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FURTHER CONTRIBUTIONS TO THE THEORY OF F-SQUARES DESIGN[1]

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The main purpose of this paper is four-fold. First, to prove that the best upper bound on the number of mutually orthogonal $F(n, \lambda)$ squares is $(n-1)^2/(m-1)$, where $m = n/\lambda$. Second, to show that this upper bound is achievable if m is a prime or prime power and $\lambda = m^h$. Third, to present a set of four mutually orthogonal $F(6; 2)$ squares design. This latter design is important because there are no orthogonal Latin squares of order 6 which could be used for this purpose. Fourth, to indicate a method of composing orthogonal F-squares designs. In addition, we have pointed out the way one may construct orthogonal fractional factorial designs and orthogonal arrays from these designs.

1. Introduction and summary. To save space, the reader is referred to Hedayat and Seiden (1970) and Raghavarao (1971) for details and definitions of terms used here. Familiarity with the algebra of statistical designs is assumed. Our main purpose in this paper is to obtain the best possible bound on the number of orthogonal F-squares with certain parameters and to give a construction method for some families of orthogonal F-squares which achieve this bound. Also, we present a set of four mutually orthogonal F-squares of order 6 based on three symbols. This later design is important because there are no orthogonal Latin squares of order 6 which could be used for this purpose as has been pointed out by Hedayat and Seiden (1970). We indicate a method of composing orthogonal F-squares. Finally, we will indicate under what condition a set of orthogonal F-squares can be transformed into an orthogonal array, a structure which is useful for factorial experimentation.

2. Maximal number of orthogonal F-squares. Analogous to the result that the maximal number of mutually orthogonal Latin squares of order n is $n-1$, we have the following:

THEOREM 2.1. *The maximal number, t, of orthogonal F-squares of the type $F(n; \lambda)$, where $n = \lambda m$, satisfies the inequality*

$$t \leqq (n-1)^2/(m-1) . \tag{2.1}$$

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PROOF. Let $F_1, F_2, \cdots, F_t$ be a set of t mutually orthogonal F-squares of the type $F(n; \lambda)$. Based on F_α we define an $n^2 \times m$ matrix $N_\alpha = (n_{\alpha,ij,k})$, where $n_{\alpha,ij,k} = 1$, if the kth symbol occurs in the (i, j)th cell ($i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, n$) of F_α and 0, otherwise. Let

$$M = (N_1 | N_2 | \cdots | N_t) .$$

Using the property of F-squares, we can easily see that the number of independent rows in M are at most $(n - 1)^2 + 1$ and

$$R(M) \leqq \min ((n-1)^2 + 1, tm) ,$$

where $R(M)$ denotes the rank of the matrix M.

Again,

$$M'M = \begin{bmatrix} n\lambda I_m & \lambda^2 J_m & \cdots & \lambda^2 J_m \\ \lambda^2 J_m & n\lambda I_m & \cdots & \lambda^2 J_m \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^2 J_m & \lambda^2 J_m & \cdots & n\lambda I_m \end{bmatrix},$$

where I_m is the identity matrix of order m and J_m is the $m \times m$ matrix with the element 1 everywhere. The eigenvalues of $M'M$ are $n\lambda t$, $n\lambda$ and 0 with respective multiplicities 1, $t(m - 1)$ and $t - 1$. Thus

$$tm - t + 1 = R(M'M) = R(M) \leqq \min ((n-1)^2 + 1, tm) ,$$

from which the required inequality (2.1) follows.

Clearly, when $\lambda = 1$, we obtain the following:

COROLLARY 2.1. *The maximal number of orthogonal $F(n; 1)$ squares, that is, the maximal number of orthogonal Latin squares of order n, is $n - 1$.*

The method of proof of Theorem 2.1 can be applied to prove Theorem 2.2.4 of Raghavarao (1971, page 16) on the maximal number of constraints in an orthogonal array $(\lambda s^2, k, s, 2)$.

Theorem 2.1 suggests the following definition.

DEFINITION 2.1. A set of t mutually orthogonal $F(n; \lambda)$ squares design is said to be complete if $t = (n-1)^2/(m-1)$, where $m = n/\lambda$.

A reader familiar with properties of fractional factorial designs may find the following proof of Theorem 2.1 simpler.

A set of t mutually orthogonal $F(n; \lambda)$ squares design can be utilized to construct an orthogonal fractional factorial design for two factors at n levels and t factors at m levels in n^2 treatment combinations. To see this let the rows $0, 1, \cdots, n - 1$ correspond to the n levels of the first factor, the columns $0, 1, \cdots, n - 1$ to the n levels of the second factor, the m symbols of the ith $F(n; \lambda)$ square to the m levels of $(i + 2)$th factor, $i = 1, 2, \cdots, t$. Then the treatment combinations corresponding to the n^2 cells of the superimposed t squares give the required orthogonal fractional factorial design. Such a fraction can provide unbiased estimators for general mean and the main effects if we associate a usual linear additive model with these n^2 treatment combinations. Therefore the n^2 degrees of freedom can be partitioned as $1 + 2(n - 1) + t(m - 1)$, which implies $t \leqq (n - 1)^2/(m - 1)$.

3. Construction of a complete set of orthogonal F-squares design.

THEOREM 3.1. *A complete set of mutually orthogonal $F(n; \lambda)$ squares design exists if $m = n/\lambda$ is a prime power and $\lambda = m^h$.*

PROOF. By construction. Consider a symmetrical factorial design in $2h + 2$ factors, each factor being at m levels and let the m^{2h+2} treatment combinations be arranged in an $m^{h+1} \times m^{h+1}$ square array, A, such that between the rows the effects or interactions corresponding to pencils $P_1, P_2, \cdots, P_{h+1}$ and their generalized interactions are confounded; and between the columns the effects or interactions corresponding to pencils $P_1', P_2', \cdots, P'_{h+1}$ and their generalized interactions are confounded. The pencils, which are not confounded either between the rows or columns, are $s = (m^{h+1} - 1)^2/(m - 1)$ in number and let them be $Q_1, Q_2, \cdots, Q_s$. Each of these determine an F-square. We construct F_α from Q_α by mapping the treatment combinations of A to the number of the $(m - 1)$-flat of Q_α to which that treatment combination belongs. Since the pencils $Q_1, Q_2, \cdots, Q_s$ belong to orthogonal contrasts, $F_1, F_2, \cdots, F_s$ are mutually orthogonal F-squares and that set has the maximal number of orthogonal F-squares.

The above construction method will be elucidated with the following example:

EXAMPLE 3.1. Let $m = 2$ and $\lambda = 2$ so that we want to construct nine mutually orthogonal F-squares of the type $F(4; 2)$.

Consider a 2^4 factorial experiment in factors a, b, c and d and let the treatment combinations be written in a 4×4 array A, confounding the interactions A, B and AB between rows and C, D and CD between columns. Such an A is exhibited below:

$$A = \begin{bmatrix} 1 & c & d & cd \\ a & ac & ad & acd \\ b & bc & bd & bcd \\ ab & abc & abd & abcd \end{bmatrix}.$$

The interactions which are not confounded between rows and columns of A are 9 in number and they are AC, BC, AD, BD, ABC, ABD, ACD, BCD, $ABCD$. Now by mapping the treatment combinations of A into 1 or 0 according as it has a plus sign in the interaction AC, or not, we have

$$F_1 = \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}.$$

Analogously, from the other interactions, we obtain the following F-squares

$$F_2 = \begin{matrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}, \quad F_3 = \begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{matrix}, \quad F_4 = \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{matrix},$$

$$F_5 = \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}, \quad F_6 = \begin{matrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}, \quad F_7 = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix},$$

$$F_8 = \begin{matrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{matrix}, \quad F_9 = \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix}.$$

It can be verified that $\{F_1, F_2, \cdots, F_9\}$ forms a complete set of mutually orthogonal F-squares of the type $F(4; 2)$.

4. On mutually orthogonal F-squares for a composite order. Let F_1 be an $F(n; \lambda_1, \cdots, \lambda_t)$ and F_2 be an $F(m; \xi_1, \cdots, \xi_k)$. Then the following proposition can be easily verified.

PROPOSITION 4.1. *$F_1 \otimes F_2$ is an $F(mn; \alpha_{11}, \cdots, \alpha_{tk})$ where $\alpha_{ij} = \lambda_i \xi_k$.*

PROPOSITION 4.2. *If $F_1 \perp F_2$ and $F_3 \perp F_4$, then $F_1 \otimes F_3 \perp F_2 \otimes F_4$.*

However, the above propositions or the method described in Section 3 will not hold to get the maximal number of mutually orthogonal F-squares. Even for the smallest possible n, i.e., $n = 6$, the problem is complicated. However, from the orthogonal array (36, 13, 3, 2) constructed by Seiden (1954) four mutually orthogonal $F(6; 2)$ squares were obtained and are exhibited below.

$$\begin{matrix} 0&0&1&1&2&2 \\ 0&0&1&1&2&2 \\ 1&1&2&2&0&0 \\ 1&1&2&2&0&0 \\ 2&2&0&0&1&1 \\ 2&2&0&0&1&1 \end{matrix} \quad \begin{matrix} 0&0&2&2&1&1 \\ 1&2&0&1&2&0 \\ 1&1&0&0&2&2 \\ 2&0&1&2&0&1 \\ 2&2&1&1&0&0 \\ 0&1&2&0&1&2 \end{matrix} \quad \begin{matrix} 0&2&1&0&1&2 \\ 2&1&2&1&0&0 \\ 1&0&2&1&2&0 \\ 0&2&0&2&1&1 \\ 2&1&0&2&0&1 \\ 1&0&1&0&2&2 \end{matrix} \quad \begin{matrix} 0&1&2&1&0&2 \\ 2&2&1&0&1&0 \\ 1&2&0&2&1&0 \\ 0&0&2&1&2&1 \\ 2&0&1&0&2&1 \\ 1&1&0&2&0&2 \end{matrix}$$

It is not known at this stage whether the above set of mutually orthogonal $F(6; 2)$ squares can be embedded in a larger set. It may be noted that the above set of F-squares has a special structure and this might be the reason for the difficulty in extending the set.

Closing remarks. It is well known that the existence of a set of t mutually orthogonal $F(n; 1)$ squares is equivalent to the existence of an orthogonal array $(n^2, t + 2, n, 2)$. Therefore, it is useful to find out what relationship, if any, exists between arbitrary orthogonal F-squares and orthogonal arrays or partially balanced arrays. It may be noted that the existence of a set of t mutually orthogonal $F(n; \lambda)$ squares implies the existence of an orthogonal array $(n^2, t + 2, n/\lambda, 2)$.

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