : f(t) > 0\}$ and $B = \{t \in K : f(t) = 0\}$.

Observe that A is an open set, B is a closed set, $A \cup B = K$ and:

$$0 \geq \int_K f \mathrm{d}\mu = \int_A f \mathrm{d}\mu + \int_B f \mathrm{d}\mu = \int_A f \mathrm{d}\mu.$$

Let $L \subseteq A$ be a compact set, there exists $\beta > 0$ such that $f(t) \geq \beta$ if $t \in L$ and so

$$0 \geq \int_A f \mathrm{d}\mu \geq \int_L f \mathrm{d}\mu \geq \beta\mu(L)$$

This implies that $\mu(L) = 0$ for every $L \subseteq A$ compact, and we have

$$\mu(A) = \sup\{\mu(L) : L \subseteq A,\ \text{L compact}\} = 0.$$

We conclude that $f = 0$ for almost every $t \in K$. □

Lemma 2.8. *Let K be a compact set, $t_0 \in K$, $(y, y^*) \in \Pi_{\mathcal{C}(K)}$. If $|y(t_0)| < 1$, then $\|y^* + \delta_{t_0}\| = 2$.*

Proof. Because of the Riesz' Representation Theorem, the following is an isometric isomorphism:

$$\begin{array}{rcl} T : \mathcal{M}_r(K) & \to & \mathcal{C}(K)^* \\ \mu & \mapsto & \int_K \cdot\, \mathrm{d}\mu : \begin{array}{ccc} \mathcal{C}(K) & \to & \mathbb{R} \\ f & \mapsto & \int_K f \mathrm{d}\mu \end{array} \end{array}$$

where $\mathcal{M}_r(K)$ is the regular Borel measure space over K. Take $A \subseteq K$ and consider the measure

$$\mu_{t_0}(A) = \begin{cases} 1 & \text{if } t_0 \in A. \\ 0 & \text{if } t_0 \notin A. \end{cases}$$

It is well known that $T(\mu_{t_0}) = \delta_{t_0}$, so we have to prove the following:

If $\mu \in \mathcal{M}_r(K)$ with $\|\mu\| = 1$, $f_0 \in \mathsf{S}_{\mathcal{C}(K)}$, $t_0 \in K$ and $|f_0(t_0)| < 1$ with $\displaystyle\int_K f_0 \mathrm{d}\mu = 1$, then $\|\mu + \mu_{t_0}\| = 2$.

It is clear that $P_1 = \{t_0\}$, $P_2 = K \setminus \{t_0\}$ is a Hahn decomposition of μ_{t_0}. So we have the positive measures $\mu_{t_0}^+ = \mu_{t_0}$ and $\mu_{t_0}^- = 0$ such that for every $S \in \Sigma$,

$$\begin{aligned} \mu_{t_0}^+(A) &= \mu_{t_0}(S \cap P_1), \\ \mu_{t_0}^-(A) &= -\mu_{t_0}(S \cap P_2), \\ \mu_{t_0} &= \mu_{t_0}^+ - \mu_{t_0}^-, \\ |\mu_{t_0}| &= \mu_{t_0}^+ + \mu_{t_0}^-. \end{aligned}$$

And observe that

$$\begin{aligned} \|\mu_{t_0}\| &= |\mu_{t_0}|(K) \\ &= \mu_{t_0}^+(K) + \mu_{t_0}^-(K) \\ &= \mu_{t_0}(K \cap P_1) \\ &= \mu_{t_0}(P_1) = 1. \end{aligned}$$

Let $\mu \in \mathcal{M}_r(K)$ be with $\|\mu\| = 1$ and Q_1, $Q_2 = Q_1^c$ a Hahn decomposition of μ, that is, for every $S \in \Sigma$,

$$\begin{aligned} 0 \le \mu(S \cap Q_1) &\le 1, \\ -1 \le \mu(S \cap Q_2) &\le 0. \end{aligned}$$

Then,

$$\begin{aligned} 1 = \|\mu\| &= |\mu|(K) \\ &= \mu^+(K) + \mu^-(K) \\ &= \mu(K \cap Q_1) - \mu(K \cap Q_2) \\ &= \mu(Q_1) - \mu(Q_2). \end{aligned}$$

Consider now R_1, R_2 a Hahn decomposition for $\mu + \mu_{t_0}$. We distinguish two cases.

- If $t_0 \in Q_1$:
  For every $S \in \Sigma$,

  $$\begin{aligned} (\mu + \mu_{t_0})(S \cap Q_1) &\ge 0, \\ \mu(S \cap Q_2) = (\mu + \mu_{t_0})(S \cap Q_2) &\le 0, \end{aligned}$$

  so we can take $R_1 = Q_1$ and $R_2 = Q_2$ as a Hahn decomposition of $\mu + \mu_{t_0}$ and notice that

  $$\begin{aligned} \|\mu + \mu_{t_0}\| &= (\mu + \mu_{t_0})^+(K) + (\mu + \mu_{t_0})^-(K) \\ &= (\mu + \mu_{t_0})(K \cap Q_1) - (\mu + \mu_{t_0})(K \cap Q_2) \\ &= \mu(Q_1) + 1 - \mu(Q_2) = 2. \end{aligned}$$

- If $t_0 \in Q_2$:

For every $S \in \Sigma$,

$$\begin{aligned}
(\mu+\mu_{t_0})(S\cap(Q_1\cup\{t_0\})) &= (\mu+\mu_{t_0})(S\cap Q_1)+(\mu+\mu_{t_0})(S\cap\{t_0\})\\
&= \mu(S\cap Q_1)+\mu(S\cap\{t_0\})+\mu_{t_0}(S\cap\{t_0\})\\
&= \underbrace{\mu(S\cap Q_1)}_{\geq 0}+\underbrace{\begin{cases}\mu(\{t_0\})+1, & \text{if } t_0\in S\\ 0, & \text{if } t_0\notin S.\end{cases}}_{\geq 0}\\
&\geq 0
\end{aligned}$$

since $\mu(\{t_0\}) = \mu(\{t_0\}\cap Q_2) \geq -1$.

On the other hand,

$$\begin{aligned}
(\mu+\mu_{t_0})(S\cap(Q_2\setminus\{t_0\})) &= \mu(S\cap(Q_2\setminus\{t_0\}))\\
&= \mu(S\setminus\{t_0\}\cap Q_2)\leq 0,
\end{aligned}$$

so we can take $R_1 = Q_1\cup\{t_0\}$ and $R_2 = Q_2\setminus\{t_0\}$ as a Hahn decomposition of $\mu+\mu_{t_0}$.

Observe that if $f_0 \in \mathsf{S}_{\mathcal{C}(K)}$ with $|f_0(t_0)| < 1$ and $\int_K f_0 \mathrm{d}\mu = 1$,

$$\begin{aligned}
1 = \int_K f_0\mathrm{d}\mu &= \int_{Q_1} f_0\mathrm{d}\mu + \int_{Q_2} f_0\mathrm{d}\mu\\
&\leq \mu(Q_1) + \int_{Q_2} f_0\mathrm{d}\mu
\end{aligned}$$

and so,

$$\int_{Q_2} f_0\mathrm{d}\mu \geq 1-\mu(Q_1) = -\mu(Q_2).$$

Or equivalently,

$$-\int_{Q_2} f_0\mathrm{d}\mu \leq \mu(Q_2) = \int_{Q_2}\xi_1\mathrm{d}\mu.$$

and it will be,

$$0 \leq \int_{Q_2}(\xi_1+f_0)\mathrm{d}\mu = \int_{Q_2}(\xi_1+f_0)\mathrm{d}\mu^+ - \int_{Q_2}(\xi_1+f_0)\mathrm{d}\mu^-. \tag{2.1}$$

Now, we have that $\xi_1+f_0 \geq 0$, is continuous and:

$$\begin{aligned}
\mu^+(\{t\in Q_2 : (\xi_1+f_0)(t) > 0\}) &= \mu^+(Q_2\cap\{t\in K : 1+f_0(t)>0\})\\
&= \mu(Q_2\cap\{t\in K : 1+f_0(t)>0\}\cap Q_1)\\
&= \mu(\varnothing) = 0.
\end{aligned}$$

This leads to

$$\int_{Q_2}(\xi_1+f_0)\mathrm{d}\mu^+ = 0.$$

If we substitute this in (2.1), we deduce that

$$\int_{Q_2}(\xi_1+f_0)\mathrm{d}\mu^- \leq 0.$$

By Lemma 2.7, since $\xi_1 + f_0 \geq 0$ and f_0 is continuous, we have $f_0 + \xi_1 = 0$ for almost every $t \in Q_2$. That is, $f_0(t) = -1$ for almost every $t \in Q_2$. This means that if $H = \{t \in Q_2 : f_0(t) > -1\}$, then

$$\mu(H) = \mu(\{t \in Q_2 : f_0(t) > -1\}) = 0.$$

Using the hypothesis, $|f_0(t_0)| < 1$, so $H \subseteq Q_2$ and $t_0 \in H$.
On the other hand,

$$\begin{aligned} 0 &= \mu(H) \\ &= \mu(H \setminus \{t_0\}) + \mu(\{t_0\}) \\ &= \underbrace{\mu(H \setminus \{t_0\} \cap Q_2)}_{\leq 0} + \underbrace{\mu(\{t_0\} \cap Q_2)}_{\leq 0}, \end{aligned}$$

so necessarily it will be $\mu(\{t_0\}) = 0$.
Finally,

$$\begin{aligned} \|\mu + \mu_{t_0}\| &= (\mu + \mu_{t_0})^+(K) + (\mu + \mu_{t_0})^-(K) \\ &= (\mu + \mu_{t_0})(K \cap R_1) - (\mu + \mu_{t_0})(K \cap Q_2) \\ &= \mu(Q_1 \cup \{t_0\}) + 1 - \mu(Q_2 \setminus \{t_0\}) \\ &= \mu(Q_1) + \mu(\{t_0\}) + 1 - \mu(Q_2) - \mu(\{t_0\}) \\ &= \mu(Q_1) + 1 - \mu(Q_2) = 2. \end{aligned}$$

□

Theorem 2.9. *Let K be a compact set with card $(K) \geq 2$. Then,*

$$C_{\mathcal{C}(K)}(\varepsilon) = \frac{\varepsilon^2}{2}, \text{ for every } \varepsilon \in (0, 1/2).$$

Proof. Take $\varepsilon \in (0, 1/2)$. Let $\delta_0 := \frac{\varepsilon^2}{2}$

$\leq$ First we will show that $\delta_0 \notin a_{\mathcal{C}(K)}(\varepsilon)$ which will prove that $\sup a_{\mathcal{C}(K)}(\varepsilon) \leq \frac{\varepsilon^2}{2}$. Consider $t_0, t_1 \in K$ $(t_0 \neq t_1)$ and take $x \in \mathsf{S}_{\mathcal{C}(K)}$ such that

$$\begin{aligned} x(t_0) &= 1 - \varepsilon, \\ x(t_1) &= 1 \end{aligned}$$

and

$$x^* = \frac{\varepsilon}{2}\delta_{t_0} + \left(1 - \frac{\varepsilon}{2}\right)\delta_{t_1} \in \mathsf{S}_{\mathcal{C}(K)^*}.$$

We have $x^*(x) = 1 - \frac{\varepsilon^2}{2}$. For every $(y, y^*) \in \Pi_{\mathcal{C}(K)}$, if $\|x - y\| > \varepsilon$, then we are done. Suppose then that

$$\|x - y\| = \sup\{|x(t) - y(t)| : t \in K\} < \varepsilon.$$

This implies that $|y(t_0) - 1 + \varepsilon| < \varepsilon$, so $|y(t_0)| < 1$. Observe that $|-y(t_0)| < 1$ and $-y^*(-y) = 1$, so $(-y, -y^*) \in \Pi_{\mathcal{C}(K)}$. If we use Lemma 2.8, we have $\|y^* - \delta_{t_0}\| = 2$.
But then,

$$\|x^* - y^*\| = \|x^* - \delta_{t_0} - y^* + \delta_{t_0}\| \geq \|x^* - \delta_{t_0}\| - \|y^* - \delta_{t_0}\| = 2 - \left(-\tfrac{\varepsilon}{2} + 1 + 1 - \tfrac{\varepsilon}{2}\right) = \varepsilon.$$

This means that $\sup a_{\mathcal{C}(K),\mathbb{R}}(\varepsilon) \leq \frac{\varepsilon^2}{2}$.

$\geq$ Suppose now that $0 < \delta < \delta_0$ and we will prove that $\delta \in a_{\mathcal{C}(K)}(\varepsilon)$, which will show that $C_{\mathcal{C}(K)}(\varepsilon) \geq \frac{\varepsilon^2}{2}$. Let $(x, x^*) \in \mathsf{S}_{\mathcal{C}(K)} \times \mathsf{S}_{\mathcal{C}(K)^*}$ be with $x^*(x) \geq 1 - \delta > 1 - \frac{\varepsilon^2}{2}$.

Suppose first that $x^* = \sum_{n=0}^{k} \alpha_n \delta_{t_n}$ for some $k \in \mathbb{N}$, $\alpha_0, \ldots, \alpha_n \in [-1, 1]$, $\sum_{n=0}^{k} |\alpha_n| = 1$ and some different $t_0, \ldots, t_k \in K$. We will find $y \in \mathcal{C}(K)$ and $y^* = \sum_{n=0}^{k} \beta_n \delta_{t_n}$ such that:

- $y \in S_{\mathcal{C}(K)}$.
- $\|y^*\| = \sum_{n=0}^{k} |\beta_n| = 1$.
- $y^*(y) = \sum_{n=0}^{k} \beta_n y(t_n) = 1$.
- $|x(t) - y(t)| < \varepsilon$ for every $t \in K$.
- $\|x^* - y^*\| = \sum_{n=0}^{k} |\beta_n - \alpha_n| < \varepsilon$.

Take $y_0, y_1, ..., y_k$ and $\beta_0, \beta_1, ..., \beta_k$ this way:

$$y_n = \begin{cases} 1 & \text{if} \quad x(t_n) > 1 - \varepsilon \\ -1 & \text{if} \quad x(t_n) < \varepsilon - 1 \\ x(t_n) & \text{if} \quad x(t_n) \in [\varepsilon - 1, 1 - \varepsilon] \end{cases}, \quad \beta_n = \begin{cases} 0 & \text{if} \quad n \in C \\ \alpha_n & \text{if} \quad n \in H^c \setminus C \\ \frac{-S \cdot \operatorname{sgn}(\alpha_n)}{\operatorname{card}(H \setminus C)} & \text{if} \quad n \in H \setminus C \end{cases}$$

where

$$\begin{aligned} C &= \{n \in \{0, \ldots, k\} : |x(t_n)| \leq 1 - \varepsilon\}, \\ H &= \{n \in \{0, \ldots, k\} : \alpha_n x(t_n) \leq 0\}, \\ S &= \sum_{n \in H \cup C} |\alpha_n|. \end{aligned}$$

We have:

(a) $\|y^*\| = \sum_{n=0}^{k} |\beta_n| = 1.$

$$\begin{aligned}\sum_{n=0}^{k} |\beta_n| &= \sum_{n\in C} |\beta_n| + \sum_{n\in H^c\setminus C} |\beta_n| + \sum_{n\in H\setminus C} |\beta_n| \\ &= 0 + \sum_{n\in H^c\setminus C} |\alpha_n| + \sum_{n\in H\setminus C} \frac{S}{\operatorname{card}(H\setminus C)} \\ &= \sum_{n\in H^c\setminus C} |\alpha_n| + \sum_{n\in H\cup C} |\alpha_n| \\ &= 1.\end{aligned}$$

(b) $\sum_{n=0}^{k} \beta_n y_n = 1.$
If $\beta_n \neq 0$, then $n \notin C$ and

$$\operatorname{sgn}(\beta_n) = \begin{cases} \operatorname{sgn}(\alpha_n) = \operatorname{sgn}(x(t_n)) & \text{if} \quad n \in H^c \\ -\operatorname{sgn}(\alpha_n) = \operatorname{sgn}(x(t_n)) & \text{if} \quad n \in H \end{cases}$$

so we have

$$\beta_n y_n = \beta_n \operatorname{sgn}(x(t_n)) = |\beta_n|, \text{ if } n = 0, \dots, k$$

and

$$\sum_{n=0}^{k} \beta_n y_n = \sum_{n=0}^{k} |\beta_n| = 1.$$

(c) $|x(t_n) - y_n| < \varepsilon$ for every $n \in \{0, \dots k\}$.
- If $x(t_n) \in [\varepsilon - 1, 1 - \varepsilon]$, we have $|x(t_n) - y_n| = |x(t_n) - x(t_n)| = 0 < \varepsilon$.
- If $x(t_n) > 1 - \varepsilon$, then $0 < 1 - x(t_n) < \varepsilon$, and $|x(t_n) - y_n| = |x(t_n) - 1| < \varepsilon$.
- If $x(t_n) < \varepsilon - 1$, then $0 < 1 + x(t_n) < \varepsilon$, $|x(t_n) - y_n| = |x(t_n) + 1| < \varepsilon$.

(d) $\|x^* - y^*\| = \sum_{n=0}^{k} |\alpha_n - \beta_n| < \varepsilon$.

$$\begin{aligned}\|x^* - y^*\| = \sum_{n=0}^{k} |\alpha_n - \beta_n| &= \sum_{n\in C} |\alpha_n| + \sum_{n\in H\setminus C} (|\alpha_n| + |\beta_n|) \\ &= 2 - \sum_{n\in H^c\setminus C} |\alpha_n| - \sum_{n\in H^c\cup C} |\beta_n| \\ &= 2 - \sum_{n\in H^c\setminus C} |\alpha_n| - \sum_{n\in H^c\setminus C} |\alpha_n| \\ &= 2\sum_{n\in H\cup C} |\alpha_n| = 2S.\end{aligned}$$

Remember that $x^*(x) = \sum_{n=0}^{k} \alpha_n x(t_n) > 1 - \frac{\varepsilon^2}{2}$. Now,

$$\begin{aligned}\|x^* - y^*\| &= 2\sum_{n\in H\cup C} |\alpha_n| = \frac{2}{\varepsilon}\left(\sum_{n\in C} \varepsilon|\alpha_n| + \sum_{n\in H\setminus C} \varepsilon|\alpha_n|\right) \\ &< \frac{2}{\varepsilon}\left(\varepsilon\sum_{n\in C} |\alpha_n| + \sum_{n\in H\setminus C} (1 + |x(t_n)|)|\alpha_n|\right) \\ &= \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C} |\alpha_n|(1-\varepsilon) + \Big(1 - \sum_{n\in C} |\alpha_n| - \sum_{n\in H\setminus C} |\alpha_n|\Big)\right.\right. \\ &\qquad \left.\left. - \sum_{n\in H\setminus C} |\alpha_n x(t_n)|\right)\right] \\ &= \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C} |\alpha_n|(1-\varepsilon) + \sum_{n\in H^c\setminus C} |\alpha_n| + \sum_{n\in H\setminus C} \alpha_n x(t_n)\right)\right] \\ &\leq \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C} \alpha_n x(t_n) + \sum_{n\in H^c\setminus C} \alpha_n x(t_n) + \sum_{n\in H\setminus C} \alpha_n x(t_n)\right)\right] \\ &= \frac{2}{\varepsilon}\Big(1 - \sum_{n=0}^{k} \alpha_n x(t_n)\Big) \\ &< \frac{2}{\varepsilon}\left[1 - \Big(1 - \frac{\varepsilon^2}{2}\Big)\right] = \varepsilon,\end{aligned}$$

so $\|x^* - y^*\| < \varepsilon$.

Summarizing, given $(x,x^*) \in \mathsf{S}_{\mathcal{C}(K)} \times \mathsf{S}_{\mathcal{C}(K)^*}$ with

$$\begin{aligned} x^*(x) &> 1 - \frac{\varepsilon^2}{2}, \\ x^* &= \sum_{n=0}^{k} \alpha_n \delta_{t_n}, \text{ where } \sum_{n=0}^{k} |\alpha_n| = 1, \end{aligned}$$

there exists $y^* = \displaystyle\sum_{n=0}^{k} \beta_n \delta_{t_n}$ with

$$\begin{aligned} \|x^* - y^*\| &< \varepsilon, \\ \|y^*\| &= 1 \end{aligned}$$

and $y_0, \ldots, y_k$ such that

$$\begin{aligned} \sum_{n=0}^{k} \beta_n y_n &= 1, \\ \max_{n=0,\ldots,k} |y_n| &= 1, \\ |x(t_n) - y_n| &< \varepsilon, \text{ for every } n = 0, \ldots, k. \end{aligned}$$

Consider $m = \max\{|x(t_n) - y_n| : n \in \{0, \ldots, k\}\}$ and $F = \{t_0, \ldots, t_k\}$, which is a closed set in K. Take

$$\begin{aligned} a : F &\longrightarrow [-m, m] \\ t_n &\mapsto x(t_n) - y_n \end{aligned}$$

Using the Tietze's theorem, there exists $\widetilde{a} : K \to [-m,m]$ continuous with $\widetilde{a}|_F = a$ and $\widetilde{a} \in \varepsilon B_{\mathcal{C}(K)}$. Take $y = \min\{\max\{-1, \widetilde{a} + x\}, 1\}$. We have:

(a) $y^*(y) = \displaystyle\sum_{n=0}^{k} \beta_n y(t_n) = \sum_{n=0}^{k} \beta_n y_n = 1, \big(\text{since } y(t_n) = y_n\big).$

(b) $\|y\| = 1$.

(c) $\|x - y\| < \varepsilon$.

Finally, given any $(x, x^*) \in \mathsf{S}_{\mathcal{C}(K)} \times \mathsf{S}_{\mathcal{C}(K)^*}$ with $x^*(x) \geq 1 - \delta > 1 - \frac{\varepsilon^2}{2}$, by the Krein-Milman Theorem, there exists a net

$$(x_i^*)_{i \in I} \subseteq \mathcal{L}(\mathrm{ex}(B_{\mathcal{C}(K)^*})) \cap \mathsf{S}_{\mathcal{C}(K)^*}$$

such that $x_i^* \xrightarrow{w^*} x^*$. In particular,

$$x_i^*(x) \to x^*(x) > 1 - \frac{\varepsilon^2}{2},$$

so there exists $i_0 \in I$ such that if $i \geq i_0$, $x_i^*(x) > 1 - \frac{\varepsilon^2}{2}$.

With these x_i^*, $i \geq i_0$, with the previous procedure, we obtain $(y, y_i^*) \in \Pi_{\mathcal{C}(K)}$ such that $\|x - y\| < \varepsilon$ and $\|x_i^* - y_i^*\| < \varepsilon$.

For y_i^*, there exists a subnet $(y_{i_j}^*)_{j\in I} \xrightarrow{w^*}$ some $y^* \in B_{\mathcal{C}(K)^*}$, and we have $x^* - y^* = w^* - \lim_j (x_{i_j}^* - y_{i_j}^*)$, so

$$\|x^* - y^*\| \leq \liminf_j \|x_{i_j}^* - y_{i_j}^*\| < \varepsilon.$$

Observe that $y_{i_j}^*(y) = 1$ for every $j \in I$, so $y^*(y) = 1$, since y^* is the w^* limit, and so $\|y^*\| = 1$.

We finally conclude that $C_{\mathcal{C}(K)}(\varepsilon) = \frac{\varepsilon^2}{2}$.

□

Corollary 2.10. *Let X be one the following spaces: c, ℓ_∞ and $\mathbb{R}^n_\infty$ for $n \geq 2$. Then*

$$C_X(\varepsilon) = \frac{\varepsilon^2}{2}$$

for every $\varepsilon \in (0, 1/2)$.

By bearing in mind Theorem 2.4, we deduce the following scholium.

Scholium 2.11. $C_{\mathbb{R}^n_1}(\varepsilon) = \frac{\varepsilon^2}{2}$ *for every $\varepsilon \in (0, 1/2)$ and $n \geq 2$.*

2.4. $\boldsymbol{C_{\ell_1}}$.

Theorem 2.12. *For every $\varepsilon \in (0, 1/2)$, $C_{\ell_1}(\varepsilon) = \frac{\varepsilon^2}{2}$.*

Proof. Let $\varepsilon \in (0, 1/2)$. Consider $\delta_0 := \frac{\varepsilon^2}{2}$.

$\leq$ We will show that $\delta_0 \notin a_{\ell_1}(\varepsilon)$, which will prove that $\sup a_{\ell_1}(\varepsilon) \leq \frac{\varepsilon^2}{2}$. Take

$$x = \left(1 - \frac{\varepsilon}{2}\right)e_1 + \frac{\varepsilon}{2}e_2,$$
$$x^* = e_1^* + (1-\varepsilon)e_2^*.$$

We have $x^*(x) = 1 - \frac{\varepsilon^2}{2}$. Given any $(y, y^*) \in \Pi_{\ell_1}$, if $\|x^* - y^*\| < \varepsilon$, then

$$|y_1^* - 1| < \varepsilon,$$
$$|y_2^* - 1 + \varepsilon| < \varepsilon,$$
$$|y_k^*| < \varepsilon, \text{ for every } k \geq 3.$$

Since $0 \leq 1 - 2\varepsilon < y_2^* < 1$ and $|y_k^*| < \varepsilon$ for every $k \geq 3$, necessarily $y_1^* = 1$ (because it is $\|y^*\| = 1$).
On the other hand $y^*(y) = 1$, so

$$y_1 = 1 - \sum_{n=2}^{\infty} y_n^* y_n \geq 1 - \sum_{n=2}^{\infty} |y_n^*||y_n| \geq 1 - \sum_{n=2}^{\infty} |y_n| \geq 0,$$

and this leads to $y_1 \geq 0$.
But then, $1 = y_1 + \sum_{n=2}^{\infty} y_n^* y_n = y_1 + \sum_{n=2}^{\infty} |y_n|$ and we have $\sum_{n=2}^{\infty}(|y_n| - y_n^* y_n) = 0$.

Since $|y_n^*| \leq 1$ for every n, we have that $y_n^* y_n \leq |y_n|$ for every $n \in \mathbb{N}$ and so $|y_n| - y_n^* y_n \geq 0$.
This implies that $|y_n| = y_n^* y_n$ for every $n \geq 2$, so $y_n = 0$ for every $n \geq 2$ and necessarily $y = e_1$.
This way,

$$\|x - y\| = \sum_{n=1}^{\infty} |x_n - y_n| = \left|1 - 1 + \frac{\varepsilon}{2}\right| + \frac{\varepsilon}{2} = \varepsilon.$$

And we conclude that $\delta_0 = \frac{\varepsilon^2}{2} \notin a_{\ell_1}(\varepsilon)$, so $\sup a_{\ell_1}(\varepsilon) \leq \frac{\varepsilon^2}{2}$.

$\geq$ Now, if $0 < \delta < \delta_0$, we will prove that $\delta \in a_{\ell_1}(\varepsilon)$ which will show that $\sup a_{\ell_1}(\varepsilon) \geq \frac{\varepsilon^2}{2}$.

Let $(x, x^*) \in \mathsf{S}_{\ell_1} \times \mathsf{S}_{\ell_\infty}$ be such that $x^*(x) = \displaystyle\sum_{n=1}^{\infty} x_n^* x_n \geq 1 - \delta > 1 - \frac{\varepsilon^2}{2}$.
We give $(y, y^*) \in \mathsf{S}_{\ell_1} \times \mathsf{S}_{\ell_\infty}$ with $y^*(y) = 1$, $\|x - y\| < \varepsilon$ and $\|x^* - y^*\| < \varepsilon$.
Take $y = \displaystyle\sum_{n=1}^{\infty} y_n e_n$ and $y^* = \displaystyle\sum_{n=1}^{\infty} y_n^* e_n^*$ in the following way:

$$y_n = \begin{cases} 0 & \text{if} \quad n \in C \\ x_n & \text{if} \quad n \in H^c \setminus C \\ \dfrac{-S \cdot \operatorname{sgn}(x_n)}{\operatorname{card}(H \setminus C)} & \text{if} \quad n \in H \setminus C \end{cases}, \quad y_n^* = \begin{cases} 1 & \text{if} \quad x_n^* > 1 - \varepsilon \\ -1 & \text{if} \quad x_n^* < \varepsilon - 1 \\ x_n^* & \text{if} \quad x_n^* \in [\varepsilon - 1, 1 - \varepsilon] \end{cases}$$

where

$$\begin{aligned} C &= \{n \in \mathbb{N} : |x_n^*| \leq 1 - \varepsilon\}, \\ H &= \{n \in \mathbb{N} : x_n^* x_n \leq 0\}, \\ S &= \sum_{n \in H \cup C} |x_n|. \end{aligned}$$

We have:

(a) $\|y\| = 1$. Indeed,

$$\begin{aligned} \|y\| = \sum_{n=1}^{\infty} |y_n| &= \sum_{n \in C} |y_n| + \sum_{n \in H^c \setminus C} |y_n| + \sum_{n \in H \setminus C} |y_n| \\ &= 0 + \sum_{n \in H^c \setminus C} |x_n| + \sum_{n \in H \setminus C} \frac{S}{\operatorname{card}(H \setminus C)} \\ &= \sum_{n \in H^c \setminus C} |x_n| + \sum_{n \in H \cup C} |x_n| = 1. \end{aligned}$$

(b) $\|y^*\| = 1$. If $|x_n^*| \leq 1 - \varepsilon$ for every n, then $\|x^*\| \leq 1 - \varepsilon < 1$ and it could not be $\|x^*\| = 1$, so $\|y^*\| = 1$.

(c) $y^*(y) = 1$. Indeed, if $n \notin C$,

$$y_n^* = \begin{cases} \operatorname{sgn}(x_n^*) &= \operatorname{sgn}(x_n) &= \operatorname{sgn}(y_n) & \text{if} \quad n \in H^c \\ \operatorname{sgn}(x_n^*) &= -\operatorname{sgn}(x_n) &= \operatorname{sgn}(y_n) & \text{if} \quad n \in H \end{cases}$$

so

$$|y_n| = y_n \mathrm{sgn}(y_n) = y_n y_n^*, \text{ if } n \notin C,$$

and this leads to

$$y^*(y) = \sum_{n\in C} y_n^* y_n + \sum_{n\notin C} |y_n| = 0 + \sum_{n\notin C} |y_n| = 1.$$

(d) $\|x^* - y^*\| = \sup\{|x_n^* - y_n^*| : n \in \mathbb{N}\} < \varepsilon$. Indeed:
- If $x_n^* \in [\varepsilon - 1, 1 - \varepsilon]$, $|y_n^* - x_n^*| = |x_n^* - x_n^*| = 0 < \varepsilon$.
- If $x_n^* > 1-\varepsilon$, then $0 < 1-x_n^* < \varepsilon$, and so $|y_n^* - x_n^*| = 1 - x_n^* < \varepsilon$.
- If $x_n^* < \varepsilon - 1$, then $0 < 1 + x_n^* < \varepsilon$, and so $|y_n^* - x_n^*| = |-1-x_n^*| = 1 + x_n^* < \varepsilon$.

(e) $\|x - y\| = \sum_{n\in\mathbb{N}} |x_n - y_n| < \varepsilon$. Indeed,

$$\begin{aligned} \|x-y\| &= \sum_{n\in\mathbb{N}} |x_n - y_n| \qquad (2.2)\\ &= \sum_{n\in C} |x_n - 0| + \sum_{n\in H^c\setminus C} |x_n - x_n| + \sum_{n\in H\setminus C} \left| x_n + \frac{S\cdot \mathrm{sgn}(x_n)}{\mathrm{card}(H\setminus C)} \right| \end{aligned}$$

Observe that, if $x_n > 0$,

$$\left| x_n + \frac{S\cdot \mathrm{sgn}(x_n)}{\mathrm{card}(H\setminus C)} \right| = \left| x_n + \frac{S\, x_n}{\mathrm{card}(H\setminus C)} \right| = x_n + \frac{S\, x_n}{\mathrm{card}(H\setminus C)} = |x_n| + |y_n|.$$

Similarly, if $x_n < 0$,

$$\left| x_n + \frac{S\cdot \mathrm{sgn}(x_n)}{\mathrm{card}(H\setminus C)} \right| = \left| x_n - \frac{S\, x_n}{\mathrm{card}(H\setminus C)} \right| = -x_n + \frac{-S\, x_n}{\mathrm{card}(H\setminus C)} = |x_n| + |y_n|.$$

So if we substitute this in the expression (2.2),

$$\begin{aligned} \|x^* - y^*\| &= \sum_{n\in C} |x_n| + \sum_{n\in H\setminus C} (|y_n| + |x_n|)\\ &= \sum_{n\in H\cup C} |x_n| + \sum_{n\in H\setminus C} |y_n|\\ &= 2 \sum_{n\in H\cup C} |x_n|\\ &= \frac{2}{\varepsilon}\left(\sum_{n\in C} \varepsilon |x_n| + \sum_{n\in H\setminus C} \varepsilon |x_n| \right). \end{aligned}$$

And so,

$$\begin{aligned}
\|x^* - y^*\| &< \frac{2}{\varepsilon}\left(\sum_{n\in C}\varepsilon|x_n| + \sum_{n\in H\setminus C}(1+|x_n^*|)|x_n|\right)\\
&= \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}|x_n|(1-\varepsilon) + \left(1 - \sum_{n\in C}|x_n| - \sum_{n\in H\setminus C}|x_n|\right)\right.\right.\\
&\quad\left.\left. - \sum_{n\in H\setminus C}|x_n^*||x_n|\right)\right]\\
&= \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}|x_n|(1-\varepsilon) + \sum_{n\in H^c\setminus C}|x_n| + \sum_{n\in H\setminus C}x_n^*x_n\right)\right]\\
&\le \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}|x_n^*||x_n| + \sum_{n\in H^c\setminus C}|x_n^*||x_n| + \sum_{n\in H\setminus C}x_n^*x_n\right)\right]\\
&\le \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}x_n^*x_n + \sum_{n\in H^c\setminus C}x_n^*x_n + \sum_{n\in H\setminus C}x_n^*x_n\right)\right]\\
&= \frac{2}{\varepsilon}\left[1 - \sum_{n\in\mathbb{N}}x_n^*x_n\right] < \frac{2}{\varepsilon}\left[1 - \left(1 - \frac{\varepsilon^2}{2}\right)\right] = \varepsilon.
\end{aligned}$$

Then we conclude that $C_{\ell_1}(\varepsilon) = \frac{\varepsilon^2}{2}$. □

2.5. $\boldsymbol{C_{c_0}}$.

Theorem 2.13. *For every* $\varepsilon \in (0, 1/2)$, $C_{c_0}(\varepsilon) = \frac{\varepsilon^2}{2}$.

Proof. Let $\varepsilon \in (0, 1/2)$. Consider $\delta_0 := \frac{\varepsilon^2}{2}$.

$\le$ We will show that $\delta_0 \in a_{c_0}(\varepsilon)$, which will prove that $\sup a_{c_0}(\varepsilon) \le \frac{\varepsilon^2}{2}$. Take

$$\begin{aligned}
x &= e_1 + (1-\varepsilon)e_2,\\
x^* &= \left(1 - \frac{\varepsilon}{2}\right)e_1^* + \frac{\varepsilon}{2}e_2^*.
\end{aligned}$$

We have, $x^*(x) = 1 - \frac{\varepsilon^2}{2}$. Given any $(y, y^*) \in \Pi_{c_0}$, if $\|y - x\| < \varepsilon$, then

$$\begin{aligned}
|y_1 - 1| &< \varepsilon,\\
|y_2 - 1 + \varepsilon| &< \varepsilon,\\
|y_k| &< \varepsilon, \text{ for every } k \ge 3.
\end{aligned}$$

Since $\|y\| = 1$, it must happen that $y_1 = 1$ (because $0 \le 1 - 2\varepsilon < y_2 < 1$). Next, since $y_1 = 1$ and $y^*(y) = 1$, we obtain

$$\sum_{n=1}^{\infty} y_n^* y_n = y_1^* + \sum_{n=2}^{\infty} y_n^* y_n = 1,$$

and then

$$y_1^* = 1 - \sum_{n=2}^{\infty} y_n^* y_n \geq 1 - \sum_{n=2}^{\infty} |y_n^*||y_n| \geq 1 - \sum_{n=2}^{\infty} |y_n^*| \geq 0,$$

thus $y_1^* \geq 0$. On the other hand, $\|y^*\| = 1$ and hence $y_1^* = 1 - \sum_{n=2}^{\infty} |y_n^*|$. We deduce then that $\sum_{n=2}^{\infty} |y_n^*| = \sum_{n=2}^{\infty} y_n^* y_n$. Because $|y_n| \leq 1$ for all n, we have

$$y_n^* y_n \leq |y_n^*| \text{ for all } n \in \mathbb{N}$$

therefore, $|y_n^*| - y_n^* y_n \geq 0$. This implies that

$$\sum_{n=2}^{\infty} (|y_n^*| - y_n^* y_n) = 0,$$

and then

$$|y_n^*| = y_n^* y_n \text{ for all } n \geq 2.$$

As a consequence, $y_n^* = 0$ for every $n \geq 2$ which forces that $y^* = e_1^*$. Then we have,

$$\|y^* - x^*\| = \sum_{n=1}^{\infty} |y_n^* - x_n^*| = \left|1 - 1 + \frac{\varepsilon}{2}\right| + \frac{\varepsilon}{2} = \varepsilon.$$

And we conclude that $\delta_0 = \frac{\varepsilon^2}{2} \notin a_{c_0}(\varepsilon)$, so $\sup a_{c_0}(\varepsilon) \leq \frac{\varepsilon^2}{2}$.

$\leq$ Now, if $0 < \delta < \delta_0$, then we will prove that $\delta \in a_{c_0}(\varepsilon)$ which will show that $\sup a_{c_0}(\varepsilon) \geq \frac{\varepsilon^2}{2}$. Let $(x, x^*) \in \mathsf{S}_{c_0} \times \mathsf{S}_{\ell_1}$ with $x^*(x) = \sum_{n=1}^{\infty} x_n x_n^* \geq 1 - \dfrac{\varepsilon^2}{2}$. We will find $(y, y^*) \in \mathsf{S}_{c_0} \times \mathsf{S}_{\ell_1}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$. Take

$$y = \sum_{n=1}^{\infty} y_n e_n$$

$$y^* = \sum_{n=1}^{\infty} y_n^* e_n^*$$

given by

$$y_n = \left\{ \begin{array}{rcl} 1 & \text{if} & x_n > 1 - \varepsilon \\ -1 & \text{if} & x_n < \varepsilon - 1 \\ x_n & \text{if} & x_n \in [\varepsilon - 1, 1 - \varepsilon] \end{array} \right. , \quad y_n^* = \left\{ \begin{array}{ccc} 0 & \text{if} & n \in C \\ x_n^* & \text{if} & n \in H^c \setminus C \\ \dfrac{-S \cdot \operatorname{sgn}(x_n^*)}{\operatorname{card}(H \setminus C)} & \text{if} & n \in H \setminus C \end{array} \right. ,$$

where

$$\begin{aligned}
C &= \{n \in \mathbb{N} : |x_n| \leq 1 - \varepsilon\}, \\
H &= \{n \in \mathbb{N} : x_n^* x_n \leq 0\}, \\
S &= \sum_{n \in H \cup C} |x_n^*|.
\end{aligned}$$

Note that

- $y \in c_0$. Indeed, since $z \in c_0$, there exists $n_0 \in \mathbb{N}$ so that if $n \geq n_0$, then $|x_n| \leq 1 - \varepsilon$, thus $y_n = x_n$ for $n \geq n_0$.
- $\|y\| = 1$. Indeed, if $|x_n| \leq 1-\varepsilon$ for every n, then $1 = \|x\| \leq 1-\varepsilon < 1$.
- $\|y^*\| = 1$. Indeed,

$$\begin{aligned}
\|y^*\| = \sum_{n=1}^{\infty} |y_n^*| &= \sum_{n \in C} |y_n^*| + \sum_{n \in H^c \setminus C} |y_n^*| + \sum_{n \in H \setminus C} |y_n^*| \\
&= 0 + \sum_{n \in H^c \setminus C} |x_n^*| + \sum_{n \in H \setminus C} \frac{S}{\operatorname{card}(H \setminus C)} \\
&= \sum_{n \in H^c \setminus C} |x_n^*| + \sum_{n \in H \cup C} |x_n^*| = 1.
\end{aligned}$$

- $y^*(y) = 1$. Indeed, if $y_n^* \neq 0$, then $n \notin C$ and thus,

$$\operatorname{sgn}(y_n^*) = \begin{cases} \operatorname{sgn}(x_n^*) = \operatorname{sgn}(x_n) = \operatorname{sgn}(y_n) & \text{if} \quad n \in H^c \\ -\operatorname{sgn}(x_n^*) = \operatorname{sgn}(x_n) = \operatorname{sgn}(y_n) & \text{if} \quad n \in H \end{cases},$$

  which means

$$y_n^* y_n = y_n^* \operatorname{sgn}(y_n^*) = |y_n^*|, \text{ for } n \notin C,$$

  and then

$$y^*(y) = \sum_{n \in C} y_n^* y_n + \sum_{n \notin C} |y_n^*| = 0 + \sum_{n \notin C} |y_n^*| = 1.$$

- $\|y - x\| = \sup\{|y_n - x_n| : n \in \mathbb{N}\} < \varepsilon$. Indeed,
  (a) If $x_n \in [\varepsilon - 1, 1 - \varepsilon]$,

$$|y_n - x_n| = |x_n - x_n| = 0 < \varepsilon.$$

  (b) if $x_n > 1 - \varepsilon$, then $0 < 1 - x_n < \varepsilon$,

$$|y_n - x_n| = |1 - x_n| < \varepsilon.$$

  (c) If $x_n < \varepsilon - 1$, then $0 < 1 + x_n < \varepsilon$,

$$|y_n - x_n| = |-1 - x_n| < \varepsilon.$$

- $\|y^* - x^*\| = \sum_{n\in\mathbb{N}} |y_n^* - x_n^*| < \varepsilon$. Indeed, observe that

$$\begin{aligned}
\|y^* - x^*\| &= \sum_{n\in\mathbb{N}} |y_n^* - x_n^*| \qquad (2.3)\\
&= \sum_{n\in C} |0 - x_n^*| + \sum_{n\in H^c\setminus C} |x_n^* - x_n^*| + \sum_{n\in H\setminus C} \left| \frac{-S\cdot \operatorname{sgn}(x_n^*)}{\operatorname{card}(H\setminus C)} - x_n^* \right|
\end{aligned}$$

Now, if $x_n^* > 0$,

$$\left| \frac{-S\cdot \operatorname{sgn}(x_n^*)}{\operatorname{card}(H\setminus C)} - x_n^* \right| = \left| \frac{-S\ x_n^*}{\operatorname{card}(H\setminus C)} - x_n^* \right| = \frac{S\ x_n^*}{\operatorname{card}(H\setminus C)} + x_n^* = |y_n^*| + |x_n^*|.$$

Analogously, if $x_n^* < 0$,

$$\left| \frac{-S\cdot \operatorname{sgn}(x_n^*)}{\operatorname{card}(H\setminus C)} - x_n^* \right| = \left| \frac{S\ x_n^*}{\operatorname{card}(H\setminus C)} - x_n^* \right| = \frac{-S\ x_n^*}{\operatorname{card}(H\setminus C)} - x_n^* = |y_n^*| + |x_n^*|.$$

Therefore, by substituting this in the expression (2.3),

$$\begin{aligned}
\|y^* - x^*\| &= \sum_{n\in C} |y_n^*| + \sum_{n\in H\setminus C} (|x_n^*| + |y_n^*|)\\
&= \sum_{n\in C} |x_n^*| + \sum_{n\in H\setminus C} |x_n^*| + 1 - \sum_{n\in H^c\setminus C} |y_n^*|\\
&= \sum_{n\in H\cup C} |x_n^*| + 1 - \sum_{n\in H^c\setminus C} |x_n^*|\\
&= 2 \sum_{n\in H\cup C} |x_n^*|\\
&= \frac{2}{\varepsilon} \left(\sum_{n\in C} \varepsilon |x_n^*| + \sum_{n\in H\setminus C} \varepsilon |x_n^*| \right).
\end{aligned}$$

And then,

$$
\begin{aligned}
&\|y^* - x^*\| \\
&< \frac{2}{\varepsilon}\left(\sum_{n\in C}\varepsilon|x_n^*| + \sum_{n\in H\setminus C}(1+|x_n|)|x_n^*|\right) \\
&= \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}|x_n^*|(1-\varepsilon) + \Big(1 - \sum_{n\in C}|x_n^*| - \sum_{n\in H\setminus C}|x_n^*|\Big) - \sum_{n\in H\setminus C}|x_n^*||x_n|\right)\right] \\
&= \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}|x_n^*|(1-\varepsilon) + \sum_{n\in H^c\setminus C}|x_n^*| + \sum_{n\in H\setminus C}x_n^*x_n\right)\right] \\
&\le \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}|x_n^*||x_n| + \sum_{n\in H^c\setminus C}|x_n^*||x_n| + \sum_{n\in H\setminus C}x_n^*x_n\right)\right] \\
&\le \frac{2}{\varepsilon}\left[1 - \left(\sum_{n\in C}x_n^*x_n + \sum_{n\in H^c\setminus C}x_n^*x_n + \sum_{n\in H\setminus C}x_n^*x_n\right)\right] \\
&= \frac{2}{\varepsilon}\left[1 - \sum_{n\in\mathbb{N}}x_n^*x_n\right] \\
&< \frac{2}{\varepsilon}\left[1 - \left(1 - \frac{\varepsilon^2}{2}\right)\right] = \varepsilon.
\end{aligned}
$$

Then we conclude that $C_{c_0}(\varepsilon) = \frac{\varepsilon^2}{2}$. $\square$

Acknowledgement. The first author was supported by MTM2014-58984-P (this project has been funded by the Spanish Ministry of Economy and Competitivity and by the European Fund for Regional Development FEDER).

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