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ASYMPTOTIC EXPANSIONS FOR THE ERROR PROBABILITIES OF SOME REPEATED SIGNIFICANCE TESTS

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Asymptotic expansions for the error probabilities of repeated significance tests about a normal mean are developed. Use of the expansions appears to result in substantially improved numerical accuracy, when compared to the use of the leading term, at least in some important special cases. The expansions are sufficiently refined to show the effect of some simple modifications of the basic procedure, such as requiring an initial sample size.

1. Introduction. Let $X_1, X_2, \cdots$ be i.i.d normally distributed random variables with unknown mean θ, $-\infty < \theta < \infty$, and unit variance; and consider the problem of testing H_0: $\theta = 0$. In recent work on sequential tests of H_0, there has been continuing interest in repeated significance tests, which were introduced by Armitage (1975) and by Robbins (1970), in a slightly different form. These may be described as follows: for $a > 0$ and $c \geq 0$, let

$$\tau = \tau_{a,c} = \inf\{n \geq 1 : |S_n| > \sqrt{2a(n+c)}\},$$

where $S_n = X_1 + \cdots + X_n$, $n \geq 1$, and the infimum of the empty set is ∞; if $N > 1$, then repeated significance tests with parameters a, c, and N take $T = \min(\tau, N)$ observations and reject H_0 iff $\tau \leq N$.

Exact expressions for the error probabilities and expected sample sizes of repeated significance tests are extremely complicated: but asymptotic values as $a \to \infty$ may be deduced from non-linear renewal theory, as developed by Woodroofe (1976a) and Lai and Siegmund (1977, 1979). In particular, Siegmund (1977) has developed the following approximation to the Type I error: let P_θ denote the probability measure under which X_1, $X_2, \cdots$ are i.i.d normally distributed random variables with mean θ, $-\infty < \theta < \infty$. If $N = N_a \to \infty$ as $a \to \infty$ with $N \sim 2a/\delta_0^2$, where $0 < \delta_0 < \infty$, then

$$P_0\{\tau \leq N\} \sim K\sqrt{a}e^{-a} \tag{1}$$

as $a \to \infty$, where

$$K = \frac{2}{\sqrt{\pi}} \int_{\delta_0}^{\infty} \nu(x)\exp(-cx^2/2)\,\frac{dx}{x}$$

$$\nu(x) = 2x^{-2}\exp\left\{-2\sum_{k=1}^{\infty} \frac{1}{k}\,\Phi\left(-\frac{1}{2}x\sqrt{k}\right)\right\}, \quad x > 0,$$

and Φ denotes the standard normal distribution function.

Siegmund (1977) conducted a substantial study of the numerical accuracy of (1) and related approximations. In the study, he compared his approximations to exact values computed by repeated numerical integrations by McPherson and Armitage (1971) for selected special cases. Siegmund expressed general satisfaction with the approximation (1) when the offset parameter is $c = 0$, but found that it could substantially underestimate the

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real Type I error probability when $c \neq 0$. He then suggested including an additional term, derived from a Wiener process approximation, in (1).

Here we continue the study of numerical accuracy by presenting an asymptotic expansion for the Type I error probability, up to terms which are of order $a^{-1/2}e^{-a}$ as $a \to \infty$. For selected $c \neq 0$, the asymptotic expansion provides a substantial improvement over the basic approximation (1). In addition, we consider two modifications of τ, requiring an initial sample size and adding a lower stopping boundary. The latter is suggested by Schwarz's (1962, 1968) asymptotic shapes. In both cases, the effect of the modification is of order $a^{-1/2}e^{-a}$, the same as the correction to (1).

The paper proceeds as follows: in Section 2, we present an asymptotic expansion which was recently developed by Takahashi and Woodroofe (1981) for certain conditional probabilities; in Section 3, we show how this asymptotic expansion may be converted into an asymptotic expansion for the Type I error probability; in Section 4, we investigate the Type I error probability of the related procedure in which one is required to take m observations initially; in Section 5, we adapt our techniques to study the error probabilities which result from using Schwarz's (1962) asymptotic shapes; and Section 6 contains remarks.

2. Conditional probabilities. It is simpler to work with the one-sided version of τ,

$$t = t_{a,c} = \inf\{n \geq 1 : S_n > \sqrt{2a(n+c)}\}. \tag{2}$$

Let

$$\psi_a(n, r) = P\{t \geq n \mid S_n = \sqrt{2a(n+c)} + r\}, \quad -\infty < r < \infty,\ a > 0,\ n \geq 1;$$

then

$$\psi_a(n, r) = P_0\{S_{nk} \leq \beta^a_{nk},\ 1 \leq k \leq n-1\},$$

where

$$S_{nk} = S_k - \frac{k}{n} S_n, \quad 1 \leq k \leq n,$$

and

$$\beta^a_{nk} = \sqrt{2a(n-k+c)} - \left(1 - \frac{k}{n}\right)[\sqrt{2a(n+c)} + r], \quad 1 \leq k \leq n.$$

Indeed, this follows from the independence of S_n from the vector S_{nk}, $k = 1, \cdots, n-1$, and obvious symmetries of the latter. Now, as $n = n_a \to \infty$ and $a \to \infty$ with $\sqrt{(2a/n)} \to \varepsilon$, $0 < \varepsilon < \infty$, $\beta^a_{nk} \to \frac{1}{2}\varepsilon k - r$, and $S_{nk} \to S_k$ w.p.1 (P_0) for each fixed $k \geq 1$. This suggests approximating the conditional probabilities $\psi_a(n, r)$ by

$$\psi(\varepsilon, r) = P_0\{S_k \leq \tfrac{1}{2}\varepsilon k - r, \quad \text{for all} \quad k \geq 1\}, \quad -\infty < r < \infty,\ \varepsilon > 0.$$

Moreover, while it is not obvious from Siegmund's (1977) derivation, (1) follows easily from this approximation (cf. Woodroofe, 1976b).

There are two major sources of error in approximating $\psi_a(n, r)$ by $\psi(\varepsilon, r) : \beta^a_{nk}$ is not quite equal to $\frac{1}{2}\varepsilon k - r$; and S_{nk} is not quite equal to S_k. It is straightforward to generate an asymptotic expansion for β^a_{nk}; and it is not difficult to obtain an asymptotic expansion for the likelihood ratio of the distribution of $S_{n1}, \cdots, S_{nb}$ to that of $S_1, \cdots, S_b$ for suitable chosen b. If these expansions are used in place of the simple limits above, then the following expansion for $\psi_a(n, r)$ results: if $n = n_a \to \infty$ as $a \to \infty$ with $\varepsilon_n = \sqrt{(2a/n)}$ bounded away from 0 and bounded above by $O(a^{1/8})$, then

$$\psi_a(n, r) = \psi(\varepsilon_n^*, r) - \frac{1}{n}\rho_c(\varepsilon_n, r) + o\left(\frac{1}{n}\right), \tag{3}$$

where

$$\varepsilon_n = \sqrt{(2a/n)}, \quad \varepsilon_n^* = \varepsilon_n + \frac{2r}{n}, \quad \rho_c(\varepsilon, r) = \left[\frac{\psi'(\varepsilon, 0)}{\psi(\varepsilon, 0)} + \frac{1}{2}r - \frac{3}{2}c\varepsilon\right]\psi'(\varepsilon, r) + \psi''(\varepsilon, r),$$

and $'$ denotes differentiation with respect to ε. The relation (3) holds uniformly on compacts in r, $-\infty < r < \infty$. Moreover, letting $\lambda_0 > 0$, $0 < \lambda_n = o[\exp(\sqrt{\log n}\,)]$, and

$$D_a(n, r) = n[\psi_a(n, r) - \psi(\varepsilon_n^*, r)] + \rho_c(\varepsilon_n, r),$$

$$|D_a(n, r)| \le C\varepsilon_n, \quad -\infty < r < \lambda_n,$$

and

$$|D_a(n, r)| \le C\exp(-\eta\varepsilon_n^2) + \bar{o}(n^{-\infty}), \quad -\infty < r < \lambda_0$$

for some constants C and η, where $\bar{o}(n^{-\infty})$ denotes a term of smaller order of magnitude n^{-k} for every $k \ge 1$.

The proof of these assertions when $c = 0$ is given by Takahashi and Woodroofe (1981). The extension to $c > 0$ is straightforward and has been omitted.

3. The Type I Error. Given an integer $N > 1$, let

$$\alpha = \alpha(a, c, N) = P_0\{t \le N\}.$$

Then the Type I error of the repeated significance tests is easily seen to be $2\alpha + o(\alpha^{\Delta})$ as $a \to \infty$ for some $\Delta > 1$. In this section we obtain an asymptotic expansion for α as N and $a \to \infty$ with N/a bounded away from 0 and ∞.

Our approach to the expansion is quite simple. Write

$$(4) \qquad \alpha = \textstyle\sum_{n=1}^N P_0\{t = n\} = \sum_{n=1}^N \displaystyle\int_0^\infty \psi_a(n, r)\frac{1}{\sqrt{n}}\phi[\sqrt{\{2a(1 + cn^{-1})\}} + rn^{-1/2}]\,dr,$$

where ϕ denotes the standard normal density function. If the expansion (3) for $\psi_a(n, r)$ is substituted into (4), then an expansion for α results. To describe the latter expansion, let

$$U(\varepsilon, s) = \int_0^\infty \psi(\varepsilon, r)e^{-sr}\,dr, \quad \varepsilon, s > 0,$$

and

$$V_c(\varepsilon) = \frac{\psi'(\varepsilon, 0)}{\psi(\varepsilon, 0)}U_{10}(\varepsilon, \varepsilon) + \frac{3}{2}U_{11}(\varepsilon, \varepsilon) + \frac{1}{2}U_{02}(\varepsilon, \varepsilon) + U_{20}(\varepsilon, \varepsilon) - \frac{1}{2}c\varepsilon[U_{01}(\varepsilon, \varepsilon) + 3U_{10}(\varepsilon, \varepsilon)], \quad \varepsilon > 0,$$

where $U_{ij}(\varepsilon, s) = \partial^{i+j}U(\varepsilon, s)/\partial\varepsilon^i\partial s^j$ for $\varepsilon, s > 0$. Then it may be shown that

$$(5) \qquad \alpha = \frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{a}}e^{-a}\left\{\textstyle\sum_{n=1}^N \varepsilon_n U(\varepsilon_n, \varepsilon_n)e^{-(1/2)c\varepsilon_n^2} - \dfrac{1}{2a}\sum_{n=1}^N \varepsilon_n^3 V_c(\varepsilon_n)e^{-(1/2)c\varepsilon_n^2} + o(1)\right\},$$

where $\varepsilon_n = \sqrt{(2a/n)}$; see Appendix 1. Moreover, the summations may be compared to integrals to yield the alternative form

$$(6) \qquad \alpha = \frac{1}{\sqrt{\pi}}\sqrt{a}e^{-a}\left\{\int_{e_0}^{e_1} 2U(\varepsilon, \varepsilon)e^{-(1/2)c\varepsilon^2}\varepsilon^{-2}\,d\varepsilon - \frac{1}{a}\int_{e_0}^{e_1} V_c(\varepsilon)e^{-(1/2)c\varepsilon^2}\,d\varepsilon + o\left(\frac{1}{a}\right)\right\},$$

where

$$e_0^2 = 2a/\left(N + \frac{1}{2}\right) \quad \text{and} \quad e_1^2 = 4a.$$

TABLE 1
*The Type 1 Error**

c = 4	N	a	exact	1st order	2nd order	diffusion
	185	3.65	0.050	0.045	0.048	0.048
	100	3.35	0.050	0.043	0.048	0.049
	42	2.88	0.049	0.037	0.047	0.050
	24	2.51	0.048	0.033	0.046	0.052
c = 1						
	140	5.54	0.0098	0.0095	0.0097	0.0095
	61	5.18	0.0096	0.0089	0.0095	0.0094
	33	4.87	0.0093	0.0083	0.0091	0.0093
	21	4.66	0.0093	0.0081	0.0092	0.0096

* *The exact figure is from the numerical integrations of McPherson and Armitage (1971); the first order approximation is computed from (1) with $\delta_0 = a/N$; the second order approximation is computed from (6); and the diffusion approximation is taken from Siegmund (1977).*

The function U may be computed from Spitzer's Formula as

$$U(\varepsilon, s) = \frac{1}{s} \exp\left\{ - \sum_{k=1}^{\infty} \frac{1}{k} \left[\Phi\left(-\frac{1}{2} \varepsilon \sqrt{k} \right) + \mathscr{F}^k \Phi\left(\left(\frac{1}{2} \varepsilon - s \right) \sqrt{k} \right) \right] \right\},$$

where

$$\mathscr{F} = \mathscr{F}(\varepsilon, s) = \exp\left(-\frac{1}{2} \varepsilon s + \frac{1}{2} s^2 \right), \quad \varepsilon, s > 0;$$

and V_c may then be computed by straightforward differentiation. See Takahashi and Woodroofe (1981) for details. Finally, it is straightforward to compute (6) by numerical integration, or to compute (5) directly.

Table 1 compares the approximations (1) and (6) with the exact values of McPherson and Armitage (1971) and with Siegmund's (1977) corrected approximation. Observe that (6) provides a much better approximation than (1) when $c \neq 0$, especially for small values of N.

4. An initial sample size. In some cases, one may want to take a fixed sample of size m initially and then perform repeated significance tests. Then one is led to study

$$t_m = t_{m,a,c} = \inf\{n \geq m : S_n > \sqrt{2a(n + c)}\}.$$

Let

$$\alpha_m = \alpha_m(a, c, N) = P_0\{t_m \leq N\}$$

for integers $N > m$. Then t_1 and α_1 are the t and α of the previous section. Next, let

$$\alpha_m^* = P_0\{m \leq t \leq N\}.$$

Then α_m^* is given by (6) with e_1 replaced by $e_1^* = \sqrt{\{2a/(m - ½)\}}$. Thus, α_m^* may be regarded as known. Now

$$\alpha_m - \alpha_m^* = \sum_{n=m}^{N} \int_0^{\infty} \psi_{a,m}(n, r) \frac{1}{\sqrt{n}} \phi[\sqrt{\{2a(1 + cn^{-1})\}} + rn^{-1/2}]\, dr,$$

where

$$\begin{aligned} \psi_{a,m}(n, r) &= P\{t_m \geq n \text{ and } t < m \mid S_n = \sqrt{2a(n + c)} + r\} \\ &= P_0\{S_{nj} \leq \beta_{nj}^a \text{ for } 1 \leq j \leq n - m, \text{ and} \\ &\qquad S_{nj} > \beta_{nj}^a \text{ for some } j > n - m\}. \end{aligned}$$

If m, N, and $n \to \infty$ as $a \to \infty$ in such a manner that

$$m \sim 2a/\delta_1^2, \quad N \sim 2a/\delta_0^2, \quad \text{and} \quad n - m \to k \tag{7}$$

where $0 < \delta_0 < \delta_1 < \infty$ and $k \geq 0$ is an integer, then $\psi_{a,m}(n, r)$ converges to

$$\psi_k(\delta_1, r) = P_0\{S_j \leq \tfrac{1}{2}\delta_1 j - r \quad \text{for} \quad j \leq k \quad \text{and} \quad S_j > \tfrac{1}{2}\delta_1 j - r \quad \text{for some} \quad j > k\}.$$

Since

$$\phi[\sqrt{\{2a(1 + cn^{-1})\}} + rn^{-1/2}] \sim \frac{1}{\sqrt{2\pi}} \exp\left(-a - \frac{1}{2} c\delta_1^2 - \delta_1 r\right)$$

under the limiting operation (7), the asymptotic relation

$$\alpha_m - \alpha_m^* \sim \frac{1}{\sqrt{(2\pi m)}} \exp\left\{-a - \frac{1}{2} c\delta_1^2\right\} \textstyle\sum_{k=0}^{\infty} \displaystyle\int_0^{\infty} \psi_k(\delta_1, r)e^{-\delta_1 r}\, dr \tag{8}$$

is suggested; and (8) may be proved rigorously along the lines of Woodroofe (1976b). See also Section 5 for a related proof.

Denote the summation on the right side of (8) by $K_1 = K_1(\delta_1)$. Then, as explained below,

$$K_1 = \delta_1^{-1} S_1(\delta_1) \exp\{-S_0(\delta_1)\} \tag{9}$$

where
$$S_0(\varepsilon) = \textstyle\sum_{k=1}^{\infty} \displaystyle\frac{1}{k} \mathscr{F}(\varepsilon)^k \Phi\left\{\left(\frac{1}{2}\delta_1 - \varepsilon\right)\sqrt{k}\right\}$$

and
$$S_1(\varepsilon) = \textstyle\sum_{k=1}^{\infty} \displaystyle\mathscr{F}(\varepsilon)^k \Phi\left\{\left(\frac{1}{2}\delta_1 - \varepsilon\right)\sqrt{k}\right\}$$

with $\mathscr{F}(\varepsilon) = \exp(-\tfrac{1}{2}\delta_1\varepsilon + \tfrac{1}{2}\varepsilon^2)$ for $\varepsilon > 0$. Observe that $S_0(\varepsilon)$ and $S_1(\varepsilon)$ simplify when $\varepsilon = \delta_1$. It is straightforward to compute K_1 from (9).

In the proof of (9) we write δ for δ_1, let F denote the normal distribution with mean $-\tfrac{1}{2}\delta$ and unit variance, and write P for $P_{-(1/2)\delta}$. Thus, $X_1, X_2, \cdots$ are i.i.d. with common distribution F, and P. Observe that $\mathscr{F}$ is the moment generating function of F. Let

$$\tau_r = \inf\{n \geq 1 : S_n > r\}, \quad G_0(r) = P\{\tau_r < \infty\}, \quad G_1(r) = \int_{\tau_r < \infty} \tau_r\, dP, \quad -\infty < r < \infty.$$

Then
$$K_1 = \int_0^{\infty} G_1(-r)e^{-\delta r}\, dr = \int_{-\infty}^{0} G_1(r)e^{\delta r}\, dr.$$

Now
$$G_1(r) = 1 - F(r) + \int_0^{\infty} \{G_0(s) + G_1(s)\} f(r - s)\, ds$$

for all r, $-\infty < r < \infty$, where $f = F'$ denotes the density of F. It follows easily that, for $0 < \varepsilon < \delta$,

$$\begin{aligned}\int_{-\infty}^{\infty} G_1(r)e^{\varepsilon r}\, dr &= \varepsilon^{-1}\mathscr{F}(\varepsilon) + \int_0^{\infty} \{G_0(s) + G_1(s)\}e^{\varepsilon s}\, ds \cdot \mathscr{F}(\varepsilon) \\ &= \varepsilon^{-1}\mathscr{F}(\varepsilon) + \{\mathscr{G}_0(\varepsilon) + \mathscr{G}_1(\varepsilon)\}\mathscr{F}(\varepsilon),\end{aligned}$$

say. Thus,

$$\begin{aligned}\int_{-\infty}^{0} G_1(r)e^{\varepsilon r}\, dr &= \int_{-\infty}^{\infty} G_1(r)e^{\varepsilon r}\, dr - \mathscr{G}_1(\varepsilon) \\ &= \varepsilon^{-1}\mathscr{F}(\varepsilon) + \mathscr{G}_0(\varepsilon)\mathscr{F}(\varepsilon) - \{1 - \mathscr{F}(\varepsilon)\}\, \mathscr{G}_1(\varepsilon).\end{aligned} \tag{10}$$

Let σ denote the first (strict, ascending) ladder epoch; let Y denote the first ladder height; let H denote the (defective) distribution of Y; and let

$$L(r) = \int_{Y \le r} \sigma \, dP, \quad r > 0.$$

Then

$$G_0(r) = H(\infty) - H(r) + \int_0^r G_0(r-s)\, dH(s)$$

and

$$G_1(r) = L(\infty) - L(r) + \int_0^r G_0(r-s)\, dL(s) + \int_0^r G_1(r-s)\, dH(s)$$

for $r > 0$ by the renewal equation. Thus, letting $\mathscr{H}$ and $\mathscr{L}$ denote the moment generating functions of the distributions H and L,

$$\mathscr{G}_0(\varepsilon) = \varepsilon^{-1}\mathscr{H}(\varepsilon) + \mathscr{G}_0(\varepsilon)\mathscr{H}(\varepsilon)$$

and

$$\mathscr{G}_1(\varepsilon) = \varepsilon^{-1}\mathscr{L}(\varepsilon) + \mathscr{G}_0(\varepsilon)\mathscr{L}(\varepsilon) + \mathscr{G}_1(\varepsilon)\mathscr{H}(\varepsilon), \quad 0 < \varepsilon < \delta.$$

Next,

$$1 - \mathscr{H}(\varepsilon) = \{1 - \mathscr{F}(\varepsilon)\}\exp\{S_0(\varepsilon)\} \quad \text{and} \quad \mathscr{L}(\varepsilon) = [\mathscr{F}(\varepsilon) - \{1 - \mathscr{F}(\varepsilon)\}S_1(\varepsilon)]\exp\{S_0(\varepsilon)\}$$

for $0 < \varepsilon < \delta$, where S_0 and S_1 are as in (9), by a simple application of Spitzer's Formula. See, for example, Feller (1966, pages 569–570). Substituting these relations into (10) and rearranging the terms, we find

$$\int_{-\infty}^0 G_1(r)e^{\varepsilon r}\, dr = \frac{1}{\varepsilon}\frac{1}{1 - \mathscr{H}(\varepsilon)}\left\{\mathscr{F}(\varepsilon) - \mathscr{L}(\varepsilon)\frac{1 - \mathscr{F}(\varepsilon)}{1 - \mathscr{H}(\varepsilon)}\right\}$$

$$= \frac{1}{\varepsilon}\left\{\frac{1 - \mathscr{F}(\varepsilon)}{1 - \mathscr{H}(\varepsilon)}\right\}S_1(\varepsilon) \to \frac{1}{\delta} S_1(\delta)\exp\{-S_0(\delta)\}, \quad \text{as } \varepsilon \uparrow \delta.$$

This completes the proof of (9).

Table 2 lists values of $S_0(\delta_1)$, $S_1(\delta_1)$, and $K_1(\delta_1)$ for selected δ_1, and Table 3 compares the approximation (8) with some simulations for selected values of a, N, and m.

The column entitled "1st order" lists the leading term in (6), with $e_1 = \sqrt{\{2a/(m - \frac{1}{2})\}}$; the column entitled "2nd order" lists all of (6) plus (8). Observe that the second order approximations are closer to the simulated values in all cases considered, and substantially closer in most cases. Observe also that the first order approximations are uniformly less than the simulated values, while the second order approximations are frequently greater. The next column lists an approximation suggested by David Siegmund (personal communication), which consists of adding

$$\frac{\phi(\sqrt{2a})}{\sqrt{2a}}\left(2 - \frac{1}{2}\log\frac{m}{N}\right) \tag{11}$$

to the leading term in (6), with $e_0 = \sqrt{(2a/N)}$ and $e_1 = \sqrt{(2a/m)}$. The latter approximation works exceptionally well. It would be interesting to have a theoretical explanation of this excellent agreement.

5. Asymptotic shapes. Relation (3) may be used to refine Woodroofe's (1976b) approximations to the error probabilities which result from using Schwarz's (1962) asymptotic shapes. The latter may be defined as follows. Let $\delta > 0$ and consider the problem of testing $H_0: \theta \le -\delta$ vs. $H_1: \theta \ge \delta$. Suppose that there is a prior distribution π with full support, a positive bounded loss for a wrong decision when $|\theta| \ge \delta$, no loss for either decision when $|\theta| < \delta$, and a cost $c > 0$ for each observation $X_1, X_2, \cdots$. Then there is an optimal Bayesian procedure which continues sampling as long as the sufficient sequence (n, S_n), $n \ge 1$, stays in a subset $\mathscr{B}_\pi(c)$, called the Bayesian continuation region. Schwarz

TABLE 2
*Values of $S_0(\delta)$, $S_1(\delta)$, and $K(\delta)$**

δ	S_0	S_1	K
.10	2.6783	199.7542	137.1908
.15	2.2874	88.6452	60.0010
.20	2.0143	49.1583	33.1936
.25	1.8057	31.7604	20.8812
.30	1.6378	21.9847	14.2453
.35	1.4983	16.0910	10.2760
.40	1.3793	12.2666	7.7206
.45	1.2760	9.6452	5.9832
.50	1.1852	7.7707	4.7510
.55	1.1043	6.3843	3.8472
.60	1.0318	5.3304	3.1659
.65	0.9662	4.5106	2.6406
.70	0.9066	3.8606	2.2276
.75	0.8520	3.3366	1.8977
.80	0.8019	2.9081	1.6303
.85	0.7556	2.5533	1.4109
.90	0.7128	2.2563	1.2290
.95	0.6731	2.0053	1.0768
1.00	0.6362	1.7913	0.9482
1.25	0.4842	1.0814	0.5331
1.50	0.3723	0.7004	0.3218
1.75	0.2876	0.4745	0.2034
2.00	0.2225	0.3312	0.1326

* *The normal distribution function was computed using subroutine* MDNORD *of* IMSL. *The first* $100/\delta^3$ *terms in each series was computed for* $0.1 \le \delta \le 1.0$ *and the first* 100 *terms for* $\delta \ge 1.0$.

TABLE 3
*Error probabilities with an initial sample**

a	N	m	1st Order	2nd Order	Siegmund	Monte Carlo ± S.D.
4.35	200	20	.0141	.0162	.0154	.0155 ± .00030
4.18	111	11	.0152	.0169	.0170	.0170 ± .00032
3.73	28	3	.0183	.0199	.0202	.0204 ± .00038
3.65	185	19	.0261	.0310	.0292	.0299 ± .00049
3.35	100	10	.0321	.0368	.0362	.0372 ± .00060
2.88	42	4	.0445	.0495	.0500	.0516 ± .00083
2.51	24	2	.0590	.0652	.0654	.0692 ± .00108
4.35	200	40	.0103	.0140	.0122	.0122 ± .00019
4.18	111	22	.0114	.0145	.0137	.0137 ± .00021
3.73	28	6	.0136	.0162	.0172	.0172 ± .00027
3.65	185	38	.0188	.0278	.0234	.0233 ± .00033
3.35	100	20	.0235	.0321	.0297	.0293 ± .00042
2.88	42	8	.0332	.0425	.0429	.0433 ± .00060
2.51	24	4	.0456	.0557	.0586	.0542 ± .00076

* *Approximations to* α_m *are reported. The first order approximations are the leading term in (6) with* e_1 *replaced by* $\sqrt{\{2a/(m-\frac{1}{2})\}}$; *the second order approximations use (6) plus (8); Siegmund's approximations use the leading term in (6) plus (11); the simulations used a version of Siegmund's (1976) importance sampling.*

(1962) showed that

$$\mathscr{B}_\pi(c)/\log c^{-1} \to \{(x, y): |y| \le \sqrt{(2x)} - \delta x, x \ge 0\}$$

as $c \to 0$. Note the independence of the limit from π and the loss function. Of course, this suggests the approximate procedure which takes

$$\sigma = \inf\{n \ge 1: |S_n| > \sqrt{(2an)} - n\delta\}$$

observations, where $a = \log c^{-1}$, and decides in favor of H_1 iff $S_\sigma > 0$. Observe that σ is bounded by the least integer which is greater than or equal to $2a\delta^{-2}$.

Here a is regarded as a design parameter; and approximations to the error probabilities are developed for large a. Let

$$\beta_a(\theta) = P_\theta(S_\sigma > 0), \quad -\infty < \theta < \infty, \quad a > 0,$$

denote the power function of the test. Then $\beta_a(\theta)$ is continuous and strictly increasing in θ for each $a > 0$ (cf. Lehmann, 1959, pages 101–102); so, the maximum error probabilities are $\beta_a(-\delta) = 1 - \beta_a(\delta)$, $a > 0$. Suppose, for simplicity, that $N = 2a\delta^{-2}$ is an integer. Then

$$\beta_a(-\delta) = P_0(t \le N) - P_0(s < t \le N), \tag{12}$$

where $t = \inf\{n \ge 1: S_n > \sqrt{(2an)}\}$, as in (2) with $c = 0$, and

$$s = \inf\{n \ge 1: S_n < -\sqrt{(2an)} + 2\delta n\}.$$

The first term on the right side of (12) was computed up to terms which are $o(1/\sqrt{a})\exp(-a)$ in (6). Thus, the second term is of primary interest.

As in (4), we may write

$$P_0\{s < t \le N\} = \textstyle\sum_{n=2}^{N} \displaystyle\int_0^\infty \psi_a^*(n, r) \frac{1}{\sqrt{n}} \phi\left(\sqrt{2a} + \frac{r}{\sqrt{n}}\right) dr,$$

where

$$\begin{aligned}\psi_a^*(n, r) &= P\{s < n \le t \mid S_n = \sqrt{(2an)} + r\} \\ &= P(S_{nj} \le \beta_{nj}^a, 1 \le j \le n-1, \text{ and } S_{nj} < \gamma_{nj}^a, \exists j \le n-1)\end{aligned}$$

with

$$\gamma_{nj}^a = -\sqrt{\{2a(n-j)\}} + 2\delta(n-j) - \left(1 - \frac{j}{n}\right)\{\sqrt{(2an)} + r\},$$

$\beta_{nj}^a = \sqrt{\{2a(n-j)\}} - (1 - j/n)\{\sqrt{(2an)} + r\}$, and $S_{nj} = S_j - jS_n/n$, $1 \le j \le n-1$, as in Section 2. It is easily seen that if $n = n_a \to \infty$ as $a \to \infty$ with $N - n \to k \ge 0$, then

$$-\gamma_{nj}^a \to \frac{1}{2}\delta j + \delta k + r$$

for each fixed $j \ge 1$. Since $S_{nj} \to S_j$ w.p.1 (P_0), this certainly suggests

$$\begin{aligned}\psi_a^*(n, r) \to P_0\Big\{&S_j \le \frac{1}{2}\delta j - r, \forall j \ge 1, \text{ and} \\ &S_j < -\frac{1}{2}\delta j - \delta k - r, \exists j \ge 1\Big\} = \alpha_k(r), \quad \text{say},\end{aligned} \tag{13}$$

and

$$P(s < t \le N) \sim K^* a^{-1/2} e^{-a} \tag{14}$$

with

$$K^* = \frac{1}{2}\delta\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\int_0^{\infty}\alpha_k(r)e^{-\delta r}\,dr.$$

The justification for (13) and (14) is more complicated than that required for (8) and is discussed below. When combined with (6), (14) provides refined approximations to the error probabilities which result from using Schwarz's (1962) asymptotic shapes.

The approximation (14) is of theoretical interest, since the first order asymptotics neglect the lower boundary entirely, and (14) shows the interplay between the two boundaries (cf. Woodroofe, 1976b). Observe that effect of the lower boundary is of order $a^{-1/2}e^{-a}$, the same as the correction to (1) in (6). Unfortunately, the constant K^* is extremely complicated; so, the numerical usefulness of (14) may be limited.

We use a diffusion approximation to approximate K^* of (14). Let $W(t)$, $0 \le t < \infty$, denote a standard Wiener process; and, for positive b, c, c_1, and c_2, let

$$g(b, c_1, c_2) = P\{-bt - c_1 \le W(t) \le bt + c_2, \forall\, t \ge 0\}$$

and

$$h(b, c) = P\{W(t) \le bt + c, \forall\, t \ge 0\}.$$

Then $h(b, c) = 1 - \exp(-2bc)$ and an infinite series expansion of g is given by Doob (1949). The Wiener process is used to approximate the random walk $S_j - S_1$, $j \ge 1$, as follows:

$$\begin{aligned}\alpha_k(r) = &\int_{-\infty}^{-(1/2)\delta-\delta k-r} P_0\left\{S_j - S_1 \le \frac{1}{2}\delta j - r - x, \forall\, j \ge 1\right\}\phi(x)\,dx\\ &+ \int_{-(1/2)\delta-\delta k-r}^{(1/2)\delta-r} P\left\{S_j - S_1 \le \frac{1}{2}\delta j - r - x, \forall\, j \ge 1, \text{ and}\right.\\ &\left.\cdot S_j - S_1 < -\frac{1}{2}\delta j - \delta k - r - x, \exists\, j \ge 1\right\}\phi(x)\,dx\\ \approx &\int_{-\infty}^{-(1/2)\delta-\delta k-r} h\left(\frac{1}{2}\delta, \frac{1}{2}\delta - r - x\right)\phi(x)\,dx\\ &+ \int_{-(1/2)\delta-\delta k-r}^{(1/2)\delta-r}\left\{h\left(\frac{1}{2}\delta, \frac{1}{2}\delta - r - x\right)\right.\\ &\left.- g\left(\frac{1}{2}\delta, \frac{1}{2}\delta + \delta k + r + x, \frac{1}{2}\delta - r - x\right)\right\}\phi(x)\,dx.\end{aligned}$$

Using this approximation, with the result of Doob (1949) used to compute g, it is straightforward to compute numerical approximations to K^*.

Table 4 lists the values of K^* for selected values of δ; and Table 5 compares the first

TABLE 4

Approximate value of $K^* = \frac{1}{2}\delta\sqrt{1/\pi}\sum_0^\infty\int_0^\infty \alpha_k(y)e^{-\delta y}\,dy$

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
K^*	0.1339	0.1335	0.1298	0.1262	0.1225	0.1188	0.1151	0.1113	0.1075	0.1037
δ	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
K^*	0.09976	0.09560	0.09119	0.08653	0.08169	0.07674	0.07174	0.06693	0.06222	0.05769

TABLE 5
*Approximate error probabilities for asymptotic shapes**

a	δ	1st Order	2nd Order	Monte Carlo ± S.D.
3.45	.1	.0767	.0702	.0662 ± .0017
3.45	.2	.0554	.0515	.0499 ± .0015
3.45	.3	.0437	.0411	.0400 ± .0014
3.45	.5	.0301	.0287	.0285 ± .0012
3.45	1.0	.0155	.0149	.0146 ± .0009
2.50	.1	.1669	.1469	.1354 ± .00242
2.50	.2	.1204	.1033	.1040 ± .00216
2.50	.3	.0945	.0863	.0850 ± .00192
2.50	.5	.0651	.0605	.0622 ± .00171

* *The first order approximation is the leading term in (12), $P_0(t \le N)$; the second order approximation is $P_0(t \le N) - P_0(s < t \le N)$.*

and second order asymptotics with some simulations. Observe that the second order asymptotics yield approximations which are substantially closer to the simulations than do the first order asymptotics for small values of δ, although even the second order approximations are frequently larger than the simulations.

It remains to prove (13) and (14). First, a simple picture shows that

$$P_0\left(s < t \le \frac{1}{4}N\right) = o(a^{-1/2}e^{-a}), \quad \text{as } a \to \infty;$$

so, it suffices to prove (14) with $P_0(s < t \le N)$ replaced by $P_0(s < t \le N, t > \frac{1}{4}N)$. Now,

$$\begin{aligned} P_0\left(s < t \le N, t > \frac{1}{4}N\right) &= \sum_{(1/4)N < n \le N} \int_0^\infty \psi_a^*(n, r) \frac{1}{\sqrt{n}} \phi\left(\sqrt{2a} + \frac{r}{\sqrt{n}}\right) dr \\ &= \frac{1}{2\sqrt{\pi a}} e^{-a} \sum_{(1/4)N < n \le N} \int_0^\infty \psi_a^*(n, r) \\ &\quad \cdot \sqrt{\left(\frac{2a}{n}\right)} \exp\left\{-r\sqrt{\left(\frac{2a}{n}\right)} - \frac{r^2}{2n}\right\} dr. \end{aligned}$$

Let

$$\psi_{a,l}^*(n, r) = P\{s \le n - l, t \ge n \mid S_n = \sqrt{(2an)} + r\}$$

for integers $l \le 1$. Then, clearly, $\psi_a^*(n, r) - \psi_{a,l}^*(n, r) \to \alpha_k(r)$ as first $a \to \infty$ with $N - n \to k$ and then $l \to \infty$; and $\psi_a^*(n, r) = \psi_{a,1}^*(n, r)$. Thus, it suffices to show that there are constants C and $\varepsilon > 0$ for which

$$\psi_{a,l}^*(n, r) \le C \exp\{-\varepsilon(N - n + l)\} \tag{15}$$

for all $r \ge 0$, $l \ge 1$, and $n > \frac{1}{4}N$; for (15) implies that $\psi_{a,l}^*(n, r) \to 0$ as first $a \to \infty$ and then $l \to \infty$ and that $\psi_a^*(n, r) \le C' \exp\{-\varepsilon(N - n)\}$. To establish (15), let $\sigma_{nj}^2 = j(n - j)/n$ be the variance of S_{nj}, $1 \le j \le n - 1$. Then, since $\sqrt{2an} \ge \delta n$ for $n \le N$,

$$\begin{aligned} -\sigma_{nj}^{-1}\gamma_{nj} &\ge \sigma_{nj}^{-1}[\sqrt{\{2a(n - j)\}} - \delta(n - j)] \\ &= \sqrt{\left(\frac{n}{j}\right)}\{\sqrt{2a} - \delta\sqrt{(n - j)}\} \ge \frac{1}{4}\delta\left(\frac{N - n + j}{\sqrt{j}}\right), \end{aligned}$$

for all $j \le n - 1$, $n \ge \frac{1}{4}N$, and sufficiently large a; so

$$\psi_{a,l}^*(n, r) \le \sum_{j=l}^{n-1} P(S_{nj} < \gamma_{nj}) \le C \sum_{j=l}^{n-1} \exp\{-\varepsilon(N - n + j)\}$$

for appropriate C and ε. The inequality (15) follows immediately.

6. Remarks. The expansion (3) may be used to develop expansions for the probability of a Type II error for repeated significance tests too. This application was included in Takahashi and Woodroofe (1981).

Similarly, the expansion (3) may be used to generate asymptotic expansions for the distribution of $(S_t - t\theta)/\sqrt{t}$, where t is as in (2). Siegmund (1978) noted that the asymptotic normality of $(S_t - t\theta)/\sqrt{t}$ was not fast enough for use in setting confidence intervals after sequential testing; and he proposed using the relationship between tests and confidence intervals instead. The asymptotic expansion for the distribution of $(S_t - t\theta)/\sqrt{t}$ may be sufficiently accurate to compute (approximate) confidence coefficients of intervals of the form $\bar{X}_t - c_1 \le \theta \le \bar{X}_t + c_2$. This question is still under investigation.

The assumed normality has been used extensively, and may appear to have been used crucially. However, a careful reading of the derivations, including Takahashi and Woodroofe (1981), reveals the major uses to have been the following: (i) to assert that S_n is sufficient for $X_1, \cdots, X_n$; (ii) to compute the conditional distribution of $X_1, \cdots, X_n$; and (iii) to bound some tail probabilities. In addition, normality was used to simplify certain complicated expressions which occurred in the coefficients of the expansions. Clearly, (i) and (iii) are common to all exponential families; and expansions for the conditional distribution of $X_1, \cdots, X_n$ given S_n may be generated from expansions for the density of S_n, provided that the distribution of S_n is sufficiently smooth. See Feller (1966, Section 16.2). Thus, our results may be extendable to smooth exponential families.

APPENDIX

The proof of (5) and (6) is outlined. Let $N = N(a) \to \infty$ as $a \to \infty$ in such a manner that $a^{-1}N$ remains bounded away from zero and ∞; write $\varepsilon_n = \sqrt{(2an^{-1})}$ and $\varepsilon_n^* = \varepsilon_n + 2n^{-1}r$; and divide α into two parts

$$\alpha = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \int_0^\infty \{\psi_a(n, r) - \psi(\varepsilon_n^*, r)\}\phi\left\{\sqrt{2a\left(1 + \frac{c}{n}\right)} + \frac{r}{\sqrt{n}}\right\} dr$$

$$(A.1)\qquad + \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \int_0^\infty \psi(\varepsilon_n^*, r)\phi\left\{\sqrt{2a\left(1 + \frac{c}{n}\right)} + \frac{r}{\sqrt{n}}\right\} dr$$

$$= \text{Sum}_1 + \text{Sum}_2,$$

say. To analyse Sum_1, write it in the form

$$\text{Sum}_1 = \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{a}} e^{-a}\left[\sum_{n=1}^{N} \varepsilon_n \int_0^\infty \{\psi_a(n, r) - \psi(\varepsilon_n^*, r)\}\right.$$

$$\left.\cdot\exp\left(-\frac{1}{2}c\varepsilon_n^2 - r\varepsilon_n\right)\right]\exp\left[-\frac{1}{2}\frac{r^2}{n} - r\varepsilon_n\left\{\sqrt{\left(1 + \frac{c}{n}\right)} - 1\right\}\right] dr$$

$$= \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{a}} e^{-a}\text{Sum}_1^*, \quad \text{say,}$$

and then we divide the range of summation into $n \le m$ and $n > m$, where $m = m(a) = [a^{3/4}]$. Observe that $\varepsilon_n \le 2a^{1/8}$ for $n > m$ and that $\varepsilon_n > a^{1/8}$ for $n \le m$ for sufficiently large $a > 0$.

LEMMA 1. $\sum_{n=1}^{m} \varepsilon_n \int_0^\infty \{\psi_a(n, r) - \psi(\varepsilon_n^*, r)\}e^{-\varepsilon_n r}\, dr = \bar{o}(a^{-\infty})$ *as* $a \to \infty$.

PROOF. First observe that

$$\int_0^\infty |\psi_a(n, r) - \psi(\varepsilon_n^*, r)|\, e^{-\varepsilon_n r}\, dr \le \int_0^1 |\psi_a(n, r) - \psi(\varepsilon_n^*, r)|\, e^{-\varepsilon_n r}\, dr + \frac{1}{\varepsilon_n} e^{-\varepsilon_n}$$

for all $n \geq 1$. Since $\varepsilon_n > a^{1/8}$ for $n \leq m$ and sufficiently large a, the second term is at most $\exp(-a^{1/8}) = \bar{o}(a^{-\infty})$. To estimate the first, observe that for $n \leq m$, $0 \leq r \leq 1$, and sufficiently large a,

$$\psi_a(n, r) - \psi(\varepsilon_n^*, r) \leq 1 - \psi(\varepsilon_n^*, r) = P\left(S_k > \frac{1}{2}\varepsilon_n k - r, \exists\, k \geq 1\right) \leq C \exp\left(-\frac{1}{32}\varepsilon_n^2\right)$$

for some constant $C > 0$. A similar lower bound may also be obtained. The lemma follows easily.

LEMMA 2. *As $a \to \infty$*

$$\text{Sum}_1^* = -\left(\frac{1}{2a}\right)\sum_{n=1}^N \varepsilon_n^3 \left\{\frac{\psi'(\varepsilon_n, 0)}{\psi(\varepsilon_n, 0)} U_{10}(\varepsilon_n, \varepsilon_n) - \frac{1}{2} U_{11}(\varepsilon_n, \varepsilon_n) - \frac{3}{2} c\varepsilon_n U_{10}(\varepsilon_n, \varepsilon_n) + U_{20}(\varepsilon_n, \varepsilon_n)\right\} e^{-(1/2)c\varepsilon_n^2} + o(1), \tag{A.2}$$

where $U(\varepsilon, s)$ and $U_{ij}(\varepsilon, s)$, $i, j = 0, 1, 2$ are as in Section 3.

PROOF. By Lemma 1 and by (3),

$$\begin{aligned}
\text{Sum}_1^* &= -\left(\frac{1}{2a}\right)\sum_{n=m+1}^N \varepsilon_n^3 \int_0^\infty \rho_c(\varepsilon_n, r) e^{-r\varepsilon_n} \exp\left[-\frac{1}{2}\frac{r^2}{n} - r\varepsilon_n \right. \\
&\quad \left. \cdot \left\{\sqrt{\left(1 + \frac{c}{n}\right)} - 1\right\}\right] dr \cdot e^{-(1/2)c\varepsilon_n^2} \\
&\quad + \left(\frac{1}{2a}\right)\sum_{n=m+1}^N \varepsilon_n^3 \int_0^\infty D_a(n, r) e^{-r\varepsilon_n} \exp\left[-\frac{1}{2}\frac{r^2}{n} \right. \\
&\quad \left. - r\varepsilon_n \left\{\sqrt{\left(1 + \frac{c}{n}\right)} - 1\right\}\right] dr \cdot e^{-(1/2)c\varepsilon_n^2} + \bar{o}(a^{-\infty}) \\
&= \text{Sum}_{11}^* + \text{Sum}_{12}^* + \bar{o}(a^{-\infty}),
\end{aligned}$$

say, where ρ_c and D_a are as in Section 2. Since $n^{-1}r^2 \to 0$ and $n^{-1}c \to 0$ as $a \to \infty$ for $n > m$, the dominated convergence theorem yields

$$\text{Sum}_{11}^* = -\left(\frac{1}{2a}\right)\sum_{n=m+1}^N \varepsilon_n^3 \int_0^\infty \rho_c(\varepsilon_n, r) e^{-\varepsilon_n r}\, dr \cdot e^{-(1/2)c\varepsilon_n^2} + o(1). \tag{A.3}$$

Moreover, the sums in (A.2) and (A.3) differ by at most $o(1)$; so, it suffices to show that $\text{Sum}_{12}^* \to 0$ as $a \to \infty$. To see this divide the range of integration into three subintervals, $0 \leq r \leq \lambda_0$, $\lambda_0 < r \leq \lambda_n$, and $r > \lambda_n$, where $\lambda_0 > 0$ and $\lambda_n = \exp\{½\sqrt{(\log n)}\}$, $n \geq 1$. It is clear that the integral over $(0, \lambda_0]$ tend to zero as $a \to \infty$, since $D_a|n, r| \to 0$ uniformly on compacts. By the inequalities following (3),

$$\int_{\lambda_0}^{\lambda_n} |D_a(n, r)| e^{-\varepsilon_n r}\, dr \leq K \exp(-\lambda_0 \varepsilon_n)$$

for $m < n \leq N$. Finally there is a constant $K^* > 0$ for which $|D_a(n, r)| \leq K^*(n + r + \varepsilon_n)$ for all $r \geq 0$ and $m < n \leq N$; see Takahashi and Woodroofe (1981) for the infinite series representation of ρ_c. Therefore

$$\int_{\lambda_n}^\infty |D_a(n, r)| e^{-\varepsilon_n r}\, dr \leq K^*\left(\frac{1}{\varepsilon_n^2} + \frac{\lambda_n + n}{\varepsilon_n} + 1\right) e^{-\varepsilon_n \lambda_n}$$

for $m < n \leq N$. Combining the latter three bounds and letting $a \to \infty$, we find

$$\limsup \text{Sum}^*_{12} \leq 2K \int_0^\infty \exp(-\lambda_0 \varepsilon)\, d\varepsilon$$

which tends to zero as $\lambda_0 \to \infty$.

LEMMA 3. *As* $a \to \infty$

$$\text{Sum}_2 = \frac{1}{2}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{a}} e^{-a}\Big\{\textstyle\sum_{n=1}^N \varepsilon_n U(\varepsilon_n, \varepsilon_n) e^{-c\varepsilon_n^2/2} - \frac{1}{2a}\sum_{n=1}^N \varepsilon_n^3\Big[2U_{11}(\varepsilon_n, \varepsilon_n) + \frac{1}{2}U_{02}(\varepsilon_n, \varepsilon_n) - \frac{1}{2}c\varepsilon_n U_{01}(\varepsilon_n, \varepsilon_n)\Big]e^{-c\varepsilon_n^2/2} + o(1)\Big\}, \tag{A.4}$$

where $U(\varepsilon, s)$ *and* $U_{ij}(\varepsilon, s)$ $i, j = 0, 1, 2$ *are as in Section* 3.

PROOF. By Taylor's theorem,

$$\begin{aligned}
\text{Sum}_2 = {} & \frac{1}{\sqrt{\pi}}\sqrt{a}\, e^{-a}\bigg(\frac{1}{2a}\textstyle\sum_{n=1}^N \varepsilon_n \displaystyle\int_0^\infty \psi(\varepsilon_n, r) e^{-\varepsilon_n r}\, dr \cdot e^{-c\varepsilon_n^2/2} \\
& + \left(\frac{1}{2a}\right)^2 \textstyle\sum_{n=1}^N \varepsilon_n^3 \displaystyle\int_0^\infty \left[2r\psi'(\varepsilon_n^+, r) - \frac{1}{2}\left\{r^2 + \frac{c\varepsilon_n}{\sqrt{(1+\theta_n)}} r\right\}\right. \\
& \left.\cdot \psi(\varepsilon_n, r) e^{-r_n^+}\right] e^{-\varepsilon_n r}\, dr \cdot e^{-c\varepsilon_n^2/2} \\
& - \left(\frac{1}{2a}\right)^3 \textstyle\sum_{n=1}^N \varepsilon_n^5 \displaystyle\int_0^\infty \left\{r^3 + \frac{c\varepsilon_n}{\sqrt{(1+\theta_n)}} r^2\right\}\psi'(\varepsilon_n^+, r) e^{-r_n^+} e^{-\varepsilon_n r}\, dr\, e^{-c\varepsilon_n^2/2}\bigg) \\
= {} & \frac{1}{\sqrt{\pi}}\sqrt{a}\, e^{-a}(\text{Sum}_{21} + \text{Sum}_{22} + \text{Sum}_{23}),
\end{aligned}$$

say, where $\varepsilon_n \leq \varepsilon_n^+ \leq \varepsilon_n^*$, $0 \leq r_n^+ \leq \frac{1}{2} n^{-1} r^2 + r\varepsilon_n\{\sqrt{(1 + n^{-1}c)} - 1\}$ and $0 \leq \theta_n \leq n^{-1}c$. By Remark 3 of Takahashi and Woodroofe (1981), $\psi'(\varepsilon, r) \leq K(\sqrt{\varepsilon^{-1} r} + \varepsilon^{-1})$ for some constant $K > 0$. Hence,

$$\begin{aligned}
|\text{Sum}_{23}| & \leq \left(\frac{1}{2a}\right)^3 \textstyle\sum_{n=1}^N \varepsilon_n^5 \displaystyle\int_0^\infty K\{\sqrt{(r/\varepsilon_n)} + 1/\varepsilon_n\}(r^3 + c\varepsilon_n r^2) e^{-\varepsilon_n r}\, dr \cdot e^{-c\varepsilon_n^2/2} \\
& = \left(\frac{1}{2a}\right)^3 \textstyle\sum_{n=1}^N K\left[\Gamma\left(\frac{9}{2}\right) + \Gamma(4) + \left\{\Gamma\left(\frac{7}{2}\right) + \Gamma(3)\right\}\varepsilon_n^2\right] e^{-c\varepsilon_n^2/2} \\
& = \left(\frac{1}{2a}\right)^2 \left(\int_{e_0}^{e_1} 2K\left[\Gamma\left(\frac{9}{2}\right) + \Gamma(4) + \left\{\Gamma\left(\frac{7}{2}\right) + \Gamma(3)\right\}\varepsilon^2\right]\varepsilon^{-3} \cdot e^{-c\varepsilon^2/2}\, d\varepsilon + o(a^{-2})\right) \\
& = O(a^{-2}),
\end{aligned}$$

where $e_0^2 = 2a(N + \frac{1}{2})^{-1}$ and $e_1^2 = 4a$. To estimate Sum_{22} divide the range of the summation into $n \leq m$ and $m < n \leq N$. As in Lemma 1, it is not difficult to see that the contribution of $n \leq m$ is of the order $o(a^{-1})$, so by the dominated convergence theorem

$$\text{Sum}_{22} = -\left(\frac{1}{2a}\right)^2 \textstyle\sum_{n=1}^N \varepsilon_n^3\left\{2U_{11}(\varepsilon_n, \varepsilon_n) + \frac{1}{2}U_{02}(\varepsilon_n, \varepsilon_n) - \frac{1}{2}c\varepsilon_n U_{01}(\varepsilon_n, \varepsilon_n)\right\} e^{-c\varepsilon_n/2} + o(a^{-1}).$$

Since $\text{Sum}_{21} = (2a)^{-1}\sum_{n=1}^N \varepsilon_n U(\varepsilon_n, \varepsilon_n)\exp(-\frac{1}{2}c\varepsilon_n^2)$, the lemma follows by substitution.

Equation (5) now follows immediately from (A.2) and (A.4). Equation (6) follows easily from (5) by trapezoid rule. For example in Sum_{21}, make the change of variables $z_n = \varepsilon_n^{-2}$. Let $W(z) = z^{-1/2}U(z^{-1/2}, z^{-1/2})$, $z > 0$ and observe that $W''(z)$ remains bounded as $z \to 0$.

Then the trapezoid rule yields

$$\text{Sum}_{21} = \left(\frac{1}{2a}\right) \sum_{n=1}^{N} W(z_n) = \int_{e_1^{-2}}^{e_0^{-2}} W(z)\, dz + O(a^{-2}) = 2 \int_{e_0}^{e_1} U(\varepsilon, \varepsilon)\varepsilon^{-2}\, d\varepsilon + O(a^{-2}).$$

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