Adv. Oper. Theory 3 (2018), no. 3, 655–681
https://doi.org/10.15352/aot.1710-1241
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

C^*-ALGEBRA DISTANCE FILTERS

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Communicated by M. Mathieu

Abstract. We use nonsymmetric distances to give a self-contained account of C^*-algebra filters and their corresponding compact projections, simultaneously simplifying and extending their general theory.

1. Introduction

Quantum filters were introduced by Farah and Weaver to analyze pure states on C^*-algebras and various conjectures concerning them, like Anderson's conjecture and the Kadison–Singer conjecture (which has since become the Marcus–Speilman–Srivastava theorem – see [18]). They were also considered more recently in [14] in relation to quantum analogs of certain large cardinals, and they even make an appearance much earlier in [7] as faces of the positive unit ball. While their basic theory was fleshed out in [8] (as 'norm filters') and [16], there remained some fundamental questions which we aim to address in this paper.

The first such question is why they should be considered as filters at all. Filters in the classical sense are defined from a transitive relation, as the downwards directed upwards closed subsets, but in general there is no such relation defining quantum filters. Indeed, it can even happen that every maximal quantum filter in a C^*-algebra fails to be a filter in the traditional order theoretic sense – see [8, Corollary 6.6]. While it might be intuitively clear that quantum filters are the 'right' quantum analog, and their utility in analyzing states justifies their

Date: Received: Oct. 10, 2017; Accepted: Mar. 4, 2018.
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2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 06A75,, 46L85, 54E99.
Key words and phrases. C^*-algebra, filter, compact projection, nonsymmetric distance.

study, regardless of whether they are considered as filters or not, a more precise connection to order theory would of course be desirable.

The key here is to replace classical transitive relations with 'continuous' ones. These are the nonsymmetric distances, binary functions $\mathbf{D}$ to $[0,\infty]$ satisfying the continuous version of transitivity, namely the triangle inequality

$$\mathbf{D}(x,y) \leq \mathbf{D}(x,z) + \mathbf{D}(z,y).$$

The first order sentence defining a classical filter also has a continuous version, as given in ($\mathbf{D}$-filter) of Definition 3.1. This is in keeping with the general philosophy of continuous model theory (see [21]) where binary relations are replaced by binary functions taking values in $[0,\infty]$ and the quantifiers $\forall$ and $\exists$ are replaced by suprema and infima, respectively. Then quantum filters are indeed the continuous filters with respect to the appropriate distance $\mathbf{d}$ on the positive unit ball A^1_+, namely

$$\mathbf{d}(a,b) = \|a - ab\|.$$

This simple observation, which is expanded upon in Theorem 3.2, allows for a markedly different approach to the theory.

In section 2 we start off by examining the relationship between various distances and distance-like functions. We move on to $\mathbf{d}$-filters in section 3, using these relationships to provide characterizations using the distance, order, multiplicative, and convex structure of A^1_+. In section 4 we then show how $\mathbf{d}$-filters in A represent compact projections in A^{**} (just as hereditary C^*-subalgebras in A represent open projections in A^{**}). We finish by examining interior containment of compact projections and its relation to the reverse Hausdorff distance on $\mathbf{d}$-filters.

2. Distances

We will deal with a number of binary functions $\mathbf{D}$ from some set X to $[0,\infty]$. We view these as 'generalized' or 'continuous' relations on X. More precisely, the zero-set of any $\mathbf{D} : X \times X \to [0,\infty]$ defines a classical relation $\mathbf{D}^0 \subseteq X \times X$ by

$$x\mathbf{D}^0 y \quad \Leftrightarrow \quad \mathbf{D}(x,y) = 0.$$

To emphasize that we are viewing $\mathbf{D}$ as a continuous version of $\mathbf{D}^0$, we say that $\mathbf{D}$ *quantifies* $\mathbf{D}^0$. Conversely, every relation $R \subseteq X \times X$ has a trivial quantification given by its characteristic function, which we also denote by R, specifically

$$R(x,y) = \begin{cases} 0 & \text{if } xRy, \\ \infty & \text{otherwise.} \end{cases}$$

Definition 2.1. The *composition* $\mathbf{D} \circ \mathbf{E}$ of $\mathbf{D}, \mathbf{E} : X \times X \to [0,\infty]$ is given by

$$(\mathbf{D} \circ \mathbf{E})(x,y) = \inf_{z \in X} (\mathbf{D}(x,z) + \mathbf{E}(z,y)).$$

For any $Z \subseteq X$, we define the *composition in* Z by

$$(\mathbf{D} \circ_Z \mathbf{E})(x,y) = \inf_{z \in Z} (\mathbf{D}(x,z) + \mathbf{E}(z,y)).$$

Note we are using the standard infix notation xRy to mean $(x,y) \in R$.

Note how taking the composition in a smaller domain can make the resulting function larger; that is,

$$\mathbf{D} \circ \mathbf{E} \leq \mathbf{D} \circ_Z \mathbf{E}.$$

Also note that when R and S are relations (identified with their characteristic functions as mentioned above);

$$x(R \circ S)y \quad \Leftrightarrow \quad \exists z \in X \ (xRzSy),$$

so '$\circ$' extends the usual composition of classical relations. Moreover,

$$\mathbf{D}^0 \circ \mathbf{E}^0 \subseteq (\mathbf{D} \circ \mathbf{E})^0.$$

We say that $\mathbf{D}$ is $\mathbf{E}$-*invariant* when $\mathbf{D} = \mathbf{D}\circ\mathbf{E} = \mathbf{E}\circ\mathbf{D}$. In particular, it is natural to consider $\mathbf{E}$-invariance for metric $\mathbf{E}$. Then it suffices to verify $\mathbf{D} \leq \mathbf{D}\circ\mathbf{E}, \mathbf{E}\circ\mathbf{D}$, as the reverse inequalities are immediate from the reflexivity of $\mathbf{E}^0$.

Definition 2.2. We call $\mathbf{D}$ a *distance* if it satisfies the triangle inequality

$$\mathbf{D} \leq \mathbf{D} \circ \mathbf{D}. \tag{$\triangle$}$$

We call $\mathbf{D}$ a *hemimetric* if $\mathbf{D}$ is a distance and $\mathbf{D}^0$ is reflexive (i.e. $\mathbf{D}(x,x) = 0$).

On a C^*-algebra A, the only distance usually considered is the metric given by

$$\mathbf{e}(x,y) = \|x - y\|.$$

Indeed, metrics are precisely the symmetric distances quantifying equality; that is, the symmetric distances whose zero-sets coincide with the equality relation. However, our thesis is that one should also consider various nonsymmetric distances on C^*-algebras which quantify other important order relations like

$$\begin{aligned} a \ll b \quad &\Leftrightarrow \quad a = ab, \\ a \leq b \quad &\Leftrightarrow \quad b - a \in A_+. \end{aligned}$$

Here A_+ denotes the positive elements in A, while A_{sa}, A^r, and $A^{=r}$ will denote the self-adjoints, r-ball, and r-sphere, respectively. In particular, A^1 denotes the unit ball. We also consider A embedded canonically in its enveloping von Neumann algebra A^{**} and set

$$\widetilde{A} = A + \mathbb{C}1.$$

So if A is unital then $\widetilde{A} = A$, otherwise $\widetilde{A}$ is the unitization of A (see [13, II.1.2]). In particular, for any $a \in A^1_+$, we let $a^\perp = 1 - a \in \widetilde{A}^1_+$.

Proposition 2.3.

$$\mathbf{d}(a,b) = \|a - ab\| \quad \textit{is an } \mathbf{e}\textit{-invariant distance on } A^1_+ \textit{ quantifying } \ll. \tag{2.1}$$

$$\mathbf{h}(a,b) = \|(a-b)_+\| \quad \textit{is an } \mathbf{e}\textit{-invariant hemimetric on } A_{\mathrm{sa}} \textit{ quantifying } \leq. \tag{2.2}$$

Proof.

In other words, the category **Rel** of classical relations forms a wide subcategory of **GRel**, the category of generalized relations – see [11, §1] for more details.

(2.1) Fix $a, b, c \in A^1_+$. Then

$$\mathbf{d}(a,b) = \|ab^\perp\| = \|a(c^\perp + c)b^\perp\| \leq \|ac^\perp\|\,\|b^\perp\| + \|a\|\,\|cb^\perp\| \leq \mathbf{d}(a,c) + \mathbf{d}(c,b).$$

As $\mathbf{e}$ is a metric, the $\mathbf{e}$-invariance of $\mathbf{d}$ on A^1_+ follows from

$$\mathbf{d}(a,b) = \|a - ab\| \leq \|a - c\| + \|c - cb\| = \mathbf{e}(a,c) + \mathbf{d}(c,b).$$
$$\mathbf{d}(a,b) = \|a - ab\| \leq \|a - ac\| + \|ac - ab\| \leq \mathbf{d}(a,c) + \mathbf{e}(c,b).$$

(2.2) Consider the space of quasistates A^{*1}_+ on A (i.e. positive linear functionals in the dual unit ball), and recall that, for $a \in A_{\mathrm{sa}}$,

$$\|a_+\| = \sup_{\phi \in A^{*1}_+} \phi(a). \tag{2.3}$$

Thus, for all $a, b, c \in A_{sa}$, we have

$$\mathbf{h}(a,b) \leq \sup_{\phi \in A^{*1}_+} \phi(a - c) + \sup_{\phi \in A^{*1}_+} \phi(c - b) = \mathbf{h}(a,c) + \mathbf{h}(c,b).$$

Now $\mathbf{h} \leq \mathbf{e}$, as $\|a_+\| \leq \|a\|$, so $\mathbf{h}^0$ is reflexive and $\mathbf{h} \leq \mathbf{h} \circ \mathbf{h} \leq \mathbf{h} \circ \mathbf{e}, \mathbf{e} \circ \mathbf{h}$. Again, as $\mathbf{e}$ is a metric, it follows that $\mathbf{h}$ is $\mathbf{e}$-invariant. □

Basic relationships between C^*-algebra distances reveal aspects of C^*-algebraic structure. Here are some required for our investigation of C^*-algebra filters.

Proposition 2.4. *On A^1_+,*

$$\mathbf{h} \leq 2\mathbf{d}, \tag{2.4}$$
$$\mathbf{d}^2 \leq \mathbf{d} \circ \mathbf{h}, \tag{2.5}$$
$$\mathbf{d}^2 \leq \mathbf{h} \circ \mathbf{d}. \tag{2.6}$$

In (2.5) and (2.6), we can even take the composition in A_{sa} (so $\circ$ becomes $\circ_{A_{\mathrm{sa}}}$).

Proof.

(2.4) For $a, b \in A^1_+$, $bab \leq b^2 \leq b$; so

$$\begin{aligned}\mathbf{h}(a,b) &\leq \mathbf{h}(a, bab) + \mathbf{h}(bab, b)\\ &\leq \|a - bab\|\\ &\leq \|a - ab\| + \|ab - bab\|\\ &\leq \|a - ab\| + \|a - ba\|\,\|b\|\\ &\leq 2\mathbf{d}(a,b).\end{aligned}$$

(2.5) First note that, for any $a \in A^1$ and $b \in A_{\mathrm{sa}}$,

$$\|(aba^*)_+\| = \inf_{aba^* \leq c} \|c\| \leq \|ab_+a^*\| \leq \|b_+\|\,. \tag{2.7}$$

One might naively use $(a + b)_+ \leq a_+ + b_+$ instead, but this only holds for commutative A.

As $\|(a+b)_+\| \leq \|a_+\| + \|b_+\|$ (see (2.3)), for all $a, b \in A^1_+$ and $c \in A_{\mathrm{sa}}$,

$$\begin{aligned}\mathbf{d}(a,b)^2 &= \left\|ab^{\perp 2}a\right\| \\ &\leq \left\|ab^{\perp}a\right\| = \left\|(ab^{\perp}a)_+\right\| \\ &\leq \left\|(ac^{\perp}a)_+\right\| + \left\|(a(b^{\perp}-c^{\perp})a)_+\right\| \\ &\leq \left\|ac^{\perp}\right\| \|a\| + \|(c-b)_+\| \\ &\leq \mathbf{d}(a,c) + \mathbf{h}(c,b).\end{aligned}$$

(2.6) Likewise, for $a, b \in A^1_+$ and $c \in A_{\mathrm{sa}}$,

$$\begin{aligned}\mathbf{d}(a,b)^2 &= \left\|b^{\perp}a^2b^{\perp}\right\| \\ &\leq \left\|b^{\perp}ab^{\perp}\right\| = \left\|(b^{\perp}ab^{\perp})_+\right\| \\ &\leq \left\|(b^{\perp}(a-c)b^{\perp})_+\right\| + \left\|(b^{\perp}cb^{\perp})_+\right\| \\ &\leq \|(a-c)_+\| + \left\|b^{\perp}\right\| \left\|cb^{\perp}\right\| \\ &\leq \mathbf{h}(a,c) + \mathbf{d}(c,b).\end{aligned}$$

□

We can also quantify $\leq$ using elements above and below a and b in A^1_+, defining

$$\mathbf{a}(a,b) = \inf_{a \leq c \in A^1_+} \|c-b\|,$$
$$\mathbf{b}(a,b) = \inf_{b \geq c \in A^1_+} \|a-c\|.$$

Equivalently, $\mathbf{a} = \leq \circ\, \mathbf{e}$ while $\mathbf{b} = \mathbf{e} \circ \leq$, where the composition $\circ$ is taken in A^1_+ (again identifying $\leq$ with its characteristic function). If we took the composition in A_{sa} instead, then we would end up back at $\mathbf{h}$ – see (2.12) below. Indeed, there are still times when $\mathbf{h}$ coincides with $\mathbf{a}$ and $\mathbf{b}$, as in Proposition 2.7 and Corollary 2.10 below. However, they do not coincide in general.

Example 2.5. For $A = M_2$, we have $\mathbf{a} \neq \mathbf{b}$ on A^1_+ and

$$\mathbf{a} \neq \mathbf{h} \neq \mathbf{b}.$$

In fact, $\mathbf{a}$ and $\mathbf{b}$ can fail to be distances or even $\mathbf{e}$-invariant on A^1_+.

To see this, we consider $\mathbf{b}$. As $\mathbf{a}(b^{\perp}, a^{\perp}) = \mathbf{b}(a,b)$ and $\mathbf{h}(b^{\perp}, a^{\perp}) = \mathbf{h}(a,b)$, the results for $\mathbf{a}$ follow. For convenience, we also naturally extend $\mathbf{b}$ to A_+, defining

$$\mathbf{b}(a,b) = \inf_{b \geq c \in A_+} \|a-c\|.$$

Note $\mathbf{b}$ is then homogeneous on A_+, as is $\mathbf{h}$; that is, for $r \in \mathbb{R}_+$, $\mathbf{b}(ra, rb) = r\mathbf{b}(a,b)$ and $\mathbf{h}(ra, rb) = r\mathbf{h}(a,b)$; so the results for A^1_+ then follow.

In M_2, let $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and let $b = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$; then $a - tb = \begin{bmatrix} 1-4t & 1 \\ 1 & 1 \end{bmatrix}$ and

$$\det((a-tb)-\lambda) = (1-4t-\lambda)(1-\lambda) - 1 = \lambda^2 + (4t-2)\lambda - 4t.$$

So $(a-tb)$ has eigenvalues $1 - 2t \pm \sqrt{4t^2+1}$, and hence

$$\|a-tb\| = \begin{cases} 1-2t+\sqrt{4t^2+1} & \text{for } t \leq \frac{1}{2}, \\ 2t-1+\sqrt{4t^2+1} & \text{for } t \geq \frac{1}{2}. \end{cases}$$

Thus $\mathbf{b}(a,b) = \inf_{t\in[0,1]} \|a - tb\| = \left\|a - \frac{1}{2}b\right\| = \sqrt{2}$. On the other hand, $\mathbf{h}(a,b)$ is the positive eigenvalue for $t = 1$; that is, $\sqrt{5} - 1$; so

$$\mathbf{h}(a,b) < \mathbf{b}(a,b).$$

Now let $c = a + (b-a)_+$; so $\mathbf{b}(a,c) = 0$ and $\mathbf{b}(c,b) = \|c - b\| = \|(a-b)_+\| = \mathbf{h}(a,b)$, as $a, b \leq c$. Thus $\mathbf{b}(a,b) > \mathbf{b}(a,c) + \mathbf{b}(c,b) = \mathbf{b}(a,c) + \mathbf{e}(c,b)$; so $\mathbf{b}$ is not a distance and $\mathbf{b}$ is also not $\mathbf{e}$-invariant. However, $\mathbf{b}$ is left $\mathbf{e}$-invariant, as

$$\mathbf{b} \;=\; \mathbf{e} \circ \leq \;=\; \mathbf{e} \circ \mathbf{e} \circ \leq \;=\; \mathbf{e} \circ \mathbf{b}.$$

Likewise, $\mathbf{a}$ is right but not left $\mathbf{e}$-invariant. In particular, $\mathbf{a} \neq \mathbf{b}$.

However, in general $\mathbf{a}$ and $\mathbf{b}$ are still bounded by a function of $\mathbf{h}$, as the following inequalities show. Note that these are crucial to proof of (4.6) which is needed to characterize compact projections as 'closed and bounded' in Corollary 4.8. Also (2.8) is required for several of the characterizations of $\mathbf{d}$-filters given in Theorem 3.2.

Theorem 2.6. *On A^1_+,*

$$\mathbf{h} \leq \mathbf{a} \leq 2\sqrt{\mathbf{h}}, \tag{2.8}$$

$$\mathbf{h} \leq \mathbf{b} \leq 2\sqrt{\mathbf{h}}. \tag{2.9}$$

Moreover, if B is a hereditary C^*-subalgebra of A and $a, b \in B^1_+$, then

$$\mathbf{a}(a,b) = \inf_{a \leq c \in B^1_+} \|c - b\|, \tag{2.10}$$

$$\mathbf{b}(a,b) = \inf_{b \geq c \in B^1_+} \|a - c\|. \tag{2.11}$$

Proof. First we show that $\mathbf{h} \leq \mathbf{b}$ on A^1_+. Whenever $c \leq b$, (2.3) yields

$$\mathbf{h}(a,b) = \sup_{\phi \in A^{*1}_+} \phi(a-b) \leq \sup_{\phi \in A^{*1}_+} \phi(a-c) \leq \|a - c\|.$$

Also $a - b \leq (a-b)_+$; so $a - (a-b)_+ \leq b$ and $\mathbf{h}(a,b) = \|a - (a - (a-b)_+)\|$; hence

$$\mathbf{h}(a,b) = \inf_{b \geq c \in A_{\mathrm{sa}}} \|a - c\| \leq \inf_{b \geq c \in A^1_+} \|a - c\| = \mathbf{b}(a,b). \tag{2.12}$$

For $\mathbf{b} \leq 2\sqrt{\mathbf{h}}$, take $a, b \in A^1_+$ and let $z = b + (a-b)_+$. Then $a, b \leq z \in A_+$ and

$$\sqrt{a}(\tfrac{1}{n} + z)^{-\frac{1}{2}}\sqrt{b} \to u,$$

for some $u \in A$, by [20, Lemma 1.4.4]. As $\sqrt{a}(\frac{1}{n} + z)^{-\frac{1}{2}}\sqrt{z} \to \sqrt{a}$, we have

$$\sqrt{a}(\tfrac{1}{n} + z)^{-\frac{1}{2}}(\sqrt{b} - \sqrt{z}) \to u - \sqrt{a}.$$

We claim that $\left\|\sqrt{a}(\frac{1}{n} + z)^{-\frac{1}{2}}\right\| \leq 1$, for all n, and hence $\|u - \sqrt{a}\| \leq \left\|\sqrt{b} - \sqrt{z}\right\|$. First note that $a \leq z$ implies $a + \frac{1}{n} \leq z + \frac{1}{n}$, and hence $(z + \frac{1}{n})^{-1} \leq (a + \frac{1}{n})^{-1}$, by [20, Proposition 1.3.6]. Thus $\sqrt{a}(z + \frac{1}{n})^{-1}\sqrt{a} \leq \sqrt{a}(a + \frac{1}{n})^{-1}\sqrt{a} \leq 1$, and hence $\left\|\sqrt{a}(\frac{1}{n} + z)^{-\frac{1}{2}}\right\|^2 \leq \left\|\sqrt{a}(z + \frac{1}{n})^{-1}\sqrt{a}\right\| \leq 1$.

By the claim just proved and [17, Corollary 2] with $p = \infty$,

$$\left\|u - \sqrt{a}\right\| \leq \left\|\sqrt{b} - \sqrt{z}\right\| \leq \sqrt{\|b - z\|} = \sqrt{\|(a-b)_+\|} = \sqrt{\mathbf{h}(a,b)}.$$

As in the proof of [20, Proposition 1.4.10], we have $u^*u \leq b$, and

$$\|a - u^*u\| \leq \left\|a - u^*\sqrt{a} + u^*\sqrt{a} - u^*u\right\| \leq \left\|\sqrt{a} - u^*\right\| + \left\|\sqrt{a} - u\right\| \leq 2\sqrt{\mathbf{h}(a,b)}.$$

The argument for $\mathbf{h} \leq \mathbf{a}$ is a simple analog/dual to the argument for $\mathbf{h} \leq \mathbf{b}$. For $\mathbf{a} \leq 2\sqrt{\mathbf{h}}$, take $a, b \in A^1_+$, and argue as before with $b^\perp$ and $a^\perp$ replacing a and b, respectively. Specifically, let $z = a^\perp + (a-b)_+$; so $a^\perp, b^\perp \leq z$ and

$$\sqrt{a^\perp}(\tfrac{1}{n} + z)^{-\frac{1}{2}}\sqrt{b^\perp} \to u,$$

for some $u \in \widetilde{A}$. As before, we have $u^*u \leq a^\perp$ and $\left\|u^*u - b^\perp\right\| \leq 2\sqrt{\mathbf{h}(b^\perp, a^\perp)}$. Equivalently, $a \leq (u^*u)^\perp$ and $\left\|(u^*u)^\perp - b\right\| \leq 2\sqrt{\mathbf{h}(a,b)}$. Moreover, $(u^*u)^\perp \in A$ even when A is not unital, as then $\pi(z) = 1 = \pi(u)$, and hence $\pi((u^*u)^\perp) = 0$, where π is canonical homomorphism from $\widetilde{A}$ to $\mathbb{C}$ with kernel A.

Next note (2.11) is immediate, as B contains all positive elements below b. For (2.10), take $c \geq a$ in A^1_+. As B has an approximate unit, for any $\epsilon > 0$, we have $u \in B^1_+$ with $\|uau - a\|, \|ubu - b\| < \epsilon$. As $\mathbf{h}$ is $\mathbf{e}$-invariant and $uau \leq ucu \in B^1_+$,

$$\mathbf{h}(a, ucu) \leq \|a - uau\| + \mathbf{h}(uau, ucu) < \epsilon.$$

Applying $\mathbf{a} \leq 2\sqrt{\mathbf{h}}$ within B yields $d \geq a$ in B^1_+ with $\|d - ucu\| < 2\sqrt{\epsilon}$, and hence

$$\|d - b\| \leq \|d - ucu\| + \|ucu - ubu\| + \|ubu - b\| \leq 2\sqrt{\epsilon} + \|c - b\| + \epsilon.$$

As $\epsilon > 0$ was arbitrary, $\inf_{a \leq d \in B^1_+} \|d - b\| \leq \|c - b\|$. As c was arbitrary,

$$\inf_{a \leq d \in B^1_+} \|d - b\| \leq \inf_{a \leq c \in A^1_+} \|c - b\|.$$

The reverse in equality is immediate, thus proving (2.10). ☐

In the unital case, (2.8) is immediate from (2.9) and the fact that $\mathbf{a}(b^\perp, a^\perp) = \mathbf{b}(a,b)$ and $\mathbf{h}(b^\perp, a^\perp) = \mathbf{h}(a,b)$. If (2.10) could somehow be proved first then the general case would follow from passing to the unitization, as A is hereditary in $\widetilde{A}$. However, we could not do this above because we needed (2.8) to prove (2.10).

Question 1. Can 'hereditary' be deleted above? In other words, are $\mathbf{a}$ and $\mathbf{b}$ independent of the C^*-subalgebra in which they are calculated, like $\mathbf{d}$ and $\mathbf{h}$?

In the commutative case, we have a positive answer.

Proposition 2.7. *On commutative $C \subseteq A^1_+$,*

$$\mathbf{a} = \mathbf{b} = \mathbf{h}.$$

Proof. By the Gelfand representation, the C^*-algebra B generated by C is isomorphic to $C_0(X)$ for some locally compact X. In particular, a and b have a supremum $c = a \vee b$ in B^1_+ with $c - b = (a-b)_+$; so

$$\mathbf{a}(a,b) \leq \|c - b\| = \|(a-b)_+\| = \mathbf{h}(a,b).$$

Likewise, a and b have an infimum $d = a \wedge b$ in B^1_+ with $a - d = (a-b)_+$; so

$$\mathbf{b}(a,b) \leq \|a - d\| = \|(a-b)_+\| = \mathbf{h}(a,b).$$

The reverse inequalities follow from Theorem 2.6. $\square$

Next we show that the same applies to projections

$$\mathcal{P}(A) = \{p \in A : p \ll p^*\} = \{p \in A_{sa} : p \ll p\} \subseteq A^1_+.$$

One way of proving $\mathbf{a} = \mathbf{b} = \mathbf{d} = \mathbf{h}$ on $\mathcal{P}(A)$ would be to note that, by reverting to a C^*-subalgebra if necessary, one can assume A is generated by $p, q \in \mathcal{P}(A)$. As every irreducible representation of a C^*-algebra generated by a pair of projections, is on a Hilbert space of dimension at most 2; for $\mathbf{d} = \mathbf{h}$ on $\mathcal{P}(A)$, it suffices to consider $A = \mathbb{C}$ or M_2, which can be done with some elementary calculations.

Alternatively, we can use the following, adapted from [1], which strengthens the standard result that close projections are unitarily equivalent (see [13, II.3.3.5]).

Lemma 2.8. *If $p, q \in \mathcal{P}(A)$ and $\|p - q\| < 1$, then p and q can be exchanged by a symmetry(=self-adjoint unitary); that is, we have $u \in \widetilde{A}_{\mathrm{sa}}$ with $u^2 = 1$ and $up = qu$.*

Proof. Let $a = p + q - 1 \in \widetilde{A}_{\mathrm{sa}}$; so $ap = qp = qa$ and $aq = pq = pa$. Thus $a^2 p = aqp = pqp = pqa = pa^2$. Also, $a^2 = pq + qp - p - q + 1 = 1 - (p-q)^2$; so $\|1 - a^2\| = \|p - q\|^2 < 1$, and hence a^2 is invertible. Thus we may set $u = a|a|^{-1}$; so $u^2 = 1$, as $a \in A_{\mathrm{sa}}$. Also, as p commutes with a^2 and hence with $|a|^{-1}$,

$$up = a|a|^{-1}p = ap|a|^{-1} = qa|a|^{-1} = qu. \qquad \square$$

Corollary 2.9. *If $p, q \in \mathcal{P}(A)$ and $\|p - q\| < 1$, then $\mathbf{h}(p,q) = \|p - q\|$.*

Proof. if $\mathbf{e}(p,q) < 1$, then Lemma 2.8 yields an automorphism $a \mapsto uau$ of A exchanging p and q; so $\|(p-q)_+\| = \|(q-p)_+\|$, and hence

$$\|p - q\| = \|(p-q)_+\| \vee \|(q-p)_+\| = \|(p-q)_+\| = \mathbf{h}(p,q). \qquad \square$$

Corollary 2.10. *On $\mathcal{P}(A)$,* $\mathbf{a} = \mathbf{b} = \mathbf{d} = \mathbf{h}$.

Proof. If $\mathbf{d}(p,q) < 1$, then qp is well-supported; so we have a range projection $[qp] = f(qpq) \in A$ (for continuous f on $[0,1]$ that is 1 on $\sigma(qpq)$). By [9, §2.3] (specifically equation (2.3) and the sentence after it),

$$\begin{aligned} \mathbf{d}(p,q) = \|p - [qp]\| \geq \mathbf{b}(p,q) \geq \mathbf{h}(p,q) \quad &\text{and} \\ 0 = p(q - [qp]) = (p - [qp])(q - [qp]); \quad &\text{so} \\ (p-q)_+ = (p - [qp])_+ + ([qp] - q)_+ = (p - [qp])_+, \quad &\text{and hence} \\ \mathbf{h}(p,q) = \mathbf{h}(p,[qp]) = \|p - [qp]\|, \quad &\text{by Corollary 2.9.} \end{aligned}$$

While if $\mathbf{d}(p,q) = 1$, then $1 = \left\|pq^\perp p\right\| = \left\|(pq^\perp p)_+\right\|$, and hence, by (2.7),

$$1 = \left\|(pq^\perp p)_+\right\| = \|(p(p-q)p)_+\| \leq \|(p-q)_+\| = \mathbf{h}(p,q) \leq \mathbf{b}(p,q) \leq \|p - q\| \leq 1.$$

So $\mathbf{d} = \mathbf{h} = \mathbf{b}$ on $\mathcal{P}(A)$. A similar argument with $[p^\perp q^\perp]^\perp \in A$ applies to $\mathbf{a}$. $\square$

Incidentally, there does not appear to be much room for extending Corollary 2.10 to a bigger class than $\mathcal{P}(A)$ – Example 2.5 shows that even scalar multiples of projections can witness $\mathbf{a} \neq \mathbf{h} \neq \mathbf{b}$.

We have already seen several cases where functions defined from distances do not coincide in general, but are still bounded by functions of each other. A similar situation often arises in metric space theory when dealing with various distinct but uniformly equivalent metrics. Indeed, it will be convenient to formally define general uniform (sub)equivalence relations $\precapprox$ and $\approx$ as follows.

Definition 2.11. For functions $\mathbf{F}, \mathbf{G} : X \to [0, \infty]$, we define $\precapprox$ and $\approx$ by

$$\mathbf{F} \precapprox \mathbf{G} \qquad \Leftrightarrow \qquad 0 = \lim_{r \to 0} \sup_{\mathbf{G}(x) \leq r} \mathbf{F}(x),$$
$$\mathbf{F} \approx \mathbf{G} \qquad \Leftrightarrow \qquad \mathbf{F} \precapprox \mathbf{G} \precapprox \mathbf{F}.$$

Equivalently, $\mathbf{F} \precapprox \mathbf{G}$ if and only if, for all $Y \subseteq X$,

$$\inf_{y \in Y} \mathbf{G}(y) = 0 \qquad \Rightarrow \qquad \inf_{y \in Y} \mathbf{F}(y) = 0.$$

By Theorem 2.6, $\mathbf{a} \approx \mathbf{b} \approx \mathbf{h}$ on A^1_+, which is all we really need from now on. For our characterizations of $\mathbf{d}$-filters, we will also need to consider some uniformly equivalent unary functions defined from $\mathbf{d}$ by fixing the left or right coordinate.

Definition 2.12. For $\mathbf{D}\colon X \times X \to [0, \infty]$, we define ${}_x\mathbf{D}, \mathbf{D}_y : X \to [0, \infty]$ by

$${}_x\mathbf{D}(y) = \mathbf{D}(x, y) = \mathbf{D}_y(x).$$

Proposition 2.13. *For any $a, b \in A^1_+$ and $\epsilon \in (0, 1)$,*

$$\mathbf{d}_{aba} \;\approx\; \mathbf{d}_a + \mathbf{d}_b \;\approx\; \mathbf{d}_{\epsilon a + (1-\epsilon) b}.$$

Proof. First note $\mathbf{d}_{aba} \leq 2\mathbf{d}_a + \mathbf{d}_b$, and hence $\mathbf{d}_{aba} \precapprox \mathbf{d}_a + \mathbf{d}_b$, as

$$\begin{aligned} \mathbf{d}(c, aba) &= \|c - caba\| \\ &\leq \|c - ca\| + \|ca - cba\| + \|cba - caba\| \\ &\leq 2\mathbf{d}(c, a) + \mathbf{d}(c, b). \end{aligned}$$

Next, as $aba \leq a^2 \leq a$, $\mathbf{d}(c, a)^2 \leq \mathbf{d}(c, aba) + \mathbf{h}(aba, a) = \mathbf{d}(c, aba)$, and

$$\begin{aligned} \mathbf{d}(c, b)^2 = \left\|cb^{\perp 2}c\right\| \leq \left\|cb^{\perp}c\right\| &= \left\|c^2 - cbc\right\| \\ &\leq \left\|c^2 - cabac\right\| + \|cabac - cbac\| + \|cbac - cbc\| \\ &\leq \|c - caba\|\,\|c\| + \|ca - c\|\,\|bac\| + \|cb\|\,\|ac - c\| \\ &\leq \mathbf{d}(c, aba) + 2\mathbf{d}(c, a) \\ &\leq \mathbf{d}(c, aba) + 2\sqrt{\mathbf{d}(c, aba)}. \end{aligned}$$

Thus $\mathbf{d}_a + \mathbf{d}_b \precapprox \mathbf{d}_{aba}$, and hence $\mathbf{d}_a + \mathbf{d}_b \approx \mathbf{d}_{aba}$.

In particular, for any $n \in \mathbb{N}$, setting $a = b$ above and using (2.5) yield

$$\mathbf{d}_a \precapprox \mathbf{d}_{a^n} \precapprox \mathbf{d}_{a^{3^n}} \precapprox \mathbf{d}_a. \tag{2.13}$$

Also $\sup_{x \in [0,1]} (\epsilon x + 1 - \epsilon)^n (1 - x) \leq \frac{1}{n\epsilon}$; so

$$\mathbf{d}_a - \tfrac{1}{n\epsilon} \leq \mathbf{d}_a - \mathbf{d}((\epsilon a + 1 - \epsilon)^n, a).$$

By ($\triangle$), $\mathbf{d}(x,y) \leq \mathbf{d}(x,z) + \mathbf{d}(z,y)$; so $\mathbf{d}(x,y) - \mathbf{d}(z,y) \leq \mathbf{d}(x,z)$, and hence $\mathbf{d}_y - \mathbf{d}(z,y) \leq \mathbf{d}_z$. Taking $y = a$ and $z = (\epsilon a + 1 - \epsilon)^n$ then yields

$$\begin{aligned}\mathbf{d}_a - \tfrac{1}{n\epsilon} &\leq \mathbf{d}_{(\epsilon a+1-\epsilon)^n}\\ &\precapprox \mathbf{d}_{\epsilon a+1-\epsilon} \qquad \text{by (2.13)}\\ &\precapprox \mathbf{d}_{\epsilon a+(1-\epsilon)b} \qquad \text{by (2.5)}.\end{aligned}$$

As n was arbitrary, $\mathbf{d}_a \precapprox \mathbf{d}_{\epsilon a+(1-\epsilon)b}$, and, by symmetry, $\mathbf{d}_b \precapprox \mathbf{d}_{\epsilon a+(1-\epsilon)b}$.

On the other hand, $\mathbf{d}_{\epsilon a+(1-\epsilon)b} \precapprox \mathbf{d}_a + \mathbf{d}_b$ follows from

$$\begin{aligned}\mathbf{d}(c, \epsilon a + (1-\epsilon)b) &= \|c - c(\epsilon a + (1-\epsilon)b)\|\\ &= \|\epsilon c - \epsilon ca + (1-\epsilon)c - (1-\epsilon)cb\|\\ &\leq \epsilon\,\|c - ca\| + (1-\epsilon)\,\|c - cb\|\\ &= \epsilon\mathbf{d}(c,a) + (1-\epsilon)\mathbf{d}(c,b).\end{aligned}$$

□

A slightly better substitute for multiplication than aba might be the following.

Definition 2.14. For any $a, b \in A^1_+$, define $a \odot b \in A^1_+$ by

$$a \odot b = \sqrt{a}b\sqrt{a}.$$

Indeed, note that if $ab = ba$, then $a \odot b = ab$ and $(a \odot b) \odot c = a \odot (b \odot c)$. In particular, $\odot$ is left alternative; that is, $(a \odot a) \odot b = a \odot (a \odot b)$ and also right distributive; that is, $a \odot (b + c) = a \odot b + a \odot c$. We can also quantify $\ll$ using $\odot$ by

$$\mathbf{f}(a,b) = \|a - a \odot b\|.$$

The advantage of $\mathbf{f}$ over $\mathbf{d}$ is that it determines $\mathbf{h}$ in a natural way. However, $\mathbf{f}$ has other disadvantages (see the discussion after the result), and so we will not consider $\mathbf{f}$ in the rest of the paper, instead focusing our attention on $\mathbf{d}$.

Proposition 2.15. *We have* $\mathbf{d} \approx \mathbf{f} = \mathbf{f} \circ \mathbf{h}$ *and*

$$\mathbf{h}(a,b) = \sup_{c \in A^1_+} (\mathbf{f}(c,b) - \mathbf{f}(c,a)). \tag{2.14}$$

Proof. First note that, for $a, b \in A^1_+$,

$$\left\|ab^\perp\right\|^2 = \left\|b^\perp a^2 b^\perp\right\| \leq \left\|a^2 b^\perp\right\|.$$

Thus, as binary functions on A^1_+,

$$\mathbf{d}(a,b) = \left\|ab^\perp\right\| \approx \left\|\sqrt{a}b^\perp\right\| \approx \left\|\sqrt{a}\sqrt{b^\perp}\right\| \approx \left\|\sqrt{a}\sqrt{b^\perp}\right\|^2 = \left\|\sqrt{a}b^\perp\sqrt{a}\right\| = \mathbf{f}(a,b).$$

As in the proof of (2.5), we have

$$\begin{aligned}\mathbf{f}(a,b) &= \left\|a \odot b^\perp\right\|\\ &= \left\|(a \odot b^\perp)_+\right\|\\ &\leq \left\|(a \odot c^\perp)_+\right\| + \left\|(a \odot (b^\perp - c^\perp))_+\right\|\\ &\leq \left\|(a \odot c^\perp)\right\| + \|(c - b)_+\|\\ &= \mathbf{f}(a,c) + \mathbf{h}(c,b).\end{aligned}$$

Thus $\mathbf{f} = \mathbf{f} \circ \mathbf{h}$ and $\mathbf{h}(a, b) \geq \sup_{c \in A^1_+}(\mathbf{f}(c, b) - \mathbf{f}(c, a))$. Conversely, take $a, b \in A^1_+$. If $\mathbf{h}(a, b) = 0$, then the reverse inequality is immediate from $\mathbf{h}(a, b) = \mathbf{f}(0, b) - \mathbf{f}(0, c)$. Otherwise, for any $\epsilon > 0$, we can take a pure state ϕ on A with $\mathbf{h}(a, b) < \phi(a - b) + \epsilon$. By [6, Proposition 2.2], we have $c \in A^1_+$ with

$$\phi(c) = 1 \qquad \text{and} \qquad \|c \odot a - c\phi(a)\| < \epsilon.$$

Thus $\|c \odot a^\perp - c\phi(a^\perp)\| < \epsilon$; so $\|c \odot a^\perp\| < \|c\phi(a^\perp)\| + \epsilon = \phi(a^\perp) + \epsilon$ and

$$\begin{aligned}
\mathbf{h}(a, b) &< \phi(a - b) + \epsilon \\
&= \phi(b^\perp - a^\perp) + \epsilon \\
&= \phi(c \odot b^\perp) - \phi(a^\perp) + \epsilon \\
&< \|c \odot b^\perp\| - \|c \odot a^\perp\| + 2\epsilon. \\
&= \mathbf{f}(c, b) - \mathbf{f}(c, a) + 2\epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we are done. □

The drawback of $\mathbf{f}$ is that it may not be a distance. Indeed, by (2.14), $\mathbf{f}$ is a distance if and only if $\mathbf{h} \leq \mathbf{f}$. But by Corollary 2.10, for $p, q \in \mathcal{P}(A)$,

$$\mathbf{h}(p, q) = \mathbf{d}(p, q) = \sqrt{\|pq^\perp p\|} = \sqrt{\mathbf{f}(p, q)}.$$

So $\mathbf{h} \not\leq \mathbf{f}$ whenever we have $p, q \in \mathcal{P}(A)$ with $0 < \mathbf{f}(p, q) < 1$. For example, if $A = M_2$, then we can take any noncommuting rank 1 projections for p and q.

3. Filters

The main purpose of this section is to prove Theorem 3.2, characterizing the C^*-algebra filters from [8] in various ways using the theory just developed for the distances $\mathbf{d}$ and $\mathbf{h}$. First we introduce some general terminology.

Definition 3.1. Given $\mathbf{D}\colon X \times X \to [0, \infty]$, we define the following for $Y \subseteq X$.

$$a, b \in Y \quad \Leftrightarrow \quad \inf_{c \in Y}(\mathbf{D}(c, b) + \mathbf{D}(c, a)) = 0. \tag{$\mathbf{D}$-filter}$$

$$a, b \in Y \quad \Rightarrow \quad \inf_{c \in Y}(\mathbf{D}(c, b) + \mathbf{D}(c, a)) = 0. \tag{$\mathbf{D}$-directed}$$

$$b \in Y \quad \Leftarrow \quad \inf_{c \in Y}\mathbf{D}(c, b) = 0. \tag{$\mathbf{D}$-closed}$$

$$b \in Y \quad \Rightarrow \quad \inf_{c \in Y}\mathbf{D}(c, b) = 0. \tag{$\mathbf{D}$-initial}$$

$$b \in X \quad \Rightarrow \quad \inf_{c \in Y}\mathbf{D}(c, b) = 0. \tag{$\mathbf{D}$-coinitial}$$

$$c \in X \quad \Rightarrow \quad \inf_{b \in Y}\mathbf{D}(c, b) = 0. \tag{$\mathbf{D}$-cofinal}$$

For any operation $\bullet : X^n \to X$, we also call Y $\bullet$-*closed* if $\bullet[Y^n] \subseteq Y$.

Note these terms extend a number of familiar concepts from metric, order, and C^*-algebra theory. For example, $\mathbf{d}$-cofinal $\geq$-directed subsets of A^1_+ are increasing approximate units in the usual sense, when considered as self-indexed nets.

If $\mathbf{D}$ is a metric, $\mathbf{D}$-coinitial/cofinal means dense while $\mathbf{D}$-closed means closed, with respect to the usual ball topology defined by $\mathbf{D}$. Other terms become trivial, for example, arbitrary subsets are $\mathbf{D}$-initial, while the empty and one-point subsets are the only $\mathbf{D}$-directed subsets. In particular, for C^*-algebras, $\mathbf{e}$-closed/coinitial means norm closed/dense in the usual sense.

On the other hand, for any order relation $\leq$ (again identified with its characteristic function), $\leq$-closed means upwards closed, $\leq$-directed means downwards directed and $\leq$-cofinal means cofinal in the usual sense. In particular, $\leq$-filters are the usual order-theoretic filters and, more generally,

$$\mathbf{D}\text{-filter} \quad \Leftrightarrow \quad \mathbf{D}\text{-directed and } \mathbf{D}\text{-closed}.$$

We recall that $F \subseteq A^1_+$ is a *norm filter*, according to [8, Definition 3.1], if

$$\inf_{\substack{k\in\mathbb{N}\\ a_1,\ldots,a_k\in F}} \left\| a_1 a_2 \ldots a_k b^\perp \right\| = 0 \quad \Rightarrow \quad b \in F.$$

Also recall that a subset C of a vector space X is *convex* if $\epsilon a + (1-\epsilon)b \in C$ whenever $a, b \in C$ and $\epsilon \in (0,1)$, while $F \subseteq C$ is a *face* of C if the converse also holds; that is, if, for all $a, b \in C$ and $\epsilon \in (0,1)$,

$$a, b \in F \quad \Leftrightarrow \quad \epsilon a + (1-\epsilon)b \in F$$

(for faces it actually suffices to take $\epsilon = \frac{1}{2}$ or any other fixed element of $(0,1)$).

Theorem 3.2. *For $F \subseteq A^1_+$, the following are equivalent.*

1. *F is a $\mathbf{d}$-filter.*
2. *F is a $\mathbf{d}$-initial $\mathbf{h}$-filter.*
3. *F is the norm closure of a $\mathbf{d}$-initial $\leq$-filter.*
4. *F is norm closed, $\leq$-closed and $\odot$-closed.*
5. *F is norm closed, $\leq$-closed, ^{2}-closed, and convex.*
6. *F is a norm closed $\mathbf{d}$-cofinal face.*
7. *F is a norm filter.*

If A is separable or commutative, they are also equivalent to the following.

(8) *F is the norm closure of a $\ll$-filter.*

Proof.

(1)$\Leftrightarrow$(2) For any $a, b \in A^1_+$, (2.4) yields

$$\inf_{c\in F}(\mathbf{h}(c,b) + \mathbf{h}(c,a)) \leq \inf_{c\in F}(2\mathbf{d}(c,b) + 2\mathbf{d}(c,a)).$$

Conversely, if F is $\mathbf{d}$-initial, then

$$\begin{aligned}\inf_{c\in F}(\mathbf{h}(c,b) + \mathbf{h}(c,a)) &= \inf_{c\in F}\inf_{d\in F}(2\mathbf{d}(d,c) + \mathbf{h}(c,b) + \mathbf{h}(c,a)) \\ &\geq \inf_{d\in F}(\mathbf{d}\circ\mathbf{h}(d,b) + \mathbf{d}\circ\mathbf{h}(d,a)) \\ &\geq \inf_{c\in F}(\mathbf{d}^2(c,b) + \mathbf{d}^2(c,a)),\end{aligned}$$

by (2.5), so $\inf_{c\in F}(\mathbf{h}(c,b) + \mathbf{h}(c,a)) = 0$ if and only if $\inf_{c\in F}(\mathbf{d}(c,b) + \mathbf{d}(c,a)) = 0$. Thus if F is a $\mathbf{d}$-initial $\mathbf{h}$-filter, then F is a $\mathbf{d}$-filter, and, conversely, if F is a $\mathbf{d}$-filter, and hence $\mathbf{d}$-initial, then F is an $\mathbf{h}$-filter.

Note (4)$\Rightarrow$(7) eliminates the real rank zero hypothesis from [8, Proposition 3.5].

(1)⇒(5) By (2.13), $\mathbf{d}_{a^2} \precapprox \mathbf{d}_a$ and $\mathbf{d}_{\epsilon a+(1-\epsilon)b} \precapprox \mathbf{d}_a + \mathbf{d}_b$. So, as $\mathbf{h} \leq \mathbf{e}$ quantifies $\leq$,

$$\begin{aligned} \mathbf{d}\text{-filter} \quad &\Rightarrow \quad {}^2\text{-closed and convex.} \\ \mathbf{h}\text{-closed} \quad &\Rightarrow \quad \text{norm closed and } \leq\text{-closed.} \end{aligned}$$

Thus (1)⇒(5) now follows from (1)⇒(2).

(5)⇒(1) For any $a \in A^1_+$, $\mathbf{d}(a^{2^n}, a) \to 0$ as

$$\mathbf{d}(a^n, a) = \|a^n a^\perp\| \leq \sup_{x\in[0,1]} x^n(1-x) \leq 1/n. \tag{3.1}$$

Taking $a = \frac{1}{2}(b+c)$, (2.13) yields $\mathbf{d}_b + \mathbf{d}_c \precapprox \mathbf{d}_a$. Also $\mathbf{d}(a^{2^n}, a) \to 0$ and, by (2.8), $\mathbf{a} \precapprox \mathbf{h}$ so

$$\begin{aligned} {}^2\text{-closed and convex} \quad &\Rightarrow \quad \mathbf{d}\text{-directed.} \\ \text{norm closed and } \leq\text{-closed} \quad &\Rightarrow \quad \mathbf{h}\text{-closed.} \end{aligned}$$

As $\mathbf{h} \precapprox \mathbf{d}$, $\mathbf{h}$-closed implies $\mathbf{d}$-closed.

(1)⇒(4) As (2.13) yields $\mathbf{d}_{a\odot b} \precapprox \mathbf{d}_{\sqrt{a}} + \mathbf{d}_b \precapprox \mathbf{d}_a + \mathbf{d}_b$,

$$\mathbf{d}\text{-filter} \quad \Rightarrow \quad \odot\text{-closed.}$$

By (1)⇒(2), any $\mathbf{d}$-filter is $\mathbf{h}$-closed and hence norm closed and $\leq$-closed.

(4)⇒(1) For $c = a \odot b$, $\mathbf{d}_a + \mathbf{d}_b \precapprox \mathbf{d}_{\sqrt{a}} + \mathbf{d}_b \precapprox \mathbf{d}_c$ and $\mathbf{d}(c^{2^n}, c) \to 0$ (see (3.1)); so

$$\odot\text{-closed} \quad \Rightarrow \quad \mathbf{d}\text{-directed.}$$

By (2.8) and (2.4), $\mathbf{a} \precapprox \mathbf{h} \precapprox \mathbf{d}$; so norm closed and $\leq$-closed imply that $\mathbf{d}$-closed.

(7)⇒(1) We immediately see that

$$\text{norm filter} \quad \Rightarrow \quad \mathbf{d}\text{-closed.}$$

For any $a, b \in F$, $\|(aba)^n(aba)^\perp\| \to 0$, by (3.1); so $aba \in F$. Thus $(aba)^{3^n} \in F$, too; so, as $\mathbf{d}_a + \mathbf{d}_b \precapprox \mathbf{d}_{aba}$,

$$\text{norm filter} \quad \Rightarrow \quad \mathbf{d}\text{-directed.}$$

(1)⇒(7) Assume that $F \subseteq A^1_+$ is a $\mathbf{d}$-filter, and take $b \in A^1_+$ with

$$\inf_{\substack{k\in\mathbb{N} \\ a_1,\ldots,a_k\in F}} \|a_1 a_2 \ldots a_k b^\perp\| = 0.$$

As F is $\mathbf{d}$-directed, for any $a_1, \ldots, a_k \in F$ and $\epsilon > 0$, we can find $a \in F$ with $\mathbf{d}(a, a_j) \leq \epsilon$, for all $j \leq k$, and hence

$$\begin{aligned} \mathbf{d}(a, b) &= \|a(a_k^\perp + a_k)b^\perp\| \\ &\leq \|aa_k^\perp\| + \|aa_k b^\perp\| \\ &\leq \epsilon + \|a(a_{k-1}^\perp + a_{k-1})a_k b^\perp\| \leq \cdots \\ &\leq k\epsilon + \|a_1 a_2 \ldots a_k b^\perp\|. \end{aligned}$$

Thus $\inf_{a\in F} \mathbf{d}(a, b) = 0$. As F is $\mathbf{d}$-closed, $b \in F$; so F is a norm filter.

(6)⇒(4) Assume F is a norm closed face of A^1_+. We first claim that

$$a,b\in F\quad\Rightarrow\quad[a,b]\subseteq F,$$

where $[a,b]=\{c\in A^1_+:a\leq c\leq b\}$. For if $c\in[a,b]\subseteq A^1_+$, then $d=a+b-c\in[a,b]\subseteq A^1_+$ and $\frac{1}{2}(c+d)=\frac{1}{2}(a+b)\in F$, which implies $c,d\in F$, as F is a face of A^1_+. Next we claim that

$$a\in F\quad\Rightarrow\quad f(a)\subseteq F,$$

for any continuous f on $[0,1]$ taking 0 to 0 and 1 to 1, as in [7, Lemma 2.1]. Let $g(a)=(2a-1)_+$ and $h(a)=g(a^\perp)^\perp$. Then $\frac{1}{2}(g(a)+h(a))=a$; so $g(a),h(a)\in F$. Thus $g^{(n)}(a),h^{(n)}(a)\in F$, for all n (where $g^{(n)}(a)$ means g composed with itself n times), and hence

$$f(a)\in\overline{\bigcup[g^{(n)}(a),h^{(n)}(a)]}\subseteq F.$$

Now we claim F is $\odot$-closed, as in the proof of [7, Theorem 2.9]. For given $a,b\in F$ and $\epsilon\in(0,1)$, let $c_\epsilon=(\epsilon\sqrt{a}+(1-\epsilon)b)\in F$; so

$$\epsilon c_\epsilon\sqrt{a}c_\epsilon+(1-\epsilon)c_\epsilon bc_\epsilon=c_\epsilon^3\in F,$$

and hence $c_\epsilon bc_\epsilon\in F$. As $\epsilon\to1$, $c_\epsilon bc_\epsilon\to a\odot b\in F$, as required.

To see that F is $\leq$-closed when F is also $\mathbf{d}$-cofinal, take $a\in A^1_+$ with $a\geq b\in F$ and $(c_n)\subseteq F$ with $\mathbf{d}(a,c_n)\to0$. Then $c_n\odot b\in F$ and $c_n\odot b\leq c_n\odot a\leq c_n$; so $c_n\odot a\to a\in F$, by our first claim above.

(1)⇒(6) Take a $\mathbf{d}$-filter F. By (1)⇒(5), F is norm closed and convex. As F is $\mathbf{d}$-initial, $\mathbf{d}$-closed, and $\mathbf{d}_a+\mathbf{d}_b\precapprox\mathbf{d}_{\epsilon a+(1-\epsilon)b}$, F is also a face of A^1_+.

As $\mathbf{a}\precapprox\mathbf{h}\precapprox\mathbf{d}$ on A^1_+, for any $\epsilon>0$, we have $\delta>0$ such that $\mathbf{d}(b,c)<\delta$ implies $\mathbf{a}(b,c)<\epsilon$; that is, there exists $c'\geq b$ with $\|c-c'\|<\epsilon$. As A has an approximate unit, for any $a\in A^1_+$ and $b\in F$, we have $c\in A^1_+$ with $\mathbf{d}(a,c),\mathbf{d}(b,c)<\min(\delta,\epsilon)$. Taking c' as above, we have $c'\geq b$, and hence $c'\in F$, as F is $\leq$-closed, as well as $\mathbf{d}(a,c')\leq\mathbf{d}(a,c)+\|c-c'\|<2\epsilon$, by (2.1). As $\epsilon>0$ was arbitrary, this shows that F is $\mathbf{d}$-cofinal.

(3)⇒(2) As $\mathbf{d}$ and $\mathbf{h}$ are $\mathbf{e}$-invariant and $\mathbf{a}\precapprox\mathbf{h}$, for any $F\subseteq A^1_+$,

$$\begin{aligned}F\text{ is }\mathbf{d}\text{-initial}\quad&\Rightarrow\quad\overline{F}\text{ is }\mathbf{d}\text{-initial.}\\F\text{ is }\leq\text{-directed}\quad&\Rightarrow\quad\overline{F}\text{ is }\mathbf{h}\text{-directed.}\\F\text{ is }\leq\text{-closed}\quad&\Rightarrow\quad\overline{F}\text{ is }\mathbf{h}\text{-closed.}\end{aligned}$$

(1)⇒(3) Take a $\mathbf{d}$-filter F, and assume first that A is unital. Consider the invertible elements G of F. For every $a\in F$ and $\epsilon>0$, $(1-\epsilon)a+\epsilon\in G$ so $\overline{G}=F$. As F is $\leq$-closed and $\mathbf{d}$-initial, so is G. It only remains to show that G is $\leq$-directed. So take $a,b\in G$. For some $\epsilon>0$ and $a',b'\in A^1_+$, we have $a=\epsilon+(1-\epsilon)a'$ and $b=\epsilon+(1-\epsilon)b'$. As F is a face of A^1_+ containing 1, $a',b'\in F$. As F is $\mathbf{h}$-directed, we have $c'\in F$ with $\mathbf{h}(c',a'),\mathbf{h}(c',b')<\epsilon/2$. Letting $c=\epsilon+(1-\epsilon)c'\in F$, we thus have $\mathbf{h}(c,a),\mathbf{h}(c,b)<\epsilon/2$. Thus $d=c-(c-a)_+-(c-b)_+$ is an invertible element of A^1_+ (as $c\geq\epsilon$ and $\|c-d\|<\epsilon$), and we further claim that $d\in F$, and hence $d\in G$. Indeed, for any $\delta>0$, we have $e\in F$ with $\mathbf{d}(e,a),\mathbf{d}(e,b),\mathbf{d}(e,c)<\delta$. Thus

$\|e(c-a)_+\| \leq \|e(c-a)\| < 2\delta$ and $\|e(c-b)_+\| < 2\delta$; so $\mathbf{d}(e,d) < 5\delta$. As $\delta > 0$ was arbitrary and F is $\mathbf{d}$-closed, we have $d \in F$, as required.

If A is not unital, then first extend F to a $\mathbf{d}$-filter F' in $\widetilde{A}^1_+$ by taking the (upwards) $\leq$-closure. By what we just proved, the invertible elements G' of F' are a $\leq$-filter with $\overline{G'} = F'$. In particular, G' is $\mathbf{d}$-coinitial in F'. Thus, for any $a \in F$, $aG'a$ is $\mathbf{d}$-coinitial in F. Hence the $\leq$-closure G of $aG'a$ in A^1_+ is a $\mathbf{d}$-initial $\leq$-filter in A^1_+ with $\overline{G} = F$, as $\mathbf{a} \precapprox \mathbf{h} \precapprox \mathbf{d}$, by (2.4) and Theorem 2.6.

(1)$\Rightarrow$(8) If A is separable, then we can take dense $(a_n) \subseteq F$ and let $a = \sum 2^{-n} a_n \in F$. As noted above, $f(a) \in F$, for any continuous f on $[0,1]$ taking 0 to 0 and 1 to 1. Thus by choosing such (f_n) converging pointwise to 0 everywhere except at 1 and satisfying $f_1 \gg f_2 \gg \ldots$, the (upwards) $\ll$-closure G of $(f_n(a))$ is a $\ll$-filter with $F = \overline{G}$.

Now take

$$G = \{a \in F : a \gg b \in F\}.$$

Again, if $a \in F$, then $f(a) \in F$, for any continuous f on $[0,1]$ taking 0 to 0 and 1 to 1. In particular, for any $\epsilon > 0$ we can take $f(x) = (1+\epsilon)x \wedge 1$ and $g(x) = (\epsilon^{-1}(x-1)+1)_+$; so $f(a) \gg g(a) \in F$, and hence $f(a) \in G$. As $\epsilon \to 0$, $f(a) \to a$; so $F = \overline{G}$. Likewise, if $a \gg b \in F$, then $a \gg f(b) \gg g(b) \in F$; so

$$G = \{a \in F : a \gg b \gg c \in F\}.$$

If A is commutative, $a \gg b$ and $a' \gg b'$ imply $aa' \gg bb'$. For $a, a' \in G$, we have $b, b', c, c' \in F$ with $a \gg b \gg c$ and $a' \gg b' \gg c'$; so $bb' \gg cc' \in F$, and hence $a, a' \gg bb' \in G$; that is, G is $\ll$-directed, and hence a $\ll$-filter.

(8)$\Rightarrow$(3) As $\mathbf{d}$ quantifies $\ll$ and $\ll \circ \leq \ \subseteq \ \ll \ \subseteq \ \leq$,

$$\begin{aligned} \ll\text{-initial} \quad &\Rightarrow \quad \mathbf{d}\text{-initial.} \\ \ll\text{-closed and } \ll\text{-initial} \quad &\Rightarrow \quad \leq\text{-closed.} \\ \ll\text{-directed} \quad &\Rightarrow \quad \leq\text{-directed.} \end{aligned}$$

□

In (4) and (5), we could not replace '$\leq$-closed' with '$\ll$-closed'. For example, the norm closure C of the convex combinations of the functions x^n in $C([0,1])$, for $n \in \mathbb{N}$, satisfies these conditions – as every $f \in C$ is positive on $(0,1]$, C is vacuously $\ll$-closed – however C is not $\leq$-closed, being bounded above by x. Although we could replace 'norm closed and $\leq$-closed' with '$\mathbf{h}$-closed' or '$\mathbf{d}$-closed'.

Furthermore, not every $\mathbf{d}$-filter is the norm closure of a $\ll$-filter. Indeed, if this was the case, then, for any nonunital A, the $\mathbf{d}$-filter $\{1 - a : a \in A^1_+\}$ in $\widetilde{A}^1_+$ would be the norm closure of a $\ll$-filter F. Then $1 - F$ would be a $\ll$-increasing approximate unit of A. However, a C^*-algebra was recently constructed in [12, Theorem 1.4] that does not possess such an approximate unit. But all ω_1-unital C^*-algebras have $\ll$-increasing approximate units, by [12, Corollary 4.3]; so (8) could be extended to any A with a dense subset of size $\leq \omega_1$.

We can at least say a bit more in the commutative case.

Proposition 3.3. *If A is commutative and $F \subseteq A^1_+$ is a $\mathbf{d}$-filter, then*
$$G = \{a \in F : a \gg b \in F\}$$
is the unique $\ll$-filter with $F = \overline{G}$.

Proof. The only thing left to show is uniqueness. By the Gelfand represtentation, we may assume that $A = C_0(X)$ for some locally compact Hausdorff X; so
$$f \ll g \qquad \Leftrightarrow \qquad X \setminus f^{-1}\{0\} \subseteq g^{-1}\{1\}.$$
Take a $\mathbf{d}$-filter $F \subseteq A^1_+$, and let
$$C = \bigcap_{f \in F} f^{-1}\{1\}.$$
For any $\ll$-filter G with $\overline{G} = F$, we must also have $C = \bigcap_{g \in G} g^{-1}\{1\}$. Otherwise, we could pick some $x \in \bigcap_{g \in G} g^{-1}\{1\} \setminus C$ and $f \in F$ with $f(x) \neq 1$, and then $\|f - g\| \geq g(x) - f(x) = 1 - f(x) > 0$, for all $g \in G$, contradicting $\overline{G} = F$.

Take $f \in A^1_+$ with $C \subseteq f^{-1}\{1\}^\circ$. For every $x \in X \setminus f^{-1}\{1\}^\circ$, we have $g_x \in G$ with $g_x(x) \neq 1$. Thus we can pick arbitrary $g \in G$ and cover the compact set $g^{-1}[\frac{1}{2}, 1] \setminus f^{-1}\{1\}^\circ$ with finitely many open sets $X \setminus g^{-1}_{x_1}\{1\}, \ldots, X \setminus g^{-1}_{x_k}\{1\}$. As G is $\ll$-directed, we have some $h \in G$ with $h \ll g, g_{x_1}, \ldots, g_{x_k}$, and hence
$$X \setminus h^{-1}\{0\} \subseteq g^{-1}\{1\} \cap g^{-1}_{x_1}\{1\} \cap \ldots \cap g^{-1}_{x_k}\{1\} \subseteq f^{-1}\{1\}^\circ \subseteq f^{-1}\{1\};$$
that is, $h \ll f$. Thus $f \in G$, as G is $\ll$-closed, so
$$\{f \in F : f \gg g \in F\} \subseteq \{f \in A^1_+ : C \subseteq f^{-1}\{1\}^\circ\} \subseteq G.$$
Conversely, $G \subseteq \{f \in F : f \gg g \in F\}$, as G is a $\ll$-filter contained in F. $\square$

This does not extend to noncommutative A; that is,
$$G = \{a \in F : a \gg b \in F\}$$
may fail to be a $\ll$-filter and F may contain various dense $\ll$-filters. For example, consider $A = C([0,1], M_2)$, and take everywhere rank 1 projections $p, q \in A$ with $p(0) = P = q(0)$ but $p(x) \neq q(x)$, for all $x > 0$. Also take continuous f_n on $[0,1]$ with $f_1 \gg f_2 \gg \ldots$ and $\bigcap_n f^{-1}\{1\} = \{0\}$. Then the $\ll$-closures $F, G \subseteq A^1_+$ of $(f_n p)$ and $(f_n q)$ are distinct $\ll$-filters with $\overline{F} = \overline{G} = \{a \in A^1_+ : a(0) \geq P\}$.

Definition 3.4. We say Y *generates* a $\mathbf{D}$-filter $F \subseteq X$ if F is the smallest $\mathbf{D}$-filter containing Y. We call X a $\mathbf{D}$-*semilattice* if every $Y \subseteq X$ generates a $\mathbf{D}$-filter.

For posets, $\leq$-semilattices are precisely the meet semilattices in the usual sense.

Proposition 3.5. *If $\leq$ is a partial order on X, then*
$$X \text{ is a } \leq\text{-semilattice} \qquad \Leftrightarrow \qquad \text{every } x, y \in X \text{ has an infimum } x \wedge y \in X.$$

Proof. If every $x, y \in X$ has an infimum $x \wedge y \in X$, then the $\leq$-closure of the $\wedge$-closure of any $Y \subseteq X$ is the $\leq$-filter generated by Y; so X is a $\leq$-semilattice. If some $x, y \in X$ have no infimum, then
$$\bigcap_{z \leq x, y} \{w \in X : z \leq w\}$$

is an intersection of $\leq$-filters containing x and y but no lower bound of x and y. Thus $\{x, y\}$ does not generate a $\leq$-filter, and hence X is not a $\leq$-semilattice. $\square$

Proposition 3.6. *A^1_+ is a $\mathbf{d}$-semilattice.*

Proof. For any $B \subseteq A^1_+$, let D be the $\odot$-closure of B; so D is $\mathbf{d}$-directed and every $\mathbf{d}$-filter containing B must contain D. Let F be the $\mathbf{d}$-closure of D; so F is a $\mathbf{d}$-filter and every $\mathbf{d}$-filter containing B, and hence D contains F; that is, F is the $\mathbf{d}$-filter generated by B. $\square$

Definition 3.7. We call $Y \subseteq X$ $\mathbf{D}$-*centered* if, for all $y_1, \ldots, y_k \subseteq Y$,

$$\inf_{x \in X} \mathbf{D}(x, y_1) + \ldots + \mathbf{D}(x, y_k) = 0.$$

If $\leq$ is a partial order on X, then $Y \subseteq X$ is $\leq$-centered if and only if every finite subset of Y has a lower bound in X; that is, if and only if Y is centered in the usual order theoretic sense.

As with filters, we see that the centered subset analogs for C^*-algebras considered in [8] are precisely the $\mathbf{d}$-centered subsets, this time in the positive unit sphere $A^{=1}_+$ rather than the positive unit ball A^1_+. Specifically, recall that $C \subseteq A^{=1}_+$ is *norm centered*, according to [8, Definition 2.1], if the multiplicative closure of C is contained in the unit sphere.

Proposition 3.8. *For $C \subseteq A^{=1}_+$, the following are equivalent.*

(1) *C is $\mathbf{d}$-centered in $A^{=1}_+$.*
(2) *C is $\mathbf{h}$-centered in $A^{=1}_+$.*
(3) *C is norm centered.*
(4) *C generates a proper $\mathbf{d}$-filter in A^1_+.*

Proof.

(1)⇒(2) Immediate from (2.4).

(2)⇒(1) If $\|a\| = 1$, then $\|a^n\| = 1$ and, for any $b \in A^1_+$, (2.6) yields

$$\mathbf{d}(a^n, b)^2 \leq \mathbf{d}(a^n, a) + \mathbf{h}(a, b) \to \mathbf{h}(a, b).$$

(1)⇒(3) If C is $\mathbf{d}$-centered, then, for any $a_1, \ldots, a_k \in C$ and $\epsilon > 0$, we have $b \in A^{=1}_+$ with $\mathbf{d}(b, a_1) + \ldots + \mathbf{d}(b, a_k) < \epsilon$ so

$$\begin{aligned} 1 &= \left\| b(a_1 + a_1^\perp) \right\| \\ &\leq \left\| b(a_2 + a_2^\perp)a_1 \right\| + \mathbf{d}(b, a_1) \\ &\leq \left\| b(a_3 + a_3^\perp)a_2 a_1 \right\| + \mathbf{d}(b, a_2) + \mathbf{d}(b, a_1) \leq \ldots \\ &\leq \|b a_k \ldots a_1\| + \epsilon \\ &\leq \|a_k \ldots a_1\| + \epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, C is norm centered.

(3)⇒(4) If the multiplicative closure of C is contained in the unit sphere, then the same goes for the closure D of C under the operation $(a, b) \mapsto aba$. The same then applies to the $\mathbf{d}$-closure F of D; so, in particular, F is proper. As in the proof of Proposition 3.6, F is the $\mathbf{d}$-filter generated by C.

(4)⇒(1) First note that a $\mathbf{d}$-filter F in A^1_+ is a proper subset of A^1_+ if and only if it is contained in the positive unit sphere $A^{=1}_+$. If $a \in F$ and $\|a\| < 1$, then $a^n \in F$ for all n. Thus, for any $b \in A^1_+$,

$$\mathbf{d}(a^n, b) \leq \|a^n\| = \|a\|^n \to 0,$$

and hence $b \in F$, as F is $\mathbf{d}$-closed; that is, $F = A^1_+$, contradicting properness.

If C is contained in such a $\mathbf{d}$-filter F, then, for any $c_1, \ldots, c_n \in C$,

$$\inf_{a \in A^{=1}_+} \mathbf{d}(a, c_1) + \ldots + \mathbf{d}(a, c_k) \leq \inf_{a \in F} \mathbf{d}(a, c_1) + \ldots + \mathbf{d}(a, c_k) = 0,$$

as F is $\mathbf{d}$-directed; that is, C is $\mathbf{d}$-centered in $A^{=1}_+$. □

In particular, the maximal $\mathbf{d}$-centered subsets of $A^{=1}_+$ are precisely the maximal proper $\mathbf{d}$-filters in A^1_+. These were the original quantum filters defined by Farah and Weaver to study pure states. Pure states correspond to minimal projections in A^{**}, and, more generally, $\mathbf{d}$-filters correspond to the compact projections in A^{**} introduced by Akemann (which was touched on briefly in [8, Corollary 3.4]). This is the connection we explore next.

4. Compact Projections

Let $\uparrow p$ denote the upper set in A^1_+ defined by any projection $p \in A^{**}$; that is,

$$\uparrow p = \{a \in A^1_+ : p \leq a\}.$$

Also let $\bigwedge$ below denote the infimum with respect to $\leq$ on A^{**}_{sa}.

Definition 4.1. A projection $p \in A^{**}$ is *compact* if $p = \bigwedge \uparrow p$.

Note for p to be compact it is implicit that $\uparrow p$ is nonempty (and actually has an infimum in A^{**}_{sa} – as A^{**}_{sa} is not a complete lattice, not all subsets have infima).

Theorem 4.2. *We have mutually inverse bijections*

$$p \mapsto \uparrow p \qquad \text{and} \qquad F \mapsto \bigwedge F$$

between compact projections $p \in A^{**}$ and $\mathbf{d}$-filters $F \subseteq A^1_+$. Moreover, for compact projections $p, q \in A^{**}$ and corresponding $\mathbf{d}$-filters $F, G \subseteq A^1_+$,

$$\mathbf{d}(p, q) = \sup_{b \in G} \inf_{a \in F} \mathbf{d}(a, b). \tag{4.1}$$

Proof. Take a projection $p \in A^{**}$, and consider $\uparrow p$. If $p = pa$, then $pa^2 = pa = p$; that is, $p \ll a$ implies $p \ll a^2$; so $\uparrow p$ is ^{2}-closed. Likewise, if $p = pa$, $p = pb$, and $\epsilon \in (0, 1)$, then $p(\epsilon a + (1 - \epsilon)b) = \epsilon p + (1 - \epsilon)p = p$; that is, $p \ll (\epsilon a + (1 - \epsilon)b)$; so $\uparrow p$ is convex. Also $p \ll a \leq b \in A^1_+$ implies $p \ll b$, as $\mathbf{d}^2 \leq \mathbf{d} \circ \mathbf{h}$ on A^1_+, so $\uparrow p$ is $\leq$-closed. Finally, if $a_n \to a$ and $p \ll a_n$, for all n, then $\mathbf{d}(p, a) \leq \lim_n(\mathbf{d}(p, a_n) + \mathbf{e}(a_n, a)) = 0$, as $\mathbf{d}$ is $\mathbf{e}$-invariant; that is, $p \ll a$; so $\uparrow p$ is norm closed, and thus a $\mathbf{d}$-filter, by Theorem 3.2.

Conversely, take a $\mathbf{d}$-filter $F \subseteq A^1_+$ which, by (3.2), contains a dense $\leq$-filter F'. The pointwise infimum of F' on A^{*1}_+ is an affine function and thus defines an element $p \in A^{**}$. As $\leq$ on A^{**}_{sa} is determined by A^{*1}_{+}, $p = \bigwedge F' = \bigwedge F = \bigwedge \uparrow p$. As p takes A^{*1}_{+} to $[0,1]$, p is positive and has norm at most 1; that is, $p \in A^{**1}_{+}$. As F is $\mathbf{d}$-initial,

$$\sup_{a\in F} \mathbf{d}(p,a)^2 \leq \sup_{a\in F} \inf_{b\in F} (\mathbf{h}(p,b) + \mathbf{d}(b,a)) = 0;$$

that is, for all $a \in F$, $p \ll a$; so $\sqrt{p} \ll a$. Thus $\sqrt{p} \leq \bigwedge F = p$; so p is a projection.

Now take another $\mathbf{d}$-filter G containing a dense $\leq$-filter G' and defining a compact projection $q = \bigwedge G$ which is a pointwise infimum of G on A^{*1}_{+}. Then $pG'p$ is also $\leq$-directed; so $pqp = \bigwedge pG'p = \bigwedge pGp$ is also a pointwise infimum on A^{*1}_{+}, and hence

$$\begin{aligned}
\mathbf{d}(p,q)^2 = \left\|pq^{\perp}p\right\| &= \sup_{\phi\in A^{*1}_{+}} \phi(pq^{\perp}p) \\
&= \phi(p) - \inf_{\phi\in A^{*1}_{+}} \phi(pqp) \\
&= \phi(p) - \inf_{\phi\in A^{*1}_{+}, b\in G} \phi(pbp) \\
&= \sup_{\phi\in A^{*1}_{+}, b\in G} \phi(pb^{\perp}p) \\
&= \sup_{\phi\in A^{*1}_{+}, b\in G} \phi(pb^{\perp 2}p) \qquad (4.2)\\
&= \sup_{b\in G} \left\|pb^{\perp 2}p\right\| \\
&= \sup_{b\in G} \mathbf{d}(p,b)^2.
\end{aligned}$$

For (4.2), note that $\left\|ab^{\perp}\right\|^2 = \left\|ab^{\perp 2}a\right\| \leq \left\|ab^{\perp 2}\right\| \leq \left\|ab^{\perp}\right\|$; so $\mathbf{d}_b \approx \mathbf{d}_{b^{\perp 2\perp}}$. Thus, as G is $\mathbf{d}$-initial and $\mathbf{d}$-closed, $b \in G$ if and only if $b^{\perp 2\perp} \in G$; that is,

$$\{b^{\perp} : b \in G\} = \{b^{\perp 2} : b \in G\}.$$

Fix $b \in G$, and define weak* continuous $f_a : A^{*1}_{+} \to [0,1]$ by

$$f_a(\phi) = (\phi(b^{\perp}ab^{\perp}) - \left\|b^{\perp}pb^{\perp}\right\|)_{+}.$$

Then $(f_a)_{a\in F'}$ is a decreasing net in $[0,1]^{A^{*1}_{+}}$ (with the product ordering) and converges to 0 pointwise. As A^{*1}_{+} is weak* compact, Dini's theorem says $(f_a)_{a\in F'}$ must actually converge uniformly to 0 on A^{*1}_{+}, and hence

$$\inf_{a\in F'} \left\|b^{\perp}ab^{\perp}\right\| \leq \left\|b^{\perp}pb^{\perp}\right\| \leq \inf_{a\in F} \left\|b^{\perp}ab^{\perp}\right\|.$$

As F is ^{2}-closed and $\sqrt{\ }$-closed (as F is $\leq$-closed),

$$\inf_{a\in F} \mathbf{d}(a,b)^2 = \inf_{a\in F} \left\|b^{\perp}a^2b^{\perp}\right\| = \inf_{a\in F} \left\|b^{\perp}ab^{\perp}\right\| = \inf_{a\in F'} \left\|b^{\perp}ab^{\perp}\right\| = \left\|b^{\perp}pb^{\perp}\right\| = \mathbf{d}(p,b)^2.$$

Thus, together with the above we have

$$\mathbf{d}(p,q) = \sup_{b\in G} \mathbf{d}(p,b) = \sup_{b\in G} \inf_{a\in F} \mathbf{d}(a,b). \qquad (4.3)$$

Now note that

$$\sup_{b\in G}\inf_{a\in F}\mathbf{d}(a,b)=0 \qquad \Leftrightarrow \qquad G\subseteq F. \tag{4.4}$$

Indeed, the $\mathbf{d}$-initiality of F yields $\Leftarrow$, while the fact F is $\mathbf{d}$-closed yields $\Rightarrow$. Combined with (4.3), this shows that $p=q$ implies $F=G$; that is, the map $F\mapsto\bigwedge F$ is injective on $\mathbf{d}$-filters. Thus the given maps are bijections, as required. □

In the above proof, we used dense $\leq$-filter subsets of $\mathbf{d}$-filters in a couple of places, but this was not absolutely necessary. Indeed, one could verify directly that pointwise infimums on A^{*1}_{+} of $\mathbf{h}$-directed subsets are affine and hence define elements of A^{**}. Likewise, Dini's theorem can be generalized to $\mathbf{h}$-directed subsets and even $\mathbf{h}$-Cauchy nets – see [10, Theorem 1].

The gist of Theorem 4.2 is that compact projections in A^{**} can be more concretely represented by $\mathbf{d}$-filters in A, and this extends to various relations or functions one might consider. For example, from (4.1) and (4.4) we immediately see that, for compact projections $p,q\in A^{**}$ and corresponding $\mathbf{d}$-filters $F,G\subseteq A^1_+$,

$$p\leq q \qquad \Leftrightarrow \qquad F\supseteq G.$$

Likewise, as $\|p-q\|=\max\{\|pq^\perp\|,\|qp^\perp\|\}$, (4.1) yields

$$\|p-q\|=\max(\sup_{b\in G}\inf_{a\in F}\mathbf{d}(a,b),\sup_{a\in F}\inf_{b\in G}\mathbf{d}(b,a));$$

that is, the metric on compact projections corresponds to the Hausdorff metric on $\mathbf{d}$-filters. We can also show that the natural quantification of orthogonality on compact projections is determined by the corresponding $\mathbf{d}$-filters.

Theorem 4.3. *For compact $p,q\in A^{**}$ and $\mathbf{d}$-filters $F=\uparrow p$ and $G=\uparrow q$,*

$$\|pq\|=\inf_{a\in F,b\in G}\|ab\|.$$

Proof. Let $r=\inf_{a\in F,b\in G}\|ab\|$. As $F\subseteq\uparrow p$ and $G\subseteq\uparrow q$, we immediately have

$$\|pq\|^2=\|qpq\|\leq\inf_{a\in F}\|qa^2q\|=\inf_{a\in F}\|aqa\|\leq\inf_{a\in F,b\in G}\|ab^2a\|=r^2.$$

Conversely, take a dense $\leq$-filter $F'\subseteq F$, and, for any $a\in F'$, consider

$$A^{*1}_{+a}=\{\phi\in A^{*1}_{+}:\phi[G]=\{1\}\text{ and }\phi(a)\geq r^2\}.$$

By [8, Theorem 2.2], each A^{*1}_{+a} is nonempty. So $\bigcap_{a\in F'}A^{*1}_{+a}$ is a directed intersection of nonempty weak* compact subsets and we thus have some $\phi\in\bigcap_{a\in F}A^{*1}_{+a}$. As $\phi[G]=\{1\}$, $\phi(q)=1$ and hence $\|pq\|^2=\|qpq\|\geq\phi(qpq)=\phi(p)\geq r^2$, as p is the pointwise infimum of F' on A^{*1}_{+} (see the proof of Theorem 4.2). □

A natural question to ask is if the infimum above is actually a minimum.

Question 2 ([2])**.** Do we always have $a\in F$ and $b\in G$ with $\|pq\|=\|ab\|$?

When $pq=0$ the answer is yes, by Akemann's noncommutative Urysohn lemma – see [3, Lemma III.1]. However, we feel that a truly noncommutative Urysohn lemma should apply to compact projections that do not commute.

Speaking of commutativity, note that commuting $a \in A^1_+$ and $p \in \mathcal{P}(A^{**})$ always satisfy the following (taking composition in A^1_+)

$$(\ll \circ\, \mathbf{e})(p,a) = \mathbf{d}(p,a). \tag{4.5}$$

Indeed, if $\mathbf{d}(p,a) = r < 1$, then $p \ll f(a) \in A^1_+$, for any continuous function f on $[0,1]$ taking 0 to 0 and $[1-r,1]$ to 1. Thus $\inf_{p \ll b \in A^1_+} \|a-b\| = r$ as required. In particular, for any projection $p \in A^{**}$, $\inf_{a \in A^1_+, ap=pa} \mathbf{d}(p,a)$ must either be 0 or 1. However, the commutativity of p and a here is crucial.

Example 4.4. Take a rank one projection $Q \in M_2$, and consider the C^*-algebra

$$A = \{a \in C([0,1], M_2) : a(0) \in \mathbb{C}Q\}.$$

Take any other rank one projection $P \in M_2$, and define p on $[0,1]$ by $p(0) = Q$ and $p(x) = P$ otherwise. This represents a closed projection in A^{**} (as the atomic representation is faithful on closed projections – see [20, Theorem 4.3.15]) with $\inf_{a \in A^1_+} \mathbf{d}(p,a) = \|P - Q\|$, which can be anywhere between 0 and 1.

However, even when a and p do not commute, we can still prove a weaker uniform substitute for (4.5) for certain 'closed' projections.

First, dual to compact projections, we define open projections. Specifically, let $\downarrow p$ denote the lower set in A^1_+ defined by any projection $p \in A^{**}$; that is,

$$\downarrow p = \{a \in A^1_+ : a \le p\}.$$

Also let $\bigvee$ below denote the supremum with respect to $\le$ on A^{**}_{sa}.

Definition 4.5. A projection $p \in A^{**}$ is *open* if $p = \bigvee \downarrow p$.

Definition 4.6. A projection $p \in A^{**}$ is *closed* if $p^\perp$ is open.

Note the following proof is inspired by interpolation arguments by Brown – see [15] – and adapted by Akemann and Pedersen – see [7] (although the distance-like functions they used were never formalized as such).

Theorem 4.7. *Assume that $p \in A^{**}$ is a closed projection. On A^1_+, we have*

$$_p(\ll \circ\, \mathbf{e}) \precapprox {}_p\mathbf{d} \tag{4.6}$$

Proof. If $\inf_{a \in A^1_+} \mathbf{d}(p,a) > 0$, then (4.6) holds vacuously on A^1_+. So assume that

$$\inf_{a \in A^1_+} \mathbf{d}(p,a) = 0. \tag{4.7}$$

We first claim a weakened form of (4.6) on A^1_+, namely,

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall a \in A^1_+ \tag{4.8}$$
$$\mathbf{d}(p,a) < \delta \quad \Rightarrow \quad \forall \gamma > 0\ \exists b \in A^1_+\ (\mathbf{d}(p,b) < \gamma \text{ and } \|a-b\| < \epsilon).$$

Indeed, we claim that we can take $\delta = \frac{1}{32}\epsilon^2$. To see this, take $\epsilon > 0$ and $a \in A^1_+$ with $\mathbf{d}(p,a) < \frac{1}{32}\epsilon^2$. For any $\gamma > 0$, we have $u \in A^1_+$ with $\mathbf{d}(p,u) < \frac{1}{2}\gamma^2$, by

(4.7). As A has an approximate unit, we have $v \in A^1_+$ with $\mathbf{d}(a,v) < \frac{1}{32}\gamma^4$ and $\mathbf{d}(u,v) < \frac{1}{2}\gamma^2 - \mathbf{d}(p,u)$. Thus $\mathbf{d}(p,v) \le \mathbf{d}(p,u) + \mathbf{d}(u,v) < \frac{1}{2}\gamma^2$ and, by (2.8),

$$\mathbf{a}(a,v) \le 2\sqrt{\mathbf{h}(a,v)} \le 2\sqrt{2\mathbf{d}(a,v)} < \tfrac{1}{2}\gamma^2.$$

Hence we have $w \in A^1_+$ with $a \le w$ and $\|w - v\| < \frac{1}{2}\gamma^2$. Thus

$$\mathbf{d}(p,w) \le \mathbf{d}(p,v) + \mathbf{e}(v,w) < \gamma^2 \tag{4.9}$$

and $\mathbf{h}(w - a, p^\perp) \le \mathbf{h}(a^\perp, p^\perp) = \mathbf{h}(p,a) \le 2\mathbf{d}(p,a) < \frac{1}{16}\epsilon^2$. As $w - a - p^\perp$ is the pointwise infimum on A^{*1}_+ of $(w-a-c)_{p^\perp \ge c \in A^1_+}$, Dini's theorem again yields $c \in A^1_+$ with $c \le p^\perp$ and $\mathbf{h}(w-a,c) < \frac{1}{16}\epsilon^2$. By (2.9), $\mathbf{b}(w-a,c) < \frac{1}{2}\epsilon$; that is, we have $d \in A^1_+$ with $d \le c$ and $\|w - a - d\| < \frac{1}{2}\epsilon$. Setting $b = (w-d)_+$,

$$\|w - d - b\| = \|(d-w)_+\| \le \|(d + a - w)_+\| \le \|d + a - w\| < \tfrac{1}{2}\epsilon;$$

so $\|a - b\| \le \|a + d - w\| + \|w - d - b\| < \epsilon$. As $pd = 0$ and $w - d \le b$,

$$\mathbf{d}(p,b)^2 \le \mathbf{d}(p, w-d) + \mathbf{h}(w-d,b) = \mathbf{d}(p,w) < \gamma^2,$$

by (4.9); thus proving (4.8).

Now (4.6) is saying the same thing as (4.8), just with $\mathbf{d}(p,b) < \gamma$ strengthened to $p \ll b$. To prove this, we iterate (4.8). First take positive (δ_n) satisfying (4.8) with ϵ replaced by $\epsilon/2^n$, for any fixed $\epsilon > 0$. For any $a_1 \in A^1_+$ with $\mathbf{d}(p,a_1) < \delta_1$, we can then recursively take $a_{n+1} \in A^1_+$ with $\mathbf{d}(p,a_{n+1}) < \delta_{n+1}$ and $\|a_n - a_{n+1}\| < \epsilon/2^n$. Thus (a_n) has a limit $b \in A^1_+$ with $\mathbf{d}(p,b) \le \mathbf{d}(p,a_n) + \mathbf{e}(a_n,b) \to 0$; that is, $p \ll b$. Also $\|a_1 - b\| < \sum \epsilon/2^n = \epsilon$, thus proving (4.6). □

Now we can show that 'compact' is the same as 'closed and bounded'. Indeed, this is usually taken as the definition; that is, compact projections are usually defined as closed projections satisfying some notion of boundedness, like $p \le a \in A_+$ (see [3, Definition II.1]) or $p \ll a \in A^1_+$ (see [19, §3.5]). However, these definitions make it difficult to appreciate that 'compact' and 'open' are dual to each other in a natural order theoretic way. To make this duality clear, and to avoid any debate about the most appropriate notion of boundedness, we opted to define compact and open projections independently via $\uparrow p$ and $\downarrow p$, respectively and save the discussion of boundedness until now.

Note that when A is commutative, (4.10) below actually holds for an arbitrary $p \in \mathcal{P}(A^{**})$ (see (4.5)). But in the noncommutative case, it is crucial for p to be closed, as we show in Example 4.9 below.

Corollary 4.8. *A projection $p \in A^{**}$ is compact if and only if p is closed and*

$$\inf_{a \in A^1_+} \mathbf{d}(p,a) = 0.$$

Proof. If p is compact, then $\uparrow p$ is nonempty; so certainly $\inf_{a \in A^1_+} \mathbf{d}(p,a) = 0$. To see that p is closed, consider

$$B = \{a \in A : ap = 0 = pa\},$$

which is immediately seen to be a hereditary C^*-subalgebra of A. In particular, B has an approximate unit; so B^1_+ is $\mathbf{d}$-directed, and hence has a supremum $p' = \bigvee B^1_+$ in A^{**}, which is a projection and also a pointwise supremum on A^{*1}_+.

As $bp=0$, for all $b\in B$, we also have $p'p=0$ and we claim that in fact $p'=p^\perp$. If not, then, identifying A^{**} with A'' in the universal representation of A, we would have a unit vector v with $pv+p'v=0$. Then v defines a state $\phi(a)=\langle av,v\rangle$ in A^{*1}_+. As $\phi(p)=0$ and p is compact, we have $a_n\in{\uparrow}p$ with $\phi(a_n)\to 0$, and hence $a_nv\to 0$. As $\phi(1)=1$, we have $(u_n)\subseteq A^1_+$ with $\phi(u_n)\to 1$, and hence $u_nv\to v$. Thus $b_n=a_n^\perp u_n a_n^\perp\in B^1_+$ satisfies $b_nv\to v$; so $\phi(b_n)\to 1$. But $\phi(b_n)\le\phi(p')=0$, a contradiction. Thus $p'=p^\perp$ is open; so p is closed.

If p is closed, then, by (4.6),

$$\inf_{a\in A^1_+}\mathbf{d}(p,a)=0\qquad\Rightarrow\qquad \exists a\in A^1_+\ (p\ll a). \tag{4.10}$$

So we can take $a\in{\uparrow}p$. For all $b\in{\downarrow}\,p^\perp$, we then have $ab^\perp a\in{\uparrow}p$. Also, as p is closed; that is, $p^\perp$ is open, we have $p=p^{\perp\perp}=(\bigvee{\downarrow}p^\perp)^\perp=\bigwedge({\downarrow}p^\perp)^\perp$; so

$$p=apa=\bigwedge a({\downarrow}p^\perp)^\perp a=\bigwedge\{ab^\perp a: b\in{\downarrow}p^\perp\}\ge\bigwedge{\uparrow}p\ge p.$$

(For the second equality note that, as $\phi(a\cdot a)\in A^{*1}_+$ whenever $a\in A^1_+$ and $\phi\in A^{*1}_+$, $\inf_{c\in C}\phi(c)=\phi(d)$, for all $\phi\in A^{*1}_+$, implies $\inf_{c\in C}\phi(aca)=\phi(ada)$, for all $\phi\in A^{*1}_+$). Thus p is compact. $\square$

Example 4.9. It is possible to have open $p\in A^{**}$ with

$$\inf_{a\in A^1_+}\mathbf{d}(p,a)=0\qquad\text{but}\qquad\nexists a\in A^1_+\ (p\ll a).$$

To see this, we consider a C^*-subalgebra of $C([0,1],\mathcal{B}(H))$; that is, the continuous functions from $[0,1]$ to $\mathcal{B}(H)$ for a separable infinite dimensional Hilbert space H. First let (e_n) be an orthonormal basis for H, and let (P_n) be the rank 1 projections onto $(\mathbb{C}e_n)$. Take $(r_n)\subseteq(0,1)$ with $\inf_n r_n=0$, and define $p_n:[0,1]\to\mathcal{B}(H)$ by

$$p_n(x)=\begin{cases}P_n & \text{if } x>r_n\\ 0 & \text{if } x\le r_n.\end{cases}$$

Define p on $[0,1]$ by letting $p(x)=\bigvee_n p_n(x)$ (taking the supremum in the projection lattice of $\mathcal{B}(H)$). Also let $K=C([0,1],\mathcal{K}(H))$ and $B=pKp\cap K$. Let Q be the projection onto $\mathbb{C}v$, for $v=\sum 2^{-n}e_n$, and let A be the C^*-subalgebra of $C([0,1],\mathcal{B}(H))$ generated by B and the constant projection $Q^\perp$. As $p=\bigvee B^1_+$ pointwise on $[0,1]$, we may identify p with the open projection in A^{**} defined by B. For any $n\in\mathbb{N}$ and continuous function f_n on $[0,1]$ with $f_n(0)=0$ and $f_n(x)=1$, for all $x\in[\frac{1}{n},1]$, $a_n=Q^\perp+f_nQ$ is an element of A. Moreover,

$$\mathbf{d}(p,a_n)=\|p(1-f_n)Q\|\to 0;$$

that is, $\inf_{a\in A^1_+}\mathbf{d}(p,a)=0$. However, for each $x\in(0,1)$,

$$\{a(x):a\in A\}=\mathbb{C}1+\mathcal{K}(p'(x)Hp'(x)),$$

Alternatively, note $\inf_{a\in A^1_+}\mathbf{d}(p,a)=0$ implies that the facial support of p is given by

$$\{\phi\in A^{*1}_+:\phi(p)=1\}=\bigcap_{a\in A^1_+}\{\phi\in A^{*1}_+:\phi(a)\ge 1-\mathbf{d}(p,a)\},$$

which is weak* closed in A^{*1}_+, so [5, Lemma 2.4] yields (4.10).

where $p'(x) = p(x) \vee Q$. Thus if $p \leq a \in A^1_+$, then, for all $x \in (0,1)$, we must have $a(x) = 1 - f(x)q'(x)$, where $q' = p' - p$ and f is some function on $[0,1]$. But for each $n \in \mathbb{N}$, q' is discontinuous at r_n, and so the only way a could be continuous is if $f(r_n) = 0$ and hence $a(r_n) = 1$. But then continuity yields $a(0) = 1$, contradicting the fact that $a(0) \leq Q^\perp$. Thus there is no $a \in A^1_+$ with $p \ll a$.

Incidentally, there are several other boundedness conditions on p that one might consider. However, they are all equivalent, even in a more general context.

Recall that B^r denotes the r-ball of B; that is, $B = \{b \in B : ||b|| \leq r\}$.

Proposition 4.10. *For any $a \in A^1_+$, $r > 1$ and C^*-subalgebra $B \subseteq A$, the following are equivalent .*

(1) $\exists b \in B^r_+ \ (a \leq b)$.
(2) $\inf_{b \in B_{\mathrm{sa}}} \mathbf{h}(a,b) = 0$.
(3) $\inf_{b \in B^1_+} \mathbf{d}(a,b) = 0$.
(4) $\inf_{b \in B} \mathbf{d}(a,b) = 0$.

Proof. We immediately have (1)⇒(2) and (3)⇒(4).

(2)⇒(3) By (2.6) (and the existence of an approximate unit for B in B^1_+),

$$\inf_{b \in B^1_+} \mathbf{d}(a,b)^2 \leq \inf_{c \in B_{\mathrm{sa}}} \inf_{b \in B^1_+} (\mathbf{h}(a,c) + \mathbf{d}(c,b)) = \inf_{c \in B_{\mathrm{sa}}} \mathbf{h}(a,c).$$

(4)⇒(2) If $\mathbf{d}(a,b_n) \to 0$; that is, $ab_n \to a$, and hence $b^*_n a b_n \to a$; then

$$\mathbf{h}(a, b^*_n b_n) \leq \mathbf{h}(a, b^*_n a b_n) \leq \mathbf{e}(a, b^*_n a b_n) \to 0.$$

(3)⇒(1) See [4, Theorem 1.2]. □

Another relation on compact projections one might like to quantify is 'interior containment'. Specifically, define the *interior* p° of any projection $p \in A^{**}$ to be the largest open projection below p; that is,

$$p^\circ = \bigvee {\downarrow} p.$$

We quantify the interior containment relation $p \leq q^\circ$ by

$$\mathbf{c}(p,q) = \mathbf{d}(p,q^\circ).$$

Note $\mathbf{d}(p,q^\circ) \leq \mathbf{d}(p,r^\circ) + \mathbf{d}(r^\circ,r) + \mathbf{d}(r,q^\circ) = \mathbf{d}(p,r^\circ) + \mathbf{d}(r,q^\circ)$; so $\mathbf{c}$ is also a distance. Another closely related function on compact projections comes from the 'reverse Hausdorff distance' on the corresponding $\mathbf{d}$-filters defined by

$$\mathbf{g}(p,q) = \inf_{a \in \uparrow p} \sup_{b \in \uparrow q} \mathbf{d}(a,b),$$

Proposition 4.11. *If A is commutative, then $\mathbf{c} = \mathbf{g}$ on compact projections in A^{**}.*

Proof. We may assume that $A = C_0(X)$, for some locally compact X, and identify p and q with characteristic functions of compact subsets of X; that is, $\{0,1\}$-valued functions such that $p^{-1}\{1\}$ and $q^{-1}\{1\}$ are compact. First note that

$$p \leq q^\circ \quad \Leftrightarrow \quad \exists a \in A^1_+ \ (p \leq a \leq q). \tag{4.11}$$

If $p \leq a \leq q$, then $p^{-1}\{1\} \subseteq a^{-1}(0,1] \subseteq q^{-1}\{1\}$ and hence $p^{-1}\{1\} \subseteq q^{-1}\{1\}^\circ$; that is, $p \leq q^\circ$. Conversely, if $p \leq q^\circ$, then Urysohn's lemma yields $a \in C_0(X)$ with $\overline{q^{-1}\{0\}} \subseteq a^{-1}\{0\}$ and $p^{-1}\{1\} \subseteq a^{-1}\{1\}$; so $p \leq a \leq q$. And if $p \leq a \leq q$, then

$$\inf_{f\in F}\sup_{g\in G}\mathbf{d}(f,g) \leq \sup_{g\in G}\mathbf{d}(a,g) \leq \mathbf{d}(a,q) = 0.$$

Thus $\mathbf{g}(p,q) = 0 = \mathbf{c}(p,q)$ when $p \leq q^\circ$.

Conversely, say $p \not\leq q^\circ$, and hence $\mathbf{c}(p,q) = 1$. For any $a \in F$ and $r \in (0,1)$, $a^{-1}(r,1] \subseteq q^{-1}\{1\}$ would imply $p^{-1}\{1\} \subseteq q^{-1}\{1\}^\circ$, a contradiction. Thus we have $x \in a^{-1}(r,1] \setminus q^{-1}\{1\}$ and again Urysohn's lemma (or the complete regularity of X) yields $b \in C_0(X)$ with $b(x) = 0$ and $q^{-1}\{1\} \subseteq b^{-1}\{1\}$. Thus $b \in G$ and $\mathbf{d}(a,b) \geq a(x)(1-b(x)) > r$. As r was arbitrary, $\sup_{b\in G}\mathbf{d}(a,b) = 1$, and, as a was arbitrary, $\inf_{a\in F}\sup_{b\in G}\mathbf{d}(a,b) = 1$. Thus $\mathbf{g}(p,q) = 1 = \mathbf{c}(p,q)$ when $p \not\leq q^\circ$. □

In fact, (4.11) holds for any C^*-algebra A, by Akemann's noncommutative Urysohn lemma (see [3, Lemma III.1]). However, this is not true for $\mathbf{c} = \mathbf{g}$.

Example 4.12. There can be compact $p, q \in A^{**}$ with $\mathbf{g}(p,q) = 0$ but $\mathbf{c}(p,q) = 1$.

To see this, we follow Example 4.9 where we defined A and B satisfying

$$\inf_{a\in A^1_+}\sup_{b\in B^1_+}\mathbf{d}(b,a) = 0. \tag{4.12}$$

Moreover, any $a \in A^1_+$ with $ab = 0$, for all $b \in B$, would have to be of the form fq' for some function f on $[0,1]$. Again, as q' is discontinuous at r_n, the continuity of a would imply $f'(r_n) = 0$. As long as we chose $(r_n) \subseteq (0,1)$ to be dense in $[0,1]$, the continuity of a would then imply $a = 0$. This remains true in the unitization, for if we had nonzero $a \in \widetilde{A}^1_+$ with $ab = 0$, then, as A is essential in $\widetilde{A}$, we would have $c \in A^1_+$ with $ca \neq 0$; so $0 \neq ac^2a \in A^1_+$ even though $ac^2ab = 0$, a contradiction. Hence, for the nonzero compact projections $p = (\bigvee A^1_+)^\perp \in \widetilde{A}^{**}$ and $q = (\bigvee B^1_+)^\perp \in \widetilde{A}^{**}$, we have $\downarrow q = \{0\}$; that is, $q^\circ = 0$ so $\mathbf{c}(p,q) = \mathbf{d}(p,0) = \|p\| = 1$. But (4.12) implies that $\mathbf{g}(p,q) = 0$, as $\mathbf{d}(b,a) = \mathbf{d}(a^\perp, b^\perp)$.

In general, $\mathbf{c}$ can also fail to be $\mathbf{e}$-invariant, even in a weak uniform sense.

Example 4.13. It is possible to have $\mathbf{c} \not\lesssim \mathbf{e} \circ \mathbf{c}$ on compact projections.

To see this, let $A = ([0,1], M_2)$, and let P_θ be the projection onto $\mathbb{C}(\sin\theta, \cos\theta)$,

$$P_\theta = \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$

For $\epsilon \geq 0$, consider the compact projections p_ϵ represented by

$$p_\epsilon(x) = \begin{cases} P_{\epsilon\sin(1/x)} & \text{if } x > 0, \\ 1 & \text{if } x = 0 \end{cases}$$

(this is a projection in the atomic representation of A rather than the universal representation A^{**} but again this does not matter as the atomic representation is faithful on open and closed projections, by [20, Theorem 4.3.15]). So $p(x)$ is a rank 1 projection which 'wiggles' with amplitude ϵ and increasing frequency as $x \to 0$. This means, for $\epsilon > 0$, any $a \in A$ with $a \leq p_\epsilon$ must satisfy $a(0) = 0$; so

$$p_\epsilon^\circ(x) = \begin{cases} P_{\epsilon \sin(1/\theta)} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Also let p be the compact projection defined by $p(x) = P_0$, for all $x \in [0, 1]$; so $p_0^\circ = p$. For all $\epsilon > 0$, $\mathbf{c}(p, p_\epsilon) \geq \|p(0) - p(0)p_\epsilon^\circ(0)\| = \|P_0\| = 1$ even though

$$(\mathbf{c} \circ \mathbf{e})(p, p_\epsilon) \leq \mathbf{c}(p, p_0) + \mathbf{e}(p_0, p_\epsilon) = \mathbf{e}(p_0, p_\epsilon) = \|P_0 - P_\epsilon\| \to 0, \quad \text{as } \epsilon \to 0.$$

This contrasts with $\mathbf{g}$, which is $\mathbf{e}$-invariant and even $\mathbf{h}$-invariant. Moreover, $\mathbf{g}$ can also be calculated from $\mathbf{h}$ instead of $\mathbf{d}$. These facts suggest $\mathbf{g}$ may actually be the more natural extension of interior containment to noncommutative A. This is especially so if one is to consider $\mathbf{d}$-filters in a domain theoretic way – see [11].

Theorem 4.14. $\mathbf{g}$ *is an* $\mathbf{h}$*-invariant distance on compact projections satisfying*

$$\mathbf{g}(p, q) = \inf_{a \in \uparrow p} \sup_{b \in \uparrow q} \mathbf{h}(a, b). \tag{4.13}$$

Proof. By (4.1) and the general relationships between classical and reverse Hausdorff distances given in [11, Proposition 10.2], we have

$$\mathbf{g} \ \leq \ \mathbf{d} \circ \mathbf{g}, \mathbf{g} \circ \mathbf{d} \ \leq \ \mathbf{g} \circ \mathbf{g}.$$

By Corollary 2.10, $\mathbf{d} = \mathbf{h}$ on (compact) projections. As $\mathbf{h}^0$ is reflexive, the relevant reverse inequalities are immediate so $\mathbf{g}$ is an $\mathbf{h}$-invariant distance.

For (4.13), take compact projections $p, q \in A^{**}$ with corresponding $\mathbf{d}$-filters $F = \uparrow p$ and $G = \uparrow q$. As in the proof of Theorem 4.2,

$$\inf_{a \in F} \sup_{b \in G} \mathbf{d}(a, b) = \inf_{a \in F} \mathbf{d}(a, q) \quad \text{and} \quad \inf_{a \in F} \sup_{b \in G} \mathbf{h}(a, b) = \inf_{a \in F} \mathbf{h}(a, q),$$

Let $a_S \in A^{**}$ denote the spectral projection of $a \in A^1_+$ corresponding to $S \subseteq [0, 1]$, and consider

$$P = \{a_{[1-\epsilon,1]} : a \in F \text{ and } \epsilon > 0\}.$$

Note P and F are coinitial in each other, with respect to both $\mathbf{d}$ and $\mathbf{h}$; that is,

$$0 = \sup_{a \in F} \inf_{p \in P} \mathbf{d}(p, a) = \sup_{p \in P} \inf_{a \in F} \mathbf{d}(p, a) = \sup_{a \in F} \inf_{p \in P} \mathbf{h}(p, a) = \sup_{p \in P} \inf_{a \in F} \mathbf{h}(a, p).$$

Thus

$$\begin{aligned} \inf_{a \in F} \mathbf{d}(a, q) &\leq \inf_{a \in F, p \in P} (\mathbf{d}(a, p) + \mathbf{d}(p, q)) = \inf_{p \in P} \mathbf{d}(p, q) \\ &\leq \inf_{p \in P, a \in F} \mathbf{d}(p, a) + \mathbf{d}(a, q) = \inf_{a \in F} \mathbf{d}(a, q). \end{aligned}$$

Likewise $\inf_{a \in F} \mathbf{h}(a, q) = \inf_{p \in P} \mathbf{h}(p, q)$. Now simply note that, by Corollary 2.10,

$$\inf_{p \in P} \mathbf{d}(p, q) = \inf_{p \in P} \mathbf{h}(p, q).$$

□

Acknowledgement. The first author was supported by a WCMCS grant at IMPAN (Poland). The second author was supported by a York University Susan Mann Scholarship and a PRESTIGE co-fund grant.

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