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BLOCK DESIGNS AND ELECTRICAL NETWORKS

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For a given block design, an electrical network is constructed in which blocks and treatments are represented by points and observations by connections. This network has the property that the resistance between two points representing two different treatments is equal to the variance on the estimate of the corresponding treatment contrast in the usual additive block effect plus treatment effect model. This provides a simple tool for computation of contrast variances in many examples. Further applications are made to obtain lower bounds for contrast variances and upper bounds for efficiencies. A particular application is a complete solution to the problem of finding the A-optimal design in the case where the number of blocks equals the number of treatments and the blocks are of size 2.

1. Introduction. Consider a *block design*, that is, a finite set B of *blocks*, a finite set T of *treatments* and a $B \times T$ matrix N of nonnegative integers n_{bt} indicating how many times treatment t occurs in block b. Draw a graph as follows. Each block and each treatment is represented by a point (vertex). Each *plot* (i.e., each occurrence of a treatment in a block) is represented by a connection (edge) between the points corresponding to the block and the treatment assigned to that plot. Thus, n_{bt} connections are drawn between the points b and t, whereas two block points or two treatment points are never directly connected (the graph is *bipartite*). Now, think of the graph as a diagram of an electrical network where the edges are connections of unit resistance (1 ohm). The properties of this network turn out to be closely related to the properties of the design. The most interesting relation is probably the following. Consider the standard block + treatment model, assuming that the observations (yields) on plots are independent, normally distributed with common variance σ^2 and means of the form $\alpha_t + \beta_b$. For two treatments t' and t'', let $R(t', t'')$ denote the resistance through the network between the points corresponding to t' and t''. Then

$$\mathrm{var}(\hat{\alpha}_{t'} - \hat{\alpha}_{t''}) = \sigma^2 R(t', t''),$$

where $\hat{\alpha}_{t'} - \hat{\alpha}_{t''}$ is the (maximum likelihood or least squares) estimate of the simple contrast $\alpha_{t'} - \alpha_{t''}$.

For some block designs (including balanced incomplete block designs, simple lattices and a few others) it is easy to compute the contrast variances explicitly by reference to this result, simply by drawing the network in a convenient manner and using the laws for parallel and serial combination of resistances,

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together with some straightforward rules for cancellation and short circuiting of connections that do not carry any current. This gives some insight in the way contrast variances depend on design structure. One byproduct of this is a solution to a design optimization problem put forward by Jones and Eccleston (1980). Section 5 contains a discussion of the relation of the present work to the work of other authors, in particular Borre and Meissl [(1974), on a relation between geodetic networks and potential theory], Dynkin [(1980), on a relation between Markov processes and Gaussian fields], Paterson [(1983), on properties of a related graph derived from a block design and its relation to efficiency] and Eccleston and Hedayat [(1974), on connectedness properties of designs].

2. The main result.

NOTATION. By I, B and T we denote the sets of plots, blocks and treatments, respectively. Elements of these sets are denoted $i, i', i_1, \dots \in I$, $b, b', b_1, \dots \in B$ and $t, t', t_1, \dots \in T$. Whenever convenient, we assume $I = \{1, 2, \dots, I\}$, $T = \{1, 2, \dots, T\}$ and $B = \{1, 2, \dots, B\}$, thus letting I, B and T denote both the finite sets and their cardinality. This is not likely to cause any confusion in the present context.

Formally, a design is given by two mappings φ_B: $I \to B$ and φ_T: $I \to T$, assigning factor levels to plots [cf. Tjur (1984)]. The statistical properties of the design are determined by the integers $n_{bt} = \#\{i \in I | \varphi_B(i) = b$ and $\varphi_T(i) = t\}$, constituting the $B \times T$ *incidence matrix* $N = (n_{bt})$. By k_b, we denote the size of block b and by r_t the number of replicates of treatment t.

The *design network* is defined as follows. The set of points of a graph is taken to be the disjoint union of B and T. On figures, we use signs $\cdot$ for treatments and $\circ$ for blocks, to distinguish. A connection from $b \in B$ to $t \in T$ is introduced for each occurrence of treatment t in block b. Thus, the set of connections can (and will) be identified with I. We interpret the graph as an electrical network where these connections are unit resistances (1 ohm). In all that follows, we shall assume that the design is *connected*, in the sense that any two treatment points t' and t'' can be joined by a chain $t' = t_1 b_1 t_2 b_2 \cdots b_{n-1} t_n = t''$ such that $n_{b_{i-1}t_i}$ $(i = 2, \dots, n)$ and $n_{b_i t_i}$ $(i = 1, \dots, n-1)$ are positive. This is easily seen to be equivalent to the condition that the design network is connected in the obvious graph theoretic sense, provided that all k_b and r_t are positive.

THEOREM 2.1. *Let $y = (y_i | i \in I)$ be a vector of real random variables, independent and normally distributed with common variance σ^2 and expectations given by*

$$Ey_i = \alpha_t + \beta_b \qquad (t = \varphi_T(i), b = \varphi_B(i)).$$

Let $(\hat{\alpha}_t)$ and $(\hat{\beta}_b)$ be maximum likelihood (or least squares) estimates of (α_t) and (β_b) in this statistical model. Let $R(p', p'')$ denote the resistance between *points p' and p'' of the network. Then,*

(i) *The variance of an estimated treatment contrast is given by*

$$\operatorname{var}(\hat{\alpha}_{t'} - \hat{\alpha}_{t''}) = \sigma^2 R(t', t'').$$

(ii) *The variance of a fitted value is given by*

$$\operatorname{var}(\hat{\alpha}_t + \hat{\beta}_b) = \sigma^2 R(t, b).$$

(iii) *Let $\hat{\alpha}_{t'} - \hat{\alpha}_{t''} = \sum_i a_i^{t't''} y_i$ be the expression of the estimated (t', t'')-contrast as a linear combination of the observations. Then, the coefficients $a_i^{t't''}$ have the following interpretation as potential differences in the network. Suppose that voltages $R(t', t'')$ and 0 are kept fixed at the two points t' and t'', while all other points are left untouched. Then, $a_i^{t't''}$ is the current through connection i, signed so that current from block to treatment counts negative while current from treatment to block counts positive. Or (since potential difference = current through a unit resistance),*

$$a_i^{t't''} = v_{\varphi_T(i)} - v_{\varphi_B(i)},$$

where v_p denotes the potential at the point $p \in B \cup T$.

Remarks. We have used the term fitted value for the estimate $\hat{\alpha}_t + \hat{\beta}_b$. For combinations (t, b) which do not occur in the design, a term like predicted mean of hypothetical observation would be more correct. But the result (ii) is valid in both cases.

Notice that the formula $R(t', t'') = \sum_i (v_{\varphi_T(i)} - v_{\varphi_B(i)})^2$ [which follows from (i) and (iii)] has the physical interpretation that the total energy per time unit developed by the network equals the sum of energies per time unit emerging from the single connections.

Notice also that (iii) suggests a way of solving the normal equations numerically by means of relatively simple physical equipment. This would have been a potentially useful result before the digital computer age.

Proof. Consider the $(T + B) \times (T + B)$ matrix

$$C = (c_{pq}) = \begin{bmatrix} \operatorname{diag}(r_t) & N^* \\ N & \operatorname{diag}(k_b) \end{bmatrix}.$$

Let X_T denote the $I \times T$ design matrix for the factor T [cf. Tjur (1984)], that is,

$$(X_T)_{it} = \begin{cases} 1, & \text{for } \varphi_T(i) = t, \\ 0, & \text{otherwise,} \end{cases}$$

and define $X_B(I \times B)$ similarly. The normal equations determining the maximum likelihood estimates (up to an arbitrary constant to be added to all α_t

and subtracted from all β_b) can then be written

$$\begin{bmatrix} X_T^* X_T & X_T^* X_B \\ X_B^* X_T & X_B^* X_B \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} X_T^* \\ X_B^* \end{bmatrix} y.$$

The partitioned square matrix on the left is easily seen to equal the previously defined matrix C. Now, let C^- be a symmetric $(T + B) \times (T + B)$ matrix such that $CC^-C = C$ and $C^-CC^- = C^-$ [a *reflexive generalized inverse* for C, cf. Rao (1973)]. One solution to the normal equations is then given by

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = C^- \begin{bmatrix} X_T^* \\ X_B^* \end{bmatrix} y$$

and the covariance matrix for this set of estimates is $\sigma^2 C^-$. It follows that the variance on an estimated contrast $\hat{\alpha}_{t'} - \hat{\alpha}_{t''}$ is

$$\operatorname{var}(\hat{\alpha}_{t'} - \hat{\alpha}_{t''}) = (1_{t'} - 1_{t''})^* (\sigma^2 C^-)(1_{t'} - 1_{t''}),$$

with an obvious notation for vectors in $\mathbb{R}^{B+T}$ which are 1 at a single coordinate and 0 elsewhere. Similarly, the variance of a fitted value $\hat{\alpha}_t + \hat{\beta}_b$ is given by

$$\operatorname{var}(\hat{\alpha}_t + \hat{\beta}_b) = (1_t + 1_b)^* (\sigma^2 C^-)(1_t + 1_b).$$

As to the interpretation of C in the network context, suppose that a voltage difference of $R(t', t'')$ is kept fixed between the two points t' and t''. According to Ohm's law, the current through the network from t' to t'' will then be 1 ampere. The laws of Kirchhoff, determining the potential differences between all other points of the network, can be stated as follows. Let v_p $(p \in T \cup B)$ denote the potential at the point p. The current through a connection from t to b is then $v_t - v_b$. The currents leaving t' via connections to other (block) points of the network must sum to 1, that is,

$$\sum_{b \in B} (v_{t'} - v_b) n_{bt'} = 1. \tag{2.1}$$

Similarly, the currents entering t'' must sum to 1, that is,

$$\sum_{b \in B} (v_{t''} - v_b) n_{bt''} = -1. \tag{2.2}$$

For all other points of the network, the sum of (signed) ingoing currents equals 0, that is, for $t \neq t', t''$,

$$\sum_{b \in B} (v_t - v_b) n_{bt} = 0 \tag{2.3}$$

and for $b \in B$,

$$\sum_{t \in T} (v_t - v_b) n_{bt} = 0. \tag{2.4}$$

The equations (2.1), ..., (2.4) are conveniently put together in the matrix equation

$$C\begin{bmatrix} (v_t) \\ (-v_b) \end{bmatrix} = 1_{t'} - 1_{t''}, \tag{2.5}$$

where the matrix C defined earlier occurs as the coefficient matrix. For C^- defined as before,

$$\begin{bmatrix} (v_t) \\ (-v_b) \end{bmatrix} = C^-(1_{t'} - 1_{t''}) \tag{2.6}$$

is then a solution to the network equations. It follows that

$$\begin{aligned} R(t', t'') = v_{t'} - v_{t''} &= (1_{t'} - 1_{t''})^* \begin{bmatrix} (v_t) \\ (-v_b) \end{bmatrix} \\ &= (1_{t'} - 1_{t''})^* C^- (1_{t'} - 1_{t''}). \end{aligned}$$

Comparing this with our expression for the contrast variances in the statistical model, we see that (i) has been proved. The proof of (ii) is similar. Our convention that potentials of block points occur with a minus sign in the equations means that the role of $1_{t'} - 1_{t''}$ is taken over by $1_t + 1_b$, but except for this the proof is exactly the same.

In order to prove (iii), consider the expression

$$\hat{\alpha}_{t'} - \hat{\alpha}_{t''} = (1_{t'} - 1_{t''})^* \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (1_{t'} - 1_{t''})^* C^- \begin{bmatrix} X_T^* \\ X_B^* \end{bmatrix} y$$

for the estimated contrast. By (2.6), this equals

$$\begin{aligned} \begin{bmatrix} (v_t) \\ (-v_b) \end{bmatrix}^* \begin{bmatrix} X_T^* \\ X_B^* \end{bmatrix} y &= \begin{bmatrix} (v_t) \\ (-v_b) \end{bmatrix}^* \left(\sum_i y_i (1_{\varphi_T(i)} + 1_{\varphi_B(i)}) \right) \\ &= \sum_i (v_{\varphi_T(i)} - v_{\varphi_B(i)}) y_i, \end{aligned}$$

which proves (iii). □

REMARK. It should be noticed that the design-network relation relies on a somewhat artificial way of writing the network equations. From a physical point of view, the shift of sign for block point potentials is an unnecessary complication. The canonical matrix for the network (Kirchhoff's matrix) is not C, but the matrix which can be obtained from C by change of sign of the diagonal elements. Unfortunately, this means that there is no way of extending the design-network isomorphism to additive models in designs with three or more factors. It is not even clear what the design network should be in that case.

3. Examples.

EXAMPLE 3.1 (Circular design with 3 blocks of size 2). Consider the smallest possible balanced incomplete block design (BIBD), 3 treatments arranged in 3 blocks of size 2. The design network is given by Figure 1 (notice: blocks are drawn as small circles, treatments as points).

The resistance between two treatment points, for example, 1 and 2, is easily computed. We can split this into two parallel resistances, each of which is a serial combination of unit resistances (2 and 4, respectively). By the rules for parallel and serial combination (hereby revived), we have

$$R(1,2) = \left[(1+1)^{-1} + (1+1+1+1)^{-1}\right]^{-1} = \left[\tfrac{1}{2} + \tfrac{1}{4}\right]^{-1} = \tfrac{4}{3}.$$

Thus, $\operatorname{var}(\hat{\alpha}_1 - \hat{\alpha}_2) = (\frac{4}{3})\sigma^2$.

This computation is immediately generalized to circular designs with T treatments arranged in $B = T$ blocks of size 2 in such a way that the design network has a circular form, similar to that of Figure 1.

EXAMPLE 3.2 (A more complicated BIBD). The cyclic design which has 7 treatments arranged in the 7 blocks $\{1,2,4\}, \{2,3,5\}, \dots, \{7,1,3\}$ (each block constructed from the previous one by addition of 1 modulo 7) is a BIBD (any two treatments meet in exactly one block). Figure 2 shows the design network.

When the graph is drawn like this, it is easy to see that a constant voltage difference between treatment points 1 and 2 will induce the same potential at the points in the middle of the figure. A more detailed argument for this goes as follows. Suppose that block points 3 and 4 (and their 6 connections) are removed from the network. Then it is rather obvious that the potentials at treatment points 3, 4, 5, 6 and 7 and block point 1 will be equal. Now, reintroduce the connections that were removed. This will merely create some connections between points with the same potential and since none of these new connections will carry any current, the points in the middle of the figure will still have the same potential. Hence (with v_t^T denoting the potential at treatment point t, v_b^B the potential at block point b, to avoid index confusion), $v_1^B = v_3^B = v_4^B = v_3^T = v_4^T = v_5^T = v_6^T = v_7^T$. Similarly, $v_5^B = v_7^B$ and $v_2^B = v_6^B$.

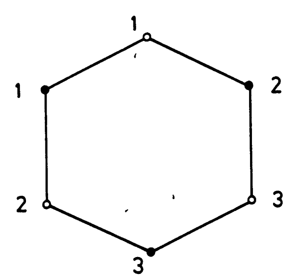

FIG. 1.

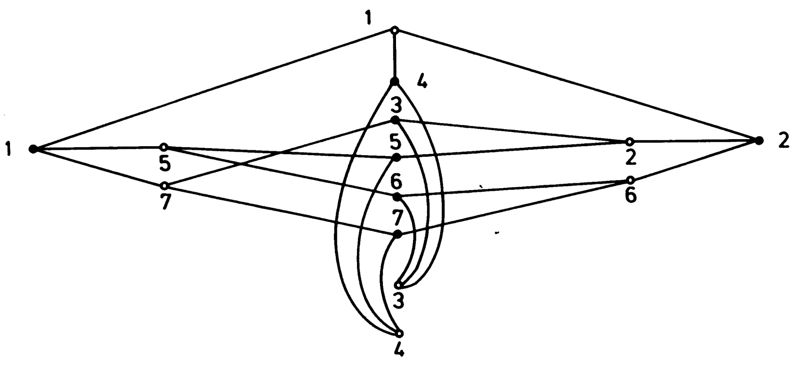

FIG. 2.

Now, notice the trivial fact that resistances of connections between points with the same potential may be changed arbitrarily, without change of the potentials. In particular, points with the same potential may be short circuited (i.e., connected by a 0-resistance or contracted to a single point), or the connection between them, if any, may be removed. In the present case, we may perform the following operations, without changing the solution to the network equations: Contract block points 5 and 7 to a single point, contract block points 2 and 6 to a single point, cut the connection between block 1 and treatment 4 and finally contract all points in the middle, except block 1, to a single point. These operations create a new network (Figure 3) for which the resistance is easily computed as

$$R(1,2) = \left[(1+1)^{-1} + \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{2}\right)^{-1}\right]^{-1} = \tfrac{6}{7}.$$

A more careful examination of this argument will show that it is valid for any BIBD. We shall not give the details, but merely notice this as one explanation of the fact that there is a simple formula for the contrast variance in a BIBD.

Other examples are the simple lattice designs and complete block designs with a single observation missing. In both cases the design network can be drawn in such a way that straightforward use of cutting, short-circuiting and

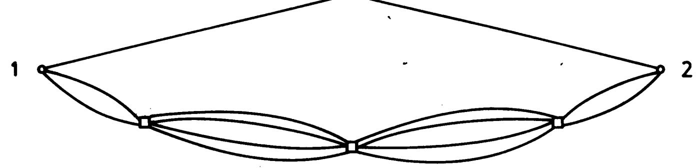

FIG. 3.

the rules for parallel and serial combination of resistances give a simple formula for the contrast variances [Tjur (1987)].

EXAMPLE 3.3. A paper by Jones and Eccleston (1980) contains an interesting remark on designs with $k = 2$ and $B = T$. It appears that the circular design (cf. Example 3.1) is not always optimal when the replicate counts are allowed to vary freely. *Optimality* here means A-optimality, that is, the property that the average of the $T(T - 1)/2$ contrast variances is as small as possible. For $B = T = 10$, 11 and 12, designs with smaller average contrast variance than the circular design (and, consequently, with unequal replicates) were found. Incidentally, this is a situation where the electrical network approach gives a complete solution to the optimization problem.

For $B = T$ and $k = 2$, the design network must necessarily be of the form illustrated by Figure 4 for the case $B = T = 10$, that is, a circular subgraph equipped with a number of (optionally branching) rays. Figure 4 is actually the variety concurrence graph [Paterson (1983)], but the design network comes out of it if we imagine a block point at the midpoint of each connection. The fact that the design network must have a structure like this follows from basic graph theory. A connected graph with $B + T$ points and $B + T - 1$ connections is a *tree*, that is, a graph without cycles, and if a single connection is added to this, a graph with exactly one cycle comes out of it. Now, it is easy to see that the operation which collects all the rays and fixes them at the same point of the cycle will improve the design. For example, the design of Figure 5 is better than that of Figure 4 because resistances between treatment points on the rays are either decreased or unchanged by this operation, while the average of the remaining contrast variances is obviously unchanged. Similarly, it is easy to see that the operation which breaks a ray into single connections and fixes these pieces as short rays at the same point of the cycle, will improve the design. Thus, the design of Figure 6 is better than that of Figure 5. These arguments show that an optimal design in this case is always of the form indicated by Figure 6, a circular design involving some of the treatments extended by a number of blocks in which the remaining treatments occur together with a selected baseline treatment from the circular design. By the rules for parallel and serial combination and some straightforward summations, it is not difficult to compute the average contrast variance of a design

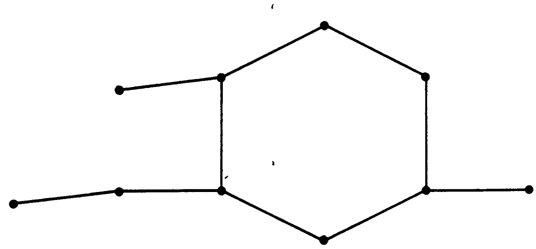

FIG. 4.

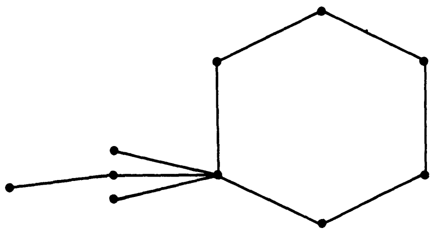

FIG. 5.

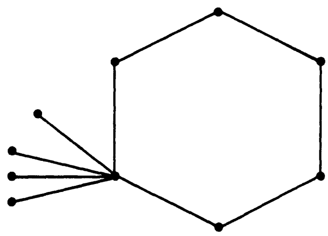

FIG. 6.

like this. It turns out to be

$$\bar{R} = \frac{(c-1)(c+1)\left(\frac{1}{3}B - \frac{1}{6}c\right) + 2(B-c)(B-1)}{\frac{1}{2}B(B-1)},$$

where c is the number of treatments in the circular design. A straightforward analysis of the behaviour of this third order polynomial in c for fixed B gives the following solution to our optimization problem:

For $B \le 8$, the circular design ($c = B$) is optimal.

For $9 \le B \le 12$, the design with $c = 4$ is optimal. The replicate counts of this design are $B - 2, 2, 2, 2, 1, 1, \ldots, 1$.

For $12 \le B$, the design with $c = 3$ is optimal. The replicate counts are $B - 1, 2, 2, 1, 1, \ldots, 1$.

The overlap for $B = 12$ means that the two designs with $c = 3$ and 4 have the same average contrast variance. The complexity of the solution confirms the impression that design optimization is a difficult matter.

4. Lower bounds for contrast variances and upper bounds for efficiencies. The main idea of this section can be illustrated by the following argument, which gives a very rough lower bound for a given contrast variance. Suppose, for an arbitrary block design, that all blocks points of the design network are contracted (short circuited) to a single point. This gives a network of the form indicated by Figure 7. Obviously, the resistance from t' to t'' in this

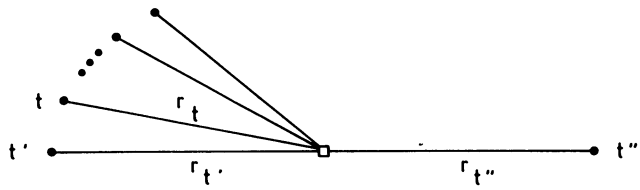

FIG. 7.

network is $1/r_{t'} + 1/r_{t''}$. Since it was obtained from the original network by the introduction of some new (0-resistance) connections, this resistance can never be larger than the original resistance. Hence, we have proved the well-known inequality

$$\operatorname{var}(\hat{\alpha}_{t'} - \hat{\alpha}_{t''}) \geq \left(\frac{1}{r_{t'}} + \frac{1}{r_{t''}} \right) \sigma^2,$$

stating that a contrast variance can never be smaller than the contrast variance for the same two treatments in a (hypothetical) design with the same treatments repeated the same number of times in a single block (and with the same σ^2).

In fact, we can say a little more than this. The situations where the previous inequality is an equality must obviously be those where a difference in potential between t' and t'' induces the same potential at all block points. With a little bit of intuition, it is not difficult to see that this happens if and only if the occurrence counts for t' and t'' in blocks are proportional, that is, $n_{bt'} = cn_{bt''}$ for some constant c independent of b. Thus, equality for *all* pairs of treatments occurs if and only if the design is *orthogonal* in the sense that all columns of N are proportional (see, e.g., Tjur (1984)).

The previous result is classical. More refined inequalities come out by less violent short circuiting of the design network. We shall prove two such inequalities, one that gives a bound similar to the expression for the contrast variance in a BIBD, and a more complicated inequality based on a short circuiting procedure which can be used for the computation of contrast variances in a simple lattice design.

PROPOSITION 4.1. *Suppose that the design is binary, that all block sizes are equal* ($k_b = k$) *and that all replicate counts are equal* ($r_t = r$). *Then*

$$R(t', t'') \geq \frac{2}{r} \left(1 - \frac{r - \lambda(t', t'')}{kr} \right)^{-1},$$

where $\lambda(t', t'')$ *denotes the number of blocks in which both* t' *and* t'' *occur.*

PROOF. For two treatments t' and t'', imagine that the design network is drawn in a way similar to that of Figure 2, with t' to the left and t'' to the right; in the middle we have all other treatment points together with all points corresponding to blocks in which neither t' nor t'' occur (bottom) and those in which both t' and t'' occur (top). Between t' and the points in the middle we place the points corresponding to blocks in which t' but not t'' occur, and similarly for t''. In this electrical network, the following groups of points are contracted to single points by short circuiting:

I The blocks containing t' but not t''.
II The blocks containing t'' but not t'.
III The blocks containing both t' and t''.
IV All treatments except t' and t'' and all blocks containing neither t' nor t''.

Finally, we cut all connections between groups III and IV (which is obviously allowed, since these two points have the same potential for reasons of symmetry) and end up with the reduced network shown by Figure 8. On this figure, the connections are bundles of parallel unit resistances and the integers assigned to connections are multiplicities, that is, inverse resistances [with the short notation λ for $\lambda(t', t'')$]. The rules for parallel and serial combination give the following expression for the resistance between t' and t'' through this network (which is then a lower bound for the corresponding contrast variance in the original design):

$$\left[\left(\frac{2}{\lambda}\right)^{-1} + \left(\frac{2}{r-\lambda} + \frac{2}{(k-1)(r-\lambda)}\right)^{-1}\right]^{-1} = \cdots = \frac{2}{r}\left(1 - \frac{r-\lambda}{kr}\right)^{-1}. \quad \square$$

PROPOSITION 4.2. *In addition to the assumptions of Proposition* 4.1, *assume that the numbers* $\lambda(t', t'')$ *are all less than or equal to* 1. *Define*

$$\Lambda(t', t'') = \#\{t \neq t', t'' | \lambda(t, t') = \lambda(t, t'') = 1\}$$

(*equals the number of other treatments that meet both t' and t''*). *Then*

$$R(t', t'') \geq \left[\frac{\lambda}{2} + \left(\frac{2r}{r(k-1)(r-\lambda-1) + \Lambda + \lambda} + \frac{2}{r-\lambda}\right)^{-1}\right]^{-1},$$

where Λ and λ are short for $\Lambda(t', t'')$ and $\lambda(t', t'')$.

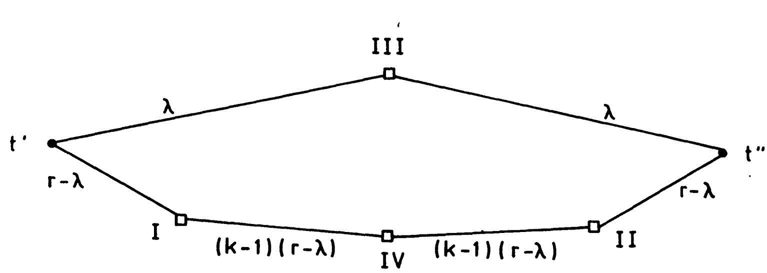

FIG. 8.

PROOF. The following sets of points are contracted to single points by short circuiting:

I The blocks containing t' and t''.
II_1 The blocks containing t' but not t''.
II_2 The blocks containing t'' but not t'.
III The blocks containing neither t' nor t''.
IV_1 The treatments occurring in a block together with t', but not with t''.
IV_2 The treatments occurring in a block together with t'', but not with t'.
V The treatments occurring in blocks with both t' and t''.
VI The treatments that never occur in a block with t' or t''.

Figure 9 shows the reduced network [still with the brief notation $\lambda = \lambda(t', t'')$ and $\Lambda = \Lambda(t', t'')$]. For reasons of symmetry, the vertical connections can be removed and it follows that we have the lower bound

$$\left[\frac{\lambda}{2} + \left(\frac{2}{r-\lambda} + \left[\left(\left(2 + \frac{2}{r-1}\right)\frac{1}{r(k-1) - (\Lambda+\lambda)}\right)^{-1} + \left(\frac{2}{\Lambda - (k-2)\lambda}\right)^{-1}\right]^{-1}\right)^{-1}\right]^{-1}$$

for $R(t', t'')$. The proposition follows after some straightforward algebra. □

The *efficiency* (or *harmonic mean efficiency*) E of a block design with equal block sizes and equal replicate counts can be defined as the harmonic mean of the $T - 1$ nonzero eigenvalues of the matrix $(1/r)C_T$, where

$$C_T = rI_{T\times T} - \frac{1}{k}N^*N$$

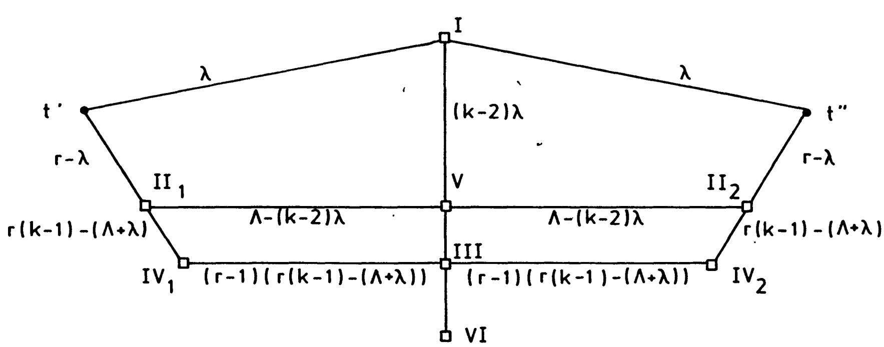

FIG. 9.

is the information matrix for the set of treatment parameters [see, e.g., Paterson (1983)]. The following (equivalent) definition gives an intuitive justification of the efficiency as a measure of design quality. Define

$$\overline{R} = \frac{2}{T(T-1)} \sum_{t'<t''} R(t', t'')$$

(equals the average resistance between pairs of treatment points in the design network). Then

$$E = \frac{2/r}{\overline{R}}.$$

Notice that $\overline{R}\sigma^2$ is the average contrast variance for the given design, while $(2/r)\sigma^2$ is the same quantity for a complete block design with each treatment repeated r times. Thus, it follows from the discussion in the beginning of this section that $E \leq 1$ (with $E = 1$ if and only if the design is orthogonal).

It follows from the last definition that any lower bound for the average contrast variance $\sigma^2\overline{R}$ gives an upper bound for the efficiency and vice versa. The lower bounds for contrast variances given by Propositions 4.1 and 4.2 are given as resistances through short circuited networks in which the inverse resistances (equals multiplicities) of connections are linear expressions in combinatorial quantities like $\lambda(t', t'')$ and $\Lambda(t', t'')$. It is easy to derive expressions for sums or averages of such combinatorial quantities over all pairs of distinct treatments. Thus, in order to average the lower bounds for contrast variances to obtain lower bounds for average contrast variances (or upper bounds for efficiencies), we need the convexity property stated by the following lemma.

LEMMA 4.3. *Consider a connected graph with P as its set of points and I as its set of connections. We think of the graph as an electrical network with variable resistances. By z_i we denote the inverse resistance of connection i. For two selected, distinct points p' and p'', let $R(z) = R((z_i))$ denote the resistance through the network from p' to p''. Then, the function R on $[0, +\infty)^I$ is convex.*

REMARK. Notice that one or more zeroes among the z_i may give the value $+\infty$ of $R(z)$, due to disconnectedness. However, R is also convex in the (obvious) extended sense.

It is possible to give a heuristic proof of this lemma, based on the kind of physical reasoning applied earlier in this section. But since the convexity property is easy to prove directly in the two situations where we are going to use it, the proof will be skipped here; see Tjur (1987).

As an example, consider the right-hand side of the inequality

$$R(t', t'') \geq \frac{1}{r_{t'}} + \frac{1}{r_{t''}}$$

derived in the beginning of this section. In this simple case, the conclusion of

Lemma 4.3 is that $1/r_{t'} + 1/r_{t''}$ is a convex function of $r = (r_t)$. By Jensen's inequality,

$$\bar{R} = \frac{2}{T(T-1)} \sum_{t'<t''} R(t', t'') \geq \frac{2}{T(T-1)} \sum_{t'<t''} \left(\frac{1}{r_{t'}} + \frac{1}{r_{t''}} \right) \geq 2/\bar{r},$$

where $\bar{r}$ is the average (in principle over all pairs of distinct treatments, but equivalently over all treatments) of the replicate counts r_t. Since this is

$$\bar{r} = \frac{1}{T} \sum_t r_t = \frac{I}{T},$$

we have proved that $\bar{R} \geq 2T/I$, that is, the average contrast variance is at least $(2T/I)\sigma^2$. This is not particularly exciting, but the example illustrates very well what is going on in the following under more complicated circumstances.

PROPOSITION 4.4. *Under the assumptions of Proposition* 4.1,

$$E \leq \frac{1 - 1/k}{1 - 1/T}.$$

REMARK. Under the definition of E in terms of the eigenvalues of the information matrix, this is merely the inequality stating that the harmonic mean of the $T - 1$ eigenvalues of $(1/r)C_T$ is less than or equal to the arithmetic mean. Hence, the result is well-known, but we include it because the proof does not refer to the spectral decomposition of C_T.

PROOF. The inequality of Proposition 4.1 was derived from a short circuited design network (Figure 8) in which the conductances are affine functions of $\lambda = \lambda(t', t'')$. It follows by Lemma 4.3 that the inequality still holds when $R(t', t'')$ on the left-hand side is replaced by $\bar{R}$ and $\lambda(t', t'')$ on the right-hand side is replaced by $\bar{\lambda}$ = the average of $\lambda(t', t'')$ over all pairs of distinct treatments. Now, for binary designs it is easy to show that

$$\bar{\lambda} = \frac{Bk(k-1)}{T(T-1)}.$$

The substitution of this expression in the averaged inequality

$$\bar{R} \geq \frac{2}{r}\left(1 - \frac{r - \bar{\lambda}}{kr}\right)^{-1}$$

followed by a little algebra, gives the desired result. □

PROPOSITION 4.5. *Under the assumptions of Proposition* 4.2,

$$E \leq \frac{k-1}{T-1} + \left(\frac{r}{(k-1)(r-1)} + \frac{T-1}{T-k} \right)^{-1}.$$

PROOF. By arguments similar to those applied in the proof of Proposition 4.4, a lower bound for $\overline{R}$ can be obtained from Proposition 4.2 by replacement of $\lambda(t', t'')$ and $\Lambda(t', t'')$ by the averages of these quantities over pairs of distinct treatments. An expression for $\overline{\lambda}$ was given in the proof of the previous proposition. Under the assumptions made, it is easy to show that

$$\overline{\Lambda} = (r(k-1)-1)\overline{\lambda}.$$

The inequality obtained from Proposition 4.2 by averaging is

$$\overline{R} \geq \left[\frac{\overline{\lambda}}{2} + \left(\frac{2r}{r(k-1)(r-\overline{\lambda}-1)+\overline{\Lambda}+\overline{\lambda}} + \frac{2}{r-\overline{\lambda}}\right)^{-1}\right]^{-1}$$

or

$$E \leq \frac{2}{r}\left[\frac{\overline{\lambda}}{2} + \left(\frac{2r}{r(k-1)(r-\overline{\lambda}-1)+\overline{\Lambda}+\overline{\lambda}} + \frac{2}{r-\overline{\lambda}}\right)^{-1}\right].$$

Noting that the relation $\overline{\Lambda} = (r(k-1)-1)\overline{\lambda}$ gives a considerable simplification of the denominator of the first fraction of the inner expression here, this inequality is easily rewritten to that stated by the proposition. □

EXAMPLE 4.1. Paterson and Wild (1986) study an α-lattice with 40 treatments arranged in 32 blocks of size 5. The efficiency of this design is 0.79048. They derive upper bounds for the efficiency under various conditions. The sharpest among those proved under the general assumptions of Proposition 4.5 is $E \leq 0.79740$. Proposition 4.5 gives $E \leq 0.79335$, which is less than half as far from the true efficiency as the bound reported by Paterson and Wild.

EXAMPLE 4.2. Jarrett (1977) gives an upper bound for the efficiency which, in the case $B = 15$, $T = 20$, $k = 4$ and $r = 3$, is $E \leq 0.7549$ (without additional assumptions, like resolvability, etc.). Our bound in this case is $E \leq 0.7505$. Jarrett gives an example (an α-lattice) with $E = 0.7447$.

These examples show that the upper bounds for efficiencies obtained by short circuiting compete well with those obtained by Jarrett (1977) and Paterson and Wild (1986) by methods based on the spectral decomposition of C_T. Bounds obtained by more refined methods [see, e.g., Fitzpatrick and Jarrett (1986), Tjur (1990)] seem to be generally better than the bounds derived here. But the fact that the bound of Proposition 4.5 does at all have a place among the strongly competing bounds of the design literature, indicates that the bound of Proposition 4.2 (from which the efficiency bound comes out by averaging) must be close to the true contrast variance in case of nearly optimal designs. The upper bounds for efficiencies given in the design literature do not have such contrast-specific counterparts.

5. Notes on related work. Borre and Meissl (1974) seems to be the first presentation of a relation between potential theory and covariance structure.

As noticed by H. Brøns (lecture around 1974), the theory presented by Borre and Meissl contains the main result of the present paper (Theorem 2.1) as a special case. The exposition of Borre and Meissl is based on the probabilistic interpretation of potential theory (the electrical network interpretation is found in an appendix) and the statistical problem investigated was the following. Suppose that measurements of height differences between certain points in a landscape are given. Denote the measurements y_i, $i \in I$, and let p_i' and p_i'' be the two corresponding points, so that y_i is a measurement of $\alpha_{p_i'} - \alpha_{p_i''}$, where α_p denotes the true level (over the sea surface, say) of point p. The classical statistical model for smoothing of such data assumes that the y_i are normally distributed, independent with $Ey_i = \alpha_{p_i'} - \alpha_{p_i''}$ and (for simplicity) known variances σ_i^2. Thus, the maximum likelihood or weighted least squares estimates $\hat{\alpha}_p$ of the true levels (given up to a common additive constant) are obtained by minimization of $\sum_i(y_i - (\alpha_{p_i'} - \alpha_{p_i''}))^2/\sigma_i^2$. Now, the connection to potential theory can be explained as follows. Draw, for each measurement y_i, a line on the map between the points p_i' and p_i'' and think of the resulting graph (which is assumed to be connected) as an electrical network where the connection corresponding to the ith measurement has a resistance of σ_i^2 ohm. Then, $\operatorname{var}(\hat{\alpha}_{p'} - \hat{\alpha}_{p''})$ equals the resistance through the network from p' to p''. The probabilistic interpretation in terms of random walks on the graph is briefly outlined later in this section.

Now consider the special case where the graph is bipartite and all σ_i^2 are equal. Denoting the two sets of points by B and T and the levels of the two different kinds of points by α_t ($t \in T$) and β_b ($b \in B$), we can write the model

$$y_i \sim N\left(\alpha_{t_i} - \beta_{b_i}, \sigma^2\right)$$

which is merely the two-way additive model with a slightly unusual (subtractive) parametrization. The previously mentioned electrical network is recognized as our design network (with resistances σ^2 instead of 1) and (i) and (ii) of Theorem 2.1 come out as special cases of the previously mentioned result.

Dynkin (1980) presented a relation between Markov processes and Gauss fields. Roughly, the idea is that the Greens function (or potential operator) of a time homogeneous symmetric Markov process is a positive definite function which can be taken as the covariance for a set of normal random variables. More recently, Ylvisaker (1987) followed up some of these ideas and noticed their relation to design and prediction problems of a more general nature. The relation between Dynkin's results and the present paper can briefly be explained as follows. Consider the random walk on the design network, performed by an electron when the potentials at all points are equal; the particle selects its next position at random among those connected to the present state and makes a jump to it after an exponential waiting time with intensity inversely proportional to the number of such connections. The Greens function of this process (in a slightly generalized sense) turns out to be the covariance matrix for the set of estimates of treatment and block parameters in the block design. For a more exhaustive exposition of the relation between Markov chains and electrical networks, see Kemeny, Snell and Knapp (1976).

Paterson (1983), for a binary design with blocks of equal size k, defined the *variety concurrence graph* as follows. The set of points of this graph is the set T of treatments and a connection between t' and t'' is introduced for each block in which both t' and t'' occur. Paterson (1983) expressed the harmonic mean efficiency in terms of combinatorial quantities related to this graph (the numbers of cycles of order $2, 3, \dots$). The variety-concurrence graph does not, as opposed to our design network, contain the full information about the structure of the design. For example, a BIBD with $\lambda = 1$ and a design consisting of a single complete block will have the same variety-concurrence graph. However, it follows from Paterson's results that this graph does (together with the common block size) hold information about contrast variances, etc. It is tempting to ask, in the present context, whether it makes sense to think of the variety-concurrence graph as an electrical network. This turns out to be the case and there are two very simple ways of seeing it. The first is a purely probabilistic argument, based on the corresponding random walk on the graph; this argument will not be given here [see Tjur (1987)]. The second is based on the so-called *star-delta transform* for electrical networks, see, for example, Bollobas (1979). According to this rule, a point in an electrical network and its connections to other points (a star) can be replaced by a set of connections between the points connected to it (a delta), without affecting the resistances between other points of the network. The formula for the resistances of the new connections implies that a star with k rays of unit resistance should be replaced by a delta consisting of $\binom{k}{2}$ edges of resistance k (cf. Example 3.3, where this principle was applied in the case $k = 2$). If we apply this transformation to all block points of the design network, we obviously end up with the variety-concurrence graph, except that all resistances will be k instead of 1. Following this line a little further, one ends up with an interpretation of Paterson's formula for the average contrast variance as an averaged form of an expression for the resistance through a network in terms of combinatorial quantities. In this way Paterson's work is closely related to classical problems in electrical network theory, see, for example, Bollobas (1979).

Eccleston and Hedayat (1974) discussed concepts of connectedness which are closely related to graph theoretical connectedness properties of the design network. Apart from small modifications, their concepts can be characterized as follows. Connectedness in the usual sense is called *local connectedness* and the following two stronger conditions are considered. *Pseudo-global connectedness* is characterized by the property that the graph will still be connected after removal of any treatment point (and its connections to block points). *Global connectedness* is the stronger property that removal of two arbitrary treatment points will leave a connected graph. This description is a considerable simplification of their definitions and the results of the present paper throws some light on their main results, stating that designs which are optimal in a certain sense must, under suitable conditions on the design constants, possess one or both of these connectedness properties.

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