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ADMISSIBILITY OF INVARIANT TESTS IN THE GENERAL MULTIVARIATE ANALYSIS OF VARIANCE PROBLEM[1]

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Necessary and sufficient conditions for an invariant test to be admissible among invariant tests in the general multivariate analysis of variance problem are presented. It is shown that in many cases the popular tests based on the likelihood ratio matrix are inadmissible. Other tests are shown admissible. Numerical work suggests that the inadmissibility of the likelihood ratio test is not serious. The results are given for the multivariate analysis of variance problem as a special case.

1. Introduction. The general multivariate analysis of variance problem (GMANOVA) is a generalization of the growth curves model of Potthoff and Roy (1964). It has as special cases the multivariate analysis of variance (MANOVA) and multivariate analysis of covariance problems. Ware and Bowden (1977) have applied the model to a circadian rhythm analysis and Zerbe and Jones (1980) to a time series analysis. Gleser and Olkin (1970) expressed the model in a convenient canonical form and used invariance considerations to reduce the problem and derive the likelihood ratio test, also found by Khatri (1966). Kariya (1978) reduced the problem even further by applying sufficiency arguments to the invariance-reduced problem. He investigated several tests and found the locally best invariant test, which is admissible among invariant tests and locally minimax. Hooper (1983) applied Kariya's work to confidence set estimation.

The admissibility of tests besides the locally best invariant has not been determined. In this paper we find a minimal complete class of invariant tests for GMANOVA under certain dimensionality restrictions. The results are used to prove particular tests admissible or inadmissible. The popular tests based on the likelihood ratio matrix T_1 (see 1.3) are shown inadmissible in many cases. We also specialize to the MANOVA problem, finding the minimal complete class of invariant tests.

The work of Gleser and Olkin and Kariya implies that when one is interested in only invariant tests, it is sufficient to consider the following multivariate analysis of covariance problem. Let $Y = (Y_1, Y_2)$ be a multivariate normal matrix with mean $\mu = (\mu_1, \mu_2)$ and independent rows with common covariance matrix Σ, where Y_1 and μ_1 are $m \times p$, Y_2 and μ_2 are $m \times q$, and Σ is $(p + q) \times (p + q)$ nonsingular. Let S, $(p + q) \times (p + q)$, be independent of Y and have a Wishart distribution on n degrees of freedom with mean Σ. That is,

$$Y \sim N_{m\times(p+q)}(\mu, I \otimes \Sigma) \quad \text{and} \quad S \sim W_{p+q}(n, \Sigma) \tag{1.1}$$

where I is the identity matrix. We assume $n \geq p + q$. The problem is to test

$$H_0: \mu_1 = 0,\ \mu_2 = 0 \quad \text{versus} \quad H_A: \mu_1 \neq 0,\ \mu_2 = 0 \quad \text{based on} \quad X \equiv (Y, S), \tag{1.2}$$

the matrix Σ being unspecified. Thus we wish to test whether $\mu_1 = 0$ knowing that $\mu_2 = 0$. It is hoped that the covariates Y_2 can be used to find better tests than those based on

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(Y_1, S_{11}) alone. (S and Σ are partitioned in concert with Y and μ.) Kariya (1978) introduces the statistic (T_1, T_2):

$$T_1 = (I + T_2)^{-1/2} Y_{1\cdot 2} S_{11\cdot 2}^{-1} Y_{1\cdot 2}' (I + T_2)^{-1/2} \quad \text{and} \quad T_2 = Y_2 S_{22}^{-1} Y_2' \tag{1.3}$$

where

$$Y_{1\cdot 2} = Y_1 - Y_2 S_{22}^{-1} S_{21} \quad \text{and} \quad S_{11\cdot 2} = S_{11} - S_{12} S_{22}^{-1} S_{21}.$$

Here and elsewhere we take a matrix square root to be symmetric. Conditional on T_2, the statistic T_1 is distributed as the usual matrix in MANOVA. Among the tests proposed for GMANOVA (see Kariya, 1978) are analogs of the popular MANOVA tests, which we will denote by $\kappa_i(T_1)$, $i = 1, 2, 3, 4$. For a symmetric $m \times m$ matrix M, the $\kappa_i(M)$ tests reject H_0 for large values of the following statistics:

$$\begin{array}{ll} \kappa_1(M): \lambda_1(M) & \text{(Roy's Maximum Root Test);} \\ \kappa_2(M): \operatorname{tr}(M) & \text{(Lawley-Hotelling Trace Test);} \\ \kappa_3(M): \operatorname{tr}[M(I+M)^{-1}] & \text{(Pillai's Trace Test);} \\ \kappa_4(M): |I + M|, & \end{array} \tag{1.4}$$

where $\lambda_i(M)$ is the ith largest characteristic root of M. The test $\kappa_4(T_1)$ is the likelihood ratio test (LRT) for problem (1.2). Other tests are $\kappa_i(T)$ tests, where

$$T = YS^{-1}Y' \equiv (I + T_2)^{1/2}(I + T_1)(I + T_2)^{1/2} - I, \tag{1.5}$$

appropriate for testing $\mu = 0$ versus $\mu \neq 0$ without assuming $\mu_2 = 0$, and $\kappa_i(T^*)$ tests, where

$$T^* = Y_1 S_{11}^{-1} Y_1', \tag{1.6}$$

appropriate for problem (1.2) when $\Sigma_{12} = 0$.

We now introduce the invariance group used. The $\kappa_i(T_1)$ and $\kappa_i(T)$ tests are invariant, whereas the $\kappa_i(T^*)$ tests are not. Let G_1 be the group of $(p + q) \times (p + q)$ nonsingular matrices A of the form

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} \text{ is } p \times p, \quad A_{22} \text{ is } q \times q,$$

and $\mathscr{O}(m)$ be the group of $m \times m$ orthogonal matrices. Problem (1.2) is invariant under the group $G_1 \times \mathscr{O}(m)$ which acts on X via

$$(A, \Gamma): (Y, S) \to (\Gamma YA, A'SA),$$

and on (μ, Σ) similarly. Kariya (1978) has shown that any invariant test is a function of only (T_1, T_2), although the statistic itself is not invariant. Banken (1983) gives an explicit representation of the maximal invariant statistic. The maximal invariant parameter is

$$\Delta \equiv \lambda^{(m \wedge p)}(\mu_1 \Sigma_{11\cdot 2}^{-1} \mu_1'), \tag{1.7}$$

where $\lambda^{(k)}(M)$ is the vector consisting of the k largest ordered characteristic roots of the symmetric matrix M. The reduced problem tests

$$\begin{gathered} H_0: \Delta = 0 \quad \text{versus} \quad H_A: \Delta \in \Omega_A \equiv \Omega - \{0\}, \\ \Omega = \{\lambda \in \mathbb{R}^{m \wedge p} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m \wedge p} \geq 0\}. \end{gathered} \tag{1.8}$$

Kariya showed that the locally best invariant (LBI) test rejects H_0 when

$$\operatorname{tr}(I + T_2)^{-1}[(n + m - q)T_1(I + T_1)^{-1} - pI] > c. \tag{1.9}$$

This test has greater power than any other essentially different invariant test of the same level for Δ in some neighborhood of 0, thus is admissible among invariant tests. For $m =$

1, Marden and Perlman (1980) found the minimal complete class of invariant tests, proved that the level α LRT is admissible among invariant tests if and only if $\alpha \le \alpha^*$ for some α^* depending on (n, p, q), and proved that the test based on T is admissible among all tests.

In Section 2 we present a class of invariant tests which are admissible among invariant tests for problem (1.2). When $p \ge m$, this class is the minimal complete class of invariant tests. In Section 3 we show that the $\kappa_i(T_1)$ tests are not in this class if $m > 1$, hence they are inadmissible even among invariant tests if $p \ge m > 1$. The $\kappa_i(T)$ tests (modified if $p < m$) and the $\kappa_i(T^*)$ tests are shown to be admissible among all tests for problem (1.2).

In Section 5 we do a small Monte Carlo study to see how serious the inadmissibility of the LRT is, and to compare some of the invariant tests. The test based on

$$|I + T_2|^{p}|I + T_1|^{n+m-q} \tag{1.10}$$

is generalized Bayes and admissible in the class of invariant tests. We expect this test to be close in power to the LRT. The calculations support this expectation, suggesting that the LRT is "ε-admissible" (i.e., there exists an admissible test of the same level which never beats it by more than ε in power) among invariant tests for reasonably small ε which decreases as n increases. For example, $\varepsilon \approx .01$ when $n = 5$ or 10, and $\varepsilon \approx .003$ when $n = 20$, for $\alpha = .05$.

The admissibility of the $\kappa_i(T)$ tests is due to having good power at alternatives far from H_0, hence there is no guarantee that they have good power for moderate values of the parameter. The calculations in Marden and Perlman (1980) and Section 5 show that the $\kappa_4(T)$ test compares very unfavorably to the LRT. The LRT can beat the $\kappa_4(T)$ test by as much as .20 to .30 for $\alpha = .05$, but rarely is beaten by as much as .001.

The LBI test (1.9) has the potential drawback, shared by the $\kappa_3(T_1)$ and $\kappa_3(T)$ tests, that there may be a sequence $\{\Delta^{(n)}\}$ of parameter points for which $\Delta_1^{(n)} \to \infty$ but the power does not approach one. See Anderson and Perlman (1982) for this behavior in MANOVA. The numerical work shows that this drawback can be serious for small n, but lessens as n increases. In comparing the LRT to the LBI test, it appears that the former is better when $\Delta = (\Delta_1, 0, 0 \cdots 0)$ and Δ_1 is large, and the latter is better when $\Delta = (\Delta_1, \Delta_1, \cdots, \Delta_1)$ and Δ_1 is small. For smaller n the differences are more pronounced. A more detailed study would be needed to pin down the relative advantages of the two tests.

2. Complete class results. Since (T_1, T_2) and (T_1, T) (see (1.3) and (1.5)) are in one-to-one correspondence, any invariant test must be a function of (T_1, T). The action of G is then $(A, \Gamma):(T_1, T) \to (\Gamma T_1 \Gamma', \Gamma T \Gamma')$. For each T, choose $\Gamma_T \in \mathscr{O}(m)$ continuously such that $\Gamma_T T \Gamma'_T$ is diagonal with nondecreasing diagonal elements. Define the statistic $(V, W) = (v(X), w(X))$ by

$$v(X) = \Gamma_T(I + T)^{-1/2}T_1(I + T)^{-1/2}\Gamma'_T, \quad w(X) = \lambda^{(m\wedge(p+q))}(T(I + T)^{-1}). \tag{2.1}$$

Let $\mathscr{W} = \{w \in \mathbb{R}^{m\wedge(p+q)} \mid 1 > w_1 > w_2 > \cdots > w_{m\wedge(p+q)} > 0\}$, and take $\mathscr{X}$ to be the space of X in (1.2) with the null set of points deleted which lead to $w(x) \notin \mathscr{W}$. Thus for $X \in \mathscr{X}$, $\mathscr{W}$ is the space of W.

The ratio of the density of the maximal invariant under a point $\Delta \in \Omega_A$ to that under H_0 can be derived from Kariya (1978), Lemma 5.1 and Equation (5.6), by changing Γ to $\Gamma\Gamma_T$. The ratio as a function of x is then

$$\begin{aligned} R_\Delta(x) = (\text{const}) \int_{\mathscr{O}(m)} &\exp\left(-\frac{1}{2}\operatorname{tr} \xi'\Gamma\Gamma_T(I + T_2)^{-1}\Gamma'_T\Gamma'\xi\right) \\ &\times \int_{Gl(p)} |B'B|^{t}\exp\left(-\frac{1}{2}\operatorname{tr} B'B + \operatorname{tr} \xi'\Gamma Q B'\right) dB\,\nu(d\Gamma), \end{aligned} \tag{2.2}$$

where ν is Haar probability measure on $\mathscr{O}(m)$, $Gl(p)$ is the group of $p \times p$ nonsingular matrices, $t = (m + n - p - q)/2$, ξ and Q are any $m \times p$ matrices such that

$$\xi\xi' = \bar{\Delta}, \qquad QQ' = V, \tag{2.3}$$

and for $\gamma \in \mathbb{R}^k$, $k \le m$, $\tilde{\gamma}$ denotes the $m \times m$ diagonal matrix with diagonal elements $(\gamma_1, \gamma_2, \cdots, \gamma_k, 0, \cdots, 0)$. Complete the square in the exponent of (2.2) and notice that $\bar{W} = I + V - \Gamma_T(I + T_2)^{-1}\Gamma'_T$ to obtain

$$R_\Delta = (\text{const})\exp\left(-\frac{1}{2}\Sigma\Delta_i\right)\int_{\mathscr{O}(m)}\exp\left(\frac{1}{2}\operatorname{tr}\bar{\Delta}\Gamma\bar{W}\Gamma'\right)E_\theta(|B'B|^t)\nu(d\Gamma) \tag{2.4}$$

where

$$B \sim N_{p\times p}(\theta, I_p \otimes I_p) \quad\text{and}\quad \theta = \xi'\Gamma Q.$$

We need some definitions before presenting the Theorem. Extend the definition of $(R_\Delta - 1)/\Sigma\Delta_i$ continuously to $\Delta = 0$ by setting

$$(R_\Delta - 1)/\Sigma\Delta_i|_{\Delta=0} = (mp)^{-1}\operatorname{tr}(tV + p\bar{W})/2 - \tfrac{1}{2} \equiv \text{LBI}, \tag{2.5}$$

which is equivalent to the statistic in the test (1.9). The definition is legitimate by the following.

KARIYA'S LEMMA. *As* $\Sigma\Delta_i \to 0$,

$$R_\Delta = 1 + (\Sigma\Delta_i)\text{LBI} + o(\Sigma\Delta_i),$$

where $o(\Sigma\Delta_i)$ *is uniform in* x *and* $\Sigma\Delta_i \le 1$. *Also, for all* x,

$$-1 \le 2\cdot\text{LBI} \le t/p.$$

PROOF. See Kariya (1978), Lemma 5.3, and note that $0 \le \operatorname{tr} V \le m$, $0 \le \Sigma w_i \le m$.

A set $C \subseteq \mathscr{W}$ is said to be nonincreasing with respect to weak submajorization ("nonincreasing$_w$") if

$$w \in C, \quad w' \in \mathscr{W} \quad\text{and}\quad w' \le_w w \Rightarrow w' \in C, \tag{2.6}$$

where $\le_w$ is the partial ordering on $\mathscr{W}$ given by

$$w' \le_w w \quad\text{if}\quad w'_1 \le w_1,\ w'_1 + w'_2 \le w_1 + w_2,\ \cdots,\ \Sigma w'_i \le w_i \tag{2.7}$$

and

$$w' <_w w \quad\text{if}\quad w'_1 < w_1,\ w'_1 + w'_2 < w_1 + w_2,\ \cdots,\ \Sigma w'_i < \Sigma w_i.$$

See Marshall and Olkin (1979), Definition 1.A.2 and Proposition 4.B.7. Define the class $\mathscr{L}$ to consist of all closed (in $\mathscr{W}$), convex and nonincreasing$_w$ subsets of $\mathscr{W}$ which depend only on the first $m \wedge p$ components of w. Let $\Omega_0 = \{\Delta \in \Omega \mid \Sigma\Delta_i \le 1\}$ and $\Omega_1 = \{\Delta \in \Omega \mid \Sigma\Delta_i \ge 1\}$, and for any finite measure π^0 on Ω_0 and locally finite measure π^1 on Ω_1 define

$$d(x; \pi^0, \pi^1) = \int_{\Omega_0}[(R_\Delta - 1)/\Sigma\Delta_i]\pi^0(d\Delta) + \int_{\Omega_1} R_\Delta\pi^1(d\Delta). \tag{2.8}$$

Denote by Φ the class of invariant tests ϕ of the form

$$\phi(x) = \begin{cases} 1 & \text{if } w(x) \notin C \\ 1 & \text{if } d(x; \pi^0, \pi^1) > c \\ 0 & \text{otherwise, a.e. } [\mu], \end{cases} \tag{2.9}$$

where $C \in \mathscr{L}$, π^0 and π^1 are as above, $|c| < \infty$, μ is Lebesgue measure on $\mathscr{X}$, and for $w(x) \in \operatorname{int} C$, $|d(x; \pi^0, \pi^1)| < \infty$.

THEOREM 2.1. *For any (p, m), all tests in Φ are admissible among invariant tests. If $p \geq m$, Φ is the minimal complete class of invariant tests.*

REMARK 2.2 Within the class of invariant tests, generalized Bayes tests of the following form are admissible. Let π be any locally finite measure on Ω_A such that $\int R_\Delta \pi(d\Delta) < \infty$ for all x. The test which rejects H_0 if and only if $\int R_\Delta \pi(d\Delta) > c'$ for some c' is in Φ: take $C = \mathscr{W}$,

$$\pi^1(d\Delta) = I_{\Omega_1}\pi(d\Delta), \quad \pi^0(d\Delta) = (\Sigma\Delta_i)I_{\{\Delta|0<\Sigma\Delta_i<1\}}\pi(d\Delta)$$

and $c = c' - \pi(\Omega_0)$, where I_A is the indicator function of A. Thus $d(x; \pi^0, \pi^1) - c = \int R_\Delta \pi(d\Delta) - c'$ for d in (2.8), and $\pi^0(\Omega_0) < \infty$ since $R_0 \equiv 1$ implies that π must have finite integral near 0 in order for $\int R_\Delta \pi(d\Delta)$ to be finite. Hence Theorem 2.1 proves the test admissible among invariant tests.

PROOF. We will use the results in Marden (1982), but find it easier to apply them to the following artificial problem. Consider (1.2) based on $X \in \mathscr{X}$ with density $R_\Delta(x)f_0(x)$ with respect to μ, where f_0 is the density of X in (1.1) when $\mu_1 = 0$ and $\Sigma = I$. Since (V, W) is a sufficient statistic in this new problem, and has the same distribution as it does in the original problem, an invariant test is admissible among invariant tests in the original problem if and only if it is admissible in the artificial problem. Below we will use asterisks to denote theorems, equations, assumptions, etc., which appear in Marden (1982). We note that in that paper, $\mathscr{X}$ is assumed to be convex. This is not true here, but Theorem 2.1* goes through without that assumption.

First let (p, m) be arbitrary. A look at Part II* of the proof of Theorem 2.1* will reveal that any test in Φ is admissible if the Local Assumption* and Equations (2.8)*, (2.12)*, (2.13)* and (2.17)* hold. The Local Assumption* holds immediately by Kariya's Lemma. It is convenient to show (2.7)* now, which states that for any $C \in \mathscr{C}$,

$$\text{int } w^{-1}(C) = w^{-1}(\text{int } C), \tag{2.10}$$

where the interiors are relative to $\mathscr{X}$ and $\mathscr{W}$ respectively.

Since w is continuous, $w^{-1}(\text{int } C) \subseteq \text{int } w^{-1}(C)$. For the reverse, suppose $x \notin w^{-1}(\text{int } C)$, so that there exists a sequence $\{w^{(k)}\}$, $w^{(k)} \notin C$, such that $w^{(k)} \to w(x)$. For $x = (y, s)$, let $ys^{-1/2} = \Gamma D\psi$, where D is diagonal of order $m \wedge (p + q)$ with diagonal elements $d_1 > d_2 > \cdots > d_{m\wedge(p+q)} > 0$, and $\Gamma(\psi)$ is appropriately dimensioned with orthogonal columns (rows). Since $T(x) = ys^{-1}y'$, $w_i = d_i^2(1 + d_i^2)^{-1}$. Define $x^{(k)} = (y^{(k)}, s)$ by letting $y^{(k)} = \Gamma D^{(k)} \psi$ where $d_i^{(k)} = [w_i^{(k)}/(1 - w_i^{(k)})]^{1/2}$. Thus $w(x^{(k)}) = w^{(k)}$, so that $x^{(k)} \notin w^{-1}(C)$. Also, $x^{(k)} \to x$, which shows $x \notin \text{int } w^{-1}(C)$. Hence (2.10) holds.

Now (2.8)* requires that μ (boundary $w^{-1}(C)) = 0$ for $C \in \mathscr{C}$. By (2.10), boundary $w^{-1}(C) = w^{-1}$ (boundary C). Since C is convex in $\mathscr{W}$, its boundary has zero $\mathscr{W}$-Lebesgue measure. Since the distribution of W is absolutely continuous with respect to that measure, μ (w^{-1} (boundary C)) = 0.

At this point we need the following lemma.

LEMMA 2.3. *Take $B \sim N_{p\times p}(\theta, I_p \otimes I_p)$. For fixed $t > 0$, there exist positive finite constants B_I and B_S such that for all $p \times p$ matrices θ,*

$$B_I < |I + \theta\theta'|^{-t}E_\theta(|BB'|^t) < B_S. \tag{2.11}$$

PROOF. Let $Y = B - \theta$, $H = (I + \theta\theta')^{-1/2}$ and $G = (I + \theta\theta')^{-1/2}\theta$. Thus

$$HH' + GG' = I_p. \tag{2.12}$$

Now

$$|I + \theta\theta'|^{-t}E_\theta(|BB'|^t) = E(|HYY'H' + GY'H' + HYG' + GG'|^t) \equiv L(H, G).$$

Consider L as a function defined for all $p \times p$ (H, G) such that (2.12) holds. For any fixed such (H, G), L is clearly finite. It is also strictly positive since the determinant in the expectation is strictly positive for $Y = I$ and a continuous function of Y. The lemma follows by noting that the set of matrices satisfying (2.12) is compact.

Thus by (2.4) and (2.11), R_Δ is bounded above and below by B_I and B_S, respectively, times

$$\text{(const)}\exp\left(-\frac{1}{2}\Sigma\Delta_i\right)\int |I + \xi'\Gamma V\Gamma'\xi|^t \exp\left(\frac{1}{2}\operatorname{tr}\bar{\Delta}\Gamma\bar{W}\Gamma'\right)\nu(d\Gamma). \tag{2.13}$$

Let $\alpha(\Delta) = |I + \bar{\Delta}|^{-t}\exp(\frac{1}{2}\Sigma\Delta_i)$. Equation (2.12)* requires that for $\eta \in \Omega_A$ and $a \in \mathbb{R}$,

$$\lim_{s\to\infty}\alpha(s\eta)\exp(-sa/2)R_{s\eta}(x) = 0(\infty) \quad \text{as} \quad \eta'w(x) < a\ (> a). \tag{2.14}$$

By (2.13), this is equivalent to

$$\lim_{s\to\infty}\int |I + s\bar{\eta}|^{-t}|I + s\bar{\eta}\Gamma V\Gamma'|^t \exp\left(\frac{s}{2}[\operatorname{tr}\bar{\eta}\Gamma\bar{W}\Gamma' - a]\right)\nu(d\Gamma) = 0(\infty) \quad \text{as} \quad \eta'w(x) < a\ (>a). \tag{2.15}$$

Since

$$\sup_{\Gamma\in\mathcal{O}(m)}\operatorname{tr}\bar{\Delta}\Gamma\bar{W}\Gamma' = \Sigma\Delta_i w_i \quad \text{for} \quad \Delta \in \Omega_A, \quad w \in \mathcal{W}, \tag{2.16}$$

(2.15) does hold: if $\eta'w(x) > a$, there exists a set of Γ's of positive ν-measure for which the integrand goes to $+\infty$, and if $\eta'w(x) < a$, the integrand is bounded by $\exp[s/2(\eta'w(x) - a)]$ because

$$|V| \le |I + \bar{\Delta}|^{-t}|I + \bar{\Delta}\Gamma V\Gamma'|^t \le 1 \quad \text{for} \quad \Delta \in \Omega_A. \tag{2.17}$$

In addition, this observation yields that

$$\sup_{s>0}\alpha(s\eta)\exp(-\tfrac{1}{2}sa)R_{s\eta}(x) \le (\text{const})B_S \quad \text{when} \quad \eta'w(x) < a,$$

which satisfies (2.13)*.

For (2.17)* we need that when $(\pi^\circ, \pi^1, c) \neq (0, 0, 0)$, $\mu(\{x \mid d(x; \pi^\circ, \pi^1) = c\}) = 0$. If $(\pi^\circ, \pi^1) = (0, 0)$, $c \neq 0$, so that the set is empty. Otherwise, note that R_Δ for $\Delta \in \Omega_A$ and LBI are strictly increasing in each w_i for fixed V. (See (1.9) and (2.4) and use $\operatorname{tr}\bar{\Delta}\Gamma\bar{w}\Gamma' = \Sigma\Sigma\Gamma_{ij}^2\Delta_i w_j$.) Thus d in (2.8) also has that property, which implies (2.17)*. Thus all tests in Φ are admissible among invariant tests.

Next suppose $p \ge m$. We will verify the remaining conditions for Theorem 2.1*. Let $\mathcal{D}$ be the class of subsets D of $\mathcal{X}$ such that for some sequence $\{\pi_k\}$ of finite measures on Ω_A,

$$D = \text{closure}\left\{x \mid \limsup_{k\to\infty}\int R_\Delta\pi_k(d\Delta) < \infty\right\}. \tag{2.18}$$

We must show that

$$\mathcal{D} = \{w^{-1}(C) \mid C \in \mathcal{C}\}. \tag{2.19}$$

Start by taking $C \in \mathcal{C}$. Let $\{\eta^{(i)}\} \subseteq \Omega_A$ and $\{a^{(i)}\} \subseteq \mathbb{R}$ be sequences such that $C = \cap_i\{w \in \mathcal{W} \mid w'\eta^{(i)} \le a^{(i)}\}$. Such sequences exist since C is convex and nonincreasing$_w$. Define the proper measure $\pi_k(d\Delta) = \sum_{i=1}^\infty 2^{-i}\alpha(k\eta^{i})\exp(-ka^{(i)}/2)\delta_{k\eta(i)}(d\Delta)$, *where* δ_τ is point mass one at $\Delta = \tau$. Hence by (2.14), D in (2.18) is $w^{-1}(C)$. For the reverse, take D, $\{\pi_k]$ as in (2.18). By (2.17) and (2.13) we have

$$B_I|V|^t e(\Delta, w(x)) \le R_\Delta \le B_S e(\Delta, w(x)), \tag{2.20}$$

where

$$e(\Delta, w) = |I + \bar{\Delta}|^t \exp\left(-\frac{1}{2}\Sigma\Delta_i\right)\int_{\mathscr{O}(m)} \exp\left(\frac{1}{2}\operatorname{tr}\bar{\Delta}\Gamma\bar{w}\Gamma'\right)\nu(d\Gamma). \tag{2.21}$$

Since $p \geq m$, $|V| > 0$, so that by (2.20), for any subsequence $\{l(k)\} \subseteq \{k\}$,

$$\lim_{l\to\infty}\int R_\Delta \pi_l(d\Delta) = 0(\infty)$$
$$\text{if and only if}\quad \lim_{l\to\infty}\int e(\Delta, w(x))\pi_l(d\Delta) = 0(\infty). \tag{2.22}$$

Hence from (2.18), $D =$ closure $w^{-1}(C')$ where

$$C' = \left\{w \in \mathscr{W} \mid \limsup_{k\to\infty}\int e(\Delta, w)\pi_k(d\Delta) < \infty\right\}.$$

For each $\Delta \in \Omega_A$, $e(\Delta, w)$ considered as a function of $w \in (0, 1)^m$ is convex, symmetric under permutations of the elements of w, and strictly increasing in each w_i. Thus Proposition 4.C.2d of Marshall and Olkin (1979) implies that $e(\Delta, w)$ is increasing$_w$, hence C' is convex and nonincreasing$_w$. Taking $C =$ closure C', we have that $C \in \mathscr{L}$, and by (2.10), that $D = w^{-1}(C)$. Thus (2.19) holds.

It remains to show (2.9)*, (2.10)* and (2.11)*. The first equation requires that if $w \in$ int C for $C \in \mathscr{L}$, then there exists $w' \in$ int C with $w' >_w w$. Since int C is open in $\mathscr{W}$, and $\mathscr{W}$ is open in $\mathbb{R}^m$, there exists $\varepsilon > 0$ for which the ε-ball around w is contained in int C. Thus $w' \equiv (w_1 + \varepsilon/2, w_2, \cdots, w_m)$ satisfies the requirement. See (2.7). For (2.10)* we need that for each D as in (2.18) and $x \notin D$, there exists a subsequence $\{l(k)\} \subseteq \{k\}$ such that

$$\lim_{l\to\infty}\int R_\Delta(x')\pi_l(d\Delta) = \infty \quad \text{for all} \quad x' \quad \text{with} \quad w(x') \geq_w w(x). \tag{2.23}$$

By the definition of D, there exists a subsequence such that the limit in (2.23) holds for $x' = x$. Since $e(\Delta, w)$ is increasing$_w$, (2.22) shows that the same subsequence can be used to show (2.23).

Equation (2.11)* requires that if $w(x') >_w w(x)$, then

$$\lim_{\gamma_0\to\infty}\sup_{\Sigma\Delta_i\geq\gamma_0}[R_\Delta(x)/R_\Delta(x')] = 0. \tag{2.24}$$

From (2.20) it suffices to show the same with $R_\Delta(x)/R_\Delta(x')$ replaced by $e(\Delta, w)/e(\Delta, w')$ for $w' >_w w$. Letting $g(\Gamma, \Delta) = \operatorname{tr}\bar{\Delta}\Gamma\bar{w}'\Gamma' - \Sigma\Delta_i w_i$, and multiplying the numerator and denominator by $\exp(-\frac{1}{2}\Sigma\Delta_i w_i)$, we have

$$\sup_{\Sigma\Delta_i\geq\gamma_0}[e(\Delta, w)/e(\Delta, w')] \leq \sup_{\Sigma\Delta_i\geq\gamma_0}\left\{\int_{\mathscr{O}(m)}\exp\left(\frac{1}{2}g(\Gamma, \Delta)\right)\nu(d\Gamma)\right\}^{-1}$$
$$\leq \left\{\int_{\mathscr{O}(m)}\exp\left[\frac{1}{2}\inf_{\Sigma\Delta_i\geq\gamma_0}g(\Gamma, \Delta)\right]\nu(d\Gamma)\right\}^{-1}. \tag{2.25}$$

Since Ω is a cone,

$$\inf_{\Sigma\Delta_i\geq\gamma_0}g(\Gamma, \Delta) = \inf_{\gamma\geq\gamma_0}\inf_{\Sigma\Delta_i=1}g(\Gamma, \gamma\Delta) = \inf_{\gamma\geq\gamma_0}\bar{g}(\Gamma),$$

where $\bar{g}(\Gamma) = \inf\{g(\Gamma, \Delta) \mid \Sigma\Delta_i = 1\}$. Note that the infimum in the definition of $\bar{g}$ can be taken over the finite number of extreme points of the convex set $\{\Delta \in \Omega \mid \sum\Delta_i = 1\}$. Thus $\bar{g}$ is continuous in Γ and is strictly positive for $\Gamma = I$ (see (2.7)) since these properties hold for $g(\Delta, \Gamma)$ for each Δ. We can therefore find a set $\Psi \subseteq \mathscr{O}(m)$ of positive ν-measure on which $\bar{g}(\Gamma) > \varepsilon$ for some $\varepsilon > 0$, that is, on which $\inf\{g(\Gamma, \Delta) \mid \Sigma\Delta_i \geq \gamma_0\} \geq \gamma_0\varepsilon$. The final expression in (2.25) is consequently bounded by $\exp(-\frac{1}{2}\varepsilon\gamma_0)(\nu(\Psi))^{-1}$, which approaches zero as $\gamma_0 \to \infty$. Equation (2.24) follows, completing the proof of the theorem.

We end this section with some properties that all tests in Φ satisfy. Theorem 2.1 implies that these properties are necessary for admissibility among invariant tests when $p \geq m$. Recall that $\lambda_i(M)$ is the ith largest characteristic root of M.

COROLLARY 2.4. *If $\phi \in \Phi$, the acceptance region of ϕ is*

(i) *for fixed W, nondecreasing in each $\lambda_i(V)$;*
(ii) *for fixed V, convex in W and nondecreasing in each W_i;*
(iii) *for fixed $U \equiv \bar{W} - V$, convex in $(\lambda_1^{1/2}(V), \cdots, \lambda_{m\wedge p}^{1/2}(V))$ and nondecreasing in each $\lambda_i(V)$;*
(iv) *for fixed T_2, nondecreasing in each $\lambda_i(T_1)$.*

PROOF. The function R_Δ satisfies the above properties for each Δ, hence so does d of (2.8), and likewise the set $\{(v, w) \mid d(x; \pi^\circ, \pi^1) \leq c\}$. Since it can be checked that each $C \in \mathscr{L}$ satisfies the conditions, the corollary is proved.

We show the conditions hold for R_Δ. Property (ii) is clear. The convexity condition in (iii) follows by writing (2.4)

$$R_\Delta = \exp\left(-\frac{1}{2}\Sigma\Delta_i\right)\int_{\mathscr{O}(m)} \exp\left(\frac{1}{2}\operatorname{tr}\bar{\Delta}\Gamma U\Gamma' + \frac{1}{2}\operatorname{tr}\bar{\Delta}\Gamma V\Gamma'\right)E_\theta(|BB'|^t)\nu(d\Gamma)$$

and noting that

$$\exp\left(\frac{1}{2}\operatorname{tr}\bar{\Delta}\Gamma V\Gamma'\right)E_\theta(|BB'|^t) = (\text{const})\int |BB'|^t\exp\left(-\frac{1}{2}\operatorname{tr}BB' + \operatorname{tr}\xi'\Gamma QB'\right)dB,$$

where we take $Q_{ii} = \lambda_i^{1/2}(V)$ for $i = 1, \cdots, m \wedge p$ and $Q_{ij} = 0$ otherwise. Property (iv) follows from (iii) since $U = \Gamma'_T(I + T_2)^{-1}\Gamma_T$ and $\Gamma'_T T_1(I + T_1)^{-1}\Gamma_T = U^{-1/2}VU^{-1/2}$. Finally the monotonicity condition in (iii) and (i) will follow from showing that

$$E_\theta(|BB'|^t) \text{ is nondecreasing in each } \lambda_i(V). \tag{2.26}$$

In fact, the following lemma implies that

$$R_\Delta \text{ is strictly increasing in each } \lambda_i(V) \text{ for fixed } W. \tag{2.27}$$

LEMMA 2.5. *Suppose B is as in Lemma 2.3. If $t \geq \frac{1}{2}$ then $E_\theta(|BB'|^t)$ is strictly increasing in each $\lambda_i(\theta\theta')$.*

PROOF. It is enough to show the result for θ diagonal and $\lambda_i(\theta\theta') = \theta_{11}^2$ with $\theta_{11} > 0$. Define $\bar{B}$ by $\bar{B}_{11} = B_{11} - \theta_{11}$ and $\bar{B}_{ij} = B_{ij}$ otherwise. Let M be B with the first row and column deleted, so that

$$|BB'|^t = [\operatorname{abs}(|\bar{B}| + \theta_{11}|M|)]^{2t}.$$

Since the first row of $|\bar{B}|$ has a distribution which is symmetric about zero, $|\bar{B}|$ and $-|\bar{B}|$ have the same distribution, hence

$$E(|BB'|^t) = \tfrac{1}{2}E\{[\operatorname{abs}(|\bar{B}| + \theta_{11}|M|)]^{2t} + [\operatorname{abs}(|\bar{B}| - \theta_{11}|M|)]^{2t}\}. \tag{2.28}$$

Since $2t \geq 1$, the expectant on the right hand side of (2.28) is nondecreasing in θ_{11} for all $(|\bar{B}|, |M|)$ and strictly increasing in θ_{11} when $|\bar{B}|$ and $|M|$ have the same sign. Hence the lemma.

3. Specific tests. In this section we verify the admissibility and inadmissibility results mentioned in the Introduction. Note that by taking $\pi^\circ = \delta_0$, $\pi^1 = 0$ and $C = \mathscr{W}$ in

(2.9), Theorem 2.1 reiterates Kariya's result that the LBI test (1.9) is admissible among invariant tests. The theorem also shows that tests of the form

$$\phi(x) = 1(0) \quad \text{as} \quad w(x) \notin C(\in C) \quad \text{for} \quad C \in \mathscr{C} \tag{3.1}$$

are admissible among invariant tests. When $p \geq m$, the $\kappa_i(T)$ tests are of this form. The same holds for $p < m$ if we replace T by $\lambda^{(p)}(T)$, that is, we only consider the p largest characteristic roots of T. The maximum root test $\kappa_1(T)$ is not affected by this modification. Proposition 3.2 shows that these tests are in fact admissible among all tests for problem (1.2), as are the $\kappa_i(T^*)$ tests. At the end of the section we derive the Bayes test (1.10) used in Section 5.

Turning to the $\kappa_i(T_1)$ tests, note that they are of the following form. Define $T_1^* = \lambda^{(m\wedge p)}(T_1)$, and let its space be $\mathscr{T} \equiv \{z \in \mathbb{R}^{m\wedge p} \mid 1 > z_1 > \cdots > z_{m\wedge p} > 0\}$. Consider tests such that

$$\phi(x) = 1(0) \quad \text{as} \quad T_1^* \notin A(\in A) \tag{3.2}$$

where $A \subseteq \mathscr{T}$ is nonincreasing in the sense that if $z^{(1)} \in A$ and $z^{(2)} \in \mathscr{T}$ with $z_i^{(2)} \leq z_i^{(1)}$ for all i, then $z^{(2)} \in A$.

PROPOSITION 3.1. *Suppose $m > 1$. No nontrivial test of the form* (3.2) *is in* Φ.

Hence by Theorem 2.1, if $p \geq m > 1$, the tests (3.2) are inadmissible even among invariant tests.

PROOF. Take ϕ nontrivial of the form (3.2), and assume $\phi \in \Phi$ as in (2.9). Since the regions where $\phi = 0$ and $\phi = 1$ are connected in $\mathscr{X}$, and $d(x; \pi^\circ, \pi^1)$ is continuous, ϕ is as in (2.9) with the "a.e.[μ]" removed. First choose an arbitrary $W \in \mathscr{W}$, and find V and T diagonal so that $T_1 \equiv (I - \bar{W})^{-1}V$ is diagonal. Since A is nonincreasing, the elements of V can be chosen small enough that $T_1^* \in A$. Thus $W \in C$ in (2.9), hence $C = \mathscr{W}$.

Next take λ on the boundary of A in $\mathscr{T}$ so that $d(x; \pi^\circ, \pi^1) = c$ when $T_1^*(x) = \lambda$. For $w \in \mathscr{W}$, define $v^{(1)}(w)$ to be the $m \times m$ diagonal matrix with diagonal elements $v_i^{(1)}(w) = \lambda_i(1 - w_i)$ if $1 \leq i \leq m \wedge p$, $v_i^{(1)}(w) = 0$ otherwise. Let $v^{(2)}(w)$ be the same as $v^{(1)}(w)$ except that $v_1^{(2)}(w) = \lambda_2(1 - w_1)$ and $v_2^{(2)}(w) = \lambda_1(1 - w_2)$. Take $w \in \mathscr{W}$ with elements close enough to one that $(v^{(k)}(w), w)$ are attainable values of the maximal invariant statistic. Considering R_Δ and LBI to be functions of (v, w), it can be seen from (1.9) and (2.4) that their definitions can be continuously extended to be functions on $\mathscr{G} = \{(v, w) \mid 0 \leq v \leq I_m,\ w \in \bar{\mathscr{W}}$ (the closure of $\mathscr{W}$ in $\mathbb{R}^{m\wedge(p+q)}$)$\}$. Also, d can be defined to be a function on $\mathscr{G}$ through (2.8). Take a sequence $\{w^{(l)}\} \subseteq \mathscr{W}$ where $w_1^{(l)} \to 1$, $w_i^{(l)} = w_i$ for $i \geq 2$, and let w° be the limit. Below we show that

$$c \equiv \lim_{l\to\infty} d(v^{(k)}(w^{(l)}), w^{(l)}; \pi^\circ, \pi^1) = d(v^{(k)}(w^\circ), w^\circ; \pi^\circ, \pi^1), \quad k = 1, 2. \tag{3.3}$$

Condition (2.27) holds also for R_Δ on $\mathscr{G}$, hence for d on $\mathscr{G}$. Thus for fixed $w = w^\circ \in \bar{\mathscr{W}}$, d is strictly increasing in each $\lambda_i(V)$. But from the definition of $v^{(k)}$, we have that $v_i^{(1)}(w^\circ) = v_i^{(2)}(w^\circ)$ for $i \neq 2$, and $v_2^{(1)}(w^\circ) = \lambda_2(1 - w_2) < \lambda_1(1 - w_2) = v_2^{(2)}(w^\circ)$. Thus the right-hand side of (3.3) for $k = 1$ is strictly less than that for $k = 2$, which contradicts (3.3). Thus $\phi \notin \Phi$.

For (3.3) we need to interchange the limit and integrals in (2.8). It is immediately allowable for the first integral since its integrand is uniformly bounded in $(v, w) \in \mathscr{G}$ and $\Delta \in \Omega_0$. See Kariya's Lemma. Since this first term is uniformly bounded and $d(v^{(k)}(w), w; \pi^\circ, \pi^1) \equiv c$, the second integral is bounded by some finite K, hence

$$K \geq \liminf_{w\to(1,\cdots,1)} \int_{\Omega_1} R_\Delta(v^{(k)}(w), w)\pi^1(d\Delta).$$

Thus Fatou's Lemma and the fact that $R_\Delta(v^{(k)}(1, \cdots, 1), (1, \cdots, 1)) \equiv 1$ shows that $\pi^1(\Omega_1)$

$\le K$. Using (2.20) we have

$$(\text{const.})^{-1}R_\Delta(v^{(k)}(w^{(l)}), w^{(l)}) \le B_S e(\Delta, w^{(l)}) \le B_S e(\Delta, w^\circ)$$

$$\le B_S(1+\Delta_1)^{mt} \int_{\mathscr{O}(m)} \exp(-\Delta_1 \textstyle\sum_{j>1} \Gamma_{1j}^2(1-w_j)/2)\nu(d\Gamma),$$

which is bounded since $w_2 < 1$ ($\text{rank}(T) = m \wedge (p+q) \ge 2$). Thus the Dominated Convergence Theorem can be used for the second term in (2.8), proving (3.3). The proof of the proposition is complete.

We now treat admissibility among all tests. Define w^*: $\mathscr{X} \to \mathscr{W}^* \equiv \{w \in \mathbb{R}^{m\wedge p} \mid 1 > w_1 > \cdots > w_{m\wedge p} > 0\}$ by

$$w^*(X) = \lambda^{(m\wedge p)}(Y_1(S_{11} + Y_1'Y_1)^{-1}Y_1'), \tag{3.4}$$

and let $\mathscr{C}^*$ be the class of closed, convex and nonincreasing$_w$ subsets of $\mathscr{W}^*$. The $\kappa_i(T^*)$ tests are of the form

$$\phi(x) = 1\ (0) \quad \text{as} \quad w^*(x) \notin C^*\ (\in C^*) \quad \text{for} \quad C^* \in \mathscr{C}^*. \tag{3.5}$$

PROPOSITION 3.2. *Any test of the form* (3.1) *or* (3.5) *is admissible among all tests for problem* (1.2).

PROOF. We use the Stein-Birnbaum technique as Schwartz (1967) did. The distribution of X in (1.1) is exponential with exponent $(\mu, \Sigma)\circ X \equiv -\frac{1}{2}\operatorname{tr}\Sigma^{-1}(S + Y'Y) + \operatorname{tr}\Sigma^{-1}\mu'Y$, and the alternative space is $\Lambda \equiv \{(\mu, \Sigma) \mid \Sigma \text{ nonsingular}, \mu = (\mu_1, 0)\}$. For any choice of $(\mu, \Sigma) \in \Lambda$ and $s > 0$, $(\mu, \Sigma/s) \in \Lambda$. Thus for any arbitrary collections $\{(\mu_i\Sigma_i)\}_{i\in I} \subseteq \Lambda$ and $\{c_i\}_{i\in I} \subseteq \mathbb{R}$, the test with acceptance region

$$D \equiv \cap_{i\in I}\{X \mid (\mu_i, \Sigma_i)\circ X \le c_i\} \tag{3.6}$$

is admissible. For $\Delta \in \Sigma_A$ and c real, take ξ as in (2.3) and define

$$\begin{aligned} D(\Delta, c) &= \cap_{(A,\Gamma)\in G}\{X \mid (\Gamma'\xi A^{-1}, (AA')^{-1}) \circ X \le c\} \\ &= \{X \mid \sup_{(A,\Gamma)\in G}(\Gamma'\xi A^{-1}, (AA')^{-1}) \circ X \le c\}. \end{aligned} \tag{3.7}$$

Since $(\Gamma'\xi A^{-1}, (AA')^{-1}) \in \Lambda$, $D(\Delta, c)$ is as in (3.6). To evaluate the supremum in (3.7), change A to A_0A for $A_0 \in G_1$ satisfying $A_0'(S + Y'Y)A_0 = I$ and $YA_0 = (\tilde{Y}_{1.2}\tilde{S}_{11.2}^{-1/2}, Y_2\tilde{S}_{22}^{-1/2})$, where $\tilde{S} = S + Y'Y$ and $\tilde{Y}_{1.2} = Y_1 - Y_2\tilde{S}_{22}^{-1}\tilde{S}_{21}$. Complete the squares with respect to A_{11} and A_{12}, and use (2.16) to show that the supremum is $\frac{1}{2} w'\Delta$. Since any set $C \in \mathscr{D}$ can be written as $\cap_{i=1}^\infty\{w \in \mathscr{W} \mid w'\eta^{(i)} \le c_i\}$ for some sequences $\{\eta^{(i)}\} \subseteq \Omega_A$ and $\{c_i\} \subseteq \mathbb{R}$, by taking $D = \cap D(\eta^{(i)}, c_i)$, (3.6) and (3.7) show (3.1) to be admissible.

The proof for the tests (3.5) is similar except that we strict $A \in G_1$ to have $A_{12} = 0$ in (3.7).

Finally we show that the test (1.10) is admissible among invariant tests. We start by defining

$$R_M^*(x) = (\text{const})\exp\left(-\frac{1}{2}\operatorname{tr} M(I + T_2)^{-1}M'\right) \cdot \int_{Gl(p)} |B'B|^t \exp\left(-\frac{1}{2}\operatorname{tr} B'B + \operatorname{tr} MQB'\right) dB$$

for $M \in \mathscr{M}$, the set of $p \times m$ matrices of full rank. A straightforward calculation will yield

$$\int R_M^*(x)\, dM = c.\, |I + T_2|^{p/2}|I + T_1|^{(n+m-q)/2}. \tag{3.8}$$

(Complete the square with respect to M and integrate it out to obtain the factor

$|I + T_2|^{p/2}$. Then change variables from B to $B(I - Q'(I + T_2)Q)^{-1/2}$, and note that $|I - Q'(I + T_2)Q| = |I + T_1|^{-1}$.) Since dM is invariant under $M \to M\Gamma$ for $\Gamma \in \mathscr{O}(m)$,

$$\begin{aligned}\int_{\mathscr{M}} R_M^*(x)\, dM &= \int_{\mathscr{M}} \int_{\mathscr{O}(m)} R_{M\Gamma}^*(x)\nu(d\Gamma)\, dM \\ &= \int_{\mathscr{M}} R_{\Delta(m)}(x)\, dM \qquad (3.9)\\ &= \int_{\Omega_A} R_\Delta(x)\pi(d\Delta).\end{aligned}$$

The second line comes from (2.4) by taking $\Delta(M) = \lambda^{(m\wedge p)}(M'M)$, and in the third line π is the measure on Ω_A induced by dM through $\Delta(M)$. Thus (3.8) and (3.9) show that the test (1.10) is generalized Bayes as in Remark 2.2, hence admissible among invariant tests.

4. MANOVA. Consider the problem testing

$$H_0{:}\mu_1 = 0 \quad \text{versus} \quad H_A{:}\mu_1 \neq 0 \quad \text{based on} \quad (Y_1, S_{11}), \tag{4.1}$$

where Y_1 and S_{11} are as in (1.1). This problem is (1.2) with $q = 0$. The invariance group here is $\bar{G} \equiv Gl(p) \times \mathscr{O}(m)$ which acts as in Section 1. Schwartz (1967) has given necessary conditions and sufficient conditions for an invariant test to be admissible among all tests.

The maximal invariant statistic and parameter are $w^*(X)$ in (3.4) and $\Delta^* \equiv \lambda^{(m\wedge p)}(\mu_1\Sigma_{11}^{-1}\mu_1')$ respectively. The ratio of densities for the reduced problem is given in James (1964, Equations (73), (74) and (23)) by

$$R_{\Delta^*}^* = (\text{const})\exp\left(-\frac{1}{2}\,\Sigma\Delta_i^*\right)\int_{\mathscr{O}(m)} {}_1F_1\left(\frac{n+m}{2};\, b;\, \frac{1}{2}\,\bar{\Delta}^*\,\Gamma^{-*}w\Gamma'\right)\nu(d\Gamma)$$

where $b = \max(p, m)$. It can also be written as R_{Δ^*} of (2.4) with (v, w) replaced by (w^*, w^*). Let Φ^* be the class of tests based on (Y_1, S_{11}) of the form (2.9) with $(R_\Delta, w, C, \mathscr{L})$ replaced by $(R_{\Delta^*}^*, w^*, C^*, \mathscr{L}^*)$.

THEOREM 4.1. *Φ^* is the minimal complete class of invariant tests for problem* (4.1).

PROOF. When $p \geq m$, the proof follows that of Theorem 2.1 with $q = 0$ and $R_{\Delta^*}^*$ instead of R_Δ. When $p < m$, the proof is similar, but Equation (2.20) needs to be replaced by

$$B_I|\bar{w}^*|^t e^*(\Delta^*, w^*) \leq R_{\Delta^*}^* \leq B_S e^*(\Delta^*, w^*)$$

where

$$e^*(\Delta^*, w^*) = \int_{\mathscr{O}(m)} |I + \Gamma_{11}'\bar{\Delta}^*\Gamma_{11}|^{-1}\exp\left(\frac{1}{2}\operatorname{tr}\bar{\Delta}^*\Gamma_{11}\bar{w}^*\Gamma_{11}'\right)\nu(d\Gamma),$$

$\bar{\Delta}^*$ and $\bar{w}^*$ are $p \times p$ diagonal matrices containing Δ^* and w^* as diagonals, and Γ_{11} is the upper left $p \times p$ part of Γ. For a fixed Δ^*, $e^*(\Delta^*, w^*)$ is clearly convex in w^*, and increasing in each w_i^*. It is also invariant under permutation of the elements of w^* since any permutation can be performed by transforming Γ_{11} alone. Hence $e^*(\Delta^*, w^*)$ is convex and increasing$_w$. The rest of the proof follows as in Theorem 2.1 with minor changes.

The $\kappa_i(T^*)$ tests (1.4) are in Φ^* since they are of the form (3.5). Schwartz has shown all such tests to be admissible among all tests. His necessary condition for an invariant test to be admissible among all tests is that the acceptance region must be convex and nonincreasing$_w$ in $(w_1^{*1/2}, \cdots, w_{m\wedge p}^{*1/2})$. That this condition is also necessary for admissibility among invariant tests follows from Theorem 4.1 by using the technique in Corollary 2.4

TABLE 5.1

n	(m, p, q)	form of Δ	Maxima: LRT beats Bayes (1.10)	Maxima: Bayes beats LRT	n	(m, p, q)	form of Δ	Maxima: LRT beats Bayes	Maxima: Bayes beats LRT
5	(2, 2, 2)	A*	.025	.005	20	(2, 2, 5)	E	.003	.002
		E*	.021	.009		(2, 5, 2)	E	.008	.002
10	(2, 2, 2)	E	.008	.003		(5, 2, 2)	E	.005	.001
	(2, 2, 5)	E	.005	.000		(2, 5, 5)	E	.004	.001
	(2, 5, 2)	E	.016	.010		(5, 2, 5)	E	.005	.000
	(5, 2, 2)	E	.014	.006		(5, 5, 2)	A	.014	.003
20	(2, 2, 2)	E	.001	.002			E	.011	.003
	(2, 2, 5)	A	.002	.001		(5, 5, 5)	E	.008	.001

* A means $\Delta = (\Delta_1, 0, \cdots, 0)$, E means $\Delta = (\Delta_1, \Delta_1, \cdots, \Delta_1)$. The standard errors for the "LRT beats Bayes" columns range from .001 to .003, and for the "Bayes beats LRT" columns from .0004 to .0014.

TABLE 5.2
Maxima (standard error)

(n, m, p, q)	Form of Δ	LRT beats $\kappa_4(T)$	$\kappa_4(T)$ beats LRT	LRT beats LBI	LBI beats LRT	Power of LBI
(5, 2, 2, 2)	$(\Delta_1, 0)$	.19 (.05)	0*	.34 (.05)	.17 (.05)	.59
	$(\Delta_1, \Delta_1/10)$	.11 (.03)	0	.21 (.05)	.14 (.05)	.78
	$(\Delta_1, \Delta_1/2)$	.09 (.04)	0	.01 (.02)	.30 (.05)	.98
	(Δ_1, Δ_1)	.066 (.004)	.005 (.001)	.009 (.002)	.220 (.005)	.97
(10, 2, 5, 2)	$(\Delta_1, 0)$	.202 (.005)	0	.324 (.005)	.040 (.004)	.69
	$(\Delta_1, \Delta_1/10)$	.21 (.05)	.01 (.01)	.28 (.05)	.05 (.04)	.86
	$(\Delta_1, \Delta_1/2)$	.159 (.005)	.001 (.001)	.030 (.002)	.087 (.004)	.986
	(Δ_1, Δ_1)	.138 (.005)	.001 (.001)	.021 (.002)	.084 (.004)	.985
(20, 2, 5, 5)	$(\Delta_1, 0, 0, 0, 0)$	.286 (.005)	0	.075 (.003)	.009 (.003)	.976
	$(\Delta_1, \Delta_1, 0, 0, 0)$	.29 (.05)	0	.02 (.02)	.06 (.03)	.998
	$(\Delta_1, \Delta_1, \cdots, \Delta_1)$	.251 (.005)	0	.001 (.001)	.046 (.003)	1
(20, 5, 2, 2)	$(\Delta_1, 0)$	.076 (.004)	.000 (.000)*	.116 (.003)	.002 (.002)	.985
	(Δ_1, Δ_1)	.057 (.004)	0	.000 (.000)	.030 (.003)	.999

* "0" indicates the difference was always negative; ".000" indicates a positive number less than .0005.

and noting that

$$R^*_{\Delta^*} = (\text{const})\exp\left(-\frac{1}{2}\Sigma\Delta_i^*\right)$$
$$\cdot \int_{\mathscr{O}(m)} \int_{Gl(p)} |B'B|^t \exp\left(-\frac{1}{2}\operatorname{tr} B'B + \operatorname{tr} \bar{\Delta}^{*1/2}\Gamma\bar{w}^{*1/2}B\right) dB\nu(d\Gamma),$$

which is convex and symmetric in $(w_1^{*1/2}, \cdots, w_m^{*1/2}{}_{\wedge p})$.

We mention that the Bayes tests analogous to the one at the end of Section 4 is here the LRT, $\kappa_4(T^*)$.

5. Numerical comparisons.

Consider the LRT, $\kappa_4(T_1)$ of (1.4), and the Bayes test (1.10). Under H_0, T_1 and T_2 are independent. See Kariya (1978), Lemma 3.1. For large n, there are constants a and b depending on (n, m, p, q) such that

$$a \log|I + T_1| \approx \chi^2_{pm} \quad \text{and} \quad b \log|I + T_2| \approx \chi^2_{qm}.$$

See, for example, Pearson and Hartley (1972), page 99. Thus the difference between the LRT and Bayes test (1.10) is similar to the difference between tests based on χ^2_{pm} and $\chi^2_{pm} + k\,\chi^2_{qm}$ where

$$k = p[n - q - \tfrac{1}{2}(p - m + 1)]/[n - \tfrac{1}{2}(q - m + 1)](n + m - q).$$

For large n, $k \approx p/n$, hence the $k\chi^2_{qm}$ part adds little to the second test, suggesting that the LRT and Bayes tests are alike.

The Monte Carlo study looked at the difference in power of the two tests for various values of Δ for $\alpha = .05$. Each difference was calculated using 10,000 pseudo-observations. In Table 5.1 we exhibit the maximum the LRT beats the Bayes test and the maximum the Bayes test beats the LRT, where each maximum was taken over between 10 and 15 parameter points. The standard errors for these differences ranged from .0004 to .003. As can be seen, even for small n the two tests are very close in power, and the LRT almost always beats the Bayes test by more than the Bayes test beats the LRT.

To compare the LRT to the LBI test (1.9) and the $\kappa_4(T)$ test (1.4), we proceeded as above, although some differences were based on only 100 replications. We include in Table 5.2 the comparisons as well as the maximum power the LBI test was observed to have. Clearly the $\kappa_4(T)$ test can be dismissed. The LBI test and LRT should be explored more carefully, as it appears that each has some advantages over the other.

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REFERENCES

[1] ANDERSON, T. W. and PERLMAN, M. D. (1982). On a consistency property of invariant tests for the multivariate analysis of variance and related problems (Abstract). *Inst. Math. Statist. Bull.* **11** 325.

[2] BANKEN, L. (1983). On the reduction of the general MANOVA model. Technical Report, Universität Trier, Trier, West Germany.

[3] GLESER, L. J. and OLKIN, I. (1970). Linear models in multivariate analysis. *Essays in Probability and Statistics* 276–292. Wiley, New York.

[4] HOOPER, P. M. (1983). Simultaneous set estimation in the general multivariate analysis of variance model. *Ann. Statist.* **11** 666–673.

[5] JAMES, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[6] KARIYA, T. (1978). The general MANOVA problem. *Ann. Statist.* **6** 200–214.

[7] KHATRI, C. G. (1966). A note on a MANOVA model applied to problems in growth curves. *Ann. Inst. Statist. Math.* **18** 75–86.

[8] MARDEN, J. I. (1982). Minimal complete classes of tests of hypotheses with multivariate one-sided alternatives. *Ann. Statist.* **10** 962–970.

[9] MARDEN, J. I. and PERLMAN, M. D. (1980). Invariant tests for means with covariates. *Ann. Statist.* **8** 25–63.

[10] MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and Its Applications.* Academic, New York.

[11] PEARSON, E. S. and HARTLEY, H. O., ed. (1972). *Biometrika Tables for Statisticians.* Cambridge Univ. Press.

[12] POTTHOFF, R. F. and ROY, S. N. (1964). A generalized multivariate analysis of variance model useful especially for growth curve problems. *Biometrika* **51** 313–326.

[13] SCHWARTZ, R. (1967). Admissible tests in multivariate analysis of variance. *Ann. Math. Statist.* **38** 698–710.

[14] WARE, J. H. and BOWDEN, R. E. (1977). Circadian rhythm analysis when output is collected at intervals. *Biometrics* **33** 566–571.

[15] ZERBE, G. O. and JONES, R. H. (1980). On application of growth curve techniques to time series data. *J. Amer. Statist. Assoc.* **75** 507–509.

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