The Annals of Statistics
1993, Vol. 21, No. 1, 391–401

ON REACHING A CONSENSUS USING DEGROOT'S ITERATIVE POOLING

BY GUSTAVO L. GILARDONI[1] AND MURRAY K. CLAYTON

University of Illinois, Chicago and University of Wisconsin-Madison

We consider a group of k experts each having a subjective probability distribution for a parameter θ. If the members of the group are allowed to know the others' opinions and they appreciate the others' skills, it is likely that each expert will modify his distribution to account for this new information. This process can be continued indefinitely leading to an iterative pooling process. The main issue is whether the experts' distributions will converge towards a common limit or consensus.

Several authors have considered this iterative process when the experts' distributions at a given stage are linear opinion pools of the distributions at the previous stage. In this paper we extend the model for the specific case where the experts use logarithmic opinion pools and, more broadly, for pools in a wide class that generalizes both the linear and the logarithmic pools. It is shown that the consensus properties in the logarithmic pool case are essentially the same as in the linear pool case, and that this fact uniquely characterizes both pools in the wide class mentioned above.

1. Introduction and notation. We consider a group of k experts $E_1, \dots, E_k$, $k \geq 2$, each having a subjective probability distribution P_{i0}, $1 \leq i \leq k$, for a certain parameter $\theta \in \Theta$. We will assume that the experts' probabilities have a Radon–Nikodym derivative with respect to a given measure ν, and denote this derivative or density by $f_{i0} = dP_{i0}/d\nu$.

DeGroot's *Iterative Pooling Scheme*, in spirit a formalization of the Delphi technique for expert panels, can be described as follows: Feedback is allowed among the experts and, after learning f_{j0}, $j \neq i$, E_i will modify his density f_{i0} to $f_{i1} = T_i(f_{10}, \dots, f_{k0})$, where T_i is a given pooling operator, $1 \leq i \leq k$. After this first stage is completed, feedback is allowed again: E_i will modify his density to $f_{i2} = T_i(f_{11}, \dots, f_{k1})$, $1 \leq i \leq k$. This process is continued indefinitely, and thus the iterative scheme is defined by the recursive equation $f_{i,n+1} = T_i(f_{1n}, \dots, f_{kn})$ for $n \geq 0$.We will say that the experts reach a consensus if there is a density f_∞ such that $\lim_n f_{in} = f_\infty$ for every i.

This kind of iterative scheme is discussed in DeGroot (1974) in the case that the expert' pools T_i are weighted arithmetic averages or *linear opinion pools* (LinOP's). Earlier discussions of a similar scheme appear in work of French

Received January 1991; revised May 1992.

[1]Research partially supported by a grant from the College of Business Administration of the University of Illinois at Chicago and by Conselho Nacional de Desenvolvimento Científico e Technológico-CNPq.

AMS 1991 *subject classifications*. Primary 62A15; secondary 60J20, 90A07, 39B12.

Key words and phrases. Logarithmic opinion pool, linear opinion pool, quasi-linear opinion pool, expert opinions, Bayesian inference.

(1956) and Harary (1959). In the LinOP case we have

$$(1.1) \qquad f_{i,n+1} = \sum_{j=1}^{k} w_{ij} f_{jn}, \qquad 1 \le i \le k, n \ge 0,$$

where the w_{ij}'s are weights that E_i assigns to E_j's opinions, $w_{ij} \ge 0$, $\Sigma_j w_{ij} = 1$, $1 \le i, j \le k$ [for a discussion of the LinOP see, e.g., Genest and Zidek (1986) and Genest and McConway (1990)]. In this setting, if $\mathbf{f}_n = (f_{1n}, \dots, f_{kn})'$ and W is the matrix with entries w_{ij}, then (1.1) becomes $\mathbf{f}_n = W\mathbf{f}_{n-1} = \cdots = W^n \mathbf{f}_0$. It is clear then that a sufficient condition for a consensus to be reached is that W^n converges to a matrix having identical rows, that is, that the Markov chain with transition probabilities W has only one recurrent class which is aperiodic. Furthermore, the final consensus will then be a LinOP of the experts' initial densities f_{i0} with weights π_i, where $\pi = (\pi_1, \dots, \pi_k)$ is the only stationary probability vector associated with W [DeGroot (1974), Theorems 2 and 3]. It is worth noting here that the same results obtain not only when the experts opinions are represented as probabilities or densities, but also when they are represented as points in a convex subset of an arbitrary linear space. The iterative model using LinOP's has been further studied by Lehrer and Wagner (1981), Chatterjee (1975), Chatterjee and Seneta (1977) and Berger (1981) among others. In this paper, we will consider extensions of DeGroot's model to pools other than LinOP's. More precisely, we examine pools T_i which are:

(i) Modified weighted geometric averages or *logarithmic opinion pools* (LogOP's) with weights w_{ij}, when the iterative process is given by

$$(1.2) \qquad f_{i,n+1} = T_i(f_{1n}, \dots, f_{kn}) = \frac{\prod_j f_{jn}^{w_{ij}}}{\int_\Theta \left(\prod_j f_{jn}^{w_{ij}}\right) d\nu} \qquad n \ge 0.$$

(ii) *g-Quasi-linear opinion pools* (g-QLOP's) with weights w_{ij},

$$(1.3) \quad f_{i,n+1} = T_i(f_{1n}, \dots, f_{kn}) = \frac{g^{-1}\left(\Sigma_j w_{ij} g(f_{jn})\right)}{\int_\Theta g^{-1}\left(\Sigma_j w_{ij} g(f_{jn})\right) d\nu} \qquad n \ge 0,$$

where g is a continuous, strictly monotone function. Note that, since $(a + bg)^{-1}(y) = g^{-1}((y - a)/b)$, the g-QLOP is identical to the $(a + bg)$-QLOP for any $b \neq 0$. Therefore, we can assume without loss of generality that g is strictly increasing. In both cases we assume that $W = (w_{ij})$ is stochastic, that is, $w_{ij} \ge 0$ and $\Sigma_j w_{ij} = 1$ for all i. [See, e.g., Genest (1984a, b) and Winkler (1968) for discussion of this latter restriction.] Also, we define $0^0 = 1$ throughout this paper.

Some properties of the g-QLOP's in (1.3) are discussed in Gilardoni (1989). For instance, by choosing $g(x) = \log(x + \varepsilon)$ for a given, small ε, they can be used as an alternative to LogOP's that avoids the extreme behavior of the LogOP, discussed below, when the experts' densities have different support.

Lehrer and Wagner (1981) considered what they called a *quasiarithmetic averaging function* of $f_{1n}, \dots, f_{kn}$ as $g^{-1}(\Sigma_j w_{ij} g(f_{j0}))$, and established a limit theorem for iterations of such pools. The g-QLOP's (1.3) deal with a modified version of the quasiarithmetic averaging functions that takes into account the restriction imposed by the fact that $\int_\Theta f_{in} = 1$. Also, note that the g-QLOP reduces to the LinOP when $g(x) = x$ and to the LogOP when $g(x) = \log x$.

Given a stochastic matrix W with entries w_{ij}, we will denote the entries of W^n by $w_{ij}^{(n)}$. Following Seneta (1981) we will say that the stochastic matrix W is *regular* if the Markov chain with first step transition probabilities W has only one recurrent class which is aperiodic or, equivalently, if W has the eigenvalue 1 with multiplicity 1. For a general reference on the Markov chain properties used in the rest of the paper the reader is referred to Karlin and Taylor (1975, 1981). Also, if g_1 and g_2 are two measurable, real valued functions on Θ, the notation $g_1 = g_2$ means that g_1 equals g_2 up to a set of ν-measure zero.

In Section 2 we show that DeGroot's results apply when the experts' pools are LogOP's. A rather surprising result in this extension of DeGroot's scheme is that the LinOP and LogOP are uniquely characterized in the class of g-QLOP in terms of the form of the final consensus. This result is shown in Section 3. Finally, a few concluding remarks are given in Section 4.

2. The LogOP case. We begin this section by considering the iterative process defined by (1.2) when the densities f_{i0} have identical supports. Without loss of generality we can assume this support to be Θ and $f_{i0}(\theta) > 0$ for every i and every $\theta \in \Theta$. In this case we have the following theorem.

THEOREM 1. *Suppose that the densities f_{i0} have support Θ, $1 \le i \le k$, and that W is regular. Then a consensus is reached. The consensual density is given by*

$$f_\infty = \frac{\prod_i f_{i0}^{\pi_i}}{\int_\Theta (\prod_i f_{i0}^{\pi_i})\, d\nu}, \tag{2.1}$$

where $\pi = (\pi_1, \dots, \pi_k)$ is the unique stationary vector associated with W, that is, the unique solution to the system of linear equations

$$\begin{aligned} \pi W &= \pi, \\ \pi_1 + \cdots + \pi_k &= 1. \end{aligned} \tag{2.2}$$

PROOF. Fix $\theta_0 \in \Theta$ and define $\phi_{in}(\theta) = \log[f_{in}(\theta)/f_{in}(\theta_0)]$. It is straightforward from (1.2) that $\phi_{i,n+1} = \Sigma_j w_{ij} \phi_{jn}$. Iterating this gives $\phi_{in} = \Sigma_j w_{ij}^{(n)} \phi_{j0}$. Therefore, from Theorems 2 and 3 in DeGroot (1974) we have that $\lim_n \phi_{in} = \Sigma_i \pi_i \phi_{i0} = \phi_\infty$, say, for every $1 \le i \le k$. Now, since $f_{in}(\theta) = f_{in}(\theta_0)\exp\{\phi_{in}(\theta)\}$, and using the fact that f_{in} integrates to 1, we have that $f_{in}(\theta_0) =$

$[\int \exp(\phi_{in})\,d\nu]^{-1}$. Therefore,

$$\lim_n f_{in} = \lim_n \frac{\exp(\phi_{in})}{\int \exp(\phi_{in})\,d\nu} = \frac{\exp(\phi_\infty)}{\lim_n \int \exp(\phi_{in})\,d\nu}, \tag{2.3}$$

provided that the limit of the denominator of the rightmost term exists. To show this we only need to prove that $\lim_n \int \exp(\phi_{in})\,d\nu = \int \exp(\phi_\infty)\,d\nu$. But this follows from dominated convergence and Hölder's inequality, since

$$\begin{aligned}\exp(\phi_{in}(\theta)) &= \exp\left[\sum_j w_{ij}^{(n)}\phi_{j0}(\theta)\right]\\ &= \frac{\prod_j [f_{j0}(\theta)]^{w_{ij}^{(n)}}}{\prod_j [f_{j0}(\theta_0)]^{w_{ij}^{(n)}}} \le \frac{\sum_j w_{ij}^{(n)} f_{j0}(\theta)}{\min_j f_{j0}(\theta_0)} \le \frac{\sum_j f_{j0}(\theta)}{\min_j f_{j0}(\theta_0)}. \qquad \square\end{aligned}$$

NOTE. By using a similar argument it is possible to state a stronger version of this theorem in terms of a sort of uniform convergence: If B is any measurable subset of Θ, then $\lim_n P_{in}(B) = \lim_n \int_B f_{in}\,d\nu = \int_B \exp(\phi_\infty)\,d\nu / \int_\Theta \exp(\phi_\infty)\,d\nu$.

Using Theorem 1, the numerical examples in DeGroot (1974) are easily extended to the LogOP case. The following example shows an application of Theorem 1 in a somewhat different setting. It makes use of an important property of the LogOP, namely, that it preserves the structure of exponential families.

EXAMPLE 1. Assume that W is regular, and let the initial densities be members of the same exponential family, $f_{i0}(\theta) \propto H(\theta)\exp(\sum_{s=1}^{t} \eta_{s_{i0}} Q_s(\theta))$, where ν is Lebesgue measure. Theorem 1 implies that the final consensus will be a LogOP of the initial densities f_{i0} with weights π_i given by (2.2). Simple algebra then shows that the consensus itself belongs to the same exponential family with "natural parameters" given by $\eta_{s_\infty} = \sum_i \pi_i \eta_{s_{i0}}$, $1 \le s \le t$.

We end this section with a few comments regarding the LogOP when the densities f_{i0} do not have identical supports. First, note that in the LinOP case or the LogOP case with identical support, the experts corresponding to transient states in the associated Markov chain are irrelevant for the determination of the consensus (since $\pi_i = 0$ if the ith state is transient). This is no longer true in the LogOP case if the hypothesis of identical support is dropped, since the LogOP possesses the *zero preservation law* "to a dramatic degree" [French (1985)]. Specifically, if E_j believes at the initial stage that a set B has probability 0, then at the next stage E_i will also assign probability 0 to B, provided that $w_{ij} > 0$. This makes it possible for a transient expert to *veto* events as long as there is at least one positive entry in the corresponding column of W^n. It should be clear that the idea we used to prove Theorem 1 cannot be applied directly, since when allowing $\phi_{in} = -\infty$ we will lose the linear structure that was needed in order to apply DeGroot's theorems. Sufficient conditions appear in the following theorem, whose proof may be found in Gilardoni (1989).

THEOREM 2. *Suppose that W is regular. Let B_{i0} be the support of the initial density f_{i0} and I the set of indices corresponding to recurrent states, and suppose that $\nu(\bigcap_{i \in I} B_{i0}) > 0$. If* (i) *$B_{j0} \supset \bigcap_{i \in I} B_{i0}$ for every $j \notin I$, or* (ii) *$w_{ij} > 0$ for every $1 \leq i, j \leq k$; then a consensus is reached. The consensual density is given by* (2.1), *where π is the unique solution to* (2.2).

EXAMPLE 1 (Continued). Assume that all the experts' weights are positive, and consider initial densities $f_{i0}(\theta) \propto H(\theta)\exp(\sum_{s=1}^{t} \eta_{s_{i0}} Q_s(\theta)) I(a_{i0} \leq \theta \leq b_{i0})$, where $I(A)$ denotes the characteristic function of a set A and ν is Lebesgue measure. Theorem 2(ii) implies that a consensus will be reached provided that $a_\infty = \max_i a_{i0} < \min_i b_{i0} = b_\infty$. The consensual density will be given by $f_\infty(\theta) \propto H(\theta)\exp(\sum_{s=1}^{t} \eta_{s_\infty} Q_s(\theta)) I(a_\infty \leq \theta \leq b_\infty)$, where $\eta_{s_\infty} = \sum_i \pi_i \eta_{s_{i0}}$, $1 \leq s \leq t$.

3. A characterization of the LinOP and the LogOP. We now turn to the problem of studying the form of the final consensus in the g-QLOP setting. [Some preliminary results on whether a consensus will be reached in the g-QLOP setting have been given in Gilardoni (1989), and will not be pursued here.] As mentioned before, in the case that $g(x) = a + bx$ or $g(x) = a + b \log x$, the g-QLOP coincides, respectively, with the LinOP and the LogOP. Therefore, *in those two cases*, Theorems 2 and 3 in DeGroot (1974) and Theorem 1 here can be expressed in the following way: If the densities f_{i0} have the same support and if in addition W is regular, then the consensus density is

$$(3.1) \qquad f_\infty = \frac{g^{-1}\left(\sum_j \pi_j g(f_{j0})\right)}{\int_\Theta g^{-1}\left(\sum_j \pi_j g(f_{j0})\right) d\nu},$$

where π is the unique solution to (2.2). This equation is good news for the experts, for then the final consensus can be seen as a LogOP (LinOP) that is a sort of compromise between the different LogOP's (LinOP's) sponsored by each individual member of the group. It is also good news for the statistician or decision-making professional who is consulting with the experts, for then he would not have to go through a numerical implementation of the iterative process in order to obtain an approximate consensual density, but he could resort instead to the exact (and easy to compute) consensus (3.1). A natural question is then whether there are any g's, other than linear and logarithmic, for which the final consensus is (3.1). The surprising answer is, under quite general conditions, *no*. In the rest of this section we will show that the above mentioned property uniquely characterizes the LinOP and the LogOP in the class of g-QLOP's.

Before proceeding we need to introduce some additional notation and terminology. The measure space $(\Theta, \mathscr{A}, \nu)$ will be assumed to be such that

$\nu(\Theta)$ is finite. We will say that $(\Theta, \mathscr{A}, \nu)$ is *at least tertiary* if there exists a measurable partition A_1, A_2, A_3 of Θ with $\nu(A_r) > 0$, $r = 1, 2, 3$. The space is *at least binary* if there exists a partition A_1, A_2 with the above property. It is *binary* if it is at least binary but it is not at least tertiary. Also, throughout this section g will be assumed to be an increasing function such that both g and g^{-1} have continuous first derivatives. The function g will be said to be *k-Markovian* if the process defined by (1.3) yields the consensus (3.1) for every regular $k \times k$ matrix W and for all initial positive densities f_{i0}, $1 \le i \le k$, where π is, as before, the unique solution to (2.2). Observe that the assertion that g is k-Markovian implicitly assumes that $\lim_n f_{in}$ exists, $1 \le i \le k$.

We will begin considering the case where $\Theta = \{\theta_1, \dots, \theta_s\}$ is finite, $s \ge 2$, and all the subsets of Θ are measurable. Let $\nu_r = \nu(\{\theta_r\})$, and assume without loss of generality that $0 < \nu_1 \le \nu_2 \le \cdots \le \nu_s$. Let $a_{irn} = f_{in}(\theta_r)$, $1 \le i \le k$, $1 \le r \le s$. Then (1.3) becomes

$$a_{ir,n+1} = \frac{g^{-1}\big(\Sigma_j w_{ij} g(a_{jrn})\big)}{\Sigma_t \nu_t g^{-1}\big(\Sigma_j w_{ij} g(a_{jtn})\big)}. \tag{3.2}$$

It is easy to show that if g is k-Markovian, then g is l-Markovian for $l \le k$. We begin, therefore, by characterizing 2-Markovian functions.

LEMMA 1. *Let Θ be finite, $s \ge 2$, $\mathscr{A} = 2^\Theta$, and suppose that g is 2-Markovian. Then*

$$\begin{aligned} &a_{120} g'(a_{120})\big[g(a_{211}) - w_{22} g(a_{210}) - (1 - w_{22}) g(a_{110})\big] \\ &\quad \equiv a_{110} g'(a_{110})\big[g(a_{221}) - w_{22} g(a_{220}) - (1 - w_{22}) g(a_{120})\big], \end{aligned} \tag{3.3}$$

where "$\equiv$" means "for every a_{ir0} such that $0 < a_{ir0} < \nu_r^{-1}$ and $\Sigma_r \nu_r a_{ir0} = 1$, $i = 1, 2$; $1 \le r \le s$, and for every $0 < w_{22} < 1$."

PROOF. The final consensus must be the same whether we start the iteration with a_{ir0} or with a_{ir1}, provided they satisfy the recursion (3.2). Therefore, we must have that

$$\begin{aligned} &\frac{g^{-1}\big(\pi_1 g(a_{1r0}) + \pi_2 g(a_{2r0})\big)}{\Sigma_t \nu_t g^{-1}\big(\pi_1 g(a_{1t0}) + \pi_2 g(a_{2t0})\big)} \\ &\qquad \equiv \frac{g^{-1}\big(\pi_1 g(a_{1r1}) + \pi_2 g(a_{2r1})\big)}{\Sigma_t \nu_t g^{-1}\big(\pi_1 g(a_{1t1}) + \pi_2 g(a_{2t1})\big)}, \end{aligned} \tag{3.4}$$

where $\pi_1 = (1 - w_{22})/(1 - w_{11} + 1 - w_{22})$ and $\pi_2 = 1 - \pi_1$. By differentiating both sides of (3.4) with respect to w_{11} and taking limits as $w_{11} \uparrow 1$ we obtain, after some straightforward but rather tedious algebra, that

$$\begin{aligned} &a_{1r0}^{-1}\left[w_{22}\frac{g(a_{1r0}) - g(a_{2r0})}{g'(a_{1r0})} - \frac{g(a_{1r0}) - g(a_{2r1})}{g'(a_{1r0})}\right] \\ &\quad \equiv w_{22}\sum_t \nu_t \frac{g(a_{1t0}) - g(a_{2t0})}{g'(a_{1t0})} - \sum_t \nu_t \frac{g(a_{1t0}) - g(a_{2t1})}{g'(a_{1t0})}. \end{aligned}$$

Since the right-hand side in this identity is independent of r, this implies that

$$a_{110}^{-1}\left[w_{22}\frac{g(a_{110})-g(a_{210})}{g'(a_{110})}-\frac{g(a_{110})-g(a_{211})}{g'(a_{110})}\right]$$
$$\equiv a_{120}^{-1}\left[w_{22}\frac{g(a_{120})-g(a_{220})}{g'(a_{120})}-\frac{g(a_{120})-g(a_{221})}{g'(a_{120})}\right],$$

which is equivalent to (3.3). □

LEMMA 2. *Let Θ be finite, $s \geq 2$ and $\mathscr{A} = 2^{\Theta}$. Define $\mu = \nu(\Theta) = \sum_r \nu_r$, and suppose that g is 2-Markovian. Then g satisfies the identity*

$$\text{(3.5)} \quad [a_{110}g'(a_{110}) - a_{120}g'(a_{120})]\left[\sum_r \nu_r g(a_{1r0}) - \mu g(\mu^{-1})\right] \equiv 0.$$

PROOF. Taking derivatives with respect to w_{22} and limits as $w_{21} \uparrow 1$ in (3.3) we obtain

$$-a_{120}g'(a_{120})a_{210}g'(a_{210})\sum_r \nu_r \frac{g(a_{2r0})-g(a_{1r0})}{g'(a_{2r0})}$$
$$\equiv -a_{110}g'(a_{110})a_{220}g'(a_{220})\sum_r \nu_r \frac{g(a_{r0})-g(a_{1r0})}{g'(a_{2r0})}.$$

The lemma now follows after choosing $a_{2r0} = \mu^{-1}$, $1 \leq r \leq s$. □

The next lemma will prove that in order for the identity (3.5) to hold, we must have that at least one of the terms in brackets be identically zero.

LEMMA 3. *Let Θ be finite, $s \geq 2$, $\mathscr{A} = 2^{\Theta}$, and suppose that g is 2-Markovian. Then either*

$$\text{(3.6)} \qquad xg'(x) = yg'(y)$$

for every $(x, y) \in S$ or

$$\text{(3.7)} \quad \nu_1 g(x) + \nu_2 g(y) + (\mu - \nu_1 - \nu_2) g\left(\frac{1 - \nu_1 x - \nu_2 y}{\mu - \nu_1 - \nu_2}\right) = \mu g(\mu^{-1}),$$

for every $(x, y) \in S$, where $S = \{(x, y)\colon 0 < x, y;\ \nu_1 x + \nu_2 y < 1\}$.

PROOF. Let $S_1 = S \cap \{(x, y)\colon x < y\}$, $S_2 = S \cap \{(x, y)\colon y < x\}$, and denote by A and B the sets of points in S such that (3.6) and (3.7) hold, respectively.

In (3.5) choose $a_{110} = x$, $a_{120} = y$ and $a_{1r0} = (\mu - \nu_1 - \nu_2)^{-1}(1 - \nu_1 x - \nu_2 y)$, $3 \leq r \leq s$. Then we have that for every $(x, y) \in S$ either (3.6) or (3.7) holds, or equivalently that $A \cup B = S$. We need to show that either $A = S$ or $B = S$. The rest of the proof will be split into several steps.

(i) $A \cup \overline{B^0} = S$, where $\overline{B^0}$ is the closure of the interior of B.

Because of the continuity of g, g^{-1}, g' and $(g^{-1})'$, both A and B are relatively closed in S. Therefore $S - A$ is an open set contained in B, and hence it is contained in B^0. It follows that $S - A \subset \overline{B^0}$ or equivalently that $S \subset A \cup \overline{B^0}$.

(ii) Either $A \cap S_1 = \varnothing$ or $\overline{B^0} \cap S_1 = \varnothing$.

Suppose neither of the two equalities is true. We know that $(A \cap S_1) \cup (\overline{B^0} \cap S_1) = S_1$, and that both $A \cap S_1$ and $\overline{B^0} \cap S_1$ are relatively closed in S_1. Since S_1 is clearly connected, there must exist $(u,v) \in A \cap \overline{B^0} \cap S_1$. Let $(u_n, v_n) \in B^0$ be such that $\lim_n (u_n, v_n) = (u,v)$. Since $(u_n, v_n) \in B^0$ we can differentiate (3.7) with respect to both x and y to obtain that $\nu_1 g'(u_n) - \nu_1 g'((1 - \nu_1 u_n - \nu_2 v_n)/(\mu - \nu_1 - \nu_2)) = 0$ and $\nu_2 g'(v_n) - \nu_2 g'((1 - \nu_1 u_n - \nu_2 v_n)/(\mu - \nu_1 - \nu_2)) = 0$. Therefore $g'(u_n) = g'(v_n)$ and by continuity $g'(u) = g'(v)$. But since $(u,v) \in A$ implies that (u,v) satisfies (3.6), it follow that $u = v$, which contradicts the fact that $(u,v) \in S_1$.

(iii) Either $A \cap S_2 = \varnothing$ or $\overline{B^0} \cap S_2 = \varnothing$.

This follows by a similar argument as in step (ii).

(iv) Either $A = S$ or $B = S$.

Suppose that $A \supset S_2$. Now (3.6) is symmetric in (x,y), and since $\nu_1 \le \nu_2$ we have that $(y,x) \in S_2$ whenever $(x,y) \in S_1$. It follows that $A \supset S_1$, and since A is closed we must have that $A = S$.

Alternatively, if A does not contain S_2, (i) and (iii) imply that $\overline{B^0}$ does. We claim now that $B = S$. For, if $B \neq S$, there must exist $(u,v) \notin B \cap S_1$. Then $(u,v) \in A \cap S_1$, and by the symmetry just discussed, $(v,u) \in A \cap S_2$. Therefore $(v,u) \in A \cap \overline{B^0} \cap S_2$, which is a contradiction. [See the argument in (ii).] □

We are now in the position to completely characterize the k-Markovian functions when Θ is finite and has at least three elements.

THEOREM 3. *Let Θ be finite, $s \ge 3$ and $\mathscr{A} = 2^\Theta$. Then g is k-Markovian, $k \ge 2$, if and only if either $g(x) \equiv a + bx$ or $g(x) \equiv a + b \log x$ for some $a, b \neq 0$, where "$\equiv$" means here "for every $x \in (0, \nu_1^{-1})$." Equivalently, g is k-Markovian if and only if the g-QLOP's are either LinOP's or LogOP's.*

PROOF. That the conditions in the theorem are sufficient for k-Markovianity follows from Theorems 2 and 3 in DeGroot (1974) (for the LinOP) and from Theorem 1 here (for the LogOP).

To prove necessity, observe that if (3.6) holds, then clearly $xg'(x) \equiv b$ for some b, so $g(x) \equiv a + b \log x$. Otherwise, (3.7) must hold. Differentiating with respect to x and with respect to y we obtain that $g'(x) = g'(y)$ for every $(x,y) \in S$. Thus, $g'(x) \equiv b$ for some b, hence $g(x) \equiv a + bx$. □

Finally, we extend Theorem 3 to general Θ [with $\nu(\Theta) < \infty$]. Suppose that there exists a measurable partition $A_1, \dots, A_s$ of Θ with $0 < \nu_1 \le \cdots \le \nu_s$, where $\nu_r = \nu(A_r)$, and define the initial densities to be $f_{i0} = \Sigma_{r=1}^{s} a_{ir0} I_{A_r}$, where the a_{ir0}'s satisfy $0 < a_{ir0}$ and $\Sigma_r \nu_r a_{ir0} = 1$, $1 \leq i \leq k$, $1 \leq r \leq s$. Then it is straightforward to show that the iterative process (1.3) mimics the finite case, in the sense that $f_{in} = \Sigma_r a_{irn} I_{A_r}$, where the a_{irn}'s are given by (3.2). Therefore, for g to be k-Markovian, $k \geq 2$, one of the identities in Theorem 3 must hold.

THEOREM 4. *Let $(\Theta, \mathscr{A}, \nu)$ be at least tertiary. Then g is k-Markovian if and only if either $g(x) \equiv a + bx$ or $g(x) \equiv a + b \log x$ for some a, b, where "$\equiv$" means "for every $0 < x < \alpha = \sup\{1/\nu(A): \nu(A) > 0\}$."*

PROOF. Again, sufficiency follows from Theorems 2 and 3 in DeGroot (1974) and from Theorem 1 here. The considerations before the theorem constitute essentially a proof of the necessity part, with the following addition. If $0 < \beta < \alpha$ is arbitrary, there must exist a partition A_1, A_2, A_3 of Θ with $0 < \nu(A_1) < \beta^{-1}$. Therefore one of the identities in Theorem 3 must hold for $0 < x < \beta$, and this is true for every $\beta < \alpha$. $\square$

It should be noted, at least as a mathematical curiosity, that although several results characterizing the LinOP and the LogOP are known [cf. McConway (1981) and Genest (1984a, b)], to our knowledge Theorem 4 is unique in the sense that it characterizes *both* the LinOP and the LogOP.

In the case that Θ is binary, a result similar to Theorem 3 holds, but the identity $g(x) \equiv a + bx$ has to be replaced by a weaker symmetry. For the sake of space, we will only state the theorem here. A proof is given in Gilardoni [(1989), pages 66–76].

THEOREM 5. *Let $\Theta = \{\theta_1, \theta_2\}$ and $\mathscr{A} = 2^{\Theta}$. Then g is k-Markovian, $k \geq 2$, if and only if either*

$$\text{(i)} \qquad \nu_1 g(x) + \nu_2 g\big(\nu_2^{-1}(1 - \nu_1 x)\big) \equiv \mu g(\mu^{-1})$$

or

$$\text{(ii)} \qquad g(x) \equiv a + b \log x \quad \textit{for some } a, b \neq 0.$$

NOTES. It is straightforward to show that if g satisfies condition (i) in Theorem 5, then g is k-Markovian (no hypothesis other than continuity and monotonicity of g are needed). The set of functions satisfying (i) clearly includes the set of linear functions. When ν is counting measure, condition (i) becomes a symmetry condition: $g(x) + g(1 - x) \equiv 1$ for $x \in (0, 1)$.

The condition in Theorems 4 and 5 that the consensus (3.1) be reached for *every* set of initial densities f_{i0} and for *every* weight matrix W (implicit in the definition of k-Markovian functions) is strong, but it is not easy to relax. The next example will address this point. In particular, note that Theorem 4 is not in general true if we define k-Markovian functions as those for which the consensus (3.1) is attained for every set of initial densities but for given, fixed weights W.

EXAMPLE 2. Using the notation for the finite case, let ν be counting measure, $k = 2$, $g(x) = x^{1/2}$ and $0 < w_{11} = w_{22} < 1$. Then the denominator of $a_{ir,n+1}$ in (3.2) is $w_{i1}^2 + (1 - w_{i1})^2 + 2w_{i1}(1 - w_{i1})\Sigma_t(a_{1tn}a_{2tn})^{1/2}$ which is independent of i. Since g (and hence g^{-1}) satisfies $g(u/v) = g(u)/g(v)$, it follows that

$$a_{irn} = \frac{\left(w_{i1}^{(n)}a_{1r0}^{1/2} + w_{i2}^{(n)}a_{2r0}^{1/2}\right)^2}{\Sigma_t\left(w_{i1}^{(n)}a_{1t0}^{1/2} + w_{i2}^{(n)}a_{2t0}^{1/2}\right)^2}.$$

Now $\lim_n w_{ij}^{(n)} = 1/2$, and by continuity it follows that the consensus will be given by $\lim_n a_{irn} \propto (0.5a_{1r0}^{1/2} + 0.5a_{2r0}^{1/2})^2$, which is exactly what the right-hand side of (3.1) gives in this situation.

4. Concluding remarks. The importance of interaction prior to the formation of group decisions has been widely recognized in the literature [see, e.g., Press (1978)]. Lehrer and Wagner [(1981), especially Chapters 1–5] have also argued that it plays an important role in the determination of consensus in society. Although a formal application of the iterative methods described here might not be practical, it forms a conceptual basis for defining the consensus as the result of iterative modifications of the densities (i.e., $f_\infty = \lim_n f_{in}$). Alternatively, consensus can be defined as the result of a consensual pooling operator T^* obtained from the experts' pools T_i [i.e., (2.1) where T^* is a LogOP with weights π_i).We have shown that the LogOP has essentially the same property as the LinOP in the sense that these two alternative ways of looking at the consensus are equivalent. Of course, although Theorem 4 states that the two representations of consensus are equivalent *only* for the LinOP and for the LogOP, one could, in any case, adopt (3.1) as the definition of consensus. [A good case for this can be made following the idea in Lehrer and Wagner (1981), page 128.] Moreover, in many numerical examples that we have considered, with g neither linear nor logarithmic, the right-hand side of (3.1) provided a surprisingly good approximation to the iterative consensus. Based on this, we conjecture that, if g is a smooth function in some appropriate sense, a tight bound can be found on the difference between the final consensus and (3.1).

Acknowledgments. We are grateful to the referees for their helpful comments.

REFERENCES

BERGER, R. L. (1981). A necessary and sufficient condition for reaching a consensus using DeGroot's method. *J. Amer. Statist. Assoc.* **76** 415–418.

CHATTERJEE, S. (1975). Reaching a consensus: Some limit theorems. In *Proceedings of the International Statistical Institute* **3** 156–160.

CHATTERJEE, S. and SENETA, E. (1977). Toward consensus: Some convergence theorems on repeated averaging. *J. Appl. Probab.* **14** 89–97.

DEGROOT, M. H. (1974). Reaching a consensus. *J. Amer. Statist. Assoc.* **69** 118–121.

FRENCH, J. R. P. (1956). A formal theory of social power. *Psychological Review* **63** 181–194.

FRENCH, S. (1985). Group consensus probability distributions: A critical survey. In *Bayesian Statistics 2* (J. M. Bernardo, M. H. DeGroot, D. V. Lindley and A. F. M. Smith, eds.) 183–201. North-Holland, Amsterdam.

GENEST, C. (1984a). A characterization theorem for externally Bayesian groups. *Ann. Statist.* **12** 1100–1105.

GENEST, C. (1984b). Pooling operators with the marginalization property. *Canad. J. Statist.* **12** 153–163.

GENEST, C. and MCCONWAY, K. J. (1990). Allocating the weights in the linear opinion pool. *Journal of Forecasting* **9** 53–73.

GENEST, C. and ZIDEK, J. V. (1986). Combining probability distributions: A critique and an annotated bibliography. *Statist. Sci.* **1** 114–148.

GILARDONI, G. L. (1989). Combining prior opinions. Ph.D. dissertation, Univ. Wisconsin-Madison.

HARARY, F. (1959). A criterion for unanimity in French's theory of social power. In *Studies in Social Power* (D. Cartwright, ed.) 168–182. Institute for Social Research, Ann Arbor, MI.

KARLIN, S. and TAYLOR, H. (1975). *A First Course in Stochastic Processes*, 2nd ed. Academic, New York.

KARLIN, S. and TAYLOR, H. (1981). *A Second Course in Stochastic Processes*. Academic, New York.

LEHRER, K. and WAGNER, C. G. (1981). *Rational Consensus in Science and Society*. Reidel, Dordrecht.

MCCONWAY, K. J. (1981). Marginalization and linear opinion pools. *J. Amer. Statist. Assoc.* **76** 410–414.

PRESS, S. J. (1978). Qualitative controlled feedback for forming group judgements and making decisions. *J. Amer. Statist. Assoc.* **73** 526–535.

SENETA, E. (1981). *Non-negative Matrices and Markov Chains*. Springer, New York.

WINKLER, R. L. (1968). The consensus of subjective probability distributions. *Management Sci.* **15** B61–B75.

DEPARTAMENTO DE MATHEMÁTICA
UNIVERSIDADE DE BRASÍLIA
70919, BRASÍLIA, D.F.
BRASIL

DEPARTMENT OF STATISTICS
UNIVERSITY OF WISCONSIN-MADISON
MADISON, WISCONSIN 53706