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WELL-POSEDNESS ISSUES FOR A CLASS OF COUPLED NONLINEAR SCHRÖDINGER EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH

HANEN HEZZI

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ABSTRACT. The initial value problem for some coupled nonlinear Schrödinger equations in two space dimensions with exponential growth is investigated. In the defocusing case, global well-posedness and scattering are obtained. In the focusing sign, global and nonglobal existence of solutions are discussed via potential well-method.

1. INTRODUCTION AND PRELIMINARIES

This paper is interested in the Cauchy problem of the nonlinear Schrödinger system

$$\begin{cases} i\partial_t u + \Delta u + \epsilon f(u) + \mu u|u|^{p-2}|v|^p = 0\,; \\ i\partial_t v + \Delta v + \epsilon f(v) + \mu v|v|^{p-2}|u|^p = 0\,; \\ (u(0,\cdot), v(0,\cdot)) = (u_0, v_0) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\,, \end{cases} \tag{1.1}$$

where $p > 2$ and u, v are two complex valued of the variable $(t, x) \in \mathbb{R} \times \mathbb{R}^2$. The nonlinearity takes the Hamiltonian form $f(z) := zF'(|z|^2)$, for some positive real function vanishing on zero $F \in C^1(\mathbb{R}_+)$. A solution (u, v) to (1.1) formally

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satisfies conservation of the mass and the energy

$$\begin{aligned}
M(u(t)) &= M_u(t) := \|u(t)\|_{L^2}^2 = M_u(0)\,;\\
M(v(t)) &= M_v(t) := \|v(t)\|_{L^2}^2 = M_v(0)\,;\\
E(u(t),v(t)) &:= \|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 - \epsilon \int_{\mathbb{R}^2} \Big(F(|u|^2) + F(|v|^2)\Big)\,dx\\
&\qquad - \frac{2\mu}{p}\int_{\mathbb{R}^2} |uv|^p\,dx\,;\\
E(u(t),v(t)) &= E(t) = E(0).
\end{aligned}$$

When ϵ and μ are negative, the energy is non-negative, and (1.1) is said to be defocusing. Otherwise, a control of the solution with the energy is no longer possible, and system (1.1) is focusing.

In recent years, intensive work has been done about coupled Schrödinger systems [8, 10, 20, 19]. These works have been mainly on 2-systems and with polynomial nonlinearities.

If $\mu = 0$, system (1.1) corresponds to the classical scalar semilinear Schrödinger equation in two space dimensions. Let us recall few historic facts about this case. In two space dimensions, the initial value problem for the nonlinear Schrödinger problem in the monomial case is of energy subcritical for all $p > 1$. So, it is natural to consider problems with exponential nonlinearities, which have several applications, such as, for example, self-trapped beams in plasmas [7]. Moreover, the two-dimensional case is interesting because of its relation to the critical Moser–Trudinger inequalities [1, 12]. The two-dimensional Schrödinger problem with exponential growth nonlinearity was studied in [9], where global well-posedness and scattering for small data were proved. Later on, the critical type nonlinearity was considered in [5]. Global well-posedness for small data and scattering were established. Decay of solutions were obtained in [16]. Unconditional global well-posedness and scattering in the energy space, for some weaker exponential nonlinearity were proved [17, 18, 15, 14].

It is the aim of this manuscript to generalize results obtained for the subcritical case in [13] about global well-posedness and scattering of the Schrödinger system (1.1) in the defocusing case and prove existence of nonglobal solutions in the focusing sign using the associate ground state solution and the potential-well theory [11], to the critical case.

The rest of the paper is organized as follows. The main results and some technical tools needed in what follows are listed in the next section. The third section is devoted to prove well-posedness of (1.1). The goal of the fourth section is to establish scattering of global solutions. In section five, the stationary problem associated to (1.1) is studied. In the sixth section, we prove either global well-posedness or blow-up in finite time of solutions to (1.1), for energy less than the ground state one. The last section concerns strong instability of standing waves.

In this note, we are interested in the two dimensional space case; so here and hereafter, we denote $\int \cdot\, dx := \int_{\mathbb{R}^2} \cdot\, dx$. For $p \geq 1$, $L^p := L^p(\mathbb{R}^2)$ is the Lebesgue space endowed with the norm $\| \cdot \|_p := \| \cdot \|_{L^p}$. In the particular case $p = 2$, we denote $\| \cdot \| := \| \cdot \|_2$. We call energy space by the usual Sobolev space H^1 endowed with the norm $\| \cdot \|_{H^1} := \Big(\| \cdot \|^2 + \| \nabla \cdot \|^2\Big)^{\frac{1}{2}}$. Let $T > 0$, and let X be an abstract functional space; we denote $C_T(X) := C([0,T], X)$, the space of continuous functions with variable in $[0,T]$ and values in X and X_{rd}, the set of radial functions in X. We mention that C is an absolute positive constant, which may vary from line to line. If A and B are non-negative real numbers, then $A \lesssim B$ means that $A \leq CB$. Finally, we define the operator $(Df)(x) := xf'(x)$.

2. Main results and background

In this section we give the main results and some technical tools needed in what follows. Let us start with some notations. For $u, v \in H^1$, we define the action

$$S(u,v) := \|u\|_{H^1}^2 - \int F(|u|^2)\, dx + \|v\|_{H^1}^2 - \int F(|v|^2)\, dx - \frac{2\mu}{p} \int |uv|^p\, dx.$$

For $\alpha, \beta \in \mathbb{R}$, we introduce the scaling $v_{\alpha,\beta}^{\lambda} := e^{\alpha\lambda} v(e^{-\beta\lambda} \cdot)$ and the operator

$$\mathcal{L}_{\alpha,\beta} : H^1 \to H^1 \quad v \mapsto \frac{1}{2}\partial_\lambda (v_{\alpha,\beta}^{\lambda})_{|\lambda=0}.$$

We extend the previous operator as follows, if $A : H^1 \to \mathbb{R}$, then

$$\mathcal{L}_{\alpha,\beta} A(v) := \frac{1}{2}\partial_\lambda (A(v_{\alpha,\beta}^{\lambda}))_{|\lambda=0}.$$

Let us also denote, for $\alpha, \beta \in \mathbb{R}$ and $u, v \in H^1$, the so-called constraint

$$\begin{aligned} K_{\alpha,\beta}(u,v) &:= \mathcal{L}_{\alpha,\beta}(S(u,v)) \\ &= \alpha\|\nabla u\|^2 + (\alpha+\beta)\|u\|^2 - \int \Big(\alpha|u|f(|u|) + \beta F(|u|^2)\Big)\, dx \\ &\quad + \alpha\|\nabla v\|^2 + (\alpha+\beta)\|v\|^2 - \int \Big(\alpha|v|f(|v|) + \beta F(|v|^2)\Big)\, dx \\ &\quad - 2\mu(\alpha + \frac{\beta}{p})\|uv\|_p^p. \end{aligned}$$

The quadratic and nonlinear parts of $K_{\alpha,\beta}(u,v)$ are

$$K_{\alpha,\beta}^Q(u,v) = \alpha\|\nabla u\|^2 + (\alpha+\beta)\|u\|^2 + \alpha\|\nabla v\|^2 + (\alpha+\beta)\|v\|^2, \quad K_{\alpha,\beta}^N := K_{\alpha,\beta} - K_{\alpha,\beta}^Q.$$

Define also for $\alpha+\beta\neq 0$, the quantity

$$\begin{aligned} H_{\alpha,\beta}(u,v) &:= (S-\frac{1}{\alpha+\beta}K_{\alpha,\beta})(u,v) \\ &= \frac{1}{\alpha+\beta}\Big[\beta\|\nabla u\|^2+\alpha\int\Big(|u|f(|u|)-F(|u|^2)\Big)\,dx\Big] \\ &\quad+\frac{1}{\alpha+\beta}\Big[\beta\|\nabla v\|^2+\alpha\int\Big(|v|f(|v|)-F(|v|^2)\Big)\,dx\Big] \\ &\quad+2\mu\frac{\alpha}{\alpha+\beta}(1-\frac{1}{p})\|uv\|_p^p. \end{aligned}$$

The following conditions fix the kind of nonlinearities to be considered in this manuscript.

$$F(0)=F'(0)=F''(0)=0\,; \tag{2.1}$$

$$\exists\varepsilon_f \quad \text{s.t.} \quad \min\{(D-1-\varepsilon_f)F,(D-1)^2F\}>0 \quad \text{on} \quad (0,\infty)\,; \tag{2.2}$$

$$\exists\alpha_0>0, \quad \text{s.t.} \quad |F'''(r)|=O(e^{\alpha_0 r}) \quad \text{as} \quad r\longrightarrow\infty. \tag{2.3}$$

Remark 2.1. *An explicit example is* $F(x)=e^x-1-x-\frac{x^2}{2}$.

Proof. Consider $DF(x)=x(e^x-1-x)$, and for $\varepsilon\in(0,2)$,

$$(D-1-\varepsilon)F(x)=(x-1-\varepsilon)e^x+(\varepsilon-1)\frac{x^2}{2}+\varepsilon x+1+\varepsilon:=\phi(x).$$

Let us compute the derivatives $\phi'(x)=(x-\varepsilon)e^x+(\varepsilon-1)x+\varepsilon$, $\phi''(x)=(x-\varepsilon+1)e^x+\varepsilon-1$, and $\phi'''(x)=(x-\varepsilon+2)e^x\geq 0$. Since $\phi(0)=\phi'(0)=0$, we have $\phi\geq 0$. Moreover,

$$(D-1)F(x)=(x-1)e^x-\tfrac{x^2}{2}+1,\; D(D-1)F(x)=x(xe^x-x);$$

$$[(D-1)^2-\varepsilon]F(x)=(x^2-x+1-\varepsilon)e^x+(\varepsilon-1)+(\varepsilon-1)\tfrac{x^2}{2}+\varepsilon x:=\psi(x).$$

The derivatives read $\psi'(x)=(x^2+x-\varepsilon)e^x+(\varepsilon-1)x+\varepsilon$, $\psi''(x)=(x^2+3x-\varepsilon+1)e^x+\varepsilon-1$, and $\psi'''(x)=(x^2+5x-\varepsilon+2)e^x\geq 0$. Since $\psi(0)=\psi'(0)=\psi''(0)=0$, we have $\psi\geq 0$. □

2.1. **Main results.**

The first result is, the following local well-posedness theorem, obtained by a classical fixed point argument.

Theorem 2.2. *Assume that f satisfies (2.3), and take $(u_0,v_0)\in H^1\times H^1$ such that* $\max\{\|\nabla u_0\|^2,\|\nabla v_0\|^2\}<\frac{4\pi}{\alpha_0}$. *Then, there exist $T>0$ and a unique solution to (1.1),*

$$(u,v)\in C([0,T],H^1)\times C([0,T],H^1).$$

Moreover,

1/ $u,v\in L^4([0,T],W^{1,4})$;
2/ (u,v) *satisfies conservation of the mass and the energy;*
3/ (u,v) *is global if* $\max\{E(u_0),E(v_0)\}\leq\frac{4\pi}{\alpha_0}$ *and* $\epsilon\leq 0,\mu\leq 0$.

In the defocusing case, scattering in the energy space is proved. Indeed, every global solution of (1.1) is asymptotic, as $t \to \pm\infty$, to a solution of the associated linear Schrödinger system ($\epsilon = \mu = 0$). In other words, the effect of the nonlinearity is negligible for large times. Precisely, the following scattering result holds.

Theorem 2.3. *Assume that f satisfies (2.3), and let $\epsilon, \mu \le 0$, and take $(u_0, v_0) \in H^1 \times H^1$ such that $\max\{E(u_0), E(v_0)\} < \frac{4\pi}{\alpha_0}$. Then, the global solution to (1.1),*

$$(u, v) \in C(\mathbb{R}, H^1) \times C(\mathbb{R}, H^1)$$

given by the previous theorem scatters and satisfies

$$(u, v) \in L^4(\mathbb{R}, W^{1,4}) \times L^4(\mathbb{R}, W^{1,4}).$$

Second, we are interested in the focusing Schrödinger problem (1.1) ($\epsilon \ge 0$ and $\mu \ge 0$). For simplicity and without loss of generality, we say that (1.1) is focusing if $\epsilon = 1$ and $\mu \ge 0$. This case is related to the associated stationary problem.

Definition 2.4. *(ϕ, ψ) is a ground state solution to (1.1) if*

$$\begin{cases} -\phi + \Delta\phi + f(\phi) + \mu\phi|\phi|^{p-2}|\psi|^p = 0\,; \\ -\psi + \Delta\psi + f(\psi) + \mu\psi|\psi|^{p-2}|\phi|^p = 0\,; \\ \quad (0,0) \neq (\phi, \psi) \in H^1 \times H^1, \end{cases} \tag{2.4}$$

and it minimizes the problem

$$m_{\alpha,\beta} := \inf_{(0,0)\neq(u,v)\in H^1\times H^1} \Big\{ S(u,v),\ s.\ t.\ K_{\alpha,\beta}(u,v) = 0 \Big\}. \tag{2.5}$$

A ground state (ϕ, ψ) is said to be a vector ground state if $\phi \neq 0$ and $\psi \neq 0$.

Denote the sets

$$A^+_{\alpha,\beta} := \{(u,v) \in H^1 \times H^1 \quad \text{s.t.} \quad S(u,v) < m \quad \text{and} \quad K_{\alpha,\beta}(u,v) \ge 0\};$$
$$A^-_{\alpha,\beta} := \{(u,v) \in H^1 \times H^1 \quad \text{s.t.} \quad S(u,v) < m \quad \text{and} \quad K_{\alpha,\beta}(u,v) < 0\}.$$

The next result ensures the existence of ground state solution to (1.1).

Theorem 2.5. *Take a couple real numbers $(0,0) \neq (\alpha, \beta) \in \mathbb{R}^2$. Assume that $\alpha_0 \in (0, 4\pi)$ and that f satisfies (2.1)–(2.3) and that $0 < \varepsilon_f < p - 1$.*

1/ *If $\beta \ge 0$ and $\alpha \ge 0$, then*
   (a) *$m := m_{\alpha,\beta}$ is nonzero and independent of (α, β);*
   (b) *there is a minimizer of (2.5), which is some nontrivial solution to (2.4).*

2/ *If $(\alpha, \beta) = (1, -1)$ and $p > 3$, then the previous result is true if we change the condition (2.2) by*

$$\min\{(D - 2 - \varepsilon_f)F, (D-2)^2 F\} > 0 \quad on \quad (0, \infty). \tag{2.6}$$

3/ *This ground state is a vector ground state if μ is large enough.*

Using the potential well method, we discuss the existence of global and non-global solution to (1.1).

Theorem 2.6. *Assume that $\alpha_0 \in (0, 4\pi)$; consider f satisfies in (2.1), (2.3), and (2.6), and suppose that $\epsilon = \mu = 1$ and that $0 < \varepsilon_f < p - 1 > 4$. Let $(u, v) \in C_{T^*}(H^1) \times C_{T^*}(H^1)$ be the maximal solution to the focusing problem (1.1).*

1/ *If there exist $(0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2_+ \cup \{(1, -1)\}$ and $t_0 \in [0, T^*)$ such that $(u(t_0), v(t_0)) \in A^-_{\alpha,\beta}$ and $(xu(t_0), xv(t_0)) \in L^2 \times L^2$, then (u, v) blows-up in finite time;*

2/ *if there exist $(0, 0) \neq (\alpha, \beta) \in \mathbb{R}^2_+ \cup \{(1, -1)\}$ and $t_0 \in [0, T^*)$ such that $(u(t_0), v(t_0)) \in A^+_{\alpha,\beta}$, then (u, v) is global and scatters.*

Finally, we obtain strong instability of standing waves.

Theorem 2.7. *Assume that $\alpha_0 \in (0, 4\pi)$; consider f satisfies in (2.1), (2.3), and (2.6), and suppose that $\epsilon = \mu = 1$ and that $0 < \varepsilon_f < p - 1 > 4$. Let (ϕ, ψ) be a ground state solution to (1.1). Then, for any $\varepsilon > 0$, there exists $(u_0, v_0) \in H^1 \times H^1$ such that $\|(u_0, v_0) - (\phi, \psi)\|_{H^1 \times H^1} < \varepsilon$ and the maximal solution to the focusing problem (1.1) is not global.*

2.2. **Tools.** Let us recall the so-called Strichartz estimate [3].

Definition 2.8. *A couple of real numbers (q, r) is said to be admissible if*

$$2 \leq q, r \leq \infty, \ (q, r) \neq (2, \infty), \ \text{and } \frac{1}{q} + \frac{1}{r} = \frac{1}{2}.$$

Proposition 2.9. *Let $T > 0$, and let two pairs (q, r) and (a, b) be admissible; then*

$$\|u\|_{L^q_T(L^r)} \lesssim \|u_0\| + \|i\partial_t u + \Delta u\|_{L^{a'}_T(L^{b'})}. \tag{2.7}$$

Recall the so-called Moser–Trudinger inequality [2].

Proposition 2.10. *Let $\alpha \in (0, 4\pi)$. There exists a constant $\mathcal{C}_\alpha$ such that for all $u \in H^1$ satisfying $\|\nabla u\| \leq 1$, we have*

$$\int \left(\mathrm{e}^{\alpha |u(x)|^2} - 1 \right) dx \leq \mathcal{C}_\alpha \|u\|^2.$$

Moreover, this is false if $\alpha \geq 4\pi$ but if we take $\|u\|_{H^1} \leq 1$ rather than $\|\nabla u\| \leq 1$, it follows that

$$\mathcal{K} := \sup_{\|u\|_{H^1(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi |u(x)|^2} - 1 \right) dx < \infty$$

and this is false for $\alpha > 4\pi$. See [12] for more details.

The following version of Moser–Trudinger type inequality [2] will be advantageous.

Proposition 2.11. *Let $\alpha \in (0, 4\pi)$, and let $p \geq 2$. There exists a constant $\mathcal{C}_{\alpha,p}$ such that for all $u \in H^1$ satisfying $\|\nabla u\| \leq 1$, we have*

$$\int |u(x)|^p \mathrm{e}^{\alpha |u(x)|^2} \, dx \leq \mathcal{C}_{\alpha,p} \int |u(x)|^p \, dx. \tag{2.8}$$

Some Logarithmic inequality reads as follows [5].

Proposition 2.12. *Let $\beta \in (0,1)$. For any $\lambda > \frac{1}{2\pi\beta}$ and $0 < \mu \leq 1$, a constant C_λ exists such that, for any function $u \in (H^1 \cap C^\beta)(\mathbb{R}^2)$,*

$$\|u\|_{L^\infty}^2 \leq \lambda \|u\|_\mu^2 \log(C_\alpha + \frac{8^\beta \|u\|_{C^\beta}}{\mu^\beta \|u\|_\mu}),$$

where

$$\|u\|_\mu^2 := \|\nabla u\|^2 + \mu^2 \|u\|^2.$$

Recall that C^β denotes the space of β-Hölder continuous functions endowed with the norm

$$\|u\|_{C^\beta} := \|u\|_{L^\infty} + \sup_{x \neq y} \frac{\|u(x) - u(y)\|}{\|x - y\|^\beta}.$$

We end this section with a Morawetz-type identity.

Definition 2.13. *Take a function $a(x)$ on $\mathbb{R}^2$. We call*

1/ *the virial potential*

$$V_a(t) := \int a(x)\Big(|u(t,x)|^2 + |v(t,x)|^2\Big)\, dx;$$

2/ *the Morawetz action*

$$M_a(t) := 2\int a_j \Im(\bar{u}u_j + \bar{v}v_j)\, dx.$$

Here we adopt the usual summation convention for the index j denoting the associated partial derivative, which means that repeated Euclidean coordinate indexes are summed.

Lemma 2.14. *Take (u, v) the solution to (1.1) given by Theorem 2.2. then, we get*

1/ *the virial identity*

$$\begin{aligned}\partial_t^2 V_a = &\int (-\Delta\Delta a)(|u|^2 + |v|^2)\, dx + 4\int a_{j,k} \Re(\bar{u}_j u_k + \bar{v}_j v_k) \\ &- 2\int \Delta a \Big[\epsilon(D-1)F(|u|^2) + \epsilon(D-1)G(|v|^2) + 2\mu(1 - \frac{1}{p})|u|^p|v|^p\Big]\, dx;\end{aligned} \tag{2.9}$$

2/ *if a is convex, then*

$$\int_0^T \int (-\Delta\Delta a)\Big[|u(t,x)|^2 + |v(t,x)|^2\Big]\, dxdt \lesssim \sup_{[0,T]} |M_a(t)|.$$

This result is known in the classical case [6, 4]. Finally, we derive that a global solution to (1.1) in the energy space, belongs to some global Strichartz space.

Lemma 2.15. *Take (u, v) a global solution to (1.1) in the energy space; then*

$$\|(u,v)\|_{L^4(L^8)\times L^4(L^8)} \lesssim \|(u,v)\|_{L^\infty(H^1)\times L^\infty(H^1)}^{\frac{1}{4}} \|(u,v)\|_{L^\infty(L^2)\times L^\infty(L^2)}^{\frac{3}{4}}. \tag{2.10}$$

Let us give some useful estimates.

Lemma 2.16. *For $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|f(U_1) - f(U_2)| \leq C_\varepsilon |U_1 - U_2| \sum_{i=1}^{2} (e^{\alpha_0(1+\varepsilon)U_i^2} - 1); \tag{2.11}$$

$$|f'(U_1) - f'(U_2)| \leq C_\varepsilon |U_1 - U_2| \sum_{i=1}^{2} (|U_i| + e^{2\alpha_0(1+\varepsilon)U_i^2} - 1). \tag{2.12}$$

Proof. Using (2.3), for any $\varepsilon > 0$,

$$|f''(x)| \lesssim \int_0^x e^{\alpha_0 t^2}\, dt \lesssim x e^{\alpha_0 x^2} \lesssim e^{\alpha_0(1+\varepsilon)x^2} - 1.$$

The proof is achieved thanks to the mean value theorem via the assumption (2.1). □

The last result reads.

Lemma 2.17. *Let $u \in C([0,T], H^1) \cap L^4([0,T], W^{1,2})$ satisfy $\|\nabla u\|^2_{L^\infty_T(L^2)} < \frac{4\pi}{\alpha_0}$. Then, there exist two real numbers $\alpha < 4$ near to 4 and $\varepsilon > 0$ near to zero such that, for any Hölder couple (p, p'),*

$$\|e^{\alpha_0(1+\varepsilon)|u|^2} - 1\|_{L^{p'}_T(L^p)} \leq T^{1-\frac{1}{p}} + \|u\|^{\alpha(1-\frac{1}{p})}_{L^4_T(W^{1,4})} T^{(1-\frac{1}{p})(1-\frac{\alpha}{4})}.$$

Proof. By the Hölder inequality, for any $\varepsilon > 0$,

$$\|e^{\alpha_0(1+\varepsilon)|u|^2} - 1\|_{L^{p'}_T(L^p)} \lesssim \|e^{\frac{1}{p'}\alpha_0(1+\varepsilon)|u|^2_{L^\infty_x}}\|_{L^{p'}_T(0,T)} \|e^{\alpha_0(1+\varepsilon)|u|^2} - 1\|^{\frac{1}{p}}_{L^\infty_T(L^1)}.$$

Take $\varepsilon > 0$ small such that

$$(1+\varepsilon)\|\nabla u\|^2_{L^\infty_T(H^1)} < \frac{4\pi}{\alpha_0}.$$

So, by the Moser–Trudinger inequality,

$$\int \Big(e^{\alpha_0(1+\varepsilon)|u|^2} - 1\Big) dx \leq \int \Big(e^{\alpha_0(1+\varepsilon)\|\nabla u\|^2_{L^\infty_T(H^1)}(\frac{|u|}{|\nabla u|})^2} - 1\Big) dx \lesssim \|u\|^2 \lesssim 1.$$

For any $\lambda > \frac{1}{\pi}$ and $\omega \in (0,1]$, by the logarithmic inequality in proposition 2.12,

$$e^{\alpha_0(1+\varepsilon)\|u\|^2_{L^\infty_x}} \leq (C + 2\sqrt{\frac{2}{\omega}} \frac{\|u\|_{C^{\frac{1}{2}}}}{\|u\|_\omega})^{\lambda\alpha_0(1+\varepsilon)\|u\|^2_\omega}.$$

Since

$$\|u\|^2_\omega = \omega^2\|u\|^2 + \|\nabla u\|^2 < \omega^2\|u\|^2_{L^\infty_T(H^1)} + \frac{4\pi}{\alpha_0},$$

we may take $0 < \omega,$, ε near to zero, and $\alpha < 4\alpha_0$ near to $4\alpha_0$ such that $(1+\varepsilon)\|u\|^2_\omega < \frac{\alpha}{\alpha_0}\pi < 4\pi$. Thus, for $\lambda > \frac{1}{\pi}$ near $\frac{1}{\pi}$,

$$\begin{aligned} e^{\alpha_0(1+\varepsilon)\|u\|^2_{L^\infty}} &\leq (C + 2\sqrt{\frac{2}{\omega}} \frac{\|u\|_{C^{\frac{1}{2}}}}{\|u\|_\omega})^{\lambda\alpha_0(1+\varepsilon)\|u\|^2_\omega} \\ &\lesssim (1 + \|u\|_{C^{\frac{1}{2}}})^\alpha \\ &\lesssim 1 + \|u\|^\alpha_{W^{1,4}}. \end{aligned}$$

It follows that

$$\begin{aligned}
\|e^{\alpha_0(1+\varepsilon)|u|^2}-1\|_{L^{p'}_T(L^p)} &\lesssim \|e^{\frac{\alpha_0}{p'}(1+\varepsilon)\|u\|^2_{L^\infty}}\|_{L^{p'}(0,T)}\|e^{(1+\varepsilon)|u|^2}-1\|^{\frac{1}{p}}_{L^\infty(L^1)}\\
&\lesssim \|e^{\frac{\alpha_0}{p'}(1+\varepsilon)\|u\|^2_{L^\infty}}\|_{L^{p'}(0,T)}\\
&\lesssim \|1+\|u\|^\alpha_{W^{1,4}}\|^{\frac{1}{p'}}_{L^1(0,T)}\\
&\lesssim T^{1-\frac{1}{p}}+\|u\|^{\alpha(1-(1/p))}_{L^4_T(W^{1,4})}T^{(1-(1/p))(1-(\alpha/4))}.
\end{aligned}$$

□

3. Proof of Theorem 2.2

In what follows, we prove the Theorem 2.2. First, we establish local existence by a fixed point argument. Since the source term sign has no local effect, without loss of generality, we take in the two next subsections $\epsilon=\mu=-1$.

3.1. **Local Existence.** For $T,r>0$, denote $\mathcal{E}_T(r)$ the ball of center zero and radius r of the space

$$[C([0,T],H^1(\mathbb{R}^2))\cap L^4([0,T],W^{1,4}(\mathbb{R}^2))]^2$$

endowed with the complete distance

$$d\Big((g_1,g_2),(h_1,h_2)\Big)=\|(h_1-g_1,h_2-g_2)\|_T,$$

where

$$\|(h_1,h_2)\|_T:=\sum_{i=1}^2\Big(\sup_{t\in[0,T]}\|h_i(t,.)\|_{L^2(\mathbb{R}^2)}+\|h_i\|_{L^4_T(L^4(\mathbb{R}^2))}\Big).$$

For $i\in\{1,2\}$, let w_i be the solution to the following free Schrödinger equation

$$i\partial_t w_i+\Delta w_i=0,$$

with respectively data $w_1(0,.)=u_0$ and $w_2(0,.)=v_0$. We consider the map ϕ on $\mathcal{E}_T(r)$ given by $\phi(v_1,v_2)=(\tilde v_1,\tilde v_2)$, such that

$$\begin{cases}
i\partial_t\tilde v_1+\Delta\tilde v_1+f(v_1+w_1)+(v_1+w_1)|v_1+w_1|^{p-2}|v_2+w_2|^p=0\,;\\
i\partial_t\tilde v_2+\Delta\tilde v_2+f(v_2+w_2)+(v_2+w_2)|v_2+w_2|^{p-2}|v_1+w_1|^p=0\,;\\
(\tilde v_1(0,.),\tilde v_2(0,.))=(0,0).
\end{cases}$$

We prove that, for $r,T>0$ sufficiently small, the map ϕ is a contraction of $\mathcal{E}_T(r)$. Applying the Strichartz estimate (2.9) to $(v_1,v_2),(u_1,u_2)\in\mathcal{E}_T(r)$, we get

$$\begin{aligned}
&d((\tilde v_1,\tilde v_2),(\tilde u_1,\tilde u_2))_T\\
&\lesssim \|f(v_1+w_1)-f(u_1+w_1)\|_{L^1_T(L^2)}+\|f(v_2+w_2)-f(u_2+w_2)\|_{L^1_T(L^2)}\\
&\quad+\|(v_1+w_1)|v_1+w_1|^{p-2}|v_2+w_2|^p-(u_1+w_1)|u_1+w_1|^{p-2}|u_2+w_2|^p\|_{L^1_T(L^2)}\\
&\quad+\|(v_2+w_2)|v_2+w_2|^{p-2}|v_1+w_1|^p-(u_2+w_2)|u_2+w_2|^{p-2}|u_1+w_1|^p\|_{L^1_T(L^2)}\\
&:=\|f(a_1)-f(b_1)\|_{L^1_T(L^2)}+\|f(a_2)-f(b_2)\|_{L^1_T(L^2)}\\
&\quad+\|a_1|a_1|^{p-2}|a_2|^p-b_1|b_1|^{p-2}|b_2|^p\|_{L^1_T(L^2)}+\|a_2|a_2|^{p-2}|a_1|^p-b_2|b_2|^{p-2}|b_1|^p\|_{L^1_T(L^2)}.
\end{aligned}$$

Let us control $(I) := \|f(a_1)-f(b_1)\|_{L^1_T(L^2)}$ and $\|f(a_2)-f(b_2)\|_{L^1_T(L^2)}$ Using Sobolev inequality and proposition 2.11,

$$\begin{aligned}
&(I)\\
&\lesssim \||(v_1+w_1-u_1-w_1)|(e^{\alpha_0(1+\varepsilon)|a_1|^2}-1+e^{\alpha_0(1+\varepsilon)|b_1|^2}-1)\|_{L^1_T(L^2)}\\
&\lesssim \||v_1-u_1|(e^{\alpha_0(1+\varepsilon)|a_1|^2}-1)\|_{L^1_T(L^2)}+\||v_1-u_1|(e^{\alpha_0(1+\varepsilon)|b_1|^2}-1)\|_{L^1_T(L^2)}\\
&\lesssim \|v_1-u_1\|_{L^4_T(L^4)}\|e^{\alpha_0(1+\varepsilon)|a_1|^2}-1\|_{L^{\frac{4}{3}}_T(L^4)}+\|v_1-u_1\|_{L^4_T(L^4)}\|e^{\alpha_0(1+\varepsilon)|b_1|^2}-1\|_{L^{\frac{4}{3}}_T(L^4)}\\
&\lesssim \|v_1-u_1\|_T\|e^{\alpha_0(1+\varepsilon)|a_1|^2}-1\|_{L^{\frac{4}{3}}_T(L^4)}+\|v_1-u_1\|_T\|e^{\alpha_0(1+\varepsilon)|b_1|^2}-1\|_{L^{\frac{4}{3}}_T(L^4)}\\
&\lesssim [T^{\frac{3}{4}}+\|v_1-u_1\|_T^{\frac{3}{4}\alpha}T^{\frac{3}{4}(1-\frac{\alpha}{4})}]\|v_1-u_1\|_T.
\end{aligned}$$

Similarly,we get

$$\sum_{i=1}^{2}\|f(a_i)-f(b_i)\|_{L^1_T(L^2)}\lesssim (T^{\frac{3}{4}}+\|(v_1,v_2)-(u_1,u_2)\|_T^{\frac{3}{4}\alpha}T^{\frac{3}{4}(1-\frac{\alpha}{4})})d\Big((v_1,v_2),(u_1,u_2)\Big). \tag{3.1}$$

Let us estimate the quantity

$$\|a_1|a_1|^{p-2}|a_2|^p-b_1|b_1|^{p-2}|b_2|^p\|_{L^1_T(L^2)}.$$

Denote the function $h:\mathbb{C}^2\to\mathbb{C}$ given by $(z_1,z_2)\mapsto z_1|z_1|^{p-2}|z_2|^p$, and compute

$$\begin{aligned}
|\partial_{z_1}h(z_1,z_2)|+|\partial_{\bar z_1}h(z_1,z_2)|&\lesssim |z_1|^{p-2}|z_2|^p;\\
|\partial_{z_2}h(z_1,z_2)|+|\partial_{\bar z_2}h(z_1,z_2)|&\lesssim |z_1|^{p-1}|z_2|^{p-1}.
\end{aligned}$$

Thus,

$$|\mathcal{D}h(z_1,z_2)|\lesssim |z_1|^{p-2}|z_2|^p+|z_1z_2|^{p-1}\lesssim (|z_1|^2+|z_2|^2)^{p-1}.$$

With the mean value theorem, it follows that

$$|h(a)-h(b)|\lesssim |a-b|(|a|^{2p-2}+|b|^{2p-2}).$$

This implies, via Sobolev embedding, that

$$\begin{aligned}
&\|h(a_1,a_2)-h(b_1,b_2)\| &(3.2)\\
&\lesssim \|a-b\|_{L^4\times L^4}\sum_{i=1}^{2}\Big(\|a_i\|_{8(p-1)}^{2(p-1)}+\|b_i\|_{8(p-1)}^{2(p-1)}\Big)\\
&\lesssim \|a-b\|_{L^\infty_T(H^1\times H^1)}\sum_{i=1}^{2}\Big(\|a_i\|_{H^1}^{2(p-1)}+\|b_i\|_{H^1}^{2(p-1)}\Big)\\
&\lesssim \|a-b\|_{L^\infty_T(H^1\times H^1)}\sum_{i=1}^{2}\Big(\|u_i\|_{H^1}^{2(p-1)}+\|v_i\|_{H^1}^{2(p-1)}+\|w_i\|_{H^1}^{2(p-1)}\Big)\\
&\lesssim \|a-b\|_{L^\infty_T(H^1\times H^1)}\sum_{i=1}^{2}\Big(r+\|u_0\|_{H^1}+\|v_0\|_{H^1}\Big)^{2(p-1)}. &(3.3)
\end{aligned}$$

Thus,

$$\begin{aligned}&\|a_1|a_1|^{p-2}|a_2|^p - b_1|b_1|^{p-2}|b_2|^p\|_{L^1_T(L^2)}\\&\quad\lesssim (r + \|u_0\|_{H^1} + \|v_0\|_{H^1})^{2(p-1)} d\Big((v_1, v_2), (u_1, u_2)\Big)T.\end{aligned} \tag{3.4}$$

Finally, using (3.1)- (3.3), and the previous inequality, for $T > 0$ small enough

$$d\Big(\phi(v_1, v_2), \phi(u_1, u_2)\Big) \le C(r, \|u_0\|_{H^1}, \|v_0\|_{H^1})T^{\frac{3}{4}} d\Big((v_1, v_2), (u_1, u_2)\Big). \tag{3.5}$$

For $(u_1, u_2) = -(w_1, w_2)$, we get $\tilde{u}_1 = \tilde{u}_2 = 0$, and the previous inequality yields

$$\begin{aligned}\|\phi(v_1, v_2)\|_T &\lesssim T^{\frac{3}{4}}\|(v_1, v_2) + (w_1, w_2)\|_T\\&\lesssim T^{\frac{3}{4}}\Big(r + \|u_0\|_{H^1} + \|v_0\|_{H^1}\Big).\end{aligned} \tag{3.6}$$

It remains to estimate $\|\nabla(f(a_1))\|_{L^1_T(L^2)} := (\mathcal{I}_1)$. Using the following conservation laws,

$$\|\nabla w_1\| = \|\nabla u_0\|, \quad \|\nabla w_2\| = \|\nabla v_0\|,$$

we get

$$\|a_1\|^2_{H^1} = \|v_1 + w_1\|^2_{H^1} \le (r + \|u_0\|_{H^1})^2.$$

Using (2.12) via Moser–Trudinger inequality and Lemma 2.17, for $\varepsilon > 0$ small enough

$$\begin{aligned}(\mathcal{I}_1) &\lesssim \|\nabla a_1 a_1(|a_1| + \mathrm{e}^{\alpha_0(1+\varepsilon)|a_1|^2} - 1\|_{L^1_T(L^2)}\\
&\lesssim \|\nabla a_1|a_1|^2\|_{L^1(L^2)} + \|\nabla a_1(a_1)(\mathrm{e}^{\alpha_0(1+\varepsilon)|a_1|^2} - 1)\|_{L^1(L^2)}\\
&\lesssim \|a_1\|_{L^\infty(H^1)}\|\nabla a_1\|_{L^4(L^4)}\Big[\|a_1\|_{L^\infty(H^1)}T^{\frac{3}{4}} + \|e^{\alpha_0(1+\varepsilon)|a_1|^2} - 1\|_{L^{\frac{4}{3}}(L^{4+\varepsilon})}\Big]\\
&\lesssim \|a_1\|^2_T\Big[\|a_1\|_T T^{\frac{3}{4}} + T^{\frac{3}{4}} + \|a_1\|^{\frac{3}{4}\alpha}_{L^4_T(W^{1,4})}T^{\frac{3}{4}(1-\frac{\alpha}{4})}\Big]\\
&\lesssim (r + \|u_0\|_{H^1})^2\Big[(r + \|u_0\|_{H^1})T^{\frac{3}{4}} + T^{\frac{3}{4}} + \|v_1 - u_1\|^{\frac{3}{4}\alpha}_T T^{\frac{3}{4}(1-\frac{\alpha}{4})}\Big].\end{aligned} \tag{3.7}$$

It remains to estimate the quantity $\|a_1|a_1|^{p-2}|a_2|^p\|_{L^1_T(\dot{H}^1)}$. Write,

$$\begin{aligned}\|\nabla(h(a))\| &= \|\mathcal{D}h(a)\nabla a\|_{L^2}\\
&\lesssim \Big(\|a_1\|^{2(p-1)}_{8(p-1)} + \|a_2\|^{2(p-1)}_{8(p-1)}\Big)\|\nabla a\|_{L^4\times L^4}\\
&\lesssim \Big(\|a_1\|^{2(p-1)}_{H^1} + \|a_2\|^{2(p-1)}_{H^1}\Big)\|\nabla a\|_{L^4\times L^4}\\
&\lesssim \Big(\|a_1\|^{2(p-1)}_{H^1} + \|a_2\|^{2(p-1)}_{H^1}\Big)\|\nabla a\|_{L^4\times L^4}\\
&\lesssim \Big(r + \|u_0\|^{2(p-1)}_{H^1} + \|v_0\|^{2(p-1)}_{H^1}\Big)\|\nabla a\|_{L^4\times L^4}.\end{aligned}$$

Integrating with respect to time, we get

$$\begin{aligned}\|\nabla(h(a))\|_{L^1_T(L^2)} &\lesssim \Big(r + \|u_0\|^{2(p-1)}_{H^1} + \|v_0\|^{2(p-1)}_{H^1}\Big)\|\nabla a\|_{L^4_T(L^4\times L^4)}T^{\frac{3}{4}}\\
&\quad + \Big(r + \|u_0\|^{2p-3}_{H^1} + \|v_0\|^{2p-3}_{H^1}\Big)\|a\|_T T^{\frac{3}{4}}\\
&\lesssim \Big(r + \|u_0\|^{2(p-1)}_{H^1} + \|v_0\|^{2(p-1)}_{H^1}\Big)(r + \|u_0\|_{H^1})T^{\frac{3}{4}}.\end{aligned} \tag{3.8}$$

By (3.7) and (3.8) we conclude that, for small $r, T > 0$, ϕ is a contraction which maps $\mathcal{E}_T(r)$ into itself. With an application of Picard fixed point theorem, the proof of existence of a local solution to (1.1) is finished.

3.2. **Uniqueness in the energy space.** In what follows, we prove the uniqueness of solution to the Cauchy problem (1.1) in the energy space. Let $T > 0$ be a positive time, and let (u_1, u_2), (v_1, v_2) be two solutions to (1.1) in $C_T(H^1) \times C_T(H^1)$. Then, for $u := u_1 - v_1$ and $v := u_2 - v_2$,

$$\begin{cases} i\partial_t u + \Delta u + f(u_1) - f(v_1) + u_1|u_1|^{p-2}|v_1|^p - v_1|u_1|^{p-2}|v_1|^p = 0\,; \\ i\partial_t v + \Delta v + f(v_1) - f(v_2) + u_2|u_2|^{p-2}|v_2|^p - 2_2|u_2|^{p-2}|v_2|^p = 0\,; \\ (u(0,.), v(0,.)) = (0,0). \end{cases}$$

By Strichartz estimate (2.7), we have

$$\begin{aligned} \|(u,v)\|_{L^\infty_T(H^1\times H^1)} \lesssim \|f(u_2) - f(v_2)\|_{L^1_T(H^1)} &+ \|f(u_1) - f(v_1)\|_{L^1_T(H^1)} \\ &+ \|h(u_1,v_1) - h(v_1,u_1)\|_{L^1_T(H^1)} \\ &+ \|h(u_2,v_2) - h(v_2,u_2)\|_{L^1_T(H^1)}. \end{aligned}$$

With a continuity argument, we may assume that

$$\max_{i=1,2} \|u_i\|_{L^\infty_T(H^1)} \le 1 + \|u_0\|_{H^1} + \|v_0\|_{H^1} \quad \text{and} \quad \max_{i=1,2} \|\nabla u_i\|^2_{L^\infty(L^2)} < \frac{4\pi}{\alpha_0}.$$

Using previous computations, we have

$$\begin{aligned} &\|f(u_1) - f(v_1)\|_{L^1_T(L^2)} + \|f(u_2) - f(v_2)\|_{L^1_T(L^2)} \\ &\qquad \lesssim (T^{\frac34} + \|(u_1,u_2) - (v_1,v_2)\|_T^{\frac34\alpha} T^{\frac34(1-\frac{\alpha}{4})})\|(u_1,u_2) - (v_1,v_2)\|_T; \end{aligned}$$

$$\begin{aligned} &\|\nabla(f(u_2) - f(v_2))\|_{L^1_T(L^2)} \\ &\qquad \lesssim \|v\|_{L^\infty(H^1)}\|\nabla u_2\|_{L^4(L^4)}\left[\|u_2\|_{L^\infty(H^1)}T^{\frac34} + \|e^{\alpha_0(1+\varepsilon)|u_2|^2} - 1\|_{L^{\frac43}(L^{4+\varepsilon})}\right] \\ &\qquad\quad + \|\nabla u\|_{L^4(L^4)}\||v_1| + e^{\alpha_0(1+\varepsilon)|v_1|^2} - 1\|_{L^{\frac43}(L^4)}; \end{aligned}$$

$$\begin{aligned} &\|h(u_1,v_1) - h(v_1,u_1)\|_{L^1_T(H^1)} \\ &\qquad \lesssim (1 + \|u_0\|_{H^1}^{2(p-1)} + \|v_0\|_{H^1}^{2(p-1)})T\|u\|_{L^\infty_T(H^1)} \\ &\qquad\quad + (1 + \|u_0\|_{H^1}^{2p-3} + \|v_0\|_{H^1}^{2p-3})[\|\nabla u_1\|_{L^4_T(L^4)}\|u\|_{L^\infty_T(H^1)} + \|u\|_{L^4_T(L^4)}]T^{\frac34}. \end{aligned}$$

By a standard translation argument, the following lemma concludes the uniqueness proof.

Lemma 3.1. *For small $T > 0$,*

$$\begin{aligned} \|\nabla u\|_{L^4_T(L^4)} \lesssim [(1 + \|u_0\|_{H^1} + \|v_0\|_{H^1})T^{\frac16} &+ T^{\frac16} \\ &+ \|(u_1,v_1)\|_{L^4_T(W^{1,4})}^{\frac16\alpha} T^{\frac16(1-\frac{\alpha}{4})}]\|u\|_{L^\infty_T(H^1)}; \end{aligned}$$

$$\|\nabla u_1\|_{L^4_T(L^4)} \lesssim [(1 + \|u_0\|_{H^1} + \|v_0\|_{H^1})T^{\frac16} + \|u_1\|_{L^4_T(W^{1,4})}^{\frac16\alpha} T^{\frac16(1-\frac{\alpha}{4})} + T^{\frac16}]\|u_1\|_{L^\infty_T(H^1)}.$$

Proof. By Strichartz estimate,

$$\|\nabla u\|_{L^4_T(L^4)} \lesssim \|\nabla(f(u_1)-f(v_1))\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})} + \|\nabla(h(u_1,v_1)-h(v_1,u_1))\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})}.$$

Moreover,

$$\begin{aligned}\|\nabla(f(u_1)-f(v_1))\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})} &\leq \|(\mathcal{D}f(u_1)-\mathcal{D}f(v_1))\nabla u_1\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})}\\ &\quad + \|\mathcal{D}f(v_1)\nabla u\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})}\\ &= \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

Using Lemma 2.17 via Sobolev embedding and arguing as previously, for $\varepsilon>0$ small enough, we get

$$\begin{aligned}\mathcal{J}_1 &\lesssim \|\nabla u_1|u_1-v_1|(|u_1|+\mathrm{e}^{\alpha_0(1+\varepsilon)|u_1|^2}-1+|v_1|+\mathrm{e}^{\alpha_0(1+\varepsilon)|v_1|^2}-1)\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})}\\ &\lesssim \|\nabla u_1|u_1-v_1||u_1|\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})} + \|\nabla u_1|u_1-v_1|(\mathrm{e}^{\alpha_0(1+\varepsilon)|u_1|^2}-1)\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})}\\ &\lesssim \|\nabla u_1\|_{L^\infty_T(L^2)}\|u_1-v_1\|_{L^\infty_T(H^1)}[T^{\frac{5}{6}}\|u_1\|_{L^\infty_T(H^1)} + \|\mathrm{e}^{\alpha_0(1+\varepsilon)|u_1|^2}-1\|_{L^{\frac{6}{5}}_T(L^{6+\varepsilon})}]\\ &\lesssim \|\nabla u_1\|_{L^\infty_T(L^2)}\|u_1-v_1\|_{L^\infty_T(H^1)}[T^{\frac{5}{6}}\|u_1\|_{L^\infty_T(H^1)} + T^{\frac{5}{6}} + \|u_1\|^{\frac{1}{6}\alpha}_{L^4_T(W^{1,4})}T^{\frac{1}{6}(1-\frac{\alpha}{4})}]\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_2 &\lesssim \|\nabla(u_1-v_1)(|v_1|+e^{\alpha_0(1+\varepsilon)|v_1|^2}-1)\|_{L^{\frac{6}{5}}_T(L^{\frac{3}{2}})}\\ &\lesssim \|\nabla(u_1-v_1)\|_{L^\infty_T(L^2)}\||v_1|+e^{\alpha_0(1+\varepsilon)|v_1|^2}-1\|_{L^{\frac{6}{5}}_T(L^6)}\\ &\lesssim \left[(1+\|u_0\|_{H^1}+\|v_0\|_{H^1})T^{\frac{5}{6}}+T^{\frac{5}{6}}+\|v_1\|^{\frac{1}{6}\alpha}_{L^4_T(W^{1,4})}T^{\frac{1}{6}(1-\frac{\alpha}{4})}\right]\|u_1-v_1\|_{L^\infty_T(H^1)}.\end{aligned}$$

So,

$$\begin{aligned}\|\nabla(f(u_1)-f(v_1))\|_{\frac{3}{2}} \lesssim \Big[&(1+\|u_0\|_{H^1}+\|v_0\|_{H^1})T^{\frac{1}{6}}\\ &+T^{\frac{1}{6}}+\|v_1\|^{\frac{1}{6}\alpha}_{L^4_T(W^{1,4})}T^{\frac{1}{6}(1-\frac{\alpha}{4})}\Big]\|u\|_{L^\infty_T(H^1)}.\end{aligned}$$

Arguing as previously for $a=(u_1,v_1)$ and $b=(v_1,u_1)$ yields

$$\begin{aligned}\|\nabla(h(a)-h(b))\|_{\frac{3}{2}} &\leq \|\mathcal{D}h(a)\nabla(a-b)\|_{\frac{3}{2}} + \|(\mathcal{D}h(a)-\mathcal{D}h(b))\nabla b\|_{\frac{3}{2}}\\ &\lesssim (\|u_1\|^{2(p-1)}_{12(p-1)}+\|v_1\|^{2(p-1)}_{12(p-1)})\|\nabla u\|\\ &\quad + \|u(|u_1|+|v_1|)^{2p-3}\|_6[\|\nabla u_1\|+\|\nabla v_1\|]\\ &\lesssim (\|u_1\|^{2(p-1)}_{H^1}+\|v_1\|^{2(p-1)}_{H^1})\|u\|_{H^1}.\end{aligned}$$

Thus,

$$\|\nabla u\|_{L^4_T(L^4)} \lesssim [(1+\|u_0\|_{H^1}+\|v_0\|_{H^1})T^{\frac{1}{6}}+T^{\frac{1}{6}}+\|v_1\|^{\frac{1}{6}\alpha}_{L^4_T(W^{1,4})}T^{\frac{1}{6}(1-\frac{\alpha}{4})}+T]\|u\|_{L^\infty_T(H^1)}.$$

The second point is obtained when taking $v_1=0$ in the first point of the lemma. $\square$

3.3. **Global well-posedness in the defocusing case $\epsilon, \mu \geq 0$.** This subsection is devoted to prove that the solution given by Theorem 2.2 is global in the case $\epsilon = \mu = -1$ and where $E(u_0, v_0) \leq \frac{4\pi}{\alpha_0}$. We recall an important fact that is the time of local existence depends only on the quantity $\|(u_0, v_0)\|_{H^1 \times H^1}$. Let (u, v) be the unique maximal solution of (1.1) in the space $\mathcal{E}_T$ for any $0 < T < T^*$ with initial data (u_0, v_0), where $0 < T^* \leq +\infty$ is the lifespan of (u, v). We shall prove that (u, v) is global. By contradiction, suppose that $T^* < +\infty$; we consider, for $0 < s < T^*$, the following problem

$$(\mathcal{P}_s) \begin{cases} i\partial_t u' + \Delta u' + f(u') + u'|u'|^{p-2}|v'|^p = 0\,; \\ i\partial_t v' + \Delta v' + f(v') + v'|v'|^{p-2}|u'|^p = 0\,; \\ \quad (u'(s,.), v'(s,.)) = (u(s,.), v(s,.)). \end{cases}$$

First, let us treat the simplest case $E(u_0, v_0) < \frac{4\pi}{\alpha_0}$. In this case,

$$\max\{\sup_{[0,T^*]} \|\nabla u(t)\|^2, \sup_{[0,T^*]} \|\nabla v(t)\|^2\} \leq E(u_0, v_0) < \frac{4\pi}{\alpha_0}.$$

Using the same arguments used in the local existence, we can find a real number $\tau > 0$ and a solution (u', v') to $(\mathcal{P}_s)$ on $[s, s+\tau]$. According to the section of local existence, and using the conservation of energy, τ does not depend on s.
Thus, if we let s be close to T^* such that $s + \tau > T^*$, then we can extend (u, v) for times higher than T^*. This fact contradicts the maximality of T^*. We obtain the result claimed in Theorem 2.2.
Second, let us treat the limiting case

$$E(u_0, v_0) = \frac{4\pi}{\alpha_0} \;\&\; \sup_{[0,T^*]} (\|\nabla u(t)\|^2 + \|v(t)\|^2) = \limsup_{T^*} (\|\nabla u(t)\|^2 + \|v(t)\|^2) = \frac{4\pi}{\alpha_0}.$$

Then, using the behaviour of the nonlinearity, we get $x^2 \lesssim F(x)$, and so

$$\liminf_{T^*} \|F(|(u(t)|^2)\|_1 = \liminf_{T^*} \|u(t)\|_4 = 0\,;$$
$$\liminf_{T^*} \|F(|(v(t)|^2)\|_1 = \liminf_{T^*} \|v(t)\|_4 = 0.$$

Global well-posedness is a consequence of the following result.

Lemma 3.2. *Let $T > 0$, and let $(u, v) \in C([0, T], H^1 \times H^1)$ be a solution to the Schrödinger equation (1.1) with $\epsilon = -1$ such that $E(u_0, v_0) + M(u_0, v_0) < \infty$. Then, a positive constant C_0, depending on (u_0, v_0), exists such that, for any $R, R' > 0$ and any $0 < t < T$,*

$$\int_{B_{R+R'}} (|u(t)|^2 + |v(t)|^2)\, dx \geq \int_{B_R} (|u_0|^2 + |v_0|^2)\, dx - C_0 \frac{t}{R'}. \tag{3.9}$$

Proof of Lemma 3.2. Let $R, R' > 0$, and let $d_R(x) := d(x, B_R)$ and consider a cut-off function $\phi := h(1 - (\frac{d_R}{R'}))$, where $h \in C^\infty(\mathbb{R})$, $0 \leq h \leq 1$, $h(t) = 1$ for $t \geq 1$, and $h(t) = 0$ for $t \leq 0$.

So, $\phi(x) = 1$ for $x \in B_R$ and $\phi(x) = 0$ for $x \notin B_{R+R'}$. Moreover,

$$\nabla\phi(x) = -\frac{x-R}{R'|x-R|}h'(1-\frac{d_R(x)}{R'})1_{R<|x|<R+R'};$$
$$\|\nabla\phi\|_{L^\infty} \leq \frac{\|h'\|_{L^\infty([0,1])}}{R'} \lesssim \frac{1}{R'}.$$

Multiplying (1.1) by $\phi^2\overline{u}$ and $\phi^2\overline{v}$, we get

$$\begin{cases} \phi^2\overline{u}(iu_t + \Delta u) = \phi^2|u|^2(F'(|u|^2) + |u|^{p-2}|v|^p); \\ \phi^2\overline{v}(iv_t + \Delta v) = \phi^2|v|^2(F'(|v|^2 + |v|^{p-2}|u|^p). \end{cases} \tag{3.10}$$

Integrating over space and then taking the imaginary part yields

$$\begin{aligned} \partial_t(\|\phi u\|^2 + \|\phi u\|^2) &= -2\Im\int \phi^2(\overline{u}\Delta u + \overline{v}\Delta v)\,dx \\ &= 2\Im\int (\nabla(\phi^2\overline{u})\nabla u + \nabla(\phi^2\overline{v})\nabla v)\,dx \\ &= 4\Im\int (\phi\nabla\phi\overline{u}\nabla u + \phi\nabla\phi\overline{v}\nabla v)\,dx \\ &\geq -\frac{C_0}{R'}. \end{aligned}$$

An integration over time achieves the proof.
Let us return to the proof of global well-posedness. With Hölder inequality, denoting $w := (u, v)$, we get

$$\|w(t)\|^2_{L^2(B_{R+R'})} \lesssim (R+R')\|w(t)\|^2_{L^4(B_{R+R'})}.$$

Taking account of (3.9) yields

$$\begin{aligned} \sqrt{\pi}(R+R')\|w(t)\|^2_{L^4(B_{R+R'})} &\geq \left(\|w_0\|^2_{L^2(B_R)} - C_0\frac{t}{R'}\right) \\ &\geq \left(\|w_0\|^2_{L^2(B_R)} - C_0\frac{T^*}{R'}\right). \end{aligned}$$

Taking the lower limit when t tends to T^* and then $R' \longrightarrow \infty$ yields the contradiction $w_0 = 0$ which ends the proof. □

4. Proof of Theorem 2.3

For easy notation and without loss of generality, in all this section we fix $\epsilon = \mu = -1$, and we prove scattering. Denote the real numbers

$$\rho := 1 + 2p, \quad \alpha := 2p - 1, \quad \frac{1}{\gamma} := \frac{1}{2} - \frac{1}{\rho}.$$

For any time slab I, take the Strichartz norm

$$\|u\|_{S(I)} := \|u\|_{L^\infty(I,H^1)} + \|u\|_{L^\gamma(I,W^{1,\rho})}.$$

The first intermediate result reads.

Lemma 4.1. *For any time slab I, we have*

$$\|(u(t)-e^{it\Delta}u_0, v(t)-e^{it\Delta}v_0)\|_{S(I)\times S(I)} \lesssim \Big(\|u\|_{L^\infty(I,L^\rho)}^{\rho-\gamma+\frac{4(1+\varepsilon)}{\rho-2}} + \|u\|_{L^\infty(I,L^\rho)}^{\rho-\gamma+2}\Big)\|u\|_{L^\gamma(I,W^{1,\rho})}^{\gamma-1}.$$

Proof. Using Strichartz estimate via the inequality $|uv|^p \leq \frac{1}{2}(|u|^{2p}+|v|^{2p})$ yields

$$\begin{aligned}
&\|(u-e^{it\Delta}u_0, v-e^{it\Delta}v_0)\|_{S(I)\times S(I)}\\
&\qquad\lesssim \|f(u)+f(v)+u^pv^p\|_{L^{\gamma'}(I,W^{1,\rho'})}\\
&\qquad\lesssim \|f(u)+|u|^{2p}\|_{L^{\gamma'}(I,W^{1,\rho'})} + \|f(v)+|v|^{2p}\|_{L^{\gamma'}(I,W^{1,\rho'})}.
\end{aligned}$$

Now, by (2.1) and (2.3), we have

$$\begin{aligned}
\|f(u) &+ |u|^{2p}\|_{W^{1,\rho'}}\\
&\lesssim \|\nabla u u^{2p-1}e^{\alpha_0(1+\varepsilon)|u|^2}\|_{L^{\rho'}} + \|u^{2p}e^{\alpha_0(1+\varepsilon)|u|^2}\|_{L^{\rho'}}\\
&\lesssim (I)+(II).
\end{aligned}$$

By (2.8),

$$\begin{aligned}
(I) &\lesssim \|\nabla u\|_{L^\rho}\|u^{2p-1}e^{\alpha_0(1+\varepsilon)|u|^2}\|_{L^{\frac{\rho}{\rho-2}}}\\
&\lesssim \|\nabla u\|_{L^\rho}\Big(\int |u|^\rho e^{\alpha_0(1+\varepsilon)|u|^2}dx\Big)^{\frac{\rho-2}{\rho}} e^{\alpha_0(1+\varepsilon)(1-\frac{\rho-2}{\rho})\|u\|_{L_x^\infty}^2}\\
&\lesssim \|\nabla u\|_{L^\rho}\|u\|_{L^\rho}^\rho e^{\alpha_0(1+\varepsilon)\frac{2}{\rho}\|u\|_{L_x^\infty}^2}.
\end{aligned}$$

Thanks to the Logarithmic inequality in Proposition 2.12,

$$\begin{aligned}
e^{\alpha_0(1+\varepsilon)\frac{2}{\rho}|u|_{L_x^\infty}^2} &\lesssim (1+\|u\|_{C^{1-\frac{2}{\rho}}})^{\frac{4(1+\varepsilon)}{\rho-2}}\\
&\lesssim (1+\|u\|_{W^{1,\rho}})^{\frac{4(1+\varepsilon)}{\rho-2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(I) &\lesssim \|\nabla u\|_{L^\rho}\|u\|_{L^\rho}^\rho(1+\|u\|_{W^{1,\rho}}^{\frac{4(1+\varepsilon)}{\rho-2}})\\
&\lesssim \|\nabla u\|_{L^\rho}\|u\|_{L^\rho}^\rho + \|u\|_{L^\rho}^\rho\|u\|_{W^{1,\rho}}^{1+\frac{4(1+\varepsilon)}{\rho-2}}.
\end{aligned}$$

Moreover, for any real number a,

$$\int_I\Big(\|u\|_{L^\rho}^\rho\|u\|_{W^{1,\rho}}^{1+\frac{4(1+\varepsilon)}{\rho-2}}\Big)^{\gamma'}dt \lesssim \|u\|_{L^\infty(I,L^\rho)}^{\gamma'(\rho-a)}\int_I\|u\|_{W^{1,\rho}}^{\gamma'(a+1+\frac{4(1+\varepsilon)}{\rho-2})}dt.$$

Taking $a+1 := \gamma-1-\frac{4(1+\varepsilon)}{\rho-2}$ yields

$$\|\|u\|_{L^\rho}^\rho\|u\|_{W^{1,\rho}}^{1+2(1+\varepsilon)\frac{\rho}{\rho-2}}\|_{L^{\gamma'}}^{\gamma'} \lesssim \|u\|_{L^\infty(I,L^\rho)}^{\gamma'(\rho-\gamma+\frac{4(1+\varepsilon)}{\rho-2})}\|u\|_{L^\gamma(I,W^{1,\rho})}^\gamma.$$

With the same way for any real number b,

$$\int_I\Big(\|u\|_{L^\rho}^\rho\|u\|_{W^{1,\rho}}\Big)^{\gamma'}dt \lesssim \|u\|_{L^\infty(I,L^\rho)}^{\gamma'(\rho-b)}\int_I\|u\|_{W^{1,\rho}}^{\gamma'(b+1)}dt.$$

Taking $b := \gamma-2$ yields

$$\|\|u\|_{L^\rho}^\rho\|u\|_{W^{1,\rho}}\|_{L^{\gamma'}}^{\gamma'} \lesssim \|u\|_{L^\infty(I,L^\rho)}^{\gamma'(\rho-\gamma+2)}\|u\|_{L^\gamma(I,W^{1,\rho})}^\gamma.$$

Thus

$$\|(I)\|_{L^{\gamma'}(I)} \lesssim \Big(\|u\|_{L^\infty(I,L^\rho)}^{\rho-\gamma+\frac{4(1+\varepsilon)}{\rho-2}} + \|u\|_{L^\infty(I,L^\rho)}^{\rho-\gamma+2}\Big)\|u\|_{L^\gamma(I,W^{1,\rho})}^{\gamma-1}.$$

Similarly,

$$\begin{aligned}(II) &\lesssim \|u\|_{L^\rho}^{1+\rho}(1+\|u\|_{W^{1,\rho}}^{2(1+\varepsilon)\frac{\rho}{\rho-2}})\\ &\lesssim \|u\|_{L^\rho}^{\rho}\|u\|_{W^{1,\rho}} + \|u\|_{L^\rho}^{\rho}\|u\|_{W^{1,\rho}}^{1+2(1+\varepsilon)\frac{\rho}{\rho-2}}.\end{aligned}$$

Finally

$$\begin{aligned}\|(u-e^{it\Delta}u_0, v-e^{it\Delta}v_0)\|_{S(I)\times S(I)} &\lesssim \|f(u)+f(v)+u^pv^p\|_{L^{\gamma'}(I,W^{1,\rho'})}\\ &\lesssim \Big(\|u\|_{L^\infty(I,L^\rho)}^{\rho-\gamma+\frac{4(1+\varepsilon)}{\rho-2}} + \|u\|_{L^\infty(I,L^\rho)}^{\rho-\gamma+2}\Big)\|u\|_{L^\gamma(I,W^{1,\rho})}^{\gamma-1}.\end{aligned}$$

□

Lemma 4.2. *We have, for any $2<p<\infty$,*

$$\lim_{t\to\infty}\|(u(t),v(t))\|_{L^p\times L^p}=0.$$

We follow the proof of the classical case [16].

Proof. Let $\chi\in C_0^\infty(\mathbb{R}^2)$ be a cut-off function, and let $(\varphi_n):=(\phi_n,\psi_n)$ be a sequence in $H^1\times H^1$ satisfying $\|\nabla\phi_n\|<\frac{4\pi}{\alpha_0}$, $\|\nabla\varphi_n\|<\frac{4\pi}{\alpha_0}$ and $\varphi_n\rightharpoonup\varphi:=(\phi,\psi)$ in $H^1\times H^1$. Let (u_n,h_n) (respectively u,h) be the solution in $C(\mathbb{R},H^1)\times C(\mathbb{R},H^1)$ to (1.1) with initial data φ_n (respectively φ). We infer that, for every $\varepsilon>0$, there exist $T_\varepsilon>0$ and $n_\varepsilon\in\mathbb{N}$ such that

$$\|\chi(u_n-u),\chi(h_n-h)\|_{L^\infty_{T_\varepsilon}(L^2\times L^2)}<\varepsilon,\quad \forall n>n_\varepsilon. \tag{4.1}$$

In fact, letting $v_n:=\chi u_n$ and $v:=\chi u$, we compute

$$\begin{aligned} i\partial_t v_n+\Delta v_n&=\Delta\chi u_n+2\nabla\chi\nabla u_n+\chi(f(u_n)+u_n|u_n|^{p-1}|v_n|^p),\quad v_n(0)=\chi\varphi_n;\\ i\partial_t v+\Delta v&=\Delta\chi u+2\nabla\chi\nabla u+\chi(f(u)+u|u|^{p-1}|v|^p),\quad v(0)=\chi\varphi.\end{aligned}$$

Denoting $w_n:=v_n-v$ and $z_n:=u_n-u$, by Strichartz estimate

$$\begin{aligned}\|w_n\|_{L^\infty_T(L^2)}\lesssim{}&\|\chi(\varphi_n-\varphi)\|+\|\Delta\chi z_n\|_{L^1_T(L^2)}+2\|\nabla\chi\nabla z_n\|_{L^1_T(L^2)}\\ &+\|\chi(f(u_n)+u_n|u_n|^{p-1}|v_n|^p-f(u)-u|u|^{p-1}|v|^p)\|_{L^1_T(L^2)}.\end{aligned}$$

Thanks to the Rellich theorem, up to subsequence extraction, we have

$$\lim_{n\to\infty}\|\chi(\varphi_n-\varphi)\|=0.$$

Moreover, by Hölder inequality, with $N:=\|u\|_{L^\infty(H^1)}+\sup_n\|u_n\|_{L^\infty(H^1)}$,

$$\begin{aligned}\|\Delta\chi z_n\|_{L^1_T(L^2)}+2\|\nabla\chi\nabla z_n\|_{L^1_T(L^2)}&\le N\Big(\|\Delta\chi\|_{L^1_T(L^4)}+2\|\nabla\chi\|_{L^1_T(L^4)}\Big)\\ &\le NT\Big(\|\Delta\chi\|_4+2\|\nabla\chi\|_4\Big)\lesssim T.\end{aligned}$$

Using (2.11) and Lemma 2.17, we have for any $\varepsilon > 0$ such that $E(u_0) \leq \frac{4\pi}{\alpha_0}$,

$$
\begin{aligned}
&(\mathcal{A})\\
&:= \|f(u_n) - f(u)\|_{L^1_T(L^2)}\\
&\lesssim \||(u_n - u)|(e^{\alpha_0(1+\varepsilon)|u_n|^2} - 1 + e^{\alpha_0(1+\varepsilon)|u|^2} - 1)\|_{L^1_T(L^2)}\\
&\lesssim \||u_n - u|(e^{\alpha_0(1+\varepsilon)|u_n|^2} - 1)\|_{L^1(L^2)} + \||u_n - u|(e^{\alpha_0(1+\varepsilon)|u|^2} - 1)\|_{L^1(L^2)}\\
&\lesssim \|u_n - u\|_{L^4(L^4)}\|e^{\alpha_0(1+\varepsilon)|u_n|^2} - 1\|_{L^{\frac{4}{3}}(L^4)} + \|u_n - u\|_{L^4(L^4)}\|e^{\alpha_0(1+\varepsilon)|u|^2} - 1\|_{L^{\frac{4}{3}}(L^4)}\\
&\lesssim \|u_n - u\|_{L^4_T(L^4)}\|e^{\alpha_0(1+\varepsilon)|u_n|^2} - 1\|_{L^{\frac{4}{3}}(L^4)} + \|u_n - u\|_{L^4_T(L^4)}\|e^{\alpha_0(1+\varepsilon)|u|^2} - 1\|_{L^{\frac{4}{3}}(L^4)}\\
&\lesssim [T^{\frac{3}{4}} + \|u_n - u\|^{\frac{3}{4}\alpha}_{L^4_T(L^4)} T^{\frac{3}{4}(1-\frac{\alpha}{4})}]\|u_n - u\|_{L^4_T(L^4)}.
\end{aligned}
$$

Arguing as previously and using (3.4), we get

$$
\begin{aligned}
(A) \quad &:= \quad \|u_n|u_n|^{p-2}|v_n|^p - u|u|^{p-2}|v|^p\|_{L^1_T(L^2)}\\
&\lesssim \quad (\|u_n\|_{H^1} + \|v_n\|_{H^1} + \|u\|_{H^1} + \|v\|_{H^1})^{2(p-1)}\|(w_n, z_n)\|_{L^\infty_T(H^1\times H^1)}T\\
&\lesssim \quad T\|(w_n, z_n)\|_{L^\infty_T(H^1\times H^1)} \lesssim T.
\end{aligned}
$$

The proof of (4.1) is achieved.
By an interpolation argument it is sufficient to prove the decay for $p = 3$. We recall the following Gagliardo–Nirenberg inequality

$$\|u(t)\|_3^3 \leq C\|u(t)\|^2_{H^1}\Big(\sup_x \|u(t)\|_{L^2(Q_1(x))}\Big), \tag{4.2}$$

where $Q_a(x)$ denotes the square centered at x whose edge has length a.
We proceed by contradiction. Assume that there exist a sequence (t_n) of positive real numbers and $\varepsilon > 0$ such that $\lim\limits_{n\to\infty} t_n = \infty$ and

$$\|u(t_n)\|_3 > \varepsilon \quad \forall n \in \mathbb{N}. \tag{4.3}$$

By (4.2) and (4.3), there exist a sequence (x_n) in $\mathbb{R}^2$ and a positive real number denoted also by $\varepsilon > 0$ such that

$$\|u(t_n)\|_{L^2(Q_1(x_n))} \geq \varepsilon \quad \forall n \in \mathbb{N}. \tag{4.4}$$

Let $\varphi_n(x) := u(t_n, x + x_n)$. Using the conservation laws, we obtain $\sup\limits_n \|\varphi_n\|_{H^1} < \infty$. Then, up to a subsequence extraction, there exists $\varphi \in H^1$ such that φ_n converges weakly to φ in H^1. By the Rellich theorem, up to a subsequence extraction, we have

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^2(Q_1(0))} = 0. \tag{4.5}$$

Now, (4.4) implies that $\|\varphi_n\|_{L^2(Q_1(0))} \geq \varepsilon$. So, using (4.5), there exists a positive real number denoted also $\varepsilon > 0$ such that

$$\|\varphi\|_{L^2(Q_1(0))} \geq \varepsilon. \tag{4.6}$$

We denote by $\bar{u} \in C(\mathbb{R}, H^1)$ the solution of (1.1) with data φ. Take a cut-off function $\chi \in C_0^\infty(\mathbb{R}^2)$ satisfying $0 \leq \chi \leq 1$, $\chi = 1$ on $Q_1(0)$ and $\mathrm{supp}(\chi) \subset Q_2(0)$.

Using a continuity argument,

$$\|\chi\bar{u}(t)\| \geq \|\bar{u}(t)\|_{L^2(Q_1(0))} \to \|\varphi\|_{L^2(Q_1(0))} \quad \text{when} \quad t \to 0.$$

So, thanks to (4.6), there exists $T > 0$ such that

$$\inf_{t\in[0,T]} \|\chi\bar{u}(t)\| \geq \frac{\varepsilon}{2}.$$

Take $u_n \in C(\mathbb{R}, H^1)$ the solution to (1.1) with data φ_n. Then

$$u_n(t,x) = u(t+t_n, x+x_n).$$

Now, by (4.1), there is a positive time denoted also T and $n_\varepsilon \in \mathbb{N}$ such that

$$\|\chi(u_n - \bar{u})\|_{L^\infty_T(L^2)} \leq \frac{\varepsilon}{4} \quad \forall n \geq n_\varepsilon.$$

Hence, for all $t \in [0,T]$ and $n \geq n_\varepsilon$,

$$\|\chi u_n(t)\| \geq \|\chi\bar{u}(t)\| - \|\chi(u_n - \bar{u})(t)\| \geq \frac{\varepsilon}{4}.$$

By the proprieties of χ and the last inequality, for all $t \in [0,T]$ and $n \geq n_\varepsilon$,

$$\|u(t+t_n)\|_{L^2(Q_2(x_n))} = \|u_n(t)\|_{L^2(Q_2(0))} \geq \frac{\varepsilon}{4}.$$

This implies that

$$\|u(t)\|_{L^2(Q_2(x_n))} \geq \frac{\varepsilon}{4} \quad \forall t \in [t_n, t_n+T] \quad \forall n \geq n_\varepsilon.$$

Since, by Hölder inequality, we have

$$\|u(t)\|_{L^2(Q_2(x_n))} \lesssim \|u(t)\|_{L^8(Q_2(x_n))};$$

then, there exists a real number $\alpha > 0$ such that

$$\|u(t)\|_{L^8(Q_2(x_n))} \geq \alpha \quad \forall t \in [t_n, t_n+T] \quad \forall n \geq n_\varepsilon.$$

Moreover, as $\lim_{n\to\infty} t_n = \infty$, we can suppose that $t_{n+1} - t_n > T$ for $n \geq n_\varepsilon$. Therefore

$$\begin{aligned}
\|u\|^4_{L^4(L^8)} &= \int_0^\infty \|u(t)\|^4_{L^8} dt \\
&\geq \sum_{n\geq n_\varepsilon} \int_{t_n}^{t_n+T} \|u(t)\|^4_{L^8} dt \\
&\geq \sum_{n\geq n_\varepsilon} \int_{t_n}^{t_n+T} \|u(t)\|^4_{L^8(Q_2(x_n))} dt \\
&\geq \sum_{n\geq n_\varepsilon} \alpha^4 T = \infty.
\end{aligned}$$

This obviously contradicts (2.10) and finishes the proof. $\square$

Finally, we are ready to prove scattering. First, with a standard Bootstrap argument, using the two previous lemmas, with the fact that

$$\min\{\gamma-2, \rho-\gamma+\frac{4}{\rho-2}, \rho-\gamma+2\}>0,$$

we have

$$w:=(u,v)\in S(\mathbb{R})\times S(\mathbb{R}).$$

Previous computation implies that

$$f(u)+f(v)+u^pv^p\in L^{\gamma'}(W^{1,\rho'}).$$

By the integral formula

$$e^{-it\Delta}u(t,x)=u_0+i\int_0^t e^{-is\Delta}\Big(f(u)+u|u|^{p-2}|v|^p\Big)\,ds.$$

Thus, by Strichartz estimate with the fact that $e^{it\Delta}$ is an isometry of H^1, we get

$$\begin{aligned}\|w(t)-w(\tau)\|_{H^1\times H^1} &= \|e^{it\Delta}\Big[\int_\tau^t e^{-is\Delta}\Big(f(u)+u|u|^{p-2}|v|^p\Big)\,ds\\ &\quad+\int_\tau^t e^{-is\Delta}\Big(f(v)+v|v|^{p-2}|u|^p\Big)\,ds\Big]\|_{H^1\times H^1}\\ &\lesssim \|f(u)+f(v)+|uv|^p\|_{L^{\gamma'}((t,\tau),W^{1,\rho'})}\overset{t,\tau\to\infty}{\longrightarrow}0.\end{aligned}$$

Scattering follows via classical arguments [3].

5. The stationary problem

This section is devoted to prove the existence of a ground state solution to (2.4); moreover, this ground state is a vector ground state for large μ.

Remark 5.1.

1/ *The proof of Theorem 2.5 is based on several lemmas;*
2/ *write, for easy notation,* $u^\lambda=u^\lambda_{\alpha,\beta}$, $K=K_{\alpha,\beta}$, $K^Q=K^Q_{\alpha,\beta}$, $K^N=K^N_{\alpha,\beta}$, $\mathcal{L}=\mathcal{L}_{\alpha,\beta}$, *and* $H=H_{\alpha,\beta}$.

Lemma 5.2. *Let* $(\alpha,\beta)\in\mathbb{R}^2_+$. *Then*

1/ $\min(\mathcal{L}H(\phi),H(\phi))\geq 0$, *for all* $(0,0)\neq\phi:=(u,v)\in H^1\times H^1$;
2/ $\lambda\mapsto H(\phi^\lambda)$ *is increasing.*

Proof. Thanks to (2.3), $H(\phi)\geq 0$. Moreover, with a direct computation

$$\begin{aligned}\mathcal{L}H(\phi) &= \mathcal{L}(1-\frac{\mathcal{L}}{\alpha+\beta})S(\phi)\\ &= \frac{-1}{\alpha+\beta}(\mathcal{L}-\alpha)(\mathcal{L}-(\alpha+\beta))S(\phi)+\alpha(1-\frac{\mathcal{L}}{\alpha+\beta})S(\phi)\\ &= \frac{-1}{\alpha+\beta}(\mathcal{L}-\alpha)(\mathcal{L}-(\alpha+\beta))S(\phi)+\alpha H(\phi).\end{aligned}$$

Now, since $(\mathcal{L}-\alpha)\|\nabla u\|^2 = (\mathcal{L}-(\alpha+\beta))\|u\|^2 = 0$, we have $(\mathcal{L}-\alpha)(\mathcal{L}-(\alpha+\beta))\|u\|_{H^1}^2 = 0$, $\mathcal{L}\int F(|u|^2)\,dx = \int(\alpha D+\beta)F(|u|^2)\,dx$, and $\mathcal{L}\int |uv|^p\,dx = (p\alpha+\beta)\int |uv|^p\,dx$. So

$$\begin{aligned}
\mathcal{L}H(\phi) &\geq \frac{1}{\alpha+\beta}\Big[(\mathcal{L}-\alpha)(\mathcal{L}-(\alpha+\beta))\int (F(|u|^2)+F(|v|^2)+\frac{2\mu}{p}|uv|^p)\,dx\Big]\\
&= \frac{\alpha}{\alpha+\beta}\int\Big([\alpha(D-1)^2+\beta(D-1)][F(|u|^2)+F(|v|^2)]\\
&\quad +\alpha(p-1)(\beta+\alpha(p-1))|uv|^p\Big)\,dx\\
&\geq 0.
\end{aligned}$$

The last inequality holds by (2.2). The last point is a consequence of the equality $\partial_\lambda H(\phi^\lambda) = \mathcal{L}H(\phi^\lambda)$. □

The next intermediate result is the following.

Lemma 5.3. *Let $\alpha > 0$, and let $\beta \in \mathbb{R}$, and let $(0,0) \neq (u_n, v_n)$ be a bounded sequence of $H^1 \times H^1$ such that $\lim_n K^Q(u_n, v_n) = 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $K(u_n, v_n) > 0$ for all $n \geq n_0$.*

Proof. We have, for some real number $p > 2$, $|rf(r)| + |F(r^2)| \lesssim r^p(e^{\alpha_0 r^2}-1)$. Indeed, by the equation (2.2), the ratio tends to zero on infinity, and using (2.3) with (2.1) the ratio is bounded near zero. Thus, for any $q \geq 1$, if $\alpha_0 q'\|v\|_{H^1}^2 < 2\pi$, by Moser–Trudinger inequality

$$\begin{aligned}
\int |v|^p(e^{\alpha_0|v|^2}-1)\,dx &\lesssim \|v\|_{qp}^p \|e^{\alpha_0|v|^2}-1\|_{q'}\\
&\lesssim \|v\|_{qp}^p \|e^{\alpha_0 q'|v|^2}-1\|_1^{\frac{1}{q'}}\\
&\lesssim \|v\|_{qp}^p.
\end{aligned}$$

Using the interpolation inequality

$$\|v\|_r \lesssim \|v\|^{\frac{2}{r}}\|\nabla v\|^{1-\frac{2}{r}} \quad \forall r \in [2,\infty), \tag{5.1}$$

we have $\|v\|_{qp}^p \lesssim \|v\|^{\frac{2}{q}}\|\nabla v\|^{p-\frac{2}{q}}$. Taking q such that $p > \frac{2}{q}+2$ yields $K^N(u_n, v_n) = o(\|\nabla u_n\|^2 + \|\nabla v_n\|^2)$ and $\|\nabla u_n\|^2 + \|\nabla v_n\|^2 \lesssim K^Q(u_n, v_n) \to 0$. Moreover, with Hölder inequality via (5.1), we have

$$\int |u_n v_n|^p\,dx \lesssim \|u_n\|\|v_n\|\|\nabla u_n\|^{p-1}\|\nabla v_n\|^{p-1} \lesssim \|u_n\|_{H^1}^p\|v_n\|_{H^1}^p.$$

Thus $K(u_n, v_n) \simeq K^Q(u_n, v_n)$. This ends the proof. □

The next lemma of this section reads.

Lemma 5.4. *Let $\alpha > 0$, and let $\beta \geq 0$. Then*

$$m_{\alpha,\beta} = \inf_{0\neq\phi\in H^1\times H^1}\{H_{\alpha,\beta}(\phi),\ s.t.\ K_{\alpha,\beta}(\phi) \leq 0\}. \tag{5.2}$$

Proof. Let m_1 be the right hand side; then it is sufficient to prove that $m_{\alpha,\beta} \leq m_1$. Take $\phi \in H^1 \times H^1$ such that $K_{\alpha,\beta}(\phi) < 0$; then, by the previous lemma and the facts that $\lim_{\lambda \to -\infty} K^Q_{\alpha,\beta}(\phi^\lambda_{\alpha,\beta}) = 0$ and $\lambda \mapsto H_{\alpha,\beta}(\phi^\lambda)$ is increasing, there exists $\lambda < 0$ such that

$$K_{\alpha,\beta}(\phi^\lambda) = 0, \ H_{\alpha,\beta}(\phi^\lambda) \leq H(\phi). \tag{5.3}$$

The proof is complete. ☐

Now, we prepare the proof of the last part of Theorem 2.5. Here and hereafter, for $\lambda > 0$ and $\phi := (u, v) \in H^1 \times H^1$ we denote $\phi_\lambda := \lambda\phi(\lambda.)$ and

$$\begin{aligned} Q(\phi) :=& K_{1,-1}(\phi) \\ =& \|\nabla u\|^2 - \int \Big(|u|f(|u|) - F(|u|^2)\Big)\, dx + \|\nabla v\|^2 \\ & - \int \Big(|v|f(|v|) - F(|v|^2)\Big)\, dx - 2\mu(1 - \frac{1}{p}) \int |uv|^p\, dx. \end{aligned}$$

Lemma 5.5. *Assume that f satisfies (2.6). Let $\phi \in H^1 \times H^1$ such that $Q(\phi) \leq 0$. Then, there exists $\lambda_0 \leq 1$ such that*

1/ $Q(\phi_{\lambda_0}) = 0$;
2/ $\lambda_0 = 1$ *if and only if* $Q(\phi) = 0$;
3/ $\frac{\partial}{\partial\lambda} S(\phi_\lambda) > 0$ *for* $\lambda \in (0, \lambda_0)$, *and* $\frac{\partial}{\partial\lambda} S(\phi_\lambda) < 0$ *for* $\lambda \in (\lambda_0, \infty)$;
4/ $\lambda \to S(\phi_\lambda)$ *is concave on* (λ_0, ∞);
5/ $\frac{\partial}{\partial\lambda} S(\phi_\lambda) = \frac{1}{\lambda} Q(\phi_\lambda)$.

Proof. We have

$$\begin{aligned} Q(\phi_\lambda) \ &= \ \int |\nabla u_\lambda(x)|^2\, dx - \int \Big(|u_\lambda|f(|u_\lambda|) - F(|u_\lambda|^2)\Big)\, dx + \int |\nabla v_\lambda(x)|^2\, dx \\ & \quad - \int \Big(|v_\lambda|f(|v_\lambda|) - F(|v_\lambda|^2)\Big)\, dx - 2\mu(1 - \frac{1}{p}) \int |u_\lambda v_\lambda|^p\, dx \\ &= \ \lambda^2\|\nabla u\|^2 - \lambda^{-2} \int \Big(\lambda|u|f(\lambda|u|) - F(|\lambda u|^2)\Big)\, dx + \lambda^2\|\nabla v\|^2 \\ & \quad -\lambda^{-2} \int \Big(\lambda|v|f(\lambda|v|) - F(|\lambda v|^2)\Big)\, dx - 2\mu(1 - \frac{1}{p})\lambda^{2(p-1)} \int |uv|^p\, dx. \end{aligned}$$

Moreover, with the previous computations

$$\begin{aligned} & \frac{1}{2}\partial_\lambda S(\phi_\lambda) \\ & = \lambda\|\nabla u\|^2 + 2\lambda^{-3} \int F(|\lambda u|^2) - 2\lambda^{-2} \int \lambda|u|f(\lambda|u|)\, dx + \lambda\|\nabla v\|^2 \\ & \quad + 2\lambda^{-3} \int F(|\lambda v|^2) - 2\lambda^{-2} \int \lambda|v|f(\lambda|v|)\, dx - 2\mu(1 - \frac{1}{p})\lambda^{2p-3} \int |uv|^p\, dx \\ & = \frac{1}{\lambda} Q(\phi_\lambda). \end{aligned}$$

which proves (5). Now

$$\begin{aligned} Q(\phi_\lambda) &= \lambda^2\|\nabla u\|^2 - \lambda^{-2}\int \Big(\lambda|u|f(\lambda|u|) - F(|\lambda u|^2)\Big)\,dx + \lambda^2\|\nabla v\|^2 \\ &\quad - \lambda^{-2}\int \Big(\lambda|v|f(\lambda|v|) - F(|\lambda v|^2)\Big)\,dx - 2\mu(1-\frac{1}{p})\lambda^{2(p-1)}\int |uv|^p\,dx \\ &= \lambda^2\Big[\|\nabla u\|^2 - \int |u|^4\Big(\frac{\lambda|u|f(\lambda|u|) - F(|\lambda u|^2)}{|\lambda u|^4}\Big)\,dx + \|\nabla v\|^2 \\ &\quad - \int |v|^4\Big(\frac{\lambda|v|f(\lambda|v|) - F(|\lambda v|^2)}{|\lambda v|^4}\Big)\,dx - 2\mu(1-\frac{1}{p})\lambda^{2(p-2)}\int |uv|^p\,dx\Big] \\ &= \lambda^2\Big(\|\nabla u\|^2 + \|\nabla v\|^2 - \int [|v|^4 h(\lambda|v|)\,dx \\ &\quad + |u|^4 h(\lambda|u|) - 2\mu(1-\frac{1}{p})\lambda^{2(p-2)}|uv|^p]\,dx\Big), \end{aligned}$$

where $s^4h(s) := sf(s) - F(s^2)$. We claim that h is strictly increasing. In fact, for $r := s^2$, we have $h'(r) = \frac{(D^2-3D+2)F(r)}{r^3} > 0$ because of (2.6). Which complete the proof of (1), (2), and (3). For (4), it is sufficient to compute using (3). $\square$

In the case $(\alpha,\beta) = (1,-1)$, we will use $T := S - K_{1,-1}$ rather than $H_{\alpha,\beta}$ which is no longer defined.

Lemma 5.6. *Assume that f satisfies (2.6). Then, for $\phi \in H^1\times H^1$, the following real function is increasing on $\mathbb{R}_+$,*

$$\lambda \mapsto T(\lambda\phi).$$

Proof. Denoting $\phi := (u,v)$, we have

$$\begin{aligned} T(\lambda\phi) &= \lambda^2(\|u\|^2 + \|v\|^2) + \int \Big((D-2)F(\lambda^2|u|^2) + (D-2)F(\lambda^2|v|^2)\Big)\,dx \\ &\quad + 2\mu(1-\frac{1}{p})\lambda^{2p}\|uv\|_p^p; \end{aligned}$$

$$\begin{aligned} \partial_\lambda T(\phi_\lambda) &= 2\lambda(\|u\|^2 + \|v\|^2) + 2\int |u|\Big([D^2-2D]F(|\lambda u|^2) \\ &\quad + [D^2-2D]F(|\lambda v|^2)\Big)\,dx + 4\mu(p-1)\lambda^{2p-1}\|uv\|_p^p. \end{aligned}$$

The proof is ended because $[D^2-2D]F = [(D-2)^2+2(D-2)]F > 0$ via (2.6). $\square$

Like in Lemma 5.4, we can express the minimizing number $m_{1,-1}$ with a negative constraint.

Proposition 5.7. *Assume that f satisfies (2.6). Then*

$$m_{1,-1} = \inf_{0\neq\phi\in H^1\times H^1}\{T(\phi), \quad K_{1,-1}(\phi) \le 0\}.$$

Proof. Let m_1 be the right hand side; then it is sufficient to prove that $m_{1,-1} \le m_1$. Take $\phi \in H^1\times H^1$ such that $K_{1,-1}(\phi) < 0$; then by Lemma 5.5 and the fact that $\lambda \mapsto T(\lambda\phi)$ is increasing, there exists $\lambda \in (0,1)$ such that $K_{1,-1}(\lambda\phi) = 0$ and $m_{1,-1} \le T(\lambda\phi) \le T(\phi)$. The proof is completed. $\square$

Proof of Theorem 2.5

• Case $\alpha \geq 0$, $\beta \geq 0$.

Let $\phi_n := (u_n, v_n)$ be a minimizing sequence, namely

$$(0,0) \neq \phi_n \in H^1 \times H^1,\ K(\phi_n) = 0,\ \text{and}\ \lim_n H(\phi_n) = \lim_n S(\phi_n) = m. \qquad (5.4)$$

With a rearrangement argument via (5.3), we can assume that u_n and v_n are radial decreasing and satisfy (5.4).

•First step :(ϕ_n) is bounded in $H^1 \times H^1$.

Assume as a first subcase that $\alpha = 0$. Without loss of generality, take $\beta = 1$. By (5.4) via the definition of $H_{0,1}$, (ϕ_n) is bounded in $\dot{H}^1 \times \dot{H}^1$. Now, because

$$K_{0,1}(u_n(\frac{\cdot}{\|u_n\|}), v_n(\frac{\cdot}{\|u_n\|})) = 0, \quad H_{0,1}(u_n(\frac{\cdot}{\|u_n\|}), v_n(\frac{\cdot}{\|u_n\|})) = H_{0,1}(\phi_n),$$

by the scaling $(u_n(\frac{\cdot}{\|u_n\|}), v_n(\frac{\cdot}{\|u_n\|}))$, we may assume that $\|u_n\| = \|v_n\| = 1$. Thus, (ϕ_n) is bounded in $H^1 \times H^1$.

Now, assume as a second subcase that $\beta\alpha \neq 0$, and denote $\lambda := \frac{\beta}{\alpha} \neq 0$. By (5.4), we get

$$\begin{aligned}\|\phi_n\|_{H^1}^2 - \int [|u_n| f(|u_n|) + |v_n| f(|v_n|)] dx - 2\mu \|u_n v_n\|_p^p \\ = \lambda (\int [F(|u_n|^2) + F(|v_n|^2)] dx - \|\phi_n\|^2 + \frac{2\mu}{p} \|u_n v_n\|_p^p);\end{aligned}$$

$$\|\phi_n\|_{H^1}^2 - \int [F(|u_n|^2) + F(|v_n|^2)]\, dx - \frac{2\mu}{p} \|u_n v_n\|_p^p \to m.$$

So, the following sequences are bounded

$$\Big(-\lambda \|\nabla \phi_n\|^2 + \|\phi_n\|_{H^1}^2 - \int [|u_n| f(|u_n|) + |v_n| f(|v_n|)]\, dx - 2\mu \|u_n v_n\|_p^p \Big);$$
$$\Big(\|\phi_n\|_{H^1}^2 - \int [F(|u_n|^2) + F(|v_n|^2)]\, dx - \frac{2\mu}{p} \|u_n v_n\|_p^p \Big).$$

Thus, for any real number a, the following sequence is also bounded

$$\Big(\lambda \|\nabla \phi_n\|^2 + (a-1)\|\phi_n\|_{H^1}^2 + \int [(D-a)F(|u_n|^2) + (D-a)F(|v_n|^2)]\, dx + 2\mu(1 - \frac{a}{p}) \|u_n v_n\|_p^p \Big).$$

Now, taking $a = 1 + \varepsilon_f < p$ and using the assumption (2.2), it follows that (ϕ_n) is bounded in $H^1 \times H^1$.

Assume as a last subcase that $\beta = 0$. Without loss of generality, we can take $\alpha = 1$. Then

$$0 = K(\phi_n) = \|u_n\|_{H^1}^2 - \int |u_n| f(|u_n|)\, dx + \|v_n\|_{H^1}^2 - \int |v_n| f(|v_n|)\, dx - 2\mu \|u_n v_n\|_p^p;$$
$$S(\phi_n) = \|u_n\|_{H^1}^2 - \int F(|u_n|^2)\, dx + \|v_n\|_{H^1}^2 - \int F(|v_n|^2)\, dx - \frac{2\mu}{p} \|u_n v_n\|_p^p.$$

Thus

$$\|u_n\|_{H^1}^2 - \int |u_n|f(|u_n|)\,dx + \|v_n\|_{H^1}^2 - \int |v_n|f(|v_n|)\,dx = 2\mu\|u_nv_n\|_p^p;$$
$$\|u_n\|_{H^1}^2 - \int F(|u_n|^2)\,dx + \|v_n\|_{H^1}^2 - \int F(|v_n|^2)\,dx - \frac{2\mu}{p}\|u_nv_n\|_p^p \to m.$$

It follows that, for any real number $a \neq 0$,

$$2\mu(a-\frac{1}{p})\|u_nv_n\|_p^p+(1-a)[\|u_n\|_{H^1}^2+\|v_n\|_{H^1}^2]+a\int[D-\frac{1}{a}][F(|u_n|^2)+F(|v_n|^2)]\,dx \to m.$$

Now taking $a = \frac{1}{1+\varepsilon_f}$ yields via (2.2),

$$\sup_n \|\phi_n\|_{H^1\times H^1} \lesssim 1.$$

Taking account of the compact injection of the radial Sobolev space $H^1_{rd}(\mathbb{R}^2)$ in the Lebesgue space $L^p(\mathbb{R}^2)$ for any $2 < p < \infty$, we take

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } H^1 \times H^1 \text{ and } (u_n, v_n) \to (u, v) \text{ in } L^p,\ \forall p \in (2, \infty).$$

• Second step:Put $\phi := (u, v) \neq (0, 0)$ and $m_{\alpha,\beta} > 0$.
Assume that $\phi = 0$. First case $\alpha \neq 0$.
Using Moser–Trudinger inequality, we have for any Hölder couple (r, r'), such that $r\alpha_0 < 4\pi$ is near to one

$$\begin{aligned}\int \Big[F(|\phi_n|^2) + |\phi_n|f(|\phi_n|)\Big]dx &\lesssim \|\phi_n^p(e^{\alpha_0|\phi_n|^2} - 1)\|_1\\ &\lesssim \|\phi_n\|_{r'p}^p\|e^{r\alpha_0|\phi_n|^2} - 1\|_1^{\frac{2}{r}}\\ &\lesssim \|\phi_n\|_{r'p}^p\|\phi_n\|^{\frac{2}{r}} \to 0.\end{aligned}$$

Thus, $K^N(\phi_n) \longrightarrow 0$ as $n \to \infty$. By Lemma 5.3, there exists $n_0 > 0$ such that $K(\phi_n) > 0$ for large n, which is absurd. So

$$\phi \neq 0.$$

Second case $\alpha = 0$.
By the Moser–Trudinger inequality via (2.2) with (2.3), the mean value theorem, the convexity of the exponential function, and Hölder inequality

$$\begin{aligned}\int |F(|u|^2) - F(|u_n|^2)|\,dx &\lesssim \int |u-u_n|(|u_n|+|u|)(e^{\alpha_0|u_n|^2} - 1 + e^{\alpha_0|u|^2} - 1)\,dx\\ &\lesssim \|u-u_n\|_{2r'}(\|u_n\|_{2r'} + \|u\|_{2r'})(\|u_n\|^{\frac{2}{r}} + \|u\|^{\frac{2}{r}}).\end{aligned}$$

Thus

$$\int F(|u_n|^2)\,dx \longrightarrow \int F(|u|^2)\,dx.$$

Now, because

$$K_{0,1}(u_n(\frac{\cdot}{\|u_n\|}), v_n(\frac{\cdot}{\|u_n\|})) = 0, \quad H_{0,1}(u_n(\frac{\cdot}{\|u_n\|}), v_n(\frac{\cdot}{\|u_n\|})) = H_{0,1}(\phi_n),$$

by the scaling $(u_n(\frac{\cdot}{\|u_n\|}), v_n(\frac{\cdot}{\|u_v\|}))$, we may assume that $\|u_n\| = \|v_n\| = 1$. Thus

$$0 = K_{0,1}(u_n, v_n) = 2 - \int F(|u_n|^2)\,dx - \int F(|v_n|^2)\,dx - \frac{2\mu}{p}\int |u_n v_n|^p\,dx.$$

On the other hand, since $p > 2$,

$$\|u_n v_n\|_p^p \le \|u_n\|_{2p}^p \|v_n\|_{2p}^p \to \|u\|_{2p}^p \|v\|_{2p}^p = 0.$$

Taking the limit in $K_{0,1}(u_n, v_n)$ when n goes to infinity yields to the contradiction

$$2 = \int \Big(F(|u|^2) + F(|v|^2)\Big)\,dx = 0.$$

Thus

$$\phi \ne 0.$$

This implies that $H(u, v) > 0$. Moreover, with lower semicontinuity of the H^1 norm, we have

$$m = \liminf_n H_{0,1}(u_n, v_n) \ge H_{0,1}(u, v) > 0.$$

• Third step: The limit $\phi = (u, v)$ is a solution to (2.4).
First case $\alpha \ne 0$. There is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'(\phi) = \eta K'(\phi)$. Then

$$\begin{aligned} 0 = K(\phi) &= \mathcal{L}S(\phi) = \langle S'(\phi), \mathcal{L}(\phi)\rangle \\ &= \eta\langle K'(\phi), \mathcal{L}(\phi)\rangle \\ &= \eta\mathcal{L}K(\phi) = \eta\mathcal{L}^2 S(\phi). \end{aligned}$$

With a previous computation and taking account of the equation of (2.2),

$$\begin{aligned} -\mathcal{L}^2 S(\phi) - \alpha(\alpha+\beta)S(\phi) &= -(\mathcal{L} - (\alpha+\beta))(\mathcal{L} - \alpha)S(\phi) \\ &= \alpha\int \Big(\alpha(D-1)^2 + \beta(D-1)\Big)F(|u|^2)\,dx \\ &\quad +\alpha\int \Big(\alpha(D-1)^2 + \beta(D-1)\Big)F(|v|^2)\,dx \\ &\quad +\frac{2\mu}{p}4\alpha(\alpha(p-1)+\beta)(p-1)\|uv\|_p^p \\ &> 0. \end{aligned}$$

Thus $\eta = 0$, and $S'(\phi) = 0$. So, ϕ is a ground state, and m is independent of α and β.
Second case $\alpha = 0$.
Without loss of generality, we assume that $\beta = 1$. With a Lagrange multiplicator $\eta \in \mathbb{R}$, we have $S'(\phi) = \eta K'(\phi)$. Then

$$\begin{cases} -\Delta u = (\eta - 1)\Big(u - f(u) - \mu u|u|^{p-2}|v|^p\Big); \\ -\Delta v = (\eta - 1)\Big(v - f(v) - \mu v|v|^{p-2}|u|^p\Big). \end{cases}$$

This implies that

$$\|\nabla\phi\|^2 = (\eta-1)\Big(K_{0,1}(\phi) - \|\phi\|^2 - \int[(D-1)F(|u|^2)+(D-1)F(|v|^2)]\,dx + (\frac{2}{p}-1)\mu\|uv\|_p^p\Big)$$
$$= (\eta-1)\Big(-\|\phi\|^2 - \int[(D-1)F(|u|^2)+(D-1)F(|v|^2)]\,dx + \mu(\frac{2}{p}-1)\int|u|^p|v|^p\,dx\Big).$$

Using the facts that $(D-1)F \geq 0$ and $p > 2$, we have $\eta < 1$. Take $1-\eta := a^2$; then $\phi_a := \phi(\frac{\cdot}{a})$ is a solution to (2.4). Finally, we have a ground state solution to (1.1), which concludes the proof.

• Case $\alpha = 1$ and $\beta = -1$.

Let $(\phi_n) := (u_n, v_n)$ be a minimizing sequence, namely

$$0 \neq \phi_n \in H^1\times H^1,\ K_{1,-1}(\phi_n) = 0,\ \text{and}\ \lim_n S(\phi_n) = m.$$

With a rearrangement argument, we can assume that ϕ_n is radial decreasing and satisfies

$$0 \neq \phi_n \in H^1\times H^1,\ K_{1,-1}(\phi_n) \leq 0,\ \text{and}\ \lim_n S(\phi_n) \leq m.$$

We can suppose that ϕ_n is radial decreasing and satisfies (5.4). Indeed, by Lemmas 5.5–5.6, there exists $\lambda \in (0,1)$ such that $K_{1,-1}(\lambda\phi_n) = 0$ and $T(\lambda\phi_n) \leq m$. Then

$$\|\nabla u_n\|^2 + \|\nabla v_n\|^2 - \int(D-1)(F(|u_n|^2)+F(|v_n|^2))\,dx = 2\mu(1-\frac{1}{p})\|u_nv_n\|_p^p;$$
$$\|u_n\|_{H^1}^2 + \|v_n\|_{H^1}^2 - \int(F(|u_n|^2)+F(|v_n|^2))\,dx - \frac{2\mu}{p}\|u_nv_n\|_p^p \to m.$$

So, for any real number $a \neq 0$,

$$\int\Big((1-a)(|\nabla u_n|^2+|\nabla v_n|^2) + |u_n|^2 + |v_n|^2 + a[D-1-\frac{1}{a}][F(|u_n|^2)+F(|v_n|^2)] + 2\mu(a-\frac{1+a}{p})|u_nv_n|^p\Big)\,dx \to m.$$

Letting $\frac{1}{a} = 1+\varepsilon_f$ yields via (2.6) that (ϕ_n) is bounded in $H^1\times H^1$. So, taking account of the compact injection of the radial Sobolev space H^1_{rd} in the Lebesgue space L^p for any $2 < p < \infty$, we take

$$\phi_n \rightharpoonup \phi := (u,v) \quad \text{in} \quad H^1\times H^1 \text{ and } \phi_n \to \phi \quad \text{in} \quad L^p\times L^p,\ \forall p \in (2,\infty). \tag{5.5}$$

Assume, by contradiction, that $\phi = 0$. Since (u_n) is bounded in H^1, using Moser–Trudinger inequality and taking account of (2.2) and (2.3), we get, for some $p > 2$,

$$\begin{aligned}\max\{\int |F(|u_n|^2)|\,dx, \int |u_n|f(u_n)|\,dx\} &\lesssim \|(u_n)^p(e^{\alpha_0|u_n|^2}-1)\|_1\\ &\lesssim \|u_n\|_{r'p}^p \|e^{r\alpha_0|u_n|^2}-1\|_1^{\frac{1}{r}}\\ &\lesssim \|u_n\|_{r'p}^p\|u_n\|^{\frac{2}{r}} \to 0.\end{aligned}$$

By Lemma 5.3, we get the absurdity $K_{1,-1}(\phi_n) > 0$. Thus

$$\phi \neq 0.$$

Thanks to lower semicontinuity of H^1 norm, we have $K_{1,-1}(\phi) \leq 0$ and $S(\phi) \leq m$. Using Lemma 5.6, we can assume that $K_{1,-1}(\phi) = 0$ and that $S(\phi) \leq m$. So ϕ is a minimizer satisfying

$$0 \neq \phi \in H^1_{rd} \times H^1_{rd}, \quad K_{1,-1}(\phi) = 0, \quad \text{and} \quad S(\phi) = m.$$

This implies, via the assumption (2.6), that

$$0 < \|\phi\|^2 \leq T(\phi) = m.$$

Now, there is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'(\phi) = \eta K'_{1,-1}(\phi)$.

Recall that

$$\mathcal{L}(\phi) := \partial_\lambda(e^\lambda\phi(e^\lambda.))_{|\lambda=0} := (\partial_\lambda\phi^\lambda)_{|\lambda=0}$$

and

$$\mathcal{L}S(\phi) := (\partial_\lambda S(\phi^\lambda_{\alpha,\beta}))_{|\lambda=0}.$$

We have

$$\begin{aligned}0 = K_{1,-1}(\phi) &= \mathcal{L}_{1,-1}S(\phi) = \langle S'(\phi), \mathcal{L}_{1,-1}(\phi)\rangle\\ &= \eta\langle K'_{1,-1}(\phi), \mathcal{L}_{1,-1}(\phi)\rangle\\ &= \eta\mathcal{L}_{1,-1}K_{1,-1}(\phi) = \eta\mathcal{L}^2_{1,-1}S(\phi).\end{aligned}$$

With a direct computation and taking account of (2.2), we have $\mathcal{L}_{1,-1}(\|u\|^2) = 0, (\mathcal{L}_{1,-1}-1)(\|\nabla u\|^2) = 0$ and $\mathcal{L}_{1,-1}(F(|u|^2)) = (D-1)F(|u|^2)$. Hence

$$\begin{aligned}-(\mathcal{L}_{1,-1}-1)^2S(\phi) = \|\phi\|^2 + \int [&(D-2)^2F(|u|^2) + (D-2)^2F(|v|^2)\\ &+ 2\mu(1-\frac{2}{p})|uv|^p]\,dx > 0.\end{aligned}$$

Then, $-\mathcal{L}^2_{1,-1}S(\phi) - S(\phi) > 0$; so $\eta = 0$ and $S'(\phi) = 0$. Finally, ϕ is a ground state solution to (2.4).

The last step is to show that $\phi := (u,v)$ is a vector ground state for large μ. First, note that if $(u,0)$ and $(0,v)$ are two ground state solutions to (2.4), then (u,v) is a ground state to (2.4) with $\mu = 0$. Assume that (u,v) is a ground state solution to (2.4) which is not a vector ground state. Compute for $\lambda := \frac{1}{t}$,

$$K_{0,1}(u_\lambda, v_\lambda) = \|u\|^2 + \|v\|^2 - t^2\int\Big(F(|\frac{u}{t}|^2) + F(|\frac{v}{t}|^2) + \frac{2\mu}{pt^{2p-2}}|uv|^p\Big)\,dx.$$

Then

$$\lim_{t\to\infty} K_{0,1}(u_\lambda, v_\lambda) = \|u\|^2 + \|v\|^2 > 0.$$

Moreover, because $\lim_{t\to 0} K_{0,1}(u_\lambda, v_\lambda) = -\infty$, there exists $\bar{t} > 0$ such that

$$K_{0,1}(u_{\bar{\lambda}}, v_{\bar{\lambda}}) = 0.$$

Hence

$$\|u\|^2 + \|v\|^2 = \bar{t}^2 \int \Big(F(|\frac{u}{\bar{t}}|^2) + F(|\frac{v}{\bar{t}}|^2) + \frac{2\mu}{p\bar{t}^{2p-2}}|uv|^p\Big)\,dx$$

and

$$\begin{aligned} S(u_{\bar{\lambda}}, v_{\bar{\lambda}}) &= \frac{1}{\bar{t}^2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \|u\|^2 + \|v\|^2 \\ &\quad - \bar{t}^2 \int \Big(F(|\frac{u}{\bar{t}}|^2) + F(|\frac{v}{\bar{t}}|^2) + \frac{2\mu}{p\bar{t}^{2p-2}}|uv|^p\Big)\,dx \\ &= \frac{1}{\bar{t}^2}(\|\nabla u\|^2 + \|\nabla v\|^2). \end{aligned}$$

Now, since

$$\lim_{\mu\to\infty} \bar{t} = \infty,$$

for large μ, if we assume for example that $u = 0$, we get the following absurdity which ends the proof

$$S(u_{\bar{\lambda}}, v_{\bar{\lambda}}) \to 0 < m = S(0, v).$$

6. Proof of Theorem 2.6

This section is devoted to obtain global or nonglobal existence of a solution to system (1.1). We start with a classical auxiliary result about stable sets.

Lemma 6.1. *The sets $A^+_{\alpha,\beta}$ and $A^-_{\alpha,\beta}$ are invariant under the flow of* (1.1).

Proof. Let $(u_0, v_0) \in A^+_{\alpha,\beta}$, and let $(u, v) \in C_{T^*}(H^1) \times C_{T^*}(H^1)$ be the maximal solution to (1.1). Assume that for some time $t_0 \in (0, T^*)$, we have $(u(t_0), v(t_0)) \notin A^+_{\alpha,\beta}$. Since the energy is conserved, $K_{\alpha,\beta}(u(t_0)) < 0$. So, with a continuity argument, there exists a positive time $t_1 \in (0, t_0)$ such that $K_{\alpha,\beta}(u(t_1)) = 0$ and $S(u(t_1)) < m$. This contradicts the definition of m. The proof is similar in the case of $A^-_{\alpha,\beta}$. □

Let us prove the main result of this section.

Proof of Theorem 2.6. 1/ With a translation argument, we assume that $t_0 = 0$. Thus, $S(u_0, v_0) < m$, and with Lemma 6.1, $(u(t), v(t)) \in A^-_{\alpha,\beta}$ for any $t \in [0, T^*)$. By contradiction, assume that $T^* = \infty$. Take the real function $Q(t) := \int |x|^2(|u(t)|^2 + |v(t)|^2)\,dx$. With Lemma 2.14, we get

$$\begin{aligned} \frac{1}{8}Q''(t) &= \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 - \int (D-1)[F(|u|^2) + G(|v|^2)]\,dx \\ &\quad - 2\mu(1 - \frac{1}{p})\|uv\|_p^p = K_{1,-1}(u, v) < 0. \end{aligned}$$

We infer that there exists $\delta > 0$ such that $K_{1,-1}(u(t), v(t)) < -\delta$ for large time. Otherwise, there exists a sequence of positive real numbers $t_n \to +\infty$ such that $K_{1,-1}(u(t_n), v(t_n)) \to 0$. Proposition 5.7 yields

$$\begin{aligned} m &\le (S-K)(u(t_n), v(t_n)) \\ &= S(u_0, v_0) - K(u(t_n), v(t_n)) \to S(u_0, , v_0) < m. \end{aligned}$$

This absurdity finishes the proof of the claim. Thus $Q'' < -8\delta$. Integrating twice, Q becomes negative for some positive time. This contradiction completes the proof.

2/ By Lemma 6.1, $u(t) \in A^+_{\alpha,\beta}$ for any $t \in [0, T^*)$. Thus

$$\begin{aligned} m &> (S - \frac{1}{2}K_{1,1})(u, v) \\ &= H_{1,1}(u, v) \\ &\ge \frac{1}{2}\Big[\|\nabla u\|^2 + \|\nabla v\|^2 + \int (D-1)[F(|u|^2) + F(|v|^2)]\, dx\Big] \\ &\ge \frac{1}{2}(\|\nabla u\|^2 + \|\nabla v\|^2). \end{aligned}$$

Thus $(u(t), v(t))$ is bounded in $\dot{H}^1 \times \dot{H}^1$. Precisely

$$\sup_{0\le t\le T^*} \Big(\|\nabla u(t)\| + \|\nabla v(t)\|\Big) \le 2m.$$

Moreover, the L^2 norm is conserved; so $(\|u(t)\|, \|v(t)\|) = (\|u_0\|, \|v_0\|)$, and

$$\sup_{0\le t\le T^*} \Big(\|u(t)\|_{H^1} + \|v(t)\|_{H^1}\Big) < \infty.$$

Thus $T^* = \infty$. The scattering proof is omitted because it is similar to the defocusing case via the previous global bound of the $H^1 \times H^1$ norm.

□

7. Proof of Theorem 2.7

In this section, we establish strong instability of standing waves. We take $\epsilon = 1$, $\mu \ge 0$, and $\varphi := (\phi, \psi)$ a ground state solution to (2.4). Let $\lambda > 1$ be close to one and $\varphi_\lambda := (\phi_\lambda, \psi_\lambda)$. We keep the notation of the previous section $Q := K_{1,-1}$. Using Theorem 2.6, it is sufficient to prove that $\varphi_\lambda \in A^-_{1,-1}$ and use the fact that $A^-_{\alpha,\beta}$ is independent of (α, β). Because $\lambda > 1$, using Lemma 5.5, yields $S(\varphi_\lambda) < S(\varphi) = m$ and $Q(\varphi_\lambda) < 0$; so $\varphi_\lambda \in A^-_{1,-1}$. This finishes the proof.

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University of Tunis El Manar, Faculty of Sciences of Tunis, LR03ES04 partial differential equations and applications, 2092 Tunis, Tunisia.

E-mail address: hezzi_82hanen@yahoo.fr