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VARIANCES OF THE KAPLAN-MEIER ESTIMATOR AND ITS QUANTILES UNDER CERTAIN FIXED CENSORING MODELS

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For fixed censoring models that contain at most one intermediate censoring point, we obtain exact algebraic expressions for the asymptotic variances of (i) the quantiles of the Kaplan-Meier (KM, 1958) survival estimator and (ii) the KM estimator itself at fixed time points. The relationship between (i) and (ii) is found to be the same as the one derived by Sander (1975) and Reid (1981b) for the random censorship model. Confidence intervals for the quantiles based on (i) are briefly discussed and compared to previously known procedures. Although Greenwood's Formula is recommended over (ii) in practice because of its (desirable) conditioning on the observed censoring pattern, (ii) is of theoretical interest as an asymptotic limit for Greenwood's Formula in closed form.

1. Introduction. Point estimates of survival quantiles can be obtained in an obvious manner from the Kaplan-Meier (KM, 1958) estimate of the entire survival function. In the random censorship model, in which censoring occurs independently of death, several methods have been proposed for obtaining confidence statements about these estimated quantiles. In particular, Efron (1981) and Reid (1981a) have developed two different bootstrap methods; Thomas and Grunkemeier (TG, 1975), Brookmeyer and Crowley (BC, 1982a), Emerson (1982), and Anderson, Bernstein and Pike (ABP, 1982) invert various two-sided tests on the survival probability at a fixed point; and Reid and Iyengar (RI, 1979) have attempted to estimate the terms in an exact asymptotic variance formula, which was derived using separate methods by Sander (1975) and Reid (1981b).

The present paper addresses the same question for experiments with fixed censoring times. In particular, we allow only a termination date for censoring all survivors and at most one intermediate censoring date for censoring all survivors among some prespecified subset. Such a design is typical of animal safety studies on experimental new drugs. In this setting, Theorem 1 displays a closed form algebraic expression for the asymptotic variance of the estimated quantiles. Theorem 2 demonstrates that the relationship between the asymptotic variance of the estimated quantiles and that of the KM estimator at fixed time points is the same as the one derived by Sander (1975) and Reid (1981b) in the random censorship model. Combining Theorems 1 and 2 yields a closed form expression for the asymptotic variance of the KM estimator at fixed time points. Hence,

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Greenwood's Formula must converge to this expression by its well-known consistency property, and we believe this to be the first known closed form limit for Greenwood's formula in any framework.

Finally, with the aid of some very limited simulations, we briefly discuss the application of these results to confidence intervals and testing. Some comparisons are made with existing procedures.

2. Notation. Let $F(t)$ be the c.d.f. of the distribution of deaths and $P(t) = 1 - F(t)$ be the corresponding survival function. We assume that F^{-1} exists and is continuous on (0, 1), and that the p.d.f. $f(t) = F'(t)$ exists. Let $t_p = F^{-1}(p)$ be the p-quantile of $F(t)$ and let $\hat{t}_p$ be the Kaplan-Meier estimate of t_p, i.e., $\hat{t}_p = \operatorname{Min}\{t: \hat{P}(t) \le q\}$, where $q = 1 - p$ and where $\hat{P}(t) = 1 - \hat{F}(t)$ is the Kaplan-Meier estimate of $P(t)$. Clearly $\hat{t}_p$ does not exist if $\hat{P}(T_2) > q$, where T_2 is the length of the experiment. Thus, it is valid to consider the properties of $\hat{t}_p$ as an estimator only if T_2 is assumed large enough that $\Pr[\hat{P}(T_2) > q]$ is miniscule enough to ignore this possibility. But then $\hat{t}_p$ is already determined before time T_2, and

$$(*)\ \begin{cases}\text{We can regard all } \textit{final} \text{ censoring as true deaths without}\\ \text{altering results about } \hat{t}_p.\end{cases}$$

Finally, let S be the total number of items tested, and let $X_{(1)}, X_{(2)}, \cdots, X_{(S)}$ represent the ordered *uncensored* death times, some of which we cannot observe. In our asymptotic analysis, we let $S \to \infty$.

3. No intermediate censoring. Let

$$\{x\} = \begin{cases} x & \text{if } x \text{ is an integer} \\ [x] + 1 & \text{otherwise,} \end{cases}$$

i.e., it is the smallest integer not less than x. Since censoring occurs only at the end, $(*)$ implies that $\hat{t}_p = X_{(\{Sp\})}$. Well-known asymptotic theory for random samples now provides that $\hat{t}_p$ has the approximate distribution $N(t_p, pq/Sf^2(t_p))$.

4. One intermediate censoring time (at T_1). Let $\pi = F(T_1) = 1 - \pi^* = 1 - P(T_1)$. Let S_1 be the number of items to be censored at T_1 if they do not die sooner, and let S_2 be the number of items to be tested until the experiment ends (equivalently, using $(*)$, until death). Let N_1 and N_2 be the number of deaths *at or before time* T_1 in each of the above groups, so that $N_i \sim \operatorname{Bin}(S_i, \pi)$ are independent for $i = 1, 2$. Let $N = N_1 + N_2$ and $S = S_1 + S_2$, so that $N \sim \operatorname{Bin}(S, \pi)$, and let $W = N/S$. Note that $\hat{P}(T_1) = 1 - W$. We assume that $S_2/S = \lambda_S$, where $\lambda_S \to \lambda$ for some $0 < \lambda \le 1$ (so that $S_2 \to \infty$); we do *not* require $S_1 \to \infty$.

4.1 *The conditional mean and variance.* We now analyze the conditional distribution of $\hat{t}_p$ given N_1 and N_2, which we will use later to derive the desired unconditional variance.

4.1A *If* $N \geq Sp$, *i.e.*, $\hat{t}_p \leq T_1$, then all censoring occurs after $\hat{t}_p$ and $\hat{t}_p = X_{(\{Sp\})}$, just as in Section 3. But the *conditional* distribution of $\hat{t}_p \mid N_1, N_2 = X_{(\{Sp\})} \mid N_1, N_2$ is the same as the unconditional distribution of $Y_{(\{Sp\})}$, the $\{Sp\}$th order statistic from $Y_1, \cdots, Y_N$, where the Y_i's are i.i.d. with p.d.f. $g(t) = f(t)/\pi$ over the interval $(0, T_1)$. The corresponding c.d.f. is $G(t) = F(t)/\pi$ for $0 < t \leq T_1$, which has inverse function $G^{-1}(u) = F^{-1}(u\pi)$ for $0 < u < 1$. Define

$$\begin{aligned}\alpha = \{Sp\}/(N+1) &= (Sp + O(1))(1/N - 1/N(N+1)) \\ &= p/W + O(1/N) = p/W + O(1/S)\end{aligned}$$

since $1/N \leq 1/Sp$.

LEMMA 1. *Suppose that for some* $\varepsilon > 0$, F^{-1} *has three derivatives in the interval* $((p-\varepsilon)\pi, \pi)$. *Then for* $N \geq Sp$,

(1) $$E(\hat{t}_p \mid N_1, N_2) = F^{-1}(\pi p/W) + V_N, \quad \textit{where} \quad V_N = O(1/S), \quad \textit{and}$$

(2) $$V(\hat{t}_p \mid N_1, N_2) = (1/N)\pi^2\alpha(1-\alpha)/f^2[F^{-1}(\pi\alpha)] + O(1/S^2).$$

PROOF. Since $G^{-1}(u) = F^{-1}(u\pi)$, it follows that $(G^{-1})^{(k)}(u) = \pi^k(F^{-1})^{(k)}(u\pi)$ where the superscript k represents the kth derivative. Since F^{-1} has three derivatives at each point in $((p-\varepsilon)\pi, \pi)$, G^{-1} has three derivatives at each point in $(p-\varepsilon, 1)$. We now refer to section 4.5, page 65, of David (1970). In our case, $r = \{Sp\}$, $n = N$, $p_r = \alpha$, and $Q_r = G^{-1}(\alpha)$. Since $Sp \leq N \leq S$, we have

$$\begin{aligned}&p(1 - 1/(S+1)) \\ &\quad = Sp/(S+1) \leq \{Sp\}/(S+1) \leq \{Sp\}/(N+1) = \alpha \leq \{Sp\}/(Sp+1) < 1.\end{aligned}$$

Hence, *for sufficiently large* S, $p - \varepsilon < \alpha < 1$, and G has three derivatives at α. We can therefore apply David's equation (4.5.3) to obtain

$$\begin{aligned}E(\hat{t}_p \mid N_1, N_2) = EY_{(\{Sp\})} &= G^{-1}(\alpha) + O(1/N) \\ &= F^{-1}(\pi p/W + O(1/S)) + O(1/S) \quad \text{since} \quad 1/N \leq 1/Sp \qquad (3)\\ &= F^{-1}(\pi p/W) + O(1/S),\end{aligned}$$

establishing (1), and we note that only two derivatives of F^{-1} were needed to do so. (The second derivative is needed for the $O(1/N)$ term since the remainder in the Taylor series involves the second derivative at some unknown point between α and $G(X_{(\{Sp\})})$.)

To verify (2), we note that $(G^{-1})'(\alpha) = 1/g[G^{-1}(\alpha)] = \pi/f[F^{-1}(\pi\alpha)]$. Since $1/(N+2) = 1/N + O(1/N^2) = 1/N + O(1/S^2)$, the desired result is a direct application of David's equation (4.5.4), with all three derivatives of F^{-1} required.

4.1B *If* $N < Sp$, *i.e.*, $\hat{t}_p > T_1$, then let $D_2 = S_2 - N_2$ represent the number of items on test beginning at time T_1. Let $W_1, \cdots, W_{D_2}$ be the *uncensored* death times of these items, with order statistics $W_{(1)}, \cdots, W_{(D_2)}$. Then given N_2, we have $W_1, \cdots, W_{D_2}$ i.i.d. with p.d.f. $h(t) = f(t)/\pi^*$ on the interval (T_1, ∞). The corresponding c.d.f. is $H(t) = (F(t) - \pi)/\pi^*$ for $T_1 < t < \infty$, which has inverse $H^{-1}(u) = F^{-1}(\pi + \pi^* u)$ for $0 < u < 1$.

LEMMA 2. *If $N < Sp$, then $\hat{t}_p \mid N_1, N_2 = W_{(\{D_2\gamma\})}$, where $\gamma = [\hat{P}(T_1) - q]/\hat{P}(T_1) = 1 - q/(1 - W)$, i.e., $\hat{t}_p \mid N_1, N_2$ is the sample γ-quantile of the W's. (Note that $0 < \gamma \leq p$ for all $0 \leq N < Sp$.)*

PROOF. For any $j = 1, 2, \cdots, D_2$, $\hat{P}(W_{(j)}) = \hat{P}(T_1)\cdot(D_2 - j)/D_2$ by definition of Kaplan-Meier estimators, so that $\hat{t}_p = W_{(i)}$, where $i = \text{Min}\{j\colon \hat{P}(T_1) \cdot (D_2 - j)/D_2 \leq q\}$. Simple algebra shows that $i = \{D_2\gamma\}$.

LEMMA 3. *Suppose that for some $\varepsilon < 0$, F^{-1} has three derivatives at each point in the interval $(\pi, \pi + \pi^*(p + \varepsilon))$. Let $\beta = \{D_2\gamma\}/(D_2 + 1)$, where γ was defined in Lemma 2. Then for $N < Sp$,*

$$E(\hat{t}_p \mid N_1, N_2) = F^{-1}(1 - \pi^* q/(1 - W)) + R_N, \quad \text{where} \quad R_N = O(1/S), \quad \text{and} \tag{4}$$

$$V(\hat{t}_p \mid N_1, N_2) = (1/D_2)(\pi^*)^2 \beta(1 - \beta)/f^2[F^{-1}(\pi + \pi^*\beta)] + O(1/S^2). \tag{5}$$

PROOF. Since $D_2 \sim \text{Bin}(S_2, \pi^*)$, we have $D_2/S_2 \to_{\text{a.s.}} \pi^*$, $D_2 \to \infty$ with probability 1, and $O(1/D_2) = O(1/S_2) = O(1/S)$. The proof now proceeds along the same lines as in Lemma 1 (with D_2 playing the role previously filled by S), and we omit the details.

4.2 *The unconditional variance.* One of our main results is

THEOREM 1. *Suppose the regularity conditions on F^{-1} from both Lemma 1 and Lemma 3 hold, and fix ε small enough to satisfy both lemmas. Also suppose that $f[F^{-1}(u)] \geq K > 0$ for $u \in (\pi(p - \varepsilon), \pi + \pi^*(p + \varepsilon))$ and some constant K. Then*

$$\lim_{S\to\infty} V(\sqrt{S}\,\hat{t}_p) = \begin{cases} pq/f^2(t_p) & \text{if } p \leq \pi \\ (q/\lambda f^2(t_p))[1 - q(1 - \lambda\pi)/\pi^*] & \text{if } p > \pi. \end{cases} \tag{6}$$

The proof is given in the Appendix. Note the intuitively satisfying results given by (6) for the special cases $\pi = 0$, $\pi = 1$, and $\lambda = 1$.

5. Relationship to the variance of the Kaplan-Meier Estimator at a fixed point and to Greenwood's Formula.

The following result is analogous to a result for the random censorship model that was proved by Sander (1975) and Reid (1981b) using separate methods:

THEOREM 2. *Regardless of whether $p \leq \pi$ or $p > \pi$,*

$$\lim_{S\to\infty} V(\sqrt{S}\hat{P}(t_p)) = f^2(t_p) \cdot \lim_{S\to\infty} V(\sqrt{S}\,\hat{t}_p). \tag{7}$$

(Thus the right side of (6) without the $f^2(t_p)$ provides a closed form expression for the asymptotic variance of the Kaplan-Meier estimator at the fixed point t_p).

We mention before the proof that we not *not* advocate using (7) in practice as a substitute for Greenwood's Formula. Once the data have been observed, Greenwood's Formula has the advantage over (7) of conditioning upon the observed censoring pattern and should therefore be better in small samples. The main practical application of (7) would be to gauge sample size by estimating the precision of $\hat{P}(t_p)$ *before* the data are observed. Nevertheless, Theorem 2 establishes a closed form expression to which Greenwood's Formula converges (by its known consistency), conferring theoretical interest upon the theorem.

PROOF OF THEOREM 2.

CASE 1. If $p \le \pi$, then $V(\hat{P}(t_p)) = V(\hat{F}(t_p)) = pq/S$, and the result follows from (6).

CASE 2. If $p > \pi$, then $t_p > T_1$. Since $D_2 \to \infty$ with probability one, $\lim V(\sqrt{S}\hat{P}(t_p)) = \lim V(\sqrt{S}\hat{P}(t_p) \mid D_2 > 0)$. (Strictly speaking, $V(\hat{P}(t_p))$ must be *defined* conditionally since $\hat{P}(t_p)$ is undefined whenever $D_2 = 0$, which has positive probability for any finite S, so that no unconditional variance exists to examine for a limiting value.) For the remainder of the proof, we therefore regard all means, variances, and covariances to be conditional upon $D_2 > 0$ (equivalently, $N_2 < S_2$) even when not explicitly stated.

Let N^* be the number of deaths in $(T_1, t_p]$, so that $N^* \sim \mathrm{Bin}(D_2, (p-\pi)/\pi^*)$, and let $W^* = N^*/D_2$. Then $\hat{P}(t_p) = (1 - W)(1 - W^*)$, and by the delta method

$$\lim V[\sqrt{S}\hat{P}(t_p)] = \lim\{(1 - EW)^2 SV(W^*) + (1 - EW^*)^2 SV(W) + (1 - EW)(1 - EW^*)S\,\mathrm{Cov}(W, W^*)\}. \tag{8}$$

Using results on truncated binomial distributions given on pages 73–74 of Johnson and Kotz (1969),

$$EW = (\pi/S)(S_1 + S_2(1 - \pi^{S_2 - 1})/(1 - \pi^{S_2})) \to \pi,$$

$$SV(W) = S_1\pi\pi^*/S + (S_2/[S(1 - \pi^{S_2})]) \cdot \{\pi\pi^* - S_2\pi^{S_2} + S_2\pi^2(1 - (1 - \pi^{S_2 - 1})^2/(1 - \pi^{S_2}))\} \to \pi\pi^*,$$

$$EW^* = (p - \pi)/\pi^*, \quad V(W^* \mid D_2) = q(p - \pi)/[(\pi^*)^2 D_2],$$

$$V(W^*) = E[V(W^* \mid D_2)] + V[E(W^* \mid D_2)] = E[V(W^* \mid D_2)], \text{ i.e.,}$$

$$SV(W^*) = \frac{Sq(p-\pi)}{(\pi^*)^2} E\left(\frac{1}{D_2} \,\middle|\, D_2 > 0\right) \to_{S\to\infty} \frac{q(p-\pi)}{(\pi^*)^2}\frac{S}{S_2\pi^* - \pi} \to \frac{q(p-\pi)}{\lambda(\pi^*)^3}. \tag{9}$$

Finally, $\mathrm{Cov}(W, W^*) = E[\mathrm{Cov}(W, W^*) \mid D_2] + \mathrm{Cov}[E(W \mid D_2), E(W^* \mid D_2)] = 0$ because (i) W and W^* are conditionally independent and (ii) $E(W^* \mid D_2)$ is a constant that does not depend on D_2. Note that W and W^* are uncorrelated but *not* independent, since their distributions both depend on D_2. The theorem now follows by substituting into (8) and comparing the result to (6).

6. Density estimation, simulations, and comparison with other procedures. Fryer (1977) surveys various density estimation techniques in the uncensored case. The obvious generalization of these techniques to the censored case is to substitute $\hat{F}(t)$ for the empirical c.d.f.; thus, $(1/N) \sum h(X_i)$ becomes $\int h(t)\, d\hat{F}(t)$. Földes, Rejtö and Winter (1981) show that the generalized versions of histogram estimators and kernel estimators are strongly consistent. For the specific case $S = 60$, $\lambda = 5/6$, and $p = .25$, a simulation study was undertaken to evaluate the performance of the generalization of a simple histogram estimator. Specifically, for both $\delta = .05$ and $\delta = .15$, we looked at $\hat{f}(t_{.25}) = 2\delta(\hat{t}_{.25+\delta} - \hat{t}_{.25-\delta})$; a wide variety of Weibull distributions (hazards increasing, constant, and decreasing) was used for the death times, and a wide range of censoring times was studied. The value $\delta = .05$ was unsatisfactory; the bias was small, but the variability in (6) was far too great. The larger bias for $\delta = .15$ was still quite acceptable, and the estimates (6) were acceptably stable in all cases. Thus, we have better results than Reid and Iyengar (1979), who agree that it is more important to control variance than bias and whose conjecture that density estimation would work better under fixed censoring appears to be correct. Finally, the tails of the standardized statistic $T = \sqrt{S}(\hat{t}_p - t_p)/\sqrt{(6)}$ were fit very well by a standard normal (not t!) distribution. This makes T easy to use in practice for confidence intervals and for one- and two-sample tests.

REMARK. Based on the discussion between Theorem 2 and its proof, it probably would be better in practice to use Greenwood's Formula divided by $\hat{f}^2(t_p)$ than (6). Also, far more sophisticated density estimation techniques are available. With neither of these, our simulations still gave satisfactory results, and we do not conjecture that these improvements would have made a large difference.

The only other two-sample (or k-sample) test based on the survival quantiles of which we are aware is Brookmeyer and Crowley (1982b), and we have no basis for comparing our procedure to theirs. Much more literature exists on confidence intervals (and hence one-sample tests); seven references were given in the introduction, and we believe that our procedure has advantages over most of them. All of the procedures (TG, BC, Emerson, and ABP) which invert tests of the form H_0: $P(t) = q$ for fixed t cannot handle t's such that $\hat{P}(t)$ is undefined. They also have difficulty when $\hat{P}(t) = 0$ or 1 (except for TG's Z_2-method); ABP admit this is a problem and propose an admittedly unsatisfying solution. Thus, these procedures can never include points beyond the termination time of the study in their confidence sets. It is also mentioned by both BC and ABP that the confidence sets determined by these procedures are not necessarily intervals. Our procedure has neither of these drawbacks. Perhaps the bootstrap methods of Efron or Reid outperform the others, but they require more computing capacity than is routinely available to most of us. Finally, all of these procedures were derived (and simulated, if applicable) under the random censorship model, and their properties under fixed censoring are not really known; in fact, our procedure with the Greenwood modification in the above remark is essentially that of RI, but ours proved more satisfactory because of this difference in the model.

7. Appendix: Proof of Theorem 1. We split the variance into its two conditional components:

$$\begin{aligned}\lim_{S\to\infty} V(\sqrt{S}\hat{t}_p) &= \lim_{S\to\infty} V[E(\sqrt{S}\hat{t}_p \mid N_1, N_2)] + \lim_{S\to\infty} E[V(\sqrt{S}\hat{t}_p \mid N_1, N_2)] \\ &= \lim_{S\to\infty} S V[E(\hat{t}_p \mid N_1, N_2)] + \lim_{S\to\infty} E[V(\sqrt{S}\hat{t}_p \mid N_1, N_2)].\end{aligned} \tag{10}$$

We will deal with the second term in (10) first. Combining (2) and (5), and multiplying by $\sqrt{S}$, we obtain

$$\begin{aligned} &V(\sqrt{S}\hat{t}_p \mid N_1, N_2) \\ &= \begin{cases}(S/N)\pi^2\alpha(1-\alpha)/f^2[F^{-1}(\pi\alpha)] + O(1/S) & \text{if } N \geq Sp \\ (S/S_2)\cdot(S_2/D_2)(\pi^*)^2\beta(1-\beta)/f^2[F^{-1}(\pi+\pi^*\beta)] + O(1/S) & \text{if } N < Sp.\end{cases}\end{aligned} \tag{11}$$

As $S \to \infty$, we note that $S/N \to_{\text{a.s.}} 1/\pi$, $\alpha \to_{\text{a.s.}} p/\pi$, $S/S_2 \to 1/\lambda$, $S_2/D_2 \to_{\text{a.s.}} 1/\pi^*$,

$$\beta = \gamma + O(1/S) \to_{\text{a.s.}} 1 - q/\pi^*,$$

and

$$P(N \geq Sp) \to \begin{cases} 1 & \text{if } p < \pi \\ 1/2 & \text{if } p = \pi. \\ 0 & \text{if } p > \pi \end{cases}$$

Applying these results to (11), we see that

$$V(\sqrt{S}\hat{t}_p \mid N_1, N_2) \to_{\text{a.s.}} \begin{cases} p(1-p/\pi)/f^2(t_p) & \text{if } p \leq \pi \\ (q/\lambda)(1-q/\pi^*)/f^2(t_p) & \text{if } p > \pi. \end{cases} \tag{12}$$

If we examine the boundedness hypothesis on $f[F^{-1}(u)]$ together with (11) and the lines that follow it, it becomes clear that $V(\sqrt{S}\hat{t}_p \mid N_1, N_2)$ is bounded for sufficiently large S. The Bounded Convergence Theorem may therefore be applied, yielding

$$\begin{aligned}\lim_{S\to\infty} E[V(\sqrt{S}\hat{t}_p \mid N_1, N_2)] &= E[\lim_{S\to\infty} V(\sqrt{S}\hat{t}_p \mid N_1, N_2)] \\ &= E[(12)] = (12),\end{aligned} \tag{13}$$

completing the evaluation of the second term in (10).

To obtain the first term in (10), we combine (1) and (4) to yield

$$\begin{aligned} Z_N &= E(\hat{t}_p \mid N_1, N_2) \\ &= \begin{cases} F^{-1}(\pi p/W) + V_N, & \text{where } V_N = O(1/S) \text{ if } N \geq Sp \\ F^{-1}(1-\pi^* q/(1-W)) + R_N, & \text{where } R_N = O(1/S) \text{ if } N < Sp. \end{cases}\end{aligned} \tag{14}$$

Working only with the first line ($N \geq Sp$) of (14), we expand (14) in a Taylor series around $W = \pi$, which yields (for some constant C)

$$Z_N = t_p - (p/\pi f(t_p))(W - \pi) + C(W - \pi)^2 + O(W - \pi)^3 + V_N \quad \text{if} \quad N \geq Sp. \tag{15}$$

Since $E(W - \pi)^3 = \pi\pi^*(\pi^* - \pi)/S^2 = O(1/S^2)$, we have

$$EZ_N = t_p + C(\pi\pi^*/S) + O(1/S^2) + EV_N \qquad \text{if} \quad N \geq Sp, \tag{16}$$

$$(EZ_N)^2 = t_p^2 + 2Ct_p(\pi\pi^*/S) + 2t_p EV_N + O(1/S^2) \quad \text{if} \quad N \geq Sp, \tag{17}$$

since $EV_N = O(1/S)$. Squaring (15), we get

$$\begin{aligned} Z_N^2 = t_p^2 - \frac{2pt_p}{\pi f(t_p)}(W - \pi) + \left[\frac{p^2}{\pi^2 f^2(t_p)} + 2Ct_p\right](W - \pi)^2 + 2t_p V_N \\ - \frac{2p}{\pi f(t_p)} V_N(W - \pi) + O(W - \pi)^3 + O(V_N(W - \pi)^2) \quad \text{if} \quad N \geq Sp, \end{aligned} \tag{18}$$

$$EZ_N^2 = t_p^2 + \left[\frac{p^2}{\pi^2 f^2(t_p)} + 2Ct_p\right]\frac{\pi\pi^*}{S} + 2t_p EV_N + O\left(\frac{1}{S^{3/2}}\right) \qquad \text{if} \quad N \geq Sp, \tag{19}$$

where the $O(1/S^{3/2})$ term comes from $E[V_N(W - \pi)]$, since V_N is $O(1/S)$, while $W - \pi$ is both (i) bounded and (ii) $O_p(1/\sqrt{S})$. Finally,

$$VZ_N = (19) - (17) = p^2\pi^*/S\pi f^2(t_p) + O(1/S^{3/2}) \quad \text{if} \quad N \geq Sp. \tag{20}$$

Repeating the steps leading to (15)–(20) on the *second* line of (14) yields

$$VZ_N = q^2\pi/S\pi^* f^2(t_p) + O(1/S^{3/2}) \quad \text{if} \quad N < Sp, \tag{21}$$

and combining (20) and (21) yields

$$\lim_{S\to\infty} S \cdot VZ_N = \begin{cases} p^2\pi^*/\pi f^2(t_p) & \text{if} \quad N \geq Sp \\ q^2\pi/\pi^* f^2(t_p) & \text{if} \quad N < Sp, \end{cases} \tag{22}$$

completing the evaluation of the first term in (10). The theorem follows immediately from (10), (13), (22), and a little algebra.

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