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ASYMPTOTIC EXPANSIONS OF THE NON-NULL DISTRIBUTIONS OF LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRICES[1]

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In this paper, asymptotic expansions of the non-null distributions of the likelihood ratio criteria are obtained for testing the hypotheses: (a) H_1: $\Sigma = \sigma^2 I$, $\sigma^2 > 0$, (b) H_2: $\Sigma_1 = \Sigma_2$, against alternatives which are close to the hypothesis. These expansions are of chi-square type. The first problem has been considered by Sugiura (1969) but because of the singularity at the hypothesis, his expansion will not be good for alternatives close to the hypothesis. The second problem has been considered by de Waal (1970) but the expansion given by him is invalid.

1. Problem (a). Let $\mathbf{x}_1, \cdots, \mathbf{x}_N$ be a p-variate random sample of size N from $N(\mathbf{x} \mid \boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$. In this section, we give two asymptotic expansions of the non-null distribution of the likelihood ratio statistic for testing sphericity against two sequences of alternative hypotheses that approach the null hypothesis. Let

$$n = N - 1\,, \qquad \bar{\mathbf{x}} = N^{-1} \textstyle\sum_1^N \mathbf{x}_i\,,$$

and

$$S = \textstyle\sum_{i=1}^N (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'\,.$$

Then the likelihood ratio statistic for testing H_1 against the general alternative $A_1 : \Sigma \neq \sigma^2 I$ is $(\lambda^*)^{N/2}$ where

$$\lambda^* = |S|/(p^{-1} \operatorname{tr} S)^p\,. \tag{1.1}$$

Let

$$n = m + 2\alpha\,, \qquad \text{where } \alpha = (2p^2 + p + 2)/12p\,.$$

Then, under H_1, the asymptotic expansion for $P\{-m\ln\lambda^* \leqq z\}$ up to $O(m^{-3})$ in terms of a linear combination of chi-square distributions has been obtained by Sugiura (1969) (same as Anderson (1958), page 263). However, under A_1, Sugiura (1969) gives an asymptotic expansion up to $O(n^{-\frac{3}{2}})$ in terms of a linear combination of normal probability and its derivatives. This breaks down for alternatives that are too close to H_1. In this section we give asymptotic expansions of $P\{-m\ln\lambda^* \leqq z\}$ in terms of linear combinations of chi-square

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distributions for the following alternatives:

$$(1)\quad (I - q\Sigma^{-1}) = (2/m)P\,, \qquad (2)\quad (I - q^{-1}\Sigma) = (2/m)Q$$

where P and Q are fixed matrices as $m \to \infty$, and $0 < q < \infty$.

In order to derive the asymptotic expansion we need the following lemmas. The notation used in this paper corresponds to that of James (1964).

LEMMA 1. *Let* $f = \frac{1}{2}p(p+1) - 1$, *and* $\beta_0 = (p+2)(p-1)(p-2)(2p^3 + 6p^2 + 3p + 2)/288$. *Then*

$$\begin{aligned} A(k) &\equiv \frac{p^{\frac{1}{2}mph}\Gamma_p(\frac{1}{2}m(1+h)+\alpha)\Gamma(\frac{1}{2}mp + p\alpha + k)}{\Gamma_p(\frac{1}{2}m+\alpha)\Gamma(\frac{1}{2}pm(1+h) + p\alpha + k)} \\ &= (1+h)^{-k-\frac{1}{2}f}\Big[1 + \Big(1 - \frac{1}{1+h}\Big)\frac{k(k-1+2\alpha p)}{pm} \\ &\quad + \frac{1}{p^2m^2}\Big\{\Big(1 - \frac{1}{(1+h)^2}\Big)(\beta_0 - \tfrac{1}{3}k(k-1)(2k-1+6\alpha p) - 2\alpha^2p^2k) \\ &\quad + \tfrac{1}{2}\Big(1 - \frac{2}{1+h} + \frac{1}{(1+h)^2}\Big)(4\alpha^2p^2k \\ &\quad + k(k-1)(k-2)((k-3) + 4(1+\alpha p)) \\ &\quad + (2 + 8\alpha p + 4\alpha^2p^2)k(k-1))\Big\} + O(m^{-3})\Big]. \end{aligned} \tag{1.2}$$

The proof can be obtained on the lines given in Anderson (1958).

LEMMA 2. *Let* $||B||$ = *maximum characteristic root of* $B \leqq 1$ *and*

$$d_i = \beta \operatorname{tr} B^i(I-B)^{-i}\,, \qquad i = 1, 2, 3, 4, \beta > 0\,.$$

Then,

(a) $$d_1(|I-B|^{-\beta}) = \textstyle\sum_{k=0}^{\infty}\sum_\kappa k(\beta)_\kappa C_\kappa(B)/k!$$

(b) $$(d_2 + d_1^2)(|I-B|^{-\beta}) = \textstyle\sum_{k=0}^{\infty}\sum_\kappa k(k-1)(\beta)_\kappa C_\kappa(B)/k!$$

(c) $$(2d_3 + 3d_2d_1 + d_1^3)(|I-B|^{-\beta}) = \textstyle\sum_{k=0}^{\infty}\sum_\kappa k(k-1)(k-2)(\beta)_\kappa C_\kappa(B)/k!$$

(d) $$\begin{aligned}&(6d_4 + 8d_1d_3 + 3d_2^2 + 6d_1^2d_2 + d_1^4)(|I-B|^{-\beta}) \\ &\qquad = \textstyle\sum_{k=0}^{\infty}\sum_\kappa k(k-1)(k-2)(k-3)(\beta)_\kappa C_\kappa(B)/k!\end{aligned}$$

PROOF. By differentiating $|I - \phi B|^{-\beta} = \sum_{k=0}^{\infty}\sum_\kappa (\beta)_\kappa C_\kappa(B)\phi^k/k!$ with respect to ϕ successively four times and putting $\phi = 1$, the lemma follows. □

It may be noted that (a) and (b) were proved by Fujikoshi (1970).

1.1 *Asymptotic expansion of* $P\{-m\ln\lambda^* \leqq z\}$. Equation (2.8) replacing n by $m + 2\alpha$ and h by $\frac{1}{2}mh$ of Khatri and Srivastava (1971, page 203) yields

$$\begin{aligned} g(h) &\equiv E(\lambda^{*\frac{1}{2}mh}) \\ &= |q\Sigma^{-1}|^{\frac{1}{2}m+\alpha}\textstyle\sum_{k=0}^{\infty}\sum_\kappa(\frac{1}{2}m(1+h)+\alpha)_\kappa C_\kappa(I - q\Sigma^{-1})A(k)/k!\,, \end{aligned} \tag{1.3}$$

where $0 < q < \infty$ (can be taken $2\lambda_1\lambda_p/(\lambda_1 + \lambda_p)$ for the rapid convergence of the series; $\lambda_1 \geqq \lambda_2 \geqq \cdots \lambda_p$ are the roots of Σ), and

$$A(k) = \frac{p^{\frac{1}{2}mph}\Gamma_p(\frac{1}{2}m(1+h)+\alpha)\Gamma(\frac{1}{2}mp+p\alpha+k)}{\Gamma_p(\frac{1}{2}m+\alpha)\Gamma(\frac{1}{2}pm(1+h)+p\alpha+k)},$$

$$(1.4)\quad (a)_\kappa = \prod_{\alpha=1}^{p}\left(a-\frac{\alpha-1}{2}\right)\left(a+1-\frac{\alpha-1}{2}\right)\cdots\left(a+k_\alpha-1-\frac{\alpha-1}{2}\right),$$

$$\sum_{\alpha=1}^{p} k_\alpha = k .$$

Case 1. $(I - q\Sigma^{-1}) = (2/m)P$.

We use in (1.3) the approximation for $A(k)$ as given by Lemma 1 and then use Lemma 2 for the summation over k. Substituting $I - 2P/m = q\Sigma^{-1}$ and neglecting the higher order terms for m, we obtain an approximate value of $g(h)$ as

$$\begin{aligned} g(h) = {} & |I-2P/m|^{\frac{1}{2}m+\alpha}\,|I-2P/m(1+h)|^{-\frac{1}{2}m(1+h)-\alpha}(1+h)^{-\frac{1}{2}f} \\ & \times\left\{1+\left(1-\frac{1}{1+h}\right)(pm)^{-1}(2\alpha p \operatorname{tr} P + (\operatorname{tr} p)^2)\right. \\ & \left. + (pm)^{-2}(\alpha_0' + \alpha_1'(1+h)^{-1} + \alpha_2'(1+h)^{-2})\right\} + O(m^{-3}) \end{aligned}$$

where

$$\begin{aligned} \alpha_0' &= \beta_0 + \tfrac{1}{2}(\operatorname{tr} P)^4 + (\tfrac{4}{3} + 2\alpha p)(\operatorname{tr} P)^3 + 2\alpha p(1+\alpha p)(\operatorname{tr} P)^2 , \\ \alpha_1' &= -(\operatorname{tr} P)^4 - 4(1+\alpha p)(\operatorname{tr} P)^3 + 2p(1 + 2\operatorname{tr} P + 2\alpha p)\operatorname{tr} P^2 \\ & \quad - (2 + 4\alpha p + 4\alpha^2 p^2)(\operatorname{tr} P)^2 \end{aligned}$$

and

$$\alpha_2' + \alpha_0' + \alpha_1' = 0 .$$

Now, using the asymptotic expansion

$$\begin{aligned} & |I-2P/m|^{\frac{1}{2}m+\alpha}\,|I-2P/m(1+h)|^{-\frac{1}{2}m(1+h)-\alpha} \\ & = 1 - \frac{1}{m}\left(1-\frac{1}{1+h}\right)(\operatorname{tr} P^2 + 2\alpha \operatorname{tr} P) \\ & \quad - \frac{4}{3m^2}\left(1-\frac{1}{(1+h)^2}\right)(\operatorname{tr} P^3 + \tfrac{3}{2}\alpha \operatorname{tr} P^2) \\ & \quad + \frac{1}{2m^2}\left(1-\frac{1}{1+h}\right)^2(\operatorname{tr} P^2 + 2\alpha \operatorname{tr} P)^2 + O(m^{-3}) \end{aligned}$$

in the expansion of $g(h)$ obtained above, the final expression is

$$(1.5)\quad \begin{aligned} g(h) = {} & (1+h)^{-\frac{1}{2}f}[1 + (c_1/pm)(1-(1+h)^{-1}) + (1/pm)^2 \\ & \times (\alpha_0 + \alpha_1(1+h)^{-1} + \alpha_2(1+h)^{-2}) + O(m^{-3})] , \end{aligned}$$

where

$$
\begin{aligned}
&f = \tfrac{1}{2}p(p+1) - 1\,, \\
(1.6)\qquad &c_1 = (\mathrm{tr}\, P)^2 - p\,\mathrm{tr}\, p^2\,; \qquad \alpha_0 = \beta_0 + \tfrac{1}{2}c_1^2 + 2\alpha p c_1 + \tfrac{4}{3}c_2\,, \\
&c_2 = (\mathrm{tr}\, P)^3 - p^2\,\mathrm{tr}\, P^3\,; \qquad \alpha_1 = -c_1^2 - 4(\mathrm{tr}\, P)c_1 - 2c_1 - 4\alpha p c_1\,; \text{ and} \\
&\alpha_2 = -(\alpha_1 + \alpha_0)\,.
\end{aligned}
$$

β_0 is defined in Lemma 1.
Hence

$$
\begin{aligned}
&P(-m\ln\lambda^* \leqq z) \\
(1.7)\qquad &= P(\chi_f^2 \leqq z) + (c_1/pm)[P(\chi_f^2 \leqq z) - P(\chi^2_{f+2} \leqq z)] \\
&\quad + (1/pm)^2[\alpha_0 P(\chi_f^2 \leqq z) + \alpha_1 P(\chi^2_{f+2} \leqq z) + \alpha_2 P(\chi^2_{f+4} \leqq z)] \\
&\quad + O(m^{-3})\,.
\end{aligned}
$$

Case 2. $I - q^{-1}\Sigma = (2/m)Q$.

We use in (1.3) the approximation for $A(k)$ as given by Lemma 1 and then use Lemma 2 for the summation over k. Substituting $I - q\Sigma^{-1} = -2Q(I - 2Q/m)^{-1}/m$ and neglecting the higher order terms for m, we obtain an approximate value of $g(h)$ as

$$
\begin{aligned}
g(h) = {} & |I - 2Q/m|^{\frac{1}{2}mh}\, |I - 2hQ/m(1+h)|^{-\frac{1}{2}m(1+h)-\alpha}\, (1+h)^{-\frac{1}{2}f} \\
& \times \Big\{1 + \Big(1 - \frac{1}{1+h}\Big)(pm)^{-1}(\mathrm{tr}\, Q)(\mathrm{tr}\, Q - 2\alpha p) \\
& + (pm)^{-2}(\alpha_0^{*\prime} + \alpha_1^{*\prime}(1+h)^{-1} + \alpha_2^{*\prime}(1+h)^{-2}) + O(m^{-3})\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_0^{*\prime} &= \beta_0 + \tfrac{1}{2}(\mathrm{tr}\, Q)^4 - (\tfrac{4}{3} + 2\alpha p)(\mathrm{tr}\, Q)^3 + 2\alpha p(1 + \alpha p)(\mathrm{tr}\, Q)^2 \\
&\quad + 4p(\mathrm{tr}\, Q^2)(\mathrm{tr}\, Q - \alpha p) \\
\alpha_1^{*\prime} &= -(\mathrm{tr}\, Q)^4 + 4(1 + \alpha p)(\mathrm{tr}\, Q)^3 - (2 + 4\alpha p + 4\alpha^2 p^2)(\mathrm{tr}\, Q)^2 \\
&\quad + 2p(\mathrm{tr}\, Q^2)(1 - 4\,\mathrm{tr}\, Q + 4\alpha p)
\end{aligned}
$$

and

$$\alpha_2^{*\prime} + \alpha_0^{*\prime} + \alpha_1^{*\prime} = 0\,.$$

Now, using the asymptotic expansion

$$
\begin{aligned}
&|I - 2Q/m|^{\frac{1}{2}mh}\, |I - 2hQ/m(1+h)|^{-\frac{1}{2}m(1+h)-\alpha} \\
&\quad = 1 + \Big(1 - \frac{1}{1+h}\Big)\frac{2\alpha\,\mathrm{tr}\, Q - \mathrm{tr}\, Q^2}{m} \\
&\qquad + \frac{1}{2m^2}\Big(1 - \frac{1}{1+h}\Big)^2 \{(2\alpha\,\mathrm{tr}\, Q - \mathrm{tr}\, Q^2)^2 + 4\alpha\,\mathrm{tr}\, Q^2\} \\
&\qquad - \frac{4}{3m^2}\Big(2 - \frac{3}{1+h} + \frac{1}{(1+h)^2}\Big)\mathrm{tr}\, Q^3 + O(m^{-3})
\end{aligned}
$$

in the expansion of $g(h)$ obtained above, the final expression becomes

$$g(h) = (1+h)^{-\frac{1}{2}f}[1 + (c_1^*/pm)(1-(1+h)^{-1}) + (1/pm)^2(\alpha_0^* + \alpha_1^*(1+h)^{-1} + \alpha_2^*(1+h)^{-2}) + O(m^{-3})], \tag{1.8}$$

where

$$\begin{aligned} c_1^* &= (\operatorname{tr} Q)^2 - p \operatorname{tr} Q^2 ; \\ \alpha_0^* &= \beta_0 + \tfrac{1}{2}(c_1^*)^2 - 4(\operatorname{tr} Q)c_1^* + 2\alpha p c_1^* + \tfrac{8}{3}c_2^* , \\ c_2^* &= (\operatorname{tr} Q)^3 - p^2 \operatorname{tr} Q^3 ; \\ \alpha_1^* &= -(c_1^*)^2 + 2c_1^* + 8(\operatorname{tr} Q)c_1^* - 4\alpha p c_1^* + 4c_2^* , \\ \alpha_2^* &= -(\alpha_0^* + \alpha_1^*) . \end{aligned} \tag{1.9}$$

β_0 is defined in Lemma 1.
Hence $P(-mln\lambda^* \leqq z)$ is given by (1.7) with c_1, α_0, α_1 and α_2 replaced by c_1^*, α_0^*, α_1^* and α_2^* respectively. Some work for Case 2 can be found in Nagao (1970).

It may be noted that when $P = Q = 0$, the expansion in the two cases reduces to that of Anderson (1958, page 263) for H_1.

2. Expansion of the distribution of the modified LR criterion for $\Sigma_1 = \Sigma_2$. The likelihood ratio criterion for testing the hypothesis $H_2: \Sigma_1 = \Sigma_2$ against the alternative $A_2: \Sigma_1 \neq \Sigma_2$ is based on the statistic

$$\lambda' = |N_1^{-1}S_1|^{\frac{1}{2}N_1} |N_2^{-1}S_2|^{\frac{1}{2}N_2}/|N^{-1}S|^{\frac{1}{2}N} , \tag{2.1}$$

where $S_1 = \sum_{i=1}^{N_1} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'$, $S_2 = \sum_{i=1}^{N_2} (\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})'$, $S = S_1 + S_2$, and $N = N_1 + N_2$. N_1 and N_2 are the sizes of the samples drawn respectively from normal populations $N(\mathbf{x} \mid \boldsymbol{\mu}_1, \Sigma_1)$ and $N(\mathbf{y} \mid \boldsymbol{\mu}_2, \Sigma_2)$ Σ_1, $\Sigma_2 > 0$. It has been shown by Das Gupta (1969) that the test based on (2.1) is unbiased if and only if $N_1 = N_2$. However, if we modify the LRC λ' to λ by reducing the sample size N_i, to the degrees of freedom $n_i = N_i - 1$, $i = 1, 2$,

$$\lambda = |n_1^{-1}S_1|^{\frac{1}{2}n_1} |n_2^{-1}S_2|^{\frac{1}{2}n_2}/|n^{-1}S|^{\frac{1}{2}n} , \qquad n = n_1 + n_2 . \tag{2.2}$$

Sugiura and Nagao (1968) have shown that the test based on this statistic is unbiased. The exact non-null distribution of λ has been obtained by Khatri and Srivastava (1971). In this section we give an asymptotic expansion for the distribution of λ under the alternative A_2; the asymptotic distribution of λ under the hypothesis is given in Anderson (1958). It may be noted that de Waal (1970) provided an asymptotic distribution but his results are invalid, for the expansions considered by him in terms of series do not satisfy the conditions of convergence or validity of the series in general. Let

$$\lambda^* = \lambda^{2/n} , \qquad r_i = n_i/n , \qquad i = 1, 2, \quad \text{and} \tag{2.3}$$

$$n = m + 2d ,$$

$$\text{where } d = \left(\frac{1}{r_1} + \frac{1}{r_2} - 1\right)(2p^2 + 3p - 1)/12(p+1) .$$

We assume that

$$(2.4)\qquad \text{(i)}\quad (I - \Sigma_1^{-1}\Sigma_2) = (2/r_1 m)P \quad \text{and } P \text{ is fixed as } m \to \infty\,.$$
$$\text{(ii)}\quad 0 < \lim r_i < 1\,, \qquad i = 1, 2\,.$$

In equations (3.4) and (3.8) of Khatri and Srivastava (1971, page 205), changing $\lambda \to \lambda^*(n/n_1^{r_1}n_2^{r_2})^p$, $n \to m + 2d$, $n_1 \to r_1(m + 2d)$, $n_2 \to r_2(m + 2d)$, and $h \to \frac{1}{2}mh$, we can write

$$(2.5)\qquad E(\lambda^{*\frac{1}{2}mh}) = |\Omega|^{-\frac{1}{2}n_1}\, g_1(h)g_2(h)\,,$$

where

$$(2.6)\qquad g_1(h) = \frac{(\frac{1}{2}n)^{\frac{1}{2}mph}}{\prod_{i=1}^2 (\frac{1}{2}n_i)^{\frac{1}{2}r_i mph}} \times \frac{\Gamma_p(\frac{1}{2}m + d)}{\Gamma_p(\frac{1}{2}m(1 + h) + d)} \prod_{i=1}^2 \left[\frac{\Gamma_p(\frac{1}{2}r_i m(1 + h) + r_i d)}{\Gamma_p(\frac{1}{2}r_i m + r_i d)}\right].$$

$$(2.7)\qquad g_2(h) = {}_2F_1((\tfrac{1}{2}m + d), (\tfrac{1}{2}r_1 m(1 + h) + r_1 d); \tfrac{1}{2}m(1 + h) + d; I - \Omega^{-1})\,,$$

and

$$(2.8)\qquad \Omega = \Sigma_1\Sigma_2^{-1}\,.$$

Let

$$(2.9)\qquad f = \tfrac{1}{2}p(p + 1)\,,$$
$$\omega_2 = p(p + 1)\left[(p - 1)(p + 2)\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} - 1\right) - 24d^2\right]\Big/48\,.$$

Then from Anderson (1958, page 255),

$$(2.10)\qquad g_1(h) = (1 + h)^{-\frac{1}{2}f}[1 + (1/m^2)\omega_2(1 + h)^{-2} - 1) + O(m^{-3})]\,.$$

In order to expand $g_2(h)$ in powers of m^{-1}, we require the following Lemma 3 whose proof can be obtained from the results of Fujikoshi (1970, page 78–79) as follows: (a) from (2.4), (b) from (2.6), (c) from (2.7), (d) from (2.9), and (e) from (2.6), (2.9), and (2.10).

LEMMA 3. *Let*

$$a_1(\kappa) = \textstyle\sum_{\alpha=1}^p k_\alpha(k_\alpha - \alpha)\,, \qquad a_2(\kappa) = \sum_{\alpha=1}^p k_\alpha(4k_\alpha^2 - 6\alpha k_\alpha + 3\alpha^2)\,,$$
$$\gamma_1(\kappa) = -a_1(\kappa)/2\,, \qquad \textit{and} \quad \gamma_2(\kappa) = [3(a_1(\kappa))^2 - a_2(\kappa) + k]/24\,.$$

Then

(a) $(n)_\kappa = n^k(1 - n^{-1}\gamma_1(\kappa) + n^{-2}\gamma_2(\kappa) + O(n^{-3}))$

(b) $\sum_{k=j}^\infty \sum_\kappa C_\kappa(P)/(k - j)! = (\operatorname{tr} P)^j \operatorname{etr} P\,, \qquad j = 0, 1, 2, \cdots$

(c) $\sum_{k=0}^\infty \sum_\kappa \gamma_1(\kappa)C_\kappa(P)/k! = -(\operatorname{tr} P^2)(\operatorname{etr} P)/2$

(d) $\sum_{k=0}^\infty \sum_\kappa \gamma_1^2(\kappa)C_\kappa(P)/k! = [(\operatorname{tr} P)^2 + \operatorname{tr} P^2 + 4 \operatorname{tr} P^3 + (\operatorname{tr} P^2)^2](\operatorname{etr} P)/4$

(e) $\sum_{k=0}^\infty \sum_\kappa \gamma_2(\kappa)C_\kappa(P)/k! = [8 \operatorname{tr} P^3 + 3(\operatorname{tr} P^2)^2](\operatorname{etr} P)/24\,.$

Now, let

$$y_1 = 1 + 2dm^{-1}, \qquad y_2 = y_1 + h, \qquad \text{and} \quad y_3 = y_1 y_2^{-2}. \tag{2.11}$$

Then, using Lemma 3(a) in (2.7), we can write

$$\begin{aligned} g_2(h) &= \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (k!)^{-1} C_\kappa(y_1 P)[1 - (2/my_1)\gamma_1(\kappa) + (2/my_1)^2\gamma_2(\kappa)] \\ &\quad \times [1 - (2/mr_1 y_2)\gamma_1(\kappa) + (2/mr_1 y_2)^2\gamma_2(\kappa)] \\ &\quad \times [1 - (2/my_2)\gamma_1(\kappa) + (2/my_2)^2\gamma_2(\kappa)]^{-1} + O(m^{-3}) \\ &= \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (k!)^{-1} C_\kappa(y_1 P)[1 - (2/my_1)(1 + r_1^{-1}y_3 - y_3)\gamma_1(\kappa) \\ &\quad + (2/my_1)^2(1 + r_1^{-2}y_3^2 - y_3^2)\gamma_2(\kappa) \\ &\quad + (2/my_1)^2 y_3(r_1^{-1} - 1)(1 - y_3)\gamma_1^2(\kappa) + O(m^{-3})]. \end{aligned} \tag{2.12}$$

Using Lemma 3(b)—3(e), we get

$$\begin{aligned} g_2(h) &= (\operatorname{etr} y_1 P)[1 + (y_1/m)(1 + r_1^{-1}y_3 - y_3) \operatorname{tr} P^2 \\ &\quad + (y_1/6m^2)(1 + r_1^{-2}y_3^2 - y_3^2)(8 \operatorname{tr} P^3 + 3y_1(\operatorname{tr} P^2)^2) \\ &\quad + (y_3/m^2)(r_1^{-1} - 1)(1 - y_3)((\operatorname{tr} P)^2 + \operatorname{tr} P^2 + 4y_1 \operatorname{tr} P^3 \\ &\quad + (y_1 \operatorname{tr} P^2)^2) + O(m^{-3})] \\ &= (\operatorname{etr} y_1 P)[1 + (y_1/m)(1 + (r_1^{-1} - 1)y_3) \operatorname{tr} P^2 \\ &\quad + (1/6m^2)(1 + (r_1^{-2} - 1)(1 + h)^{-2})(8 \operatorname{tr} P^3 + 3(\operatorname{tr} P^2)^2) \\ &\quad + (1/m^2)(r_1^{-1} - 1)((1 + h)^{-1} - (1 + h)^{-2}) \\ &\quad \times ((\operatorname{tr} P)^2 + \operatorname{tr} P^2 + 4 \operatorname{tr} P^3 + (\operatorname{tr} P^2)^2) + O(m^{-3})]. \end{aligned} \tag{2.13}$$

Now, we have, from (2.4) and (2.11)

$$|\Omega|^{-\frac{1}{2}n_1} = [|I - (2/r_1 m)P|^{\frac{1}{2}r_1 m}]^{y_1}$$

and

$$|I - (2/r_1 m)P|^{\frac{1}{2}r_1 m} = \exp[-\textstyle\sum_{j=0}^{\infty} (2/r_1 m)^j \operatorname{tr} P^{j+1}/(j + 1)].$$

This gives (see, e.g., [10] page 943, equation (1.20))

$$\begin{aligned} |\Omega|^{-\frac{1}{2}n_1} &= (\operatorname{etr} - y_1 P)[1 - (y_1/r_1 m) \operatorname{tr} P^2 \\ &\quad - (4y_1/3r_1^2 m^2) \operatorname{tr} P^3 \\ &\quad + (y_1^2/2r_1^2 m^2)(\operatorname{tr} P^2)^2] + O(m^{-3}) \\ &= \operatorname{etr}(-y_1 P)[1 - (y_1/r_1 m) \operatorname{tr} P^2 - (4/3r_1^2 m^2) \operatorname{tr} P^3 \\ &\quad + (1/2r_1^2 m^2)(\operatorname{tr} P^2)^2] + O(m^{-3}). \end{aligned} \tag{2.14}$$

Multiplying (2.13) and (2.14), we get the asymptotic expansion as

$$\begin{aligned} |\Omega|^{-\frac{1}{2}n_1} g_2(h) &= [1 + (a/m)((1 + h)^{-1} - 1) \\ &\quad + (1/m^2)(b_0 + \omega_2 + (1 + h)^{-1}b_1 + (b_2 - \omega_2)(1 + h)^{-2}) \\ &\quad + O(m^{-3})] \end{aligned} \tag{2.15}$$

where

$$
\begin{aligned}
a &= (r_1^{-1} - 1) \operatorname{tr} P^2 \\
b_0 &= -(\tfrac{4}{3})(r_1^{-2} - 1) \operatorname{tr} P^3 + (\tfrac{1}{2})(r_1^{-1} - 1)^2(\operatorname{tr} P^2)^2 \\
&\qquad - 2d(r_1^{-1} - 1) \operatorname{tr} P^2 - \omega_2 , \\
b_1 &= (r_1^{-1} - 1)[4 \operatorname{tr} P^3 + (4d + 1) \operatorname{tr} P^2 - (r_1^{-1} - 1)(\operatorname{tr} P^2)^2 + (\operatorname{tr} P)^2] \\
b_2 &= (r_1^{-1} - 1)[(\tfrac{4}{3})(r_1^{-1} - 2) \operatorname{tr} P^3 + (\tfrac{1}{2})(r_1^{-1} - 1)(\operatorname{tr} P^2)^2 \\
&\qquad - (2d + 1) \operatorname{tr} P^2 - (\operatorname{tr} P)^2] + \omega_2 ,
\end{aligned} \tag{2.16}
$$

and ω_2 is defined in (2.9).

Thus, using (2.15) and (2.10) in (2.5), we get an asymptotic expansion as

$$
\begin{aligned}
E(\lambda^{*\frac{1}{2}mh}) = (1 + h)^{-f/2}[1 &+ (a/m) \textstyle\sum_{i=0}^{1} (-1)^{i+1}(1 + h)^{-i} \\
&+ (1/m^2) \textstyle\sum_{i=0}^{2} b_i(1 + h)^{-i} + O(m^{-3})] .
\end{aligned} \tag{2.17}
$$

Hence,

$$
\begin{aligned}
&P(-m \ln \lambda^* \leqq z) \\
&\quad = P(\chi_f^2 \leqq z) + (a/m)[P(\chi^2_{f+2} \leqq z) - P(\chi_f^2 \leqq z)] \\
&\qquad + (1/m^2)[b_0 P(\chi_f^2 \leqq z) + b_1 P(\chi^2_{f+2} \leqq z) + b_2 P(\chi^2_{f+4} \leqq z)] \\
&\qquad + O(m^{-3})
\end{aligned} \tag{2.18}
$$

where, a, b_0, b_1, and b_2 are defined in (2.16).

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