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APPLYING WALD'S VARIANCE COMPONENT TEST[1]

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In this note a generalization of a variance component test that was first suggested by Wald is examined. Necessary and sufficient conditions are given for the test to be applicable in a mixed linear model. A uniqueness property of the test in terms of degrees of freedom is also obtained.

1. Introduction. For a regression model where a subset of the regression parameters are independent and normally distributed with mean zero and variance σ_B^2, Wald (1947) showed how to place a confidence interval on the ratio of σ_B^2 to the error variance σ^2 via the F-distribution. His confidence interval also provided a means to test the null hypothesis that σ_B^2 is zero. Since Wald's paper, several authors have used modifications of Wald's idea to test variance components in particular mixed linear models. Spjøtvoll (1968), however, shows by means of an example that this is not possible with all mixed linear models. The present note is an attempt to indicate the conditions under which the Wald test can be applied, and to present the test in such a way that modification of Wald's idea is not necessary to generate the test statistic.

In the sequel we obtain the Wald test via reduction sums of squares. This circumvents the necessity of transforming to independent variates and/or modifying Wald's method as discussed by Spjøtvoll (1968). We also give necessary and sufficient conditions under which the Wald test can be used in mixed linear models. These conditions are given in terms of matrix ranks and do not require the matrices involved to have any particular structure. One drawback of the conditions, however, is that they are dependent on using σ as the common scale factor for the two quadratic forms that make up the F-ratio. Lastly, we give a uniqueness property of the Wald test that allows one to immediately determine whether or not a proposed variance component test in a mixed linear model is the Wald test.

Some notation used throughout is $\underline{R}(A)$ and $\underline{r}(A)$ for the range and rank of a matrix A. The notation $\mathscr{A}^\perp$ denotes the orthogonal complement of the set $\mathscr{A}$, and $N_s(\mu, \Sigma)$ denotes the s-dimensional normal distribution with mean μ and covariance matrix Σ. Other notation will be introduced as needed.

2. Wald's test. Consider the mixed linear model

$$Y = X\pi + Bb + e \tag{2.1}$$

where π is a $p \times 1$ vector of unknown constants, $b \sim N_t(0, \sigma_B^2 I)$, $e \sim N_n(0, \sigma^2 I)$, b and e are independent, and X, B are known matrices. Wald showed how to construct a confidence interval for $\rho = \sigma_B^2/\sigma^2$ which provides a method for testing

$$H_B\colon \sigma_B^2 = 0 \quad \text{vs} \quad K_B\colon \sigma_B^2 > 0.$$

In Wald's original paper it was implicitly assumed that the partitioned matrix (X, B) was of full column rank. After examining Wald's development, it can be seen that his test statistic for H_B is identical with the test statistic for the test of "no b-effects" when b is a vector of fixed effects. Viewing Wald's test in this manner, it is straightforward to relax the full rank assumption as follows: Pretend momentarily that b is a fixed effect. Let R denote

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the residual sum of squares and let $R(b \mid \pi)$ be the sum of squares for b adjusted for π. Under the assumptions of model (2.1), it is easy to verify that $R/\sigma^2 \sim \chi^2_{(f)}$ where $f = n - \underset{\sim}{r}(X, B)$ and that R and $R(b \mid \pi)$ are independent. Furthermore, under H_B it is easy to verify that $R(b \mid \pi)/\sigma^2 \sim \chi^2_{(k)}$ where $k = \underset{\sim}{r}(X, B) - \underset{\sim}{r}(X)$. Thus, Wald's procedure leads to testing H_B via an F-ratio.

REMARK 2.2. Wald's original paper, and subsequent papers, have mainly concentrated on a confidence interval for ρ as opposed to testing H_B. The assumed full column rank of (X, B) is not necessary to form such an interval. To see this let L and F be matrices whose columns form orthonormal bases for $\underset{\sim}{R}(X, B) \cap \underset{\sim}{R}(X)^\perp$ and $\underset{\sim}{R}(X, B)^\perp$ respectively. Set $U = L'Y$ and $Z = F'Y$. Then $U \sim N_k(0, \sigma^2 I + \sigma_B^2 G)$ where $G = L'BB'L$, $Z \sim N_f(0, \sigma^2 I)$, and U and Z are independent. It follows that $Q(\rho) = k^{-1} f U'(I + \rho G)^{-1} U/Z'Z$ has an F-distribution with k and f degrees of freedom. Because G can be taken as a diagonal matrix (transform $L'Y$ via an orthogonal matrix that diagonalizes $L'BB'L$), it is easy to see that $Q(\rho)$ is a decreasing function of ρ. Let $\psi(\cdot)$ denote the cdf of the F-distribution with k and f degrees of freedom. Then a $1 - \alpha$ confidence interval $[\rho_*; \rho^*]$ can be formed for ρ by solving $Q(\rho_*) = x_2$ and $Q(\rho^*) = x_1$ where $\psi(x_1) = \alpha_1$ and $\psi(x_2) = 1 - \alpha_2$ with $\alpha = \alpha_1 + \alpha_2$. As a referee pointed out, the upper bound ρ^* can be negative which is contrary to the model assumptions. This happens when $Q(0)$, which is Wald's test statistic, is smaller than x_1. ($Q(0)$ is Wald's test statistic because $U'U = R(b \mid \pi)$ and $Z'Z = R$.) From Okamoto's (1960) corollary it can be shown that

$$\operatorname{Prob}\{Q(0) < x_1 \mid \rho\} \leq \psi(x_1/m_\rho) \quad \text{for all} \quad \rho \geq 0,$$

where m_ρ is the harmonic mean of $(1 + g_1\rho), \cdots, (1 + g_k\rho)$ when G is expressed in the diagonal form $G = \operatorname{diag}(g_1, \cdots, g_k)$. Thus, the probability that ρ^* is negative is at most α_1 and goes to zero as ρ gets large. To avoid the possibility of negative endpoints, one can use an adjusted interval suggested by Thompson (1955b). The adjusted interval is $[\max(0, \rho_*), \max(0, \rho^*)]$ which has confidence level $1 - \alpha$ for $\rho > 0$ and $1 - \alpha_2$ for $\rho = 0$. □

3. Main results. From the previous section it is clear that the Wald test is applicable to model (2.1) whenever k and f are both nonzero. For special cases of the model, the test has been derived by several authors including Thompson (1955a, 1955b), Spjøtvoll (1967), and Portnoy (1973). As illustrated by Spjøtvoll (1968) and Thomsen (1975), it is also possible to apply Wald's test in some models where there are more than two sources of variation.

Because the Wald test can sometimes be applied to more general mixed models than allowed by (2.1), we consider here the mixed linear model

$$Y = X\pi + Bb + Cc + e \tag{3.1}$$

where X, π, B, b, and e are defined as in model (2.1). We assume that interest is still in testing H_B vs K_B. We further assume that C is an $n \times s$ known matrix and that c is a random vector independent of b and e whose distribution is $N_s(0, \Gamma)$. Here Γ is an unknown covariance matrix ranging over a subset, say $\mathscr{V}$, of non-negative definite matrices. The form of $\mathscr{V} = \{\Gamma\}$ can be selected arbitrarily. For example: $\Gamma = \sigma_A^2 I$ with $\sigma_A^2 \geq 0$; or $\Gamma = \operatorname{diag}(\sigma_A^2, \cdots, \sigma_A^2, \sigma_{AB}^2, \cdots, \sigma_{AB}^2)$ with $\sigma_A^2, \sigma_{AB}^2 \geq 0$; or even Γ completely unknown. We do, however, assume that there is at least one $\Gamma \in \mathscr{V}$ that is a positive definite (p.d.) matrix.

Let us try to extend the development of Wald's test as presented in Section 2. Momentarily assume b and c are fixed effects. Let R denote the residual sum of squares and let $R(b \mid \pi, c)$ denote the sum of squares for b adjusted for π and c. Under the assumptions of model (3.1), the following facts can be established:

(3.2)

(a) $R/\sigma^2 \sim \chi^2_{(f)}$ where $f = n - r(X, B, C)$,

(b) $R(b \mid \pi, c)/\sigma^2 \sim \chi^2_{(k)}$ for all distributions under H where $k = r(X, B, C) - r(X, C)$,

(c) R and $R(b \mid \pi, c)$ are independent.

Thus, if f and k are both nonzero, then H_B can be tested via an F-ratio. We shall refer to this F-test as the Wald test.

PROPOSITION 3.3 *Let $Q_e = Y'H_eY$ where H_e is any real symmetric matrix. If $Q_e/\sigma^2 \sim \chi^2_{(v)}$ for all distributions* (3.1), *then $v \le f$. Moreover, if $v = f$, then $Q_e = R$.*

PROOF. Because $\Sigma = \text{Cov}(Y)$ is p.d., we can conclude from Theorem 9.2.1 in Rao and Mitra (1971) that $\sigma^{-2}H_e\Sigma H_e = H_e$. This equality must hold for all $\sigma^2 > 0$, all $\sigma_B^2 \ge 0$, and all $\Gamma \in \mathscr{V}$. This means H_e is idempotent; $H_eBB'H_e = 0$ which implies $B'H_e = 0$; and $H_eC\Gamma C'H_e = 0$ which implies $\Gamma C'H_e = 0$. Because there is at least one $\Gamma \in \mathscr{V}$ that is p.d. (by assumption), it must be true that $C'H_e = 0$. The same theorem in Rao and Mitra also implies $v = \sigma^{-2}\text{tr}(\Sigma H_e)$ and $\sigma^{-2}\pi'A'H_eA\pi = 0$. Using the properties of H_e already established, we get $v = \underline{r}(H_e)$ and (because π is arbitrary) $A'H_e = 0$. Hence H_e is an orthogonal projection operator satisfying $\underline{R}(H_e) \subset \mathscr{N} = \underline{R}(X, B, C)^\perp$. This implies $v \le f$ and if $v = f$ that $\underline{R}(H_e) = \mathscr{N}$ so that $Q_e = R$. □

PROPOSITION 3.4. *Let $Q_b = Y'H_bY$ where H_b is any real symmetric matrix. Assume $Q_b/\sigma^2 \sim \chi^2_{(u)}$ for all distributions under H_B. The following conclusions can be drawn*: (a) *If $k = 0$, then $Q_b/\sigma^2 \sim \chi^2_{(u)}$ for all distributions* (3.1). (b) *If Q_b and R are independent, then $u \le k$.* (c) *If Q_b and R are independent and if $u = k$, then $Q_b = R(b \mid \pi, c)$.*

PROOF. Applying the same ideas as in the proof of Proposition 3.3, we find that H_b must be an orthogonal projection operator satisfying $\underline{R}(H_b) \subset \underline{R}(X, C)^\perp$. For part (a), notice that $k = 0$ implies $\underline{R}(B) \subset \underline{R}(X, C)$, from which it follows that the distribution of Q_b does not depend on b. For parts (b) and (c) first note that independence implies $(Q_b + R)/\sigma^2 \sim \chi^2_{(u+f)}$ for all distributions under H_B. As in the proof of Proposition 3.3, this implies $(u + f) \le q = \dim[\underline{R}(X, C)^\perp]$ and that $Q_b + R = [R(b \mid \pi, c) + R]$ when $u + f = q$. Both (b) and (c) now follow after noting that $q = f + k$. □

There are several conclusions concerning the Wald test for H_B that can now be drawn:

(3.5)

(a) R and $R(b \mid \pi, c)$ are the unique quadratic forms satisfying all three conditions in (3.2);

(b) there exists a nonzero quadratic from Q_e satisfying $Q_e/\sigma^2 \sim \chi^2_{(v)}$ for all distributions (3.1) if and only if $f > 0$;

(c) there exists a nonzero quadratic form Q_b satisfying $Q_b/\sigma^2 \sim \chi^2_{(u)}$ for distributions under H_B and Q_b/σ^2 is stochastically larger than $\chi^2_{(u)}$ for all distributions under K_B if and only if $k > 0$.

The sufficiency part of (3.5c) is the only portion of these statements that does not follow directly from Propositions 3.3 and 3.4. And this can be verified from the observations in Remark 2.2 which hold true for model (3.1) when X and $R(b \mid \pi)$ in the remark are replaced by (X, C) and $R(b \mid \pi, c)$ respectively.

EXAMPLE 3.6. Portnoy (1973) considers a special case of model (2.1) and derives what he claims is an improved test over the usual test. Using our notation, Portnoy's model has $B = \text{diag}[\underline{1}_{n_1}, \cdots, \underline{1}_{n_t}]$ where $n_1, \cdots, n_t$ are integers and $\underline{1}_r$ is an $r \times 1$ vector of 1's. His development is essentially as follows: Set $Y_1 = (B'B)^{-1/2}B'Y$ and $Y_2 = L'Y$ where L is a matrix whose columns form an orthonormal basis for $\underline{R}(B)^\perp$. Let S_2 denote the residual sum of squares for the model $Y_2 \sim N_{n-t}(F_2\pi, \sigma^2 I)$. Then $S_2/\sigma^2 \sim \chi^2_{(m_2)}$ where $m_2 = (n - t) - \underline{r}(F_2)$. At this point Portnoy assumes that $n_1 = \cdots = n_t = r$. Then $Y_1 \sim N_t(F_1\pi, \phi I)$ where $\phi = \sigma^2 + r\sigma_B^2$. Let S_1 be the residual sum of squares for the Y_1 model. Then $S_1/\phi \sim \chi^2_{(m_1)}$ where $m_1 = t - r(F_1)$. Next let Λ be any $p \times m$ matrix whose columns form a basis for $\underline{R}(F'_1) \cap R(F'_2)$ and let $\theta = \Lambda'\pi$. (We have introduced θ instead of ν^* which Portnoy used. This avoids his implicit assumption that the ν^* vector is estimable in both the Y_1

and Y_2 models.) Then using the difference of the least squares estimators for θ based on the Y_1 and Y_2 models, Portnoy determines a third sum of squares T such that $T/\sigma^2 \sim \chi^2_{(m)}$ under H_B (our m is the same as Portnoy's k). Portnoy states that the usual test is based on S_1 and S_2 and suggests his improved test based on $(T + S_1)$ and S_2. It can be established that $\underset{\sim}{r}(F_2) = \underset{\sim}{r}(X, B) - \underset{\sim}{r}(B)$. Since $\underset{\sim}{r}(B) = t$, it follows that $m_2 = f$. Further,

$$m = \dim[\underset{\sim}{R}(F'_1) \cap \underset{\sim}{R}(F'_2)] = \underset{\sim}{r}(F'_1) + \underset{\sim}{r}(F'_2) - \underset{\sim}{r}(F'_1, F'_2) = \underset{\sim}{r}(F'_1) + \underset{\sim}{r}(F'_2) - \underset{\sim}{r}(X).$$

Using this last expression for m and $\underset{\sim}{r}(F_2) = \underset{\sim}{r}(X, B) - t$, it is easy to check that $m_2 + m$, the degrees of freedom for $(S_1 + T)$, is equal to k. Hence, (3.5a) implies $S_2 = R$ and $S_1 + T = R(b \mid \pi)$ so that Portnoy's test is in fact the Wald test. □

EXAMPLE 3.7. Spjøtvoll (1968) and Thomsen (1975) considered variance component testing in the completely random model

$$Y_{ijh} = \mu + a_i + d_j + g_{ij} + e_{ijh}$$

where μ is a fixed effect and the remaining terms are random with the usual assumptions. Write the model in matrix form as $Y = X\mu + Aa + Dd + Gg + e$ and suppose $i = 1, \cdots, r, j = 1, \cdots, s$, and $h = 1, \cdots, n_{ij}$. Let m denote the number of nonzero n_{ij}. To explicitly write out ranks, we assume that $r(A, D) = r + s - 1$. (This connectedness assumption is trivially satisfied in Spjøtvoll (1968) because $m = rs$ and is implicitly assumed in Thomsen (1975) because of his (iii) in equation (5.2).) Consider testing H: $\sigma^2_G = 0$. In model (3.1) take $B = G$ and $C = (A, D)$. It is well known that $r(X, A, D, G) = r(G) = m$; so $f = n - m$ and $k = m - r - s + 1$. Since both f and k agree with the degrees of freedom for the tests derived by Spjøtvoll and Thomsen, it follows from (3.5a) that their tests are based on $R(g \mid \mu, a, d)$ and R. This fact is actually apparent in Spjøtvoll's work, but required a separate proof in Thomsen's work. □

EXAMPLE 3.8. Consider testing H_A: $\sigma^2_A = 0$ in Example 3.7. Taking $B = A$ and $C = (D, G)$, we get $f = n - m$ and $k = 0$. This means $R(a \mid \mu, d, g)$ $(= 0)$ cannot be used to form a test. However, Wald type tests are derived in both Spjøtvoll (1968) and Thomsen (1975) under the additional assumption that $\sigma^2_G = 0$. In this case Gg does not appear in the model which means we now have $C = D$ so that $f = n - r - s + 1$ and $k = r - 1$. Thus, H_A can be tested via the Wald test when $\sigma^2_G = 0$. By comparing degrees of freedom we see that Spjøtvoll's test in Section 2.b (no missing cells) is the Wald test, but his test in Section 4 (missing cells) is not the Wald test, nor is the test given by Thomsen the Wald test. □

EXAMPLE 3.9. For testing H_A in Example 3.8 with a Wald test we found that $k = 0$. This does not say that it is impossible to test H_A via an F-ratio. For example, it is implied in Thomsen (1975) that this can be done if all of the nonzero n_{ij} are equal (which is a well known fact when all of the n_{ij} are nonzero and equal). To see this, suppose all of the nonzero n_{ij} are equal to v. Set $Y_1 = L'Y$ where $L' = (G'G)^{-1/2}G'$. Then

$$Y_1 = X_1\mu + A_1 a + D_1 d + g_1, \quad g_1 = L'Gg + L'e,$$

where X_1, A_1, and D_1 are defined in the obvious manner. Because $G'G = vI$, it is easy to check that $g_1 \sim N_m(0, \phi I)$ where $\phi = \sigma^2 + v\sigma^2_G$. Because of the covariance structure of g_1, we see that Y_1 satisfies the assumptions of model (3.1); so, we could attempt to apply a Wald test through the Y_1 model. Taking $B = A_1$ and $C = D_1$, we now find that $f = m - \underset{\sim}{r}(X_1, A_1, D_1) = m - r - s + 1$ and $k = r - 1$ (the matrix L' does not change the rank properties of X, A and D). Thus $R(a \mid \mu, d)$ and R based on the Y_1 model could be used to form a Wald test for H_A. Actually, even if the nonzero n_{ij} are not all equal an F-ratio for testing H_A can still be determined (although probably not recommended). For example, let v denote the minimum value of the nonzero n_{ij}. Next form an $mv \times 1$ vector Y_2 composed of v observations out of each group for which n_{ij} is nonzero. Then proceed as above to form Y_1 using the vector Y_2 in place of Y. □

Without close examination, it may appear that Example 3.9 and statements (3.5b, c) are contradictory. Statements (3.5b, c) say that $k, f > 0$ is a necessary and sufficient condition to form an F-ratio for testing H_B via quadratic forms Q_e/σ^2 and Q_b/σ^2. The common parameter σ^2 is crucial in this statement because, as Example 3.9 shows, it may happen that σ^2 can be replaced by some function of σ^2 and Γ (e.g., ϕ in Example 3.9) and still form an F-ratio to test H_B even when $k, f > 0$ is not satisfied.

REFERENCES

OKAMOTO, M. (1960). An inequality for the weighted sum of χ^2 variates. *Bull. Math. Statist.* **9** 69–70.

PORTNOY, S. (1973). On recovery of intra-block information. *J. Amer. Statist. Assoc.* **68** 384–391.

RAO, C. R. and MITRA, S. K. (1971). *Generalized Inverse of Matrices and its Applications.* Wiley, New York.

SPJØTVOLL, E. (1967). Optimum invariant tests in unbalanced variance components models. *Ann. Math. Statist.* **38** 422–428.

SPJØTVOLL, E. (1968). Confidence intervals and tests for variance ratios in unbalanced variance components models. *Rev. Internat. Statist. Inst.* **36** 37–42.

THOMPSON, W. A. (1955a). The ratio of variances in a variance components model. *Ann. Math. Statist.* **26** 325–329.

THOMPSON, W. A. (1955b). On the ratio of variances in the mixed incomplete block model. *Ann. Math. Statist.* **26** 721–733.

THOMSEN, I. (1975). Testing hypotheses in unbalanced variance components models for two-way layouts. *Ann. Statist.* **3** 257–265.

WALD, A. (1947). A note on regression analysis. *Ann. Math. Statist.* **18** 586–589.

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