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RANK TESTS FOR INDEPENDENCE FOR BIVARIATE CENSORED DATA[1]

BY DOROTA M. DABROWSKA

Carnegie–Mellon University

The paper discusses statistics that can be used to test whether two failure times, say X_1 and X_2, are independent. The two variables are subject to right censoring so that what is observed is $Y_i = \min(X_i, Z_i)$ and $\delta_i = I(X_i = Y_i)$, where (Z_1, Z_2) are censoring times independent of (X_1, X_2). Statistics that generalize the Spearman rank correlation and the log-rank correlation are considered, as well as general linear rank statistics. The Chernoff–Savage approach is adopted to show that suitably standardized versions of these statistics are asymptotically normal under both fixed and converging alternatives.

1. Introduction. Let $X_n = (X_{1n}, X_{2n})$ and $Z_n = (Z_{1n}, Z_{2n})$, $n = 1, \ldots, N$, be mutually independent sets of nonnegative bivariate random variables (rv) defined on a common probability space $(\Omega, \mathscr{F}, P)$. The X_n's and Z_n's are independent identically distributed (iid) rv's with continuous joint distribution functions (cdf) F and G, respectively, and marginal cdf's F_1, F_2 and G_1, G_2. For each $n = 1, \ldots, N$, the observable rv's are given by $Y_n = (Y_{1n}, Y_{2n})$ and $\delta_n = (\delta_{1n}, \delta_{2n})$, where $Y_{in} = \min(X_{1n}, Z_{in})$, $\delta_{in} = I(X_{in} = Y_{in})$, and $I(A)$ is the indicator function of the set A. The variables X_{1n} and X_{2n} are thought of as survival or failure times, and may represent lifetimes of twins or married couples, times from initiation of a treatment until first response in two successive courses of treatment in the same patient, etc. For each subject we observe his survival time X_{in} or censoring time Z_{in}, $i = 1, 2$, whichever occurs first, together with an rv δ_{in} indicating if he has left the study due to death or withdrawal. Further discussion of this type of censoring can be found in Campbell (1981, 1982), Clayton (1978), Hanley and Parnes (1983), Langberg and Shaked (1982), and Leurgans et al. (1982).

This paper deals with the problem of testing the hypothesis of independence of survival times $\mathscr{H}_0$: $F = F_1 F_2$. For uncensored data, tests for independence are often based on rank statistics of the form $\sum_{n=1}^{N} a(R_{1n}, R_{2n})$, where R_{1n} and R_{2n} are ranks of X_{1n}'s and X_{2n}'s and $a(i, j)$ is a real valued function. The scores a are typically generated by some functions $\mathscr{J}(u, v)$ on the unit square by taking expectations $a(i, j) = E\mathscr{J}(U_{(i)}, V_{(j)})$, where $U_{(1)} < \cdots < U_{(N)}$ and $V_{(1)} < \cdots < V_{(N)}$ are independent ordered samples from the uniform distribution on (0, 1) [see Shirahata (1974) and Ruymgaart (1973)].

Censored-data ranks of X_{in}'s can be defined as in Prentice (1978) and Kalbfleisch and Prentice (1980). Let N_i be the number of uncensored observa-

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tions among Y_{in}'s, $i = 1, 2$, and let $R_i = (R_{i1}, \dots, R_{iN})$ be given by $R_{in} = r_{in}$, where $r_{in} = \#\{m: Y_{im} \le Y_{in}, \delta_{im} = 1\}$. Here uncensored observations are ranked among themselves whereas censored observations are assigned the same rank as the nearest uncensored observation on the left. For each $d = (d_1, d_2)$, $d_i = 0$ or 1, let $A_d = \{n: \delta_{1n} = d_1, \delta_{2n} = d_2\}$. The censored data ranks of X_{in}'s are thought of as the collection of all possible rankings of X_{in}'s that are compatible with the observed values $R_{in} = r_{in}$ and A_d.

Let us assume that the joint and marginal distributions of $X_n = (X_{1n}, X_{2n})$ have densities (pdf) $f_\theta(s, t)$, $f_{\theta_1}(s)$, and $f_{\theta_2}(t)$, where the parameter θ belongs to an open subset $\Theta \subset R$ containing the origin and the hypothesis of independence is equivalent to $\mathscr{H}_0$: $\theta = 0$. Then, in the uncensored version of the experiment, the joint probability of the set of censored-data ranks of X_{in}'s is given by

$$(1.1) \qquad (N_1! N_2!)^{-1} E \prod_d \prod_{A_d} \Phi_d\Big(F_{01}^{-1}\big(U_{(r_{1n})}\big), F_{02}^{-1}\big(V_{(r_{2n})}\big); \theta\Big),$$

where $U_{(1)} < \cdots < U_{(N_1)}$ and $V_{(2)} < \cdots < V_{(N_2)}$ are independent ordered samples of sizes N_1 and N_2 from the uniform distribution on (0, 1) and

$$\begin{aligned} \Phi_d(s, t; \theta) &= f_{01}^{-1}(s) f_{02}^{-1}(t) f_\theta(s, t) \quad \text{if } d = (1,1), \\ &= f_{01}^{-1}(s) \int_t^\infty f_\theta(s, v)\, dv \quad \text{if } d = (1,0), \\ &= f_{02}^{-1}(t) \int_s^\infty f_\theta(u, t)\, du \quad \text{if } d = (0,1), \\ &= \int_s^\infty \int_t^\infty f_\theta(u, v)\, du\, dv \quad \text{if } d = (0,0). \end{aligned}$$

A locally most powerful rank test (LMPRT) for $\mathscr{H}_0$: $\theta = 0$ against $\mathscr{H}_1$: $\theta > 0$ can be based on the score statistic from (1.1). The term "LMPRT" refers here to the rank test that is LMPRT in the uncensored version of the experiment, given the observed pattern of deaths and withdrawals. A straightforward calculation shows that, under suitable regularity conditions (Hájek and Šidák, 1967, page 70), this test is based on a statistic $\sum_{n=1}^N a(R_{1n}, R_{2n}, \delta_{1n}, \delta_{2n})$, where

$$(1.2) \qquad \begin{aligned} a(i, j, d_1, d_2) &= E\mathscr{J}\big(U_{(i)}, V_{(j)}, d_1, d_2\big) \\ &\quad \times \prod_{k=1}^{N_1} m_{1k}\big(1 - U_{(k)}\big)^{a_{1k}} \prod_{l=1}^{N_2} m_{2l}\big(1 - V_{(l)}\big)^{a_{2l}}. \end{aligned}$$

Here $a_{ik} = \#\{n: R_{in} = k, \delta_{in} = 0\}$, $m_{ik} = \#\{n: R_{in} \ge k\}$ for $i = 1, 2$ and $k = 1, \dots, N_i$, and $\mathscr{J}(u, v, d_1, d_2) = \dot{\Phi}_d(F_{01}^{-1}(u), F_{01}^{-1}(v); 0) / \Phi_d(F_{01}^{-1}(u), F_{01}^{-1}(v); 0)$, where $\dot{\Phi}_d(s, t; \theta)$ is the partial derivative of $\Phi_d(s, t; \theta)$ with respect to θ.

The score generating functions $\mathscr{J}$ often take form of a product so that the corresponding test is based on a linear rank statistic $\sum_{n=1}^N a_1(R_{1n}, \delta_{1n}) a_2(R_{2n}, \delta_{2n})$, where for $i = 1, 2$

$$(1.3) \qquad a_i(j, d) = E\mathscr{J}_i\big(U_{(j)}, d\big) \prod_{k=1}^{N_i} m_{ik}\big(1 - U_{(k)}\big)^{a_{ik}}$$

and $\mathscr{J}_i(u, d)$, $d = 0, 1$, are functions satisfying

$$\int_0^u \mathscr{J}_i(v,1)\,dv = -(1-u)\mathscr{J}_i(u,0). \tag{1.4}$$

These scores were considered by Prentice (1978) and Kalbfleisch and Prentice (1980) in the two-sample problem for time transformed location models. In the context of testing for independence, Cuzick (1982) and Wu (1982) have derived (1.3) as scores of the locally most powerful tests in the so-called Bhuchongkul (1964) model.

With $J_i(u, d) = \ln(1-u) + d$ we are led to the log-rank (Savage) scores statistic

$$T_N = \sum_{n=1}^{N} (\hat{\Lambda}_1(Y_{1n}) - \delta_{1n})(\hat{\Lambda}_2(Y_{2n}) - \delta_{2n}),$$

where $\hat{\Lambda}_i$ are Nelson (1972) estimators of the marginal cumulative hazard functions $\Lambda_i = -\ln(1 - F_i)$. The choice of $\mathscr{J}_i(u, d) = d - (1 + d)u$ corresponds to the censored-data version of the Spearman test. In general the exact scores (1.3) might be hard to compute. Therefore, following Prentice (1978), Kalbfleisch and Prentice (1980), and Cuzick (1982), we shall consider approximate scores statistics

$$S_N = \sum_{n=1}^{N} \mathscr{J}_i(\hat{F}_1(Y_{1n}), \delta_{1n})\mathscr{J}_2(\hat{F}_2(Y_{2n}), \delta_{2n}),$$

where $\hat{F}_i$ are estimators close to the usual Kaplan–Meier (1958) estimators of the marginal cdf's. The exact definitions of $\hat{\Lambda}_i$ and $\hat{F}_i$ are given in Section 2. In Sections 3–5 suitably standardized versions of these statistics are shown to be asymptotically normal under both fixed and converging alternatives. The proof of the asymptotic normality of T_n and S_N patterns the Chernoff–Savage (1958) approach to the asymptotic distribution of the two-sample linear rank statistics, and extends results of Ruymgaart et al. (1972), Crowley (1973), and Crowley and Thomas (1975). The results can be used to derive Pitman efficiencies of these tests under general, not necessarily contiguous, alternatives. This problem will be considered elsewhere.

2. Asymptotic distribution of log-rank and approximate scores statistic: assumptions and results.

First let us introduce some assumptions to be used throughout this and subsequent sections.

Assumption A.2.1. *For each $N = 1, 2, \ldots$, $X_1, \ldots, X_N$, and $Z_1, \ldots, Z_N$ are mutually independent sets of iid nonnegative bivariate rv's with continuous cdf's F_N and $G_N = G$ and marginal cdf's F_{1N}, F_{2N} and G_1, G_2. For some (continuous) cdf F, $F_N \to F$ as $N \to \infty$.*

For each $N = 1, 2, \ldots$, define $L_N(s, t, d_1, d_2) = P(Y_{1n} \le s, Y_{2n} \le t, \delta_{1n} = d_1, \delta_{2n} = d_2)$, $H_N(s, t) = P(Y_{1n} \le s,\ Y_{2n} \le t)$, $H_{iN}(s) = P(Y_{in} \le s)$, and $K_{iN}(s) = 1 - P(Y_{in} > s,\ \delta_{in} = 1)$, $i = 1, 2$. Under assumption A.3.1 these cdf's may be easily expressed in terms of F_N and G. Moreover, L, H, H_i, and K_i, their limiting distributions, exist and depend on F and G only. Finally, let $\hat{L}$, $\hat{H}$, $\hat{H}_i$, and $\hat{K}_i$ denote the corresponding empiricals. In terms of these empiricals we have

$$\hat{\Lambda}_i(s) = \int_0^s \frac{d\hat{K}_i}{1-\hat{H}_i^-} \quad \text{and} \quad \hat{F}_i(s) = 1 - \prod_{t\le s}\left(1 - \frac{\Delta \hat{K}_i(t)}{1-\hat{H}_i^-(t)+N^{-1}}\right),$$

where $\hat{H}_i^-$ is the left-continuous version of $\hat{H}_i$.

The proof of the asymptotic normality of suitably standardized versions of T_N and S_n relies on a decomposition into sums of leading terms which are asymptotically normal, and remainder terms, which are asymptotically negligible. As regards the statistic S_N we assume that the score generating function $\mathscr{J}_1$ and $\mathscr{J}_2$ satisfy the following smoothness and boundedness conditions.

ASSUMPTION A.2.2. *For $i = 1, 2$ and $d = 0,1$, $\mathscr{J}_i(u,d)$ are continuously differentiable functions on $[0,1)$ such that*

$$(2.1) \qquad |\mathscr{J}_i(u,d)| \le cr(u)^{a_i} \quad \text{and} \quad |\mathscr{J}_i'(u,d)| \le cr(u)^{b_i},$$

where $r(u) = (1-u)^{-1}$ and $c > 0$, a_i, $b_i > 0$ are constants satisfying

$$(2.2) \qquad a_1 + a_2 < \tfrac{1}{2}, \qquad b_1 + a_2 < \tfrac{1}{2}, \qquad a_1 + b_2 < \tfrac{1}{2}.$$

Further, we eliminate degenerate cases by assuming

ASSUMPTION A.2.3. $K_i(0) < 1$.

With an abuse of notation, in what follows $\mathscr{J}_i(F_{iN})$, $\mathscr{J}_i(\hat{F}_i)$ denote functions $\mathscr{J}_i(F_{iN}(s), d)$ and $\mathscr{J}_i(\hat{F}_i(s), d)$, respectively. For $N = 1, 2, \dots,$ define

$$A_{0N} = \int N^{1/2}\mathscr{J}_1(F_{1N})\mathscr{J}_2(F_{2N})\,d(\hat{L} - L_N),$$

$$A_{1N} = \int N^{1/2}W_{1N}(1-F_{1N})\mathscr{J}_1'(F_{1N})\mathscr{J}_2(F_{2N})\,dL_N,$$

$$A_{2N} = \int N^{1/2}W_{2N}(1-F_{2N})\mathscr{J}_1(F_{1N})\mathscr{J}_2'(F_{2N})\,dL_N,$$

where for $i = 1, 2$

$$(2.3) \quad W_{iN}(s) = \int_0^s (\hat{H}_i^- - H_{iN})r(H_{iN})^2\,dK_{iN} + \int_0^s r(H_{iN})\,d(\hat{K}_i - K_{iN}).$$

LEMMA 2.1. *Let the assumption* A.2.1 *be satisfied and let $\mathscr{J}_1$ and $\mathscr{J}_2$ be functions such that* A.2.2 *holds with* (2.2) *replaced by*

$$(2.4) \qquad a_1 + a_2 < \tfrac{1}{2}, \qquad b_1 \le a_1 + 1, \qquad b_2 \le a_2 + 1.$$

Then with probability 1, $N^{1/2}\Sigma_{k=0}^2 A_{kn}$ *is a sum of iid rv's with mean zero and absolute moment of order* $2+\eta$, *uniformly bounded above for some* $\eta > 0$.

The proof is deferred to Section 3. To standardize T_N and S_n for location and scale define

$$\mu_N = \mu(F_N, G) = E\mathscr{J}_1(F_{1N}(Y_{1n}), \delta_{1n})\mathscr{J}_2(F_{2N}(Y_{2n}), \delta_{2n}),$$

$$\sigma_N^2 = \sigma^2(F_N, G) = \operatorname{Var}\left(\sum_{k=0}^{2} A_{kN}\right). \tag{2.5}$$

Under conditions of Lemma 2.1, σ_N^2 is well defined and converges to $\sigma_0^2 = \sigma^2(F, G) = \operatorname{Var}(\Sigma_{k=0}^2 A_{k0})$, where the variance σ_0^2 is evaluated under F and G, and the terms A_{k0} are defined as A_{kN}, $k = 0, 1, 2$, with F_N, H_{iN}, K_{iN} and L_N replaced by their limiting distributions. Further, with probability 1

$$N^{1/2}(T_N - \mu_N) = \sum_{k=0}^{2} A_{kN} + B_N \quad \text{and}$$

$$N^{1/2}(S_N - \mu_N) = \sum_{k=0}^{2} A_{kN} + C_N, \tag{2.6}$$

where B_N and C_N are remainder terms.

THEOREM 2.1. *Let the assumptions* A.2.1 *and* A.2.3 *be satisfied. Suppose that* $\sigma_0^2 > 0$ *for* $\mathscr{J}_1(u, d) = \mathscr{J}_2(u, d) = \ln(1 - u) + d$ *or* $\mathscr{J}_1$ *and* $\mathscr{J}_2$ *satisfying* A.2.2. *Then* $N^{1/2}(T_N - \mu_N)$ *and* $N^{1/2}(S_N - \mu_N)$ *converge in distribution to* $N(0, \sigma_0^2)$.

The proof of the theorem is given in subsequent sections. For uncensored data, the Chernoff–Savage approach to linear rank statistics strongly hinges on certain probability bounds for the empirical processes and the Brownian Bridge [Pyke and Shorack (1968) and Govindarajulu, Le Cam, and Raghavachari (1967)]. When censoring is present, it is not known if these bounds are satisfied by the Kaplan–Meier estimator or the estimator $\hat{F}_i$; therefore assumption A.2.2 imposes stronger boundedness conditions on the score functions than is necessary for uncensored data. Note that for uncensored data $\hat{F}_i(s) = (N + 1)^{-1}\Sigma_{n=1}^N I(X_{in} \le s)$, $i = 1, 2$, and the conclusions of Theorem 2.1 follow from results of Ruymgaart et al. (1972).

In general the asymptotic mean and variance of T_N and S_n depend on the underlying joint distributions of both survival and censoring times. If there is no censoring, formulas (2.5) reduce to the mean and variance given by Ruymagaart et al. (1972). Under the null hypothesis H_0: $F = F_1F_2$, if the condition (1.4) holds then $E[\mathscr{J}_i(F_i(Y_{in}), \delta_{in})|Z_{in}] = 0$, the asymptotic null mean is equal to zero, and $\sigma_0^2 = E\mathscr{J}_1^2(F_1(Y_{1n}), \delta_{1n})\mathscr{J}_1^2(F_2(Y_{2n}), \delta_{2n})$ [see also Cuzick (1982)]. The variance can be further simplified by applying the following result on integration by parts in two dimensions.

LEMMA 2.2 (Young, 1917). *Let* $f(s, t)$ *be a function of bounded variation such that* $f(s, 0) = f(0, t) = 0$. *Then for any bivariate cdf* G *we have* $\int_0^\infty\int_0^\infty f(s, t)\, dG(s, t) = \int_0^\infty\int_0^\infty \bar{G}(s, t)\, df(s, t)$, *where* $\bar{G}(s, t) = \int_s^\infty\int_t^\infty dG(u, v)$ *is the joint survival function corresponding to* G.

Conditioning on (Z_{1n}, Z_{2n}), applying (1.4) and Lemma 2.2 we obtain after some algebra

$$\sigma_0^2 = E\tilde{\mathscr{J}}_1^2(F_1(Y_{1n}))\tilde{\mathscr{J}}_2^2(F_2(Y_{2n}))\delta_{1n}\delta_{2n}, \tag{2.7}$$

where $\tilde{\mathscr{J}}_i(u) = \mathscr{J}_i(u,1) - \mathscr{J}_i(u,0)$. Note that (2.7) implies $\sigma_0^2(F,G) \le \sigma_0^2(F,G')$ whenever G and G' are cdf's such that $\overline{G}(s,t) \le \overline{G}'(s,t)$ for all $s, t > 0$, i.e., the asymptotic null variance increases as the dependence between the censoring times Z_{1n} and Z_{2n} increases in the sense of Lehmann's (1966) quadrant ordering. Furthermore, $\sigma_0^2(F,G) \le \int_0^1 \mathscr{J}_1^2(u,1)\,du \int_0^1 \mathscr{J}_2^2(u,1)\,du$ so that the asymptotic null variance is bounded by the asymptotic null variance of the corresponding rank statistic for uncensored data [based on scores $\mathscr{J}_i(u,1)$].

In the case of the log-rank statistic T_N, (2.7) amounts to $\sigma_0^2(F,G) = E\delta_{1n}\delta_{2n}$, so that tests based on $N^{1/2}T_N\hat{\sigma}_{nT}^{-1}$, $\hat{\sigma}_{NT}^2 = N^{-1}\sum_{n=1}^N \delta_{1n}\delta_{2n}$, are asymptotically distribution free [see also Cuzick (1982)]. Further

THEOREM 2.2. *Under assumptions of Theorem* 2.1, *if* (1.4) *holds then* $\hat{\sigma}_{NS}^2 = N^{-1}\sum_{n=1}^N \tilde{\mathscr{J}}_1^2(\hat{F}_{1N}(Y_{1n}))\tilde{\mathscr{J}}_2^2(\hat{F}_{2N}(Y_{2n}))\delta_{1n}\delta_{2n}$ *is a consistent estimator of the asymptotic null variance of* S_N.

The outline of the proof is given in Sections 3 and 5.

For most purposes it is enough to consider the null hypothesis, fixed and contiguous alternatives. If either only one variable is subject to censoring or the censoring variables are independent, exact permutation distributions are available under the null hypothesis [see Cuzick (1982) and Wu (1982)]. Furthermore, as pointed out by a referee, it should be possible to derive asymptotic normality results under the null hypothesis and contiguous alternatives by applying a suitable modification of the Aalen (1978), Gill (1980), and Anderson and Gill (1982) martingale approach to linear rank statistics. In particular, under $\mathscr{H}_0$, by conditioning on the potential censoring times (Z_1, Z_2) and the scores generated by Y_{2n}, and using the formulation of Mehrotra, Michalek, and Mihalko (1982), the conditional expectation required for the martingale property should appear straightforward, and similar to Cuzick (1985).

Theorem 2.1 can be easily extended to the case of independent but noniid continuous survival or censoring distributions. It can be also generalized to allow score generating functions with a finite number of discontinuities of the first kind and discrete underlying distributions. Assumptions needed for these extensions are similar to those in Ruymgaart (1974, 1979); due to cumbersome notation we shall not discuss this problem in more detail.

3. Proofs of Theorem 2.1 and 2.2: leading terms. The proof of Lemma 2.1 rests on a repeated application of inequalities

$$\begin{aligned} &|I(Y_{in} < s) - H_{iN}(s)|, \\ &|I(Y_{in} \le s) - H_{iN}(s)| \le r(H_{iN}(Y_{in}))^{1-\gamma} r(H_{iN}(s))^{-(1-\gamma)} \end{aligned} \tag{3.1}$$

for any $\gamma \in (0.1)$. Further, $F_{iN} \le H_{iN}$ and A.2.2. imply

$$|\mathscr{J}_i(F_{iN}(s), d)| \le r(H_{iN}(s))^{a_i}, \qquad |\mathscr{J}_i'(F_{iN}(s), d)| \le r(H_{iN}(s))^{b_i} \tag{3.2}$$

for $d = 0$ or 1. Finally, we shall need Hölder's inequality

$$\int |\phi_1(H_{1N})\phi_2(H_{2N})|\, dH_N \le \left(\int_0^1 |\phi_1(u)|^{p_1}\, du\right)^{1/p_1} \left(\int_0^1 |\phi_2(u)|^{p_2}\, du\right)^{1/p_2}, \tag{3.3}$$

where ϕ_1 and ϕ_2 are functions on $(0,1)$ and $p_1, p_2 > 1$ satisfy $1/p_1 + 1/p_2 = 1$. Note that if $a_1 + a_2 < \frac{1}{2}$ then for any η, $0 < 4\eta < \frac{1}{2} - a_1 - a_2$, there exist $p_i, q_i > 1$ such that $1/p_1 + 1/p_2 = 1$, $1/q_1 + 1/q_2 = 1$, and

$$\begin{gathered}\left(a_1 + \tfrac{1}{2} + 2\eta\right)p_1 < 1,\ a_2 p_2 < 1,\ a_1 q_1 < 1, \\ \left(a_2 + \tfrac{1}{2} + 2\eta\right)q_2 < 1\end{gathered} \tag{3.4}$$

[see Ruymgaart et al. (1972)].

Proof of Lemma 2.1. We shall show that each of the terms A_{kN}, $k = 0, 1, 2$, is a sum of iid rv's with mean zero and absolute moment of order $2 + \eta$ uniformly bounded above for some $\eta > 0$. By symmetry it is enough to consider the terms A_{0N} and A_{1N} only.

Let M denote a generic constant, independent of N and underlying cdf's. Set $a = a_1 + a_2$ and without loss of generality assume that (2.1) is satisfied with $b_i = a_i + 1$.

We have $N^{1/2}A_{0N} = \sum_{n=1}^N \mathscr{J}_1(F_{1N}(Y_{1n}), \delta_{1n})\mathscr{J}_2(F_{2N}(Y_{2n}), \delta_{2n}) - \mu_N$ which is a sum of iid mean zero rv's. Applying (3.2) and (3.3) with $p_i = a/a_i$ we obtain

$$E|(\mathscr{J}_1(F_{1N}(Y_{1n}), \delta_{1n})\mathscr{J}_2(F_{2N}(Y_{2n}), \delta_{2n})|^{2+\eta} \le M\int_0^1 r(u)^{a(2+\eta)}\, du < \infty$$

provided $\eta > 0$ is chosen so that $a(2 + \eta) < 1$. This however can always be achieved since $a = a_1 + a_2 < \frac{1}{2}$. The upper bound does not depend on N or underlying cdf's.

Further, we have $N^{1/2}A_{1N} = \sum_{n=1}^N A_{1n}$, where

$$A_{1n} = \int W_{1n}(1 - F_{1N})\mathscr{J}_1'(F_{1N})\mathscr{J}_2(F_{2N})\, dL_N.$$

The process W_{1n} is defined as W_{1N} with $\hat{H}_1$ and $\hat{K}_1$ replaced by $\hat{H}_{1n}(s) = I(Y_{1n} \le s)$ and $\hat{K}_{1n}(s) = 1 - I(Y_{1n} > s, \delta_{1n} = 1)$. Applying (3.2) and (3.1) with $\gamma = \frac{1}{2} + \eta$, we obtain after some algebra

$$|A_{1n}| \le M\left\{r(H_{1N}(Y_{1n}))^{1/2-\eta} + \int_0^1 r(u)^{1/2+\eta}\, du\right\}\int r(H_{1N})^{a_1+1/2+\eta} r(H_{2N})^{a_2}\, dH_N.$$

The $2 + \eta$ moment of the random part on the right-hand side is finite and independent of N because $(\frac{1}{2} - \eta)(2 + \eta) < 1$ for all $\eta > 0$. The second term is bounded above by

$$\left(\int_0^1 r(u)^{(a_1+1/2+2\eta)p_1}\, du\right)^{1/p_1}\left(\int_0^1 r(u)^{a_2 p_2}\, du\right)^{1/p_2} < \infty$$

provided $0 < 4\eta < \frac{1}{2} - a_1 - a_2$ and p_1 and p_2 are as in (3.4). □

Proof of Theorem 2.1. The proof of the asymptotic negligibility of the remainder terms B_N and C_n is given in Section 5. With an appropriate choice of functions $\mathscr{J}_1$ and $\mathscr{J}_2$, Lemma 2.1 and Esseen's theorem imply that $N^{1/2}(T_N - \mu_N)/\sigma_N$ and $N^{1/2}(S_N - \mu_N)/\sigma_N$ converge in distribution to $\mathscr{N}(0,1)$, provided $\liminf_{N\to\infty}\sigma_N^2 > 0$. Finally, applying Theorems 5.5 and 5.4 in Billingsley (1968) it is easy to verify that $\sigma_N^2 \to \sigma_0^2$ as $N \to \infty$. □

PROOF OF THEOREM 2.2. Let $L_{11N}(s,t) = P(Y_{1n} \le s,\ Y_{2n} \le t,\ \delta_{1n} = 1, \delta_{2n} = 1)$ and let $\hat{L}_{11}$ be the empirical counterpart of L_{11N}. Under the null hypothesis

$$\hat{\sigma}_{NS}^2 - \sigma_{0S}^2 = \int \tilde{\mathscr{J}}_1^2(F_1)\tilde{\mathscr{J}}_2^2(F_2)\,d(\hat{L}_{11} - L_{11N}) + D_N, \tag{3.5}$$

where D_N is a remainder term. The first term is an average of iid mean zero rv's whereas the second term is asymptotically negligible (see Section 5). Therefore, by the law of large numbers, $\hat{\sigma}_{NS}^2 \to \sigma_{0S}^2$ as $N \to \infty$. □

4. Decomposition of remainder terms. Set $\Delta = \Delta_1 \times \Delta_2$ where $\Delta_i = (0, \max_n Y_{in}]$, $i = 1,2$.

For $\mathscr{J}_1$ and $\mathscr{J}_2$ satisfying assumption A.2.2, the remainder term C_n in (2.6) is given by $C_N = \sum_{k=1}^3 C_{kN}$ where

$$C_{1N} = \int_\Delta N^{1/2}(\mathscr{J}_1(\hat{F}_1) - \mathscr{J}_1(F_{1N}))\mathscr{J}_2(F_{2N})\,d\hat{L} - A_{1N},$$

$$C_{2N} = \int_\Delta N^{1/2}\mathscr{J}_1(F_{1N})(\mathscr{J}_2(\hat{F}_2) - \mathscr{J}_2(F_{2N}))\,d\hat{L} - A_{2N},$$

$$C_{3N} = \int_\Delta N^{1/2}(\mathscr{J}_1(\hat{F}_1) - \mathscr{J}_1(F_{1N}))(\mathscr{J}_2(\hat{F}_2) - \mathscr{J}_2(F_{2N}))\,d\hat{L}.$$

The remainder term B_n in (2.6) is given by $B_N = \sum_{k=1}^3 B_{kN}$ where B_{kN} are defined as C_{kN} with $\mathscr{J}_i(u,d) = \ln(1-u) + d$ and $\hat{F}_i$ replaced by $1 - \exp(-\hat{\Lambda}_i)$. The terms B_{1N} and B_{2N}, C_{1N}, and C_{2N} are symmetric so in what follows we shall consider B_{1N} and C_{1N} only.

For any $\tau \in (0,1)$ let $A_\tau = A_{1\tau} \times A_{2\tau}$, where $A_{i\tau} = [0, \gamma_{i\tau}]$ and $\gamma_{i\tau} = \inf\{s: H_i(s) \ge 1 - \tau\}$, $i = 1,2$. Then $C_{1N} = \sum_{k=1}^4 C_{1k}$ and $C_{3N} = \sum_{k=1}^3 C_{3k}$, where

$$C_{11} = \int_{\Delta\cap A_\tau} N^{1/2}W_{1N}(1 - F_{1N})\mathscr{J}_1'(F_{1N})\mathscr{J}_2(F_{2N})\,d(\hat{L} - L_N),$$

$$C_{12} = -\int_{\Delta^c \cup A_\tau^c} N^{1/2}W_{1N}(1 - F_{1N})\mathscr{J}_1'(F_{1N})\mathscr{J}_2(F_{2N})\,dL_N,$$

$$C_{13} = \int_{\Delta\cap A_\tau} N^{1/2}(\mathscr{J}_1(\hat{F}_1) - \mathscr{J}_1(F_{1N}) - (1 - F_{1N})W_{1N}\mathscr{J}_1'(F_{1N}))\mathscr{J}_2(F_{2N})\,d\hat{L},$$

$$C_{14} = \int_{\Delta\cap A_\tau^c} N^{1/2}(\mathscr{J}_1(\hat{F}_1) - \mathscr{J}_1(F_{1N}))\mathscr{J}_2(F_{2N})\,d\hat{L} = -C_{33},$$

$$C_{31} = \int_{\Delta\cap A_\tau} N^{1/2}(\mathscr{J}_1(\hat{F}_1) - \mathscr{J}_1(F_{1N}))(\mathscr{J}_2(\hat{F}_2) - \mathscr{J}_2(F_{2N}))\,d\hat{L},$$

$$C_{32} = \int_{\Delta\cap A_\tau^c} N^{1/2}(\mathscr{J}_1(\hat{F}_1) - \mathscr{J}_1(F_{1N}))\mathscr{J}_2(\hat{F}_2)\,d\hat{L}$$

(the dependence of these terms on N is taken as understood). Here the process W_{1N} is given by (2.3). Analogously, $B_{1n} = \sum_{k=1}^{4} B_{1k}$ and $B_{3N} = \sum_{k=1}^{3} B_{3k}$ where B_{1k} and B_{3k} are defined as C_{1k} and C_{3k} with $\mathscr{I}_i(u, d) = \ln(1-u) + d$ and $\hat{F}_i$ replaced by $1 - \exp(-\hat{\Lambda}_i)$.

Let us recall now some properties of estimators $\hat{F}_i$ and $\hat{\Lambda}_i$, $i = 1, 2$. Both $\hat{F}_i$ and $\hat{\Lambda}_i$ are right-continuous step functions with jumps at uncensored observations. Further, for any $s > 0$

$$\hat{F}_i(s) \leq N(N+1)^{-1}\hat{H}_i(s), \qquad i = 1, 2, \tag{4.1}$$

which can be verified by applying a similar argument as in Gill (1980, page 36). If $\mathscr{I}_i$ satisfy smoothness conditions of A.2.2 then by the mean value theorem

$$\mathscr{I}_i(\hat{F}_i(s), d) - \mathscr{I}_i(F_{iN}(s), d) = (\hat{F}_i(s) - F_{iN}(s))\mathscr{I}_i'(\Phi_i(s), d) \tag{4.2}$$

for $s \in \Delta_i$ and $d = 0$ or 1. The function Φ_i is defined by $\Phi_i = F_{iN} + \theta_i(\hat{F}_i - F_{iN})$, where $\theta_i = \theta_i(\omega, N, d, s)$ is a random function valued in $(0, 1)$. Without loss of generality let us assume that θ_i does not depend on the value of d. Further, if $\mathscr{I}_1$ and $\mathscr{I}_2$ satisfy boundedness conditions of Lemma 2.1, then by (4.1) and van Zuijlen (1978), for $N = 1, 2, \ldots$

$$\sup_{\Delta_i} |\mathscr{I}_i(\hat{F}_i)| r(H_{iN})^{-a_i} = O_P(1) \quad \text{and} \quad \sup_{\Delta_i} |\mathscr{I}_i'(\Phi_i)| r(H_{iN})^{-b_i} = O_P(1) \tag{4.3}$$

uniformly in N and underlying distributions.

Finally, let us recall consistency and weak convergence results. For $i = 1, 2$

$$\sup_{A_{i\tau}} |\hat{F}_i - F_{iN}| \to_P 0, \qquad \sup_{A_{i\tau}} |\hat{\Lambda}_i - \Lambda_{iN}| \to_P 0 \tag{4.4}$$

as $N \to \infty$ [Földes and Rejtö (1978) and Gill (1980)]. The processes $N^{1/2}(\hat{\Lambda}_i - \Lambda_{iN})$ and $N^{1/2}(\hat{F}_i - F_{iN})$ converge weakly in $D(A_{i\tau})$ to W_i and $(1 - F_i)W_i$, respectively, where W_i is a mean zero Gaussian process with almost all sample paths continuous and covariance $\operatorname{cov}(W_i(s), W_i(t)) = \int_0^{\min(s,t)} r(F_i)^2 r(G_i)\, dF_i$ [Breslow and Crowley (1974) and Gill (1980)].

In the course of the proof of Theorems 2.1 and 2.2 we shall also use

$$\begin{aligned} &\sup_{A_{i\tau}} N^{1/2}|Q_{iN}| = o_P(1), \qquad \sup_{A_{i\tau}} N^{1/2}|R_{iN}| = o_P(1), \\ &\sup_{A_{i\tau}} N^{1/2}|W_{iN}| = O_P(1) \end{aligned} \tag{4.5}$$

for $Q_{iN} = \hat{F}_i - F_{iN} - (1 - F_{iN})W_{iN}$, $R_{iN} = \hat{\Lambda}_i - \Lambda_{iN} - W_{iN}$, and N sufficiently large.

5. Proofs of Theorems 2.1 and 2.2: asymptotic negligibility of remainder terms. The asymptotic negligibility of the terms B_N and C_N in Theorem 2.1 will be established by a sequence of lemma's showing that B_{14}, C_{14}, B_{32}, and C_{32} converge in probability to 0 as $\tau \to 0$ and $N \to \infty$, whereas the remaining terms tend in probability to 0 for any fixed $\tau \in (0, 1)$ and $N \to \infty$.

LEMMA 5.1. *For fixed $\tau \in (0, 1)$, C_{13}, C_{31}, B_{13}, and B_{31} converge in probability to 0 as $N \to \infty$.*

Proof. Let us consider first the term C_{13}. Using (4.2) we can write it as a sum of

$$C_{131} = \int_{\Delta\cap A_\tau} N^{1/2} Q_{1N} \mathscr{J}_1'(\Phi_1)\mathscr{J}_2(F_{2N})\, d\hat{L},$$

$$C_{132} = \int_{\Delta\cap A_\tau} N^{1/2}(1 - F_{1N})W_{1N}(\mathscr{J}_1'(\Phi_1) - \mathscr{J}_1'(F_{1N}))\mathscr{J}_2(F_{2N})\, d\hat{L}.$$

Let $\tau \in (0,1)$ and $\varepsilon > 0$ be fixed. There exists a constant $M_1 = M_1(\tau)$ such that for N large enough $\sup|F_{iN} - F_i| < \tau/3$ and $\sup_{A_{2\tau}}|\mathscr{J}_2(F_{2N})| < M_1$. Further, there exist constants $M_2 = M_2(\tau, \varepsilon$, $M_3 = M_3(\tau, \varepsilon)$ such that for N sufficiently large the sets $\Omega_1 = \{\sup_{A_{1\tau}}|\hat{F}_i - F_{iN}| < \tau/3\}$, $\Omega_2 = \{\sup_{A_{1\tau}}|\mathscr{J}_1'(\Phi_1)| < M_2\}$, $\Omega_3 = \{\sup_{A_{1\tau}} N^{1/2}|W_{1N}| < M_3\}$, and $\Omega_4 = \{A_\tau \subset \Delta\}$ have probability at least $1 - \varepsilon$. Then however

$$I\left(\bigcap_{k=1}^{4} \Omega_k\right)|C_{131}| \le M_1 M_2 \sup_{A_{1\tau}} |N^{1/2} Q_{1N}|$$

and by (4.5) this bound converges in probability to 0. Also

$$I\left(\bigcap_{k=1}^{3} \Omega_k\right)|C_{132}| \le M_1 M_3 I(\Omega_1) \sup_{A_{1\tau}} |J_1'(\Phi_1) - \mathscr{J}_1'(F_{1N})|.$$

For $d = 0, 1$ the function $\mathscr{J}_1'(u, d)$ is uniformly continuous on $[0, 1 - \tau/3]$ so that $|\Phi_1 - F_{1N}| \le |\hat{F}_1 - F_{1N}|$ and (4.4) imply that this bound tends in probability to 0. A similar argument combined with (4.3) and (4.5) shows that C_{31} converges in probability to 0 as $N \to \infty$. The asymptotic negligibility of B_{13} and B_{31} follows immediately from (4.4) and (4.5). □

Lemma 5.2. *For fixed $\tau \in (0,1)$, $B_{11} \to_P 0$ and $C_{11} \to_P 0$ as $N \to \infty$.*

Proof. The proof is similar to Ruymgaart et al. (1972). Assuming that functions $\mathscr{J}_1$ and $\mathscr{J}_2$ satisfy assumption A.2.2 with $b_1 \le a_1 + 1$, it is enough to consider the term C_{11} only.

Let $\tau \in (0,1)$ and $\varepsilon > 0$ be fixed. For any positive integer m, define $\chi_{im}(s) = \gamma_{i\tau}(k-1)/m$ for $\gamma_{i\tau}(k-1)/m < s \le \gamma_{i\tau}k/m$, $k = 1, \dots, m$, where $A_{i\tau} = [0, \gamma_{i\tau}]$, $i = 1, 2$. For arbitrary m we have $|C_{11}| \le \Sigma_{k=1}^3 C_{11km}$, where

$$C_{111m} = \int_{\Delta\cap A_\tau} N^{1/2}|W_{1N}(s) - W_{1N}(\chi_{1m}(s))||\phi(s, t, d_1, d_2)|\, d(\hat{L} + L_N),$$

$$C_{112m} = \int_{\Delta\cap A_\tau} N^{1/2}|W_{1N}(\chi_{1m}(s))| \times |\phi(s, t, d_1, d_2) - \phi(\chi_{1m}(s), \chi_{2m}(t), d_1, d_2)|\, d(\hat{L} + L_N),$$

$$C_{113m} = \left|\int_{\Delta\cap A_\tau} N^{1/2} W_{1N}(\chi_{1m}(s))\phi(\chi_{1m}(s), \chi_{2m}(t), d_1, d_2)\, d(\hat{L} - L_N)\right|,$$

and $\phi(s, t, d_1, d_2) = (1 - F_{1N}(s))\mathscr{J}_1'(F_{1N}(s), d_1)\mathscr{J}_2(F_{2N}(t), d_2)$.

There exists a constant $M_1 = M_1(\tau)$ such that for N large enough $\sup|F_{iN} - F_i| < \tau/2$ and $\sup_{A_\tau}|\phi| < M_1$. Further, there exists a constant $M_2 = M_2(\tau, \varepsilon)$ such that for N large enough the sets $\Omega_1 = \{\sup_{A_{1\tau}} N^{1/2}|W_{1N}| < M_2\}$ and $\Omega_2 = \{A_\tau \subset \Delta\}$ have probability at least $1 - \varepsilon$.

Let us consider the term C_{111m}. The process $N^{1/2}W_{1N}$ converges weakly in $\mathscr{D}(A_{1\tau})$ to W_1. Therefore, by employing a Skorohod construction,

$$\sup_{A_{1\tau}} N^{1/2}|W_{1N} - W_{1N} \circ \chi_{1m}| \to_P 0 \text{ as } N, m \to \infty$$

and there exists a sequence $\eta_{mN}, \eta_{mN} \to 0$ as $N, m \to \infty$, such that the set $\Omega_m = \{\sup_{A_{1\tau}}|W_{1N} - W_{1N} \circ \chi_{1m}| < \eta_{mN}\}$ has probability at least $1 - \varepsilon$ for all m and N sufficiently large. Combining, $I(\Omega_1 \cap \Omega_2 \cap \Omega_m)C_{111m} \le M_1\eta_{mN} \to 0$.

Further, for $d = 0, 1$, the functions $\mathscr{J}_1'(u, d)$ and $\mathscr{J}_2(u, d)$ are uniformly continuous on $[0, 1 - \tau/2]$ so that for N sufficiently large $\xi_{mN} = \sup_{A_\tau}|\phi(s, t, d_1, d_2) - \phi(\chi_{1m}(s), \chi_{2m}(t), d_1, d_2)| \to 0$ as $m \to \infty$. Hence $I(\Omega_1 \cap \Omega_2)C_{112m} \le M_2\xi_{mN} \to 0$ as $m, N \to \infty$.

Finally, for N sufficiently large and each $\omega \in \Omega_1 \cap \Omega_2$, the integrand of C_{113m} is a step function which assumes value $a_{klmd}(\omega)$ for $d = (d_1, d_2)$, $d_i = 0, 1$ and (s, t) belonging to $R_{klm} = (\gamma_{1\tau}(k-1)/m, \gamma_{1\tau}k/m] \times (\gamma_{2\tau}(l-1)/m, \gamma_{2\tau}l/m]$, $k, l = 1, \dots, m$. Therefore

$$\begin{aligned} I(\Omega_1 \cap \Omega_2)C_{113m} &= \left|\sum_{k=1}^{m}\sum_{l=1}^{m}\sum_{d_1=0}^{1}\sum_{d_2=0}^{1} a_{klmd}\int_{R_{klm}} d(\hat{L} - L_N)\right| \\ &\le 16m^2M_2(M_1 + \xi_{mN})\sup|\hat{L} - L_N| \end{aligned}$$

and the bound converges in probability to 0 as $N \to \infty$. □

LEMMA 5.3. *B_{14} and C_{14} converge in probability to 0 as $\tau \to 0$ and $N \to \infty$.*

PROOF. Assuming that functions $\mathscr{J}_1$ and $\mathscr{J}_2$ satisfy A.2.2 with $b_1 \le a_1 + 1$, it is enough to consider the term C_{14} only. Let η, $0 \le 4\eta \le \frac{1}{2} - a_1 - a_2$, and $\varepsilon > 0$ be fixed. As shown in Ruymgaart et al. (1972), applying dominated convergence theorem and Hölder's inequality (3.4) with p_1 and p_2 as in (3.5), we can find $\tilde{\tau} = \tilde{\tau}(\varepsilon)$ such that for all $\tau \le \tilde{\tau}$

$$\int_{A_\tau^c} r(H_1)^{a_1 + 1/2 + 2\eta} r(H_2)^{a_2}\, dH < \varepsilon.$$

For $\tilde{\tau}$ and N sufficiently large the set $\Omega_\tau = \{A_\tau \subset \Delta\}$ has probability at least $1 - \varepsilon$. Further, by A.2.2 (with $b_1 \le a_1 + 1$) and (3.1), $|C_{14}| \le \sum_{k=1}^{2} C_{14k}$, where

$$C_{141} = \int_{\Delta^c \cup A_\tau^c} N^{1/2}\left(\int_0^s |\hat{H}_1^- - H_{1N}| r(H_{1N})^2\, dH_{1N}\right) r(H_{1N}(s))^{a_1} r(H_{2N}(t))^{a_2}\, dH_N,$$

$$C_{142} = \int_{\Delta^c \cup A_\tau^c} N^{1/2}\left|\int_0^s r(H_{1N})\, d(\hat{K}_1 - K_{1N})\right| r(H_{1N}(s))^{a_1} r(H_{2N}(t))^{a_2}\, dH_N.$$

By Lemma 4.2 in Ruymgaart et al. (1972), there exists $M_1 = M_1(\varepsilon)$ such that the set $\Omega_1 = \{\sup N^{1/2}|\hat{H}_1^- - H_{1N}|r(H_{1N})^{1/2-\eta}r(1 - H_{1N})^{1/2-\eta} < M_1\}$ has probability at least $1 - \varepsilon$ uniformly in N. Therefore

$$I(\Omega_1)C_{141} \le M_1\int r(H_{1N})^{1-\eta}\,dH_{1N}\int_{\Delta^c\cup A_\tau^c} r(H_{1N})^{a_1+1/2+2\eta}r(H_{2N})^{a_2}\,dH_N.$$

The first term of this bound does not depend on N and the underlying cdf's. The second term is smaller than ε with probability at least $1 - \varepsilon$ for all $\tau \le \tilde{\tau}$ and N sufficiently large. This implies $C_{141} \to_P 0$ as $\tau \to 0$ and $N \to \infty$.

The proof of asymptotic negligibility of C_{142} is similar. □

Lemma 5.4. *For any c_1 and c_2, $c_1 + c_2 < \frac{1}{2}$*

$$E_{1N} = \int_{\Delta\cap A_\tau^c} N^{1/2}|\hat{\Lambda}_1 - \Lambda_{1N}|r(H_{1N})^{c_1}r(H_{2N})^{c_2}\,d\hat{H}, \tag{5.1}$$

$$E_{2N} = \int_{\Delta\cap A_\tau^c} N^{-1/2}\left(\int_0^s r(\hat{H}_1^-)r(\hat{H}_1^- - N^{-1})\,d\hat{K}_1\right) \times r(H_{1N}(s))^{c_1}r(H_{2N}(t))^{c_2}\,d\hat{H} \tag{5.2}$$

converge in probability to 0 as $\tau \to 0$ and $N \to \infty$.

Proof. Let η, $0 < 4\eta < \frac{1}{2} - c_1 - c_2$, and $\varepsilon > 0$ be fixed. Similarly as in Lemma 5.3 we have

$$\int_{\Delta\cap A_\tau^c} r(H_{1N})^{c_1+1/2+2\eta}r(H_{2N})^{c_2}\,d\hat{H} \to_P 0$$

as $\tau \to 0$ and $N \to \infty$.

Let $A_\varepsilon = \{s\colon 1 - H_{1N}(s) > \varepsilon/N\}$. By Theorem 1.4 in van Zuijlen (1978), the set $\Omega_\varepsilon = \{\Delta_1 \subset A_\varepsilon\}$ has probability at least $1 - \varepsilon$. We have $I(\Omega_\varepsilon)E_{1N} \le E_{11} + E_{12}$, where

$$E_{11} = \int_{\Delta\cap A_\tau^c} N^{1/2}\left(\int_0^s |\hat{H}_1^- - H_{1N}|r(H_{1N})r(\hat{H}_1^-)\,d\hat{K}_1\right) \times r(H_{1N}(s))^{c_1}r(H_{2N}(t))^{c_2}\,d\hat{H},$$

$$E_{12} = \int_{\Delta\cap A_\tau^c} N^{1/2}\left|\int_0^s r(H_{1N})\,d(\hat{K}_1 - K_{1N})\right|r(H_{1N}(s))^{c_1}r(H_{2N}(t))^{c_2}\,d\hat{H}$$

(the dependence of these terms on N is taken as understood).

By Theorem 1.1 and Corollary 1.1 in van Zuijlen (1978), there exist constants $M_1 = M_1(\varepsilon)$ and $M_2 = M_2(\varepsilon)$ such that the sets $\Omega_1 = \{\sup_\Delta r(\hat{H}_1^-)r(H_{1N})^{-1} < M_1\}$ and $\Omega_2 = \{\sup N^{1/2}|\hat{H}_1^- - H_{1N}|r(H_{1N})^{1/2-\eta}r(1 - H_{1N})^{1/2-\eta} < M_2\}$ have probability at least $1 - \varepsilon$ uniformly in N. Therefore

$$I(\Omega_1 \cap \Omega_2)E_{11} \le M_1M_2\int r(H_{1N})^{1-\eta}\,d\hat{H}_1\int_{\Delta\cap A_\tau^c} r(H_{1N})^{c_1+1/2+\eta}r(H_{2N})^{c_2}\,d\hat{H}$$

and the bound converges in probability to 0 as $\tau \to 0$ and $N \to \infty$. The proof of the asymptotic negligibility of E_{12} is similar. Finally

$$I(\Omega_1 \cap \Omega_\varepsilon)E_{2N} \le M_1 N^{-1/2} \int_{A_\varepsilon} r(H_{1N})^{3/2-2\eta}\, d\hat{H}_1 \int_{A_\tau^c} r(H_{1N})^{c_1+1/2+2\eta} r(H_{2N})^{c_2}\, d\hat{H}$$

$$\le N^{-1/2}(N/\varepsilon)^{1/2-\eta} \int r(H_{1N})^{1-\eta}\, d\hat{H}_1$$

$$\times \int_{A_\tau^c} r(H_{1N})^{c_1+1/2+2\eta} r(H_{2N})^{c_2}\, d\hat{H}$$

and the bound converges in probability to 0 as $\tau \to 0$ and $N \to \infty$. □

LEMMA 5.5. *B_{32} and C_{32} converge in probability to 0 as $\tau \to 0$ and $N \to \infty$.*

PROOF. Let $\varepsilon > 0$ be fixed and let Ω_1, Ω_2, and Ω_ε be defined as in Lemma 5.4. Given $\varepsilon > 0$ there exists a constant $M_3 = M_3(\varepsilon)$ such that the set $\Omega_3 = \{\sup_{\Delta_2} r(\hat{H}_2^-) r(H_{2N})^{-1} \le M_3\}$ has probability at least $1 - \varepsilon$ uniformly in N. Therefore

$$I(\Omega_3 \cap \Omega_\varepsilon)|B_{32}| \le \left(M_3 \int r(H_{2N})^{1-c_2}\, d\hat{H}_2 + 1\right) E_{1N}.$$

Here E_{1n} is given by (5.1) with $c_1 = 0$ and $c_2 < \frac{1}{2}$ chosen arbitrarily. By Lemma 5.4 the bound converges in probability to 0 as $\tau \to 0$ and $N \to \infty$.

The term C_{32} can be written as

$$C_{32} = \int_{\Delta \cap A_\tau^c} N^{1/2}(\hat{F}_1 - F_{1N}) \mathscr{J}_1'(\Phi_1) \mathscr{J}_2(F_{2N})\, d\hat{L}.$$

Applying inequalities $|x_1 - x_2| \le |\ln x_1 - \ln x_2|$ for $0 < x_1, x_2 \le 1$ and $0 < -\ln(1 - (1+x)^{-1}) - (1+x)^{-1} < (x(1+x))^{-1}$ for $x > 0$, it can be easily seen that for $\omega \in \Omega_\varepsilon$ and $s \in \Delta_1$

$$|\hat{F}_1(s) - F_{1N}(s)| \le |\hat{\Lambda}_1(s) - \Lambda_{1N}(s)| + 2N^{-1} \int_0^s r(\hat{H}_1^-) r(\hat{H}_1^- - N^{-1})\, d\hat{K}_1.$$

By (4.5), there exists a constant $M_4 = M_4(\varepsilon)$ such that the set $\Omega_4 = \{\sup_{\Delta_1} |\mathscr{J}_1'(\Phi_1)| r(H_{1N})^{-b_1} \le M_4\}$ has probability at least $1 - \varepsilon$ uniformly in N. Therefore $I(\bigcap_{k=1}^4 \Omega_k \cap \Omega_\varepsilon)|C_{32}| \le M_3 M_4 (E_{1N} + 2E_{2N})$ where E_{1N} and E_{2N} are given by (5.1) and (5.2) with $c_1 = b_1$ and $c_2 = a_2$ as in assumption A.2.2. By Lemma 5.4, the bound converges in probability to 0 as $\tau \to 0$ and $N \to \infty$ which implies the asymptotic negligibility of C_{32}. □

PROOF OF THEOREM 2.2. The remainder term D_N in (3.5) is a sum of

$$D_{1N} = \int_{\Delta \cap A_\tau} (\tilde{\mathscr{J}}_1^2(\hat{F}_1) - \tilde{\mathscr{J}}_1^2(F_{1N})) \tilde{\mathscr{J}}_2(\hat{F}_2)\, d\hat{L}_{11},$$

$$D_{2N} = \int_{\Delta \cap A_\tau^c} (\tilde{\mathscr{J}}_1^2(\hat{F}_1) - \tilde{\mathscr{J}}_1^2(F_{1N})) \tilde{\mathscr{J}}_2(\hat{F}_2)\, d\hat{L}_{11},$$

$$D_{3N} = \int_{\Delta \cap A_\tau} (\tilde{\mathscr{J}}_2^2(\hat{F}_2) - \tilde{\mathscr{J}}_2^2(F_{2N})) \tilde{\mathscr{J}}_1(F_{1N})\, d\hat{L}_{11},$$

$$D_{4N} = \int_{\Delta \cap A_\tau^c} (\tilde{\mathscr{J}}_2^2(\hat{F}_2) - \tilde{\mathscr{J}}_2^2(F_{2N})) \tilde{\mathscr{J}}_1(F_{1N})\, d\hat{L}_{11}.$$

Applying mean value theorem and a similar argument as in the proof of Theorem 2.1 it can be shown that $D_{1N} \to_P 0$ and $D_{3N} \to_P 0$ for any fixed $\tau \in (0,1)$ and $N \to \infty$, and $D_{2N} \to_P 0$ and $D_{4N} \to_P 0$ as $\tau \to 0$ and $N \to \infty$. □

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DEPARTMENT OF STATISTICS
CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213