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ESTIMATING THE SCALE PARAMETER OF THE EXPONENTIAL DISTRIBUTION WITH UNKNOWN LOCATION[1]

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Let $X_1, \cdots, X_n$ be i.i.d. random variables each with a density which is $a^{-1}\exp(b-x)$ or 0 according as $x \geqq b$ or $x < b$ where $-\infty < b < \infty$, $a > 0$ are unknown constants. Let $\bar{X} = n^{-1}\sum X_i$ and $M = \min X_i$. The maximum likelihood estimator of a is $\bar{X} - M$ and is, if quadratic loss is assumed, the best affine equivariant estimator of a. It is shown that if loss is measured by any member of a large class of "bowl-shaped" functions which includes quadratic loss, the best affine equivariant estimator is inadmissible. The proof entails an examination of the conditional expected loss given the maximal invariant under the scale group. It is carried out by exhibiting a superior alternative. In the case of quadratic loss, for example, the result is as follows.

Given any estimator $u = (\bar{X} - M)T[M(\bar{X} - M)^{-1}]$, let $T^*(y) = T(y)$ if $y < 0$ and $T^*(y) = \min\{T(y), n(n+1)^{-1}(1+y)\}$ if $y > 0$. If $T^* \neq T$ with positive probability, then the estimator, obtained from u by replacing T by T^*, has uniformly smaller risk than u.

Using a generalization of the author's conditions for admissibility [*Ann. Math. Statist.* **41** (1970) 446–457] a class, B, of generalized Bayes estimators within D, the class of scale equivariant estimators, are obtained with each member of B admissible in D. The improper measures determining members of B have densities on the orbit space R, created in the parameter space by the action of the group of scale changes. These prior densities, g, satisfy $\int_1^\infty (t^2g(t))^{-1}\,dt = \int_{-\infty}^{-1}(t^2g(t))^{-1}\,dt = \infty$.

1. Introduction. Let W be a real valued random variable with a density function which is $\sigma^{-1}\exp(-(w-\mu)/\sigma)$ or 0 according as $\mu \leqq w$ or $\mu > w$. Here μ and σ are unknown constants with $-\infty < \mu < \infty$ and $\sigma > 0$. Let $X_1, \cdots, X_n$ denote a sample consisting of independent copies of W. The joint density of these observations, evaluated at $(x_1, \cdots, x_n)$, is $\sigma^{-n}\exp(-n(\bar{x}-\mu)/\sigma)$ or 0 according as $\mu \leqq m$ or $\mu > m$, where $n\bar{x} = x_1 + \cdots + x_n$ and $m = \min x_i$. We suppose σ is to be estimated on the basis of the sample values.

A possible estimate is that given by the method of maximum likelihood, $\bar{x} - m$. The same result can be obtained, as we show below, by invoking the principle of invariance.

Define $\bar{X}$ and M by $n\bar{X} = X_1 + \cdots + X_n$ and $M = \min_i X_i$, respectively. Clearly $(\bar{X}, M)$ is a sufficient statistic. If $\hat{\theta}$ is an estimator based on this statistic, $\hat{\theta}$ is called equivariant under the affine group, J, provided $\hat{\theta}(c\bar{x} + d, cm + d) = c\hat{\theta}(\bar{x}, m) + d$ for all values of $c > 0$, d, $\bar{x}$ and m. This implies that $\hat{\theta}$ has the form $\hat{\theta}(\bar{x}, m) = K(\bar{x} - m)$ for some constant K. If loss is measured by $(a-\sigma)^2/\sigma^2$, an

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examination of the risk of $\hat{\theta}$ reveals that the best choice for K is $K = 1$. Therefore the best affine equivariant estimator under squared error loss is the same as the maximum likelihood estimator.

More generally suppose loss is measured by, say $W(a\sigma^{-1})$ with $E_{\mu,\sigma} W(\sigma^{-1}(\bar{X} - M)) < \infty$, for all μ, σ. Then the risk function of $K(\bar{X} - M)$ is independent of μ and σ. Choose $K = K_0$ so that

$$E_{0,1} W\{K_0(\bar{X} - M)\} = \inf_K E_{0,1} W\{K(\bar{X} - M)\} . \tag{1.0}$$

DEFINITION 1.1. If K_0 is chosen as in (1.0), $K_0(\bar{X} - m)$ will be called the best affine equivariant estimator of σ.

The invariance of the problem, not under J, but under a subgroup of J yields the framework which is intrinsic to the results of this paper. This subgroup, $\mathscr{G}$, acting as a transformation group, consists of the multiplicative group of positive reals. A more useful version of the sufficient statistic, for this case, is $X = (G, Y)$ where $G = \bar{X} - M$ and $Y = M/(\bar{X} - M)$. Then the action of $\mathscr{G}$ on the problem can be summarized as

$$X \to (cG, Y) , \qquad (\sigma, \mu) \to (c\sigma, c\mu) , \qquad \hat{\theta} \to c\hat{\theta} , \tag{1.1}$$

if $\hat{\theta}$ is an equivariant estimator and $0 < c < \infty$.

In Section 2 it is shown that the maximum likelihood estimator, $\bar{X} - M$, is inadmissible whenever loss is being measured by any one of a general class of loss functions which includes squared error as a special case. And since for that special case the best affine equivariant and maximum likelihood estimator are identical, the result is, perhaps, not surprising in view of the work of Brown [5], which although inapplicable in the present case, does show that the best affine equivariant estimator is inadmissible for a general class of estimation problems with varying loss function and underlying distribution.

The inadmissibility of the maximum likelihood estimator is a consequence of Theorem 2.1. This proves for the particular case of interest here, the assertion of Brown's result [5]. Brown's arguments rely on the conditional expected loss given that the maximal invariant under the scale group (defined in (1.1)) lies in a symmetric interval about the origin. We condition instead on the maximal invariant itself.

Like Brown we require, apart from mild regularity conditions, that the loss function be bowl-shaped on $(0, \infty)$ as a function of $\hat{\sigma}$, the estimate selected (A function f: $(0, \infty) \to (0, \infty)$ is (strictly) bowl-shaped if there is a constant $a \in (0, \infty)$ such that f is (strictly) decreasing on $(0, a)$ and (strictly) increasing on (a, ∞)).

The method used in Section 2 is more fully described in [4] along with other techniques of a similar character. Brewster [3] uses one of these techniques to obtain, for the problem of interest here and each of a variety of loss functions, an estimator whose risk functions is nowhere larger than that of the best affine equivariant estimator. His estimators are equal, over much of their domains, to formal Bayes estimators. Another application of the method used here is given in [11].

The main result of Section 2 is given in Theorem 2.1. Its proof entails a more general result which is stated in Corollary 2.1 and which is in form analogous to the result given by Stein [8].

When a quadratic loss function is used the consequence of Theorem 2.1 is an alternative to $\bar{X} - M$ which may be regarded as a "likelihood ratio testimator". More precisely, suppose adopted, the likelihood ratio test of $\mu = 0$ which accepts this hypothesis if and only if $0 \leqq M/(\bar{X} - M) \leqq 1/n$. The estimator in question is $\bar{X}$ or $\bar{X} - M$ depending on whether or not this preliminary test accepts $\mu = 0$. The risk of the resulting estimator is uniformly smaller than that of $\bar{X} - M$ and hence it is minimax.

While the original version of this manuscript was in its final stages of preparation its author learned that the preliminary test estimator described in the last paragraph had been obtained independently by Arnold [2]. He proved its superiority by evaluating its risk function and comparing the result with that of $\bar{X} - M$.

Section 4 is devoted to the problem of finding a family of estimators each of which is formal Bayes relative to the class of all scale equivariant estimators and admissible within that class. This family is characterized by conditions on the densities of the prior measures corresponding to its members and the main one of these conditions is stated in Lemma 4.2. That this condition (together with appropriate regularity conditions) might be sufficient in the corresponding problem for the normal law was suggested to me by C. Stein. A proof for that case is given by Brewster [3].

The general result, given in Section 3, which leads to these conditions, is related to a recent result Portnoy [7]. He gives a theorem which, in its application to the present problem, yields conditions under which a Bayes (not a formal Bayes) procedure relative to the class of scale equivariant estimators will be admissible within the class of all procedures.

The proof of Lemma 4.2 is mainly concerned with showing that for the function $M(y, \lambda)$ defined in (4,2), $M^2(y, \lambda) \leqq K(1 + y^2)$, $-\infty < y < \infty$, for some constant $K > 0$. To show that $M^2(y, \lambda) \leqq K(1 + y^{n+a})$ for some constants $K > 0$, $a > 1$ is trivial; to reduce $n + a$ to 2, as we must in order to achieve the λ^2 which appears in the statement of Lemma 4.2, requires a more delicate argument. For brevity, only a portion of this argument is given and this reveals the role of assumptions A3–A5.

We have not succeeded in showing that if a procedure is admissible within the class of scale equivariant estimators that it is admissible among all procedures. Thus we cannot assert the stronger optimality result for the members of the class of admissible equivariant rules found in Section 4.

In Section 5 are presented, some numerical results. A method for calculating certain members of the class of formal Bayes admissible equivariant estimators is given in Section 6. This method may be used on a desk calculator, at least if n is moderate, say $n \leqq 20$.

2. Inadmissibility results. In Section 1 we defined, for $n \geqq 2$, $X = (G, Y)$, where $G = \bar{X} - M$, $M = \min_i X_i$, and $Y = MG^{-1}$; the $\{X_i\}$ are independent copies of a random variable W with a density, $f(w \mid \mu, \sigma)$, which is $\sigma^{-1}\exp(-(w - \mu)/\sigma)$ or 0 according as $w \geqq \mu$ or $w < \mu$. It is easy to show that G and $M - \mu$ are independent random variables distributed, respectively, as $\sum_{i=2}^{n} W_i$ and W_1 where the $\{W_i\}$ are independent and each W_i has density $f(\cdot \mid 0, \sigma n^{-1})$. It follows immediately that X has density $f(g, y \mid \mu, 1) \propto g^{n-2}\exp(-ng) \times g\exp(-ng(y - \mu/g))$ if $y \geqq \mu g^{-1}$, $g > 0$ and 0 otherwise. Furthermore, the marginal density of G is, if $\sigma = 1$, $f(g) \propto g^{n-2}\exp(-ng)$, $g > 0$, independent of μ.

Assume σ is being estimated using X. Any scale-equivariant estimator is of the form, $GT(Y)$, for some measurable function T. Suppose the loss function is given by $W(\hat{\sigma}\sigma^{-1})$, where

(2.1) (i) $W'(u)$ exists, $0 < u < \infty$ except possibly at $u = 1$
(ii) $W(1) = 0$ and $W(u)$ is strictly decreasing (increasing) on $(0, 1)$ $((1, \infty))$ and
(iii) if $\sigma = 1$, $E|W'(cG)| < \infty$ for all $c > 0$, $n \geqq 2$.

We consider only scale-equivariant estimators in this paper. Their risk functions depend on (μ, σ) only through $\mu\sigma^{-1}$ so we assume without loss of generality that $\sigma = 1$.

LEMMA 2.1. *The functions, $c \to EW(cG)$ and $c \to E_\mu(W(cG) \mid Y = y)$, for each possible μ, y, are strictly bowl-shaped and differentiable with derivatives $EGW'(cG)$ and $E_\mu(GW'(cG) \mid Y = y)$, respectively.*

PROOF. The proof, using assumptions (2.1), is a straightforward adaptation of that given in [6] Lemma 2 (iii) page 74. □

As a consequence of Lemma 2.1, $EW(cG)$ and $E_\mu(W(cG) \mid Y = y)$ are uniquely minimized as functions of c, by say $c = c_0$ and $c = c(y, \mu)$, respectively. Moreover, $EGW'(cG) < 0\ (> 0)$ for $c < c_0\ (> c_0)$ while $E_\mu(GW'(cG) \mid Y = y) < 0\ (> 0)$ for $c < c(y, \mu)\ (> c(y, \mu))$. The best affine-equivariant estimator is given by

$$c_0 G\,. \tag{2.2}$$

THEOREM 2.1. *The estimator given in* (2.2) *is inadmissible when W in any loss function satisfying* (2.1).

PROOF. Observe that $EGW'(c_0G) = 0$ is equivalent to $\int_0^\infty g^{n-1}\exp(-ng) \times W'(c_0 g)\,dg = 0$. It follows that $\int_0^\infty g^n \exp(-ng)W'(c_0g)\,dg > c_0^{-1}\int_0^\infty g^{n-1} \times \exp(-ng)W'(c_0g)\,dg = 0$. But $F(c) = \int_0^\infty g^{n-1}\exp(-ng)W(cg)\,dg$ is strictly bowl-shaped, so we deduce that there exists c^* such that $c^* < c_0$ and $F(c^*) = \inf_c F(c) < F(c_0)$. This is readily interpreted to mean that $c(y, 0) = c^*(1 + y)$, for $y > 0$, with $c^* < c_0$. By similar reasoning it is easily shown that $c(y, \mu) \leqq$

$c(y, 0)$, $-\infty < \mu < \infty$. Let

$$T_0(y) = c_0\,, \qquad y < 0$$
$$= \min\{c_0, c(y, 0)\}\,, \qquad y \geqq 0\,.$$

Then it follows that $E_\mu(W(GT_0(Y)) \mid Y = y) \leqq E_\mu(W(c_0G) \mid Y = y)$ for all μ, y, with actual inequality for all μ when y is such that $y > 0$ and $c(y, 0) < c_0$. The last event has, for each μ, positive probability. The conclusion now follows.

COROLLARY 2.1. *For any scale-equivariant estimator $GT(Y)$, let $T^*(y) = T(y)$ if $y < 0$ and $T^*(y) = \min\{T(y), c^*(1 + y)\}$ if $y \geqq 0$. Then, if* (2.1) *holds $E_\mu W(GT^*(Y)) \leqq E_\mu W(GT(Y))$ with inequality for all μ if for $\mu = 0$, $P(T^*(Y) \neq T(Y)) > 0$.* □

COROLLARY 2.2. *The maximum likelihood estimator of σ, G, is inadmissible when the loss function satisfies* (2.1). □

Particular loss functions considered by Brown [5] are $W_1(u) = (u - 1)^2$, $W_2(u) = u - 1 - \ln u$ and $W_3(u) = (\ln u)^2$. The corresponding values of c^*, and c_0 are, respectively, $c^* = n(n + 1)^{-1}$ and $c_0 = 1$, $c^* = 1$ and $c_0 = n(n - 1)^{-1}$ and $c^* = n \exp(E \log U_{n-1}^{-1})$ and $c_0 = n \exp(E \log U_{n-2}^{-1})$, where U_α denotes the random variable with density function which is $\propto g^\alpha \exp(-g)$ for $g \geqq 0$ and 0 if $g < 0$. Observe that for W_3, $n(n - 1)^{-1} \leqq c^* \leqq n(n - 2)^{-1}$ and, if $n \geqq 3$, $n(n - 2)^{-1} \leqq c_0 \leqq n(n - 3)^{-1}$.

3. General sufficient conditions for admissibility. Let X denote a random variable whose range is a measurable space $(\mathscr{X}, \mathscr{C})$. Suppose X is distributed by a unique but unknown probability measure which is an element of $\{P_\theta : \theta \in \Theta\}$. Let $\mathscr{A}$ denote a set (the action space) from which an element is to be chosen after X is observed. If action $a \in \mathscr{A}$ is selected when X is distributed by P_θ, a loss, $L(a, \theta) \geqq 0$, is incurred. Assume $\mathscr{A}$ and Θ are endowed with σ-algebras, the former containing all the singletons of $\mathscr{A}$. We consider only nonrandomized decision rules $\delta : \mathscr{X} \to \mathscr{A}$ where δ is measurable. The risk of a decision rule, δ, when X is distributed by P_θ, say $r(\delta, \theta)$, is defined as $r(\delta, \theta) = E_\theta L(\delta(X), \theta)$.

Assume $(\mathscr{X}, \mathscr{C}, \mu)$ is a σ-finite measure space and that P_θ is absolutely continuous with respect to μ for every θ. Let $p_\theta = dP_\theta/d\mu$, and assume $p_\theta(x)$ is jointly measurable in x and θ.

Suppose $\mathscr{G}$ is a locally compact Hausdorff topological transformation group acting on the left of $\mathscr{X}$. Let l denote the left invariant Haar measure on $\mathscr{G}$. Assume $\mathscr{X} = \mathscr{G}/\mathscr{H} \times \mathscr{X}/\mathscr{G}$ where $\mathscr{H}$ is a compact subgroup of $\mathscr{G}$. Here $\mathscr{G}/\mathscr{H}$ denotes the space of left cosets of $\mathscr{H}$ and $\mathscr{X}/\mathscr{G}$ is the quotient space of $\mathscr{X}$ under the equivalence, $x_1 \sim x_2$ if and only if $x_1 = gx_2$ for some $g \in \mathscr{G}$. For simplicity let $\mathscr{G}^* = \mathscr{G}/\mathscr{H}$ and $\mathscr{X}^* = \mathscr{X}/\mathscr{G}$. Denote by τ^* the canonical mapping of $\mathscr{G}$ onto $\mathscr{G}^*$. If $x = (g_0^*, y) \in \mathscr{X}$, assume the action of $g \in \mathscr{G}$ on x is described by $gx = (gg_0^*, y)$ where if $g_0^* = \tau^*(g_0)$, gg_0^* is defined by $gg_0^* = \tau^*(gg_0)$. The random variable, X, is of the form $X = (G^*, Y)$.

The assumption that the problems remains invariant under $\mathscr{G}$ entails the existence of measurable transformation groups ([11]) $\bar{\mathscr{G}}$ and $\hat{\mathscr{G}}$ acting on the left of Θ and $\mathscr{A}$, respectively. These groups are required to act in such a way that if $\bar{g}$ and $\hat{g}$ denote the homomorphic images of $g \in \mathscr{G}$ in $\bar{\mathscr{G}}$ and $\hat{\mathscr{G}}$, respectively, then $P_{\bar{g}\theta}(gA) = P_\theta(A)$ and $L(\hat{g}a, \bar{g}\theta) = L(a, \theta)$, $g \in \mathscr{G}$, $a \in \mathscr{A}$, $\theta \in \Theta$, $A \in \mathscr{C}$. Assume $\hat{\mathscr{H}}$ acts trivially on $\mathscr{A}$.

Assume $\mu = l^* \times \rho$ where ρ is an arbitrary σ-finite measure on the Borel subsets of $\mathscr{X}^*$ and l^* is the positive, left invariant, Borel measure induced on the Borel subsets of $\mathscr{G}/\mathscr{H}$ by l through the canonical mapping of $\mathscr{G}$ onto $\mathscr{G}^*$ (the compactness of $\mathscr{H}$ is used here). Let ϕ denote a $1:1$ mapping of $\Theta/\bar{\mathscr{G}} = \Theta^*$, the quotient space of Θ under an equivalence relation $(\sim)$ analogous to that defined above for $\mathscr{X}$, into Θ which satisfies the requirement that if $\theta_1^* \neq \theta_2^*$, $\phi(\theta_1^*) \nsim \phi(\theta_2^*)$, $\theta_i^* \in \Theta^*$, $i = 1, 2$. Assume $\{g \in \mathscr{G}: \bar{g}\phi(\theta^*) = \phi(\theta^*)\}$ is compact for every point $\theta^* \in \Theta^*$. For convenience, and with no danger of confusion, we write θ^* for $\phi(\theta^*)$ in the sequel.

A nonrandomized procedure $\zeta: \mathscr{X} \to \mathscr{A}$ is called equivariant if $\hat{g}\zeta(g_0^*, y) = \zeta(gg_0^*, y)$, $g \in \mathscr{G}$ and $(g_0^*, y) \in \mathscr{X}$ (the assumption that $\hat{\mathscr{H}}$ acts trivially on $\mathscr{A}$ is necessary here). It follows that if ζ is any equivariant nonrandomized estimator is equivalent to an estimator of the form $\hat{G}T(Y)$ where $T: \mathscr{X}^* \to \mathscr{A}$ is some measurable function. We now assume that $\mathscr{A} \subset R^m$ and that

$$L(a, \theta) = c(\theta)||a - w(\theta)||^2 ,$$

where $c(\theta) > 0$, $w(\theta) \in \mathscr{A}$ for all $\theta \in \Theta$ and $||\cdot||$ denotes a norm on R^m. Let $\delta(G^*, Y) = \hat{G}T(Y)$ denote an arbitrary, equivariant nonrandomized estimator. Then the risk of δ, $r(\delta, \theta)$, is

$$E_\theta c(\theta)||\hat{G}T(Y) - w(\theta)||^2 = E_\theta c(\bar{G}^{-1}\theta)||T(Y) - w(\bar{G}^{-1}\theta)||^2 ,$$

because of the assumed invariance of the loss. It follows that

$$r(\delta, \theta) = E_\theta c(Y, \theta)||T(Y) - w(Y, \theta)||^2 + b(\theta) ,$$

where $c(Y, \theta) = E_\theta(c(\bar{G}^{-1}\theta) \mid Y)$, $w(Y, \theta) = E_\theta(c(\bar{G}^{-1}\theta) \times w(\bar{G}^{-1}\theta) \mid Y)/c(Y, \theta)$ and $b(\theta) = E_\theta(c(\bar{G}^{-1}\theta)||w(\bar{G}^{-1}\theta)||^2) - E_\theta(c(Y, \theta)||w(Y, \theta)||^2)$. Of course,

$$r(\delta, \theta) = r(\delta, \phi(\tau(\theta)))$$

as it must since δ is equivariant.

If the choice of decision rule is restricted to the class of nonrandomized equivariant procedures, say D_I, a reduced problem is obtained which can be described as follows. A random variable Y with range $\mathscr{X}^*$ is observed. The distribution of Y is a unique but unknown member of $\{P_{\phi(\theta^*)}: \theta^* \in \Theta^*\}$. The action space for the problem is $\mathscr{A}$ and loss is measured by $L^*(a, Y, \theta^*) = c(Y, \theta^*)||a - w(Y, \theta^*)||^2 + b(\theta^*)$, where for convenience we write θ^* for $\phi(\theta^*)$. The class of decision rules available, D^*, consists of all measurable functions, $T: \mathscr{X}^* \to \mathscr{A}$. We state for future reference the following obvious results.

LEMMA 3.1. *The equivariant procedure $\hat{G}T(Y)$ is admissible in D_I when loss is measured by $L(\cdot, \cdot)$ if and only if for the reduced problem, $T(Y)$ is admissible in D^*, when loss is measured by $c(\cdot, \cdot)||\cdot - w(\cdot, \cdot)||^2$.*

LEMMA 3.2. *The procedure $\hat{G}T(Y)$ is Bayes in D_I with respect to a prior measure, Π', when loss is measured by $L(\cdot, \cdot)$ if and only if, for the reduced problem, $T(Y)$ is Bayes in D^*, with respect to $\Pi' \circ \tau^{-1}$ when loss is measured by $c(\cdot, \cdot)||\cdot - w(\cdot, \cdot)||^2$.*

Assume in the reduced problem whose structure is supposed in Lemmas 3.1 and 3.2 that Θ^* is a possibly unbounded subinterval of the real line with upper and lower endpoints, θ_l and θ_u, respectively. Let Π' denote a prior probability distribution on Θ and $\Pi = \Pi' \circ \tau^{-1}$ the induced prior probability distribution on Θ^*. Then according to Lemma 3.2 $\hat{G}T(Y)$ is a Bayes procedure with respect to Π' relative to the class of equivariant procedures if

$$T(y) = E(c(y, \theta^*)w(y, \theta^*) \mid Y = y)/E(c(y, \theta^*) \mid Y = y)\,, \tag{3.1}$$

except for values of y in a measurable subset $A \subset \mathscr{X}^*$ for which

$$\int_A \int_\Theta^* p(y \mid \theta^*)\, d\Pi(\theta^*)\, d\rho(y) = 0$$

Here (Y, θ^*) is a random variable with joint distribution determined by $p(y \mid \theta^*)\, d\Pi(\theta^*)\, d\rho(y)$ and $p(y \mid \theta^*)$ is the density of Y with respect to ρ when its distribution is determined by $P_{\phi(\theta^*)}$. More explicitly,

$$p(y \mid \theta^*) = \int_{\mathscr{G}^*} p_{\phi(\theta^*)}(g^*, y)\, dl^*(g^*)\,.$$

It is useful to consider measures, Π, which are required only to be σ-finite and such that $E[\{c(y, \theta^*)||w(y, \theta^*)||^2 + 1\} \mid Y = y] < \infty$, a.e. $[\rho]$. "E" is used here and in the sequel as a convenient notational device, even though (Y, θ^*) is not necessarily a "proper" random variable. For such a measure, Π, we can define an estimator $\hat{G}T(Y)$ through (3.1). Such an estimator will be called a formal Bayes estimator with respect to Π, relative to the class of equivariant estimators.

Let $r^*(T', \cdot): \Theta^* \to [0, \infty)$ denote the risk, in the reduced problem of the estimator, $T' \in D^*$ when loss is measured by $c(\cdot, \cdot)||\cdot - w(\cdot, \cdot)||^2$.

DEFINITION 3.1. An estimator $T' \in D^*$ is called almost admissible with respect to a measure Π, if for any procedure $T^* \in D^*$ for which $r^*(T^*, \cdot) \leqq r^*(T', \cdot)$, $r^*(T^*, \theta^*) = r^*(T', \theta^*)$, a.e. $[\Pi]$.

Lemma 3.1, below, gives a sufficient condition for the almost admissibility of a formal Bayes procedure, T, with respect to Π when Π has a density π. To facilitate the statement of this result, we define a function $M^2: \mathscr{X}^* \times \Theta^* \to [0, \infty)$ by

$$M^2(y, \theta^*) = ||\int_{\theta^*}^{\theta_u} c(y, t)[T(y) - w(y, t)]p(y \mid t)\pi(t)\, dt||^2 \times (c(y, \theta^*)p(y \mid \theta^*)\pi(\theta^*))^{-2}\,.$$

We set M^2 equal to zero when its denominator is zero. Also, let $h(\theta)^* = E[M^2(Y, \theta^*) \mid \theta^*]$. Assume

A1. $\pi(\theta^*)h(\theta^*)$ is bounded away from 0 on compact subsets of Θ^* and
A2. $\{t: p(y \mid t) > 0\}$ is an interval a.e. $[\rho]$.

LEMMA 3.1. *Under assumptions* A1 *and* A2, T *is almost admissible with respect to* Π *if*

$$\int_c^{\theta_u} [\pi(t)h(t)]^{-1}\, dt = \infty \qquad \textit{when} \quad \int_c^{\theta_u} r^*(T, t)\pi(t)\, dt = \infty$$

and

$$\int_{\theta_l}^{c} [\pi(t)h(t)]^{-1}\, dt = \infty \qquad \textit{when} \quad \int_{\theta_l}^{c} r^*(T, t)\pi(t)\, dt = \infty ,$$

where $c \in (\theta_l, \theta_u)$.

PROOF. This theorem is a restatement of Theorem 3.1 ([10]) (the latter result assumes c and w are independent of y but no difficulty is encountered in showing that it applies in the present case as well).

4. Formal Bayes estimators admissible among equivariant estimators. To apply Lemma 3.1, we need to identify the objects appearing in Section 3, in the problem of interest here.

The groups $\mathscr{G}$, $\bar{\mathscr{G}}$, and $\hat{\mathscr{G}}$ are defined by (1.1) and $\mathscr{H} = \{1\}$, the identity element of $\mathscr{G}$. Furthermore:

$$\Theta^* = (-\infty, \infty)\,, \qquad \phi(\theta^*) = (1, \theta^*)\,, \qquad \mathscr{Y}^* = (-\infty, \infty)$$
$$dl^*(g) = dl(g) = dg/g\,, \qquad d\rho(y) = dy\,, \qquad p_{\phi(\theta^*)}(g, y) = f(g, y \mid \theta^*, 1)\,.$$

If we let $c^*(y, \lambda) = \int_0^\infty g p_\lambda(g, y)\, dg$, then

$$w(y, \lambda) = \int_0^\infty p_\lambda(g, y)\, dg / c^*(y, \lambda)\,,$$

and

$$h(\lambda) = \int_{-\infty}^{\infty} c^*(y, \lambda) M^2(y, \lambda)\, dy\,. \tag{4.1}$$

Define $M: (-\infty, \infty)^2 \to (-\infty, \infty)$ by

$$M(y, \lambda) = \int_\lambda^\infty \pi(t) \int_0^\infty (gT(y) - 1) p_t(g, y)\, dg\, dt \{\pi(t) \int_0^\infty g p_\lambda(g, y)\, dg\}^{-1} \tag{4.2}$$

unless $y \leqq 0 < \theta^*$ when the denominator in (4.2) becomes zero and we define $M(y, \lambda)$ to be zero. Then $M^2(y, \lambda) \equiv (M(y, \lambda))^2$.

Finally $T(y)$, defined in (3.1), is given by

$$T(y) = \int_{-\infty}^{\infty} \int_0^\infty p_\lambda(g, y)\, dg \pi(\lambda)\, d\lambda \{\int_{-\infty}^{\infty} \int_0^\infty g p_\lambda(g, y)\, dg \pi(\lambda)\, d\lambda\}^{-1}\,. \tag{4.3}$$

In order to obtain a usable consequence of Lemma 3.1, we assume of π that it satisfies

A3. $\pi(t)/\pi(\theta) \leqq a_1 + a_2|t - \theta|^\alpha$ for some positive constants a_1, a_2, α, and for all θ, t.
A4. $\pi(t\theta)/\pi(\theta) \leqq ct^{-\beta}$ for constants $\beta \leqq 2$ and $c > 0$ and for all θ and $t \in (0, 1)$.
A5. π is non-decreasing (non-increasing) on $(-\infty, 0]$ ($[0, \infty)$).

We note that assumptions A1 and A2 are satisfied in the present case, and in particular that A4 implies $\pi(\lambda) > 0$ for all λ. A family of prior densities $\{\pi_a : a \leqq 2\}$ for which A3, A4, and A5 are satisfied is defined by $\pi_a(\lambda) = (1 + |\lambda|^a)^{-1}$.

The proof of the following lemma is straightforward.

LEMMA 4.1. *If assumptions* A3–A5 *are satisfied and T is defined as in* (4.2), *then T is bounded.*

LEMMA 4.2. *Suppose assumptions* A3 A5 *are satisfied. Then T, defined in* (4.2), *is almost admissible with respect to* Π *provided*

$$\int_1^\infty [\lambda^2\pi(\lambda)]^{-1}\,d\lambda = \int_{-\infty}^{-1} [\lambda^2\pi(\lambda)]^{-1}\,d\lambda = \infty\,.$$

PROOF. The proof consists of showing that for any function, π, satisfying A3–A5

$$M^2(y,\lambda) \leqq K'(1+y^2)\,, \qquad -\infty < y < \infty,\ |\lambda| > 1\,, \tag{4.4}$$

for some constant $K' > 0$. It follows that for $|\lambda| > 1$,

$$\frac{h(\lambda)}{\lambda^2} \leqq K \int_0^\infty \int_0^\infty g^{n+1} \exp[-ng(1+y)](1+[y+1/g]^2)\,dy\,dg$$

or

$$h(\lambda) \leqq K\lambda^2\,, \qquad |\lambda| > 1\,,$$

where $K > 0$ is used here and in the sequel to denote a generic constant whose precise value is of no relevance to the argument. The conclusion is then a consequence of Lemma 3.1.

Evaluating $w(\cdot,\cdot)$ explicitly we obtain

$$\begin{aligned} w(y,t) &= \int_{t/y}^\infty g^n \exp[-ng(1+y)]\,dg \Big/ \int_{t/y}^\infty g^{n+1}\exp[-ng(1+y)]\,dg\,, \\ &\qquad\qquad t>0,\ y>0 \\ &= n(1+y)/(n+1)\,, \qquad t \leqq 0,\ y \geqq 0 \\ &= \int_0^{t/y} g^n \exp[-ng(1+y)]\,dg \Big/ \int_0^{t/y} g^{n+1}\exp[-ng(1+y)]\,dg\,, \\ &\qquad\qquad t \leqq 0,\ y \leqq 0\,. \end{aligned}$$

It is easily shown that $w(y,\cdot)$ is strictly decreasing (increasing) on $[0,\infty)$ ($(-\infty,0]$) when $y > 0$ ($y < 0$).

From (4.3) it follows that if $y > 0$

$$T(y) = \int_{-\infty}^\infty \int_{H(t/y)}^\infty p_t(g,y)\pi(t)\,dg\,dt \Big\{\int_{-\infty}^\infty \int_{H(t/y)}^\infty g p_t(g,y)\pi(t)\,dg\,dt\Big\}^{-1}\,, \tag{4.5}$$

where $H(u) = \max(0,u)$, $-\infty < u < \infty$, and if $y < 0$

$$T(y) = \int_{-\infty}^0 \int_0^{t/y} p_t(g,y)\pi(t)\,dg\,dt\Big\{\int_{-\infty}^0 \int_0^{t/y} g p_t(g,y)\pi(t)\,dg\,dt\Big\}^{-1}\,. \tag{4.6}$$

Equivalently, (4.5) and (4.6) may be written in the form of the following useful identities:

$$\int_{-\infty}^\infty \pi(t)c^*(y,t)[T(y)-w(y,t)]\,dt = 0\,, \qquad y > 0 \tag{4.7}$$

$$\int_{-\infty}^0 \pi(t)c^*(y,t)[T(y)-w(y,t)]\,dt = 0\,, \qquad y < 0\,. \tag{4.8}$$

Since $T(y) - w(y,\cdot)$ is strictly increasing on $[0,\infty)$, for $y > 0$, and constant (>0) on $(-\infty, 0]$ we conclude from (4.7) that $T(y) - w(y,\cdot)$, for $y > 0$, must have a unique zero, say y^*, in $(0,\infty)$, that $T(y) - w(y,t) > 0$ (< 0) for $t > y^*$

($t < y^*$) and consequently that

$$\int_\lambda^\infty \pi(t)c^*(y, t)[T(y) - w(y, t)]\,dt > 0\,, \qquad y > 0,\ -\infty < \lambda < \infty\,. \tag{4.9}$$

Similar reasoning implies

$$\int_{-\infty}^\lambda \pi(t)c^*(y, t)[T(y) - w(y, t)]\,dt > 0\,, \qquad y < 0,\ -\infty < \lambda < 0\,. \tag{4.10}$$

By (4.2), and inequalities (4.9) and (4.11),

$$\begin{aligned}\pi(\lambda)c^*(y, \lambda)|M(y, \lambda)| &= \int_\lambda^\infty \pi(t)c^*(y, t)[T(y) - w(y, t)]\,dt\,, \\ &\qquad\qquad y > 0,\ -\infty < \lambda < \infty \\ &= 0\,, \qquad \lambda \geqq 0\,,\ y < 0 \\ &= \int_{-\infty}^\lambda \pi(t)c^*(y, t)[T(y) - w(y, t)]\,dt\,, \\ &\qquad\qquad y < 0,\ -\infty < \lambda < \infty\,.\end{aligned} \tag{4.11}$$

That $\pi(\lambda)c^*(y, \lambda)M(y, \lambda) = 0$ for $\lambda \geqq 0$, $y < 0$ is a consequence of $c^*(\lambda, y) = 0$ (in this case) and the definition of M.

The proof of inequality (4.4) is carried out by considering the separate cases which naturally arise from the requirement that $gy > \lambda$ in order that $p_\lambda(g, y) > 0$. These cases are, (i): $\lambda > 1, y > 0$, (ii): $\lambda < -1, y > 0$, and (iii): $\lambda < -1, y < 0$. For technical reasons (iii) is most easily handled by considering the subcases (iii a): $1 + y \geqq 0$, $\lambda < -1$, (iii b): $1 + y < 0$, $\lambda < -1$, $|\lambda/y||1 + y| < \frac{1}{2}$, and (iii c): $1 + y < 0$, $\lambda < -1$, $|\lambda/y||1 + y| > \frac{1}{2}$. The proof in case (iii c) is similar to that involved in case (i). Whereas in case (i) we rely on identity (4.7), (iii c) requires (4.8). Cases (ii), (iii a), and (iii b) are straightforward so for brevity we present only the proof of case (i).

For convenience we define a function $G: (-\infty, \infty)^2 \to [0, \infty)$ by

$$G(y, \lambda) = \pi(\lambda)c^*(y, \lambda)|M(y, \lambda)|\,. \tag{4.12}$$

Equation (4.11) implies

$$G(y, \lambda) = \int_\lambda^\infty \pi(t)\{T(y) \int_{t/y}^\infty g p_t(g, y)\,dg - \int_{t/y}^\infty p_t(g, y)\,dg\}\,dt\,.$$

After integrating by parts the first integral which appears in the integrand of the last expression, we obtain

$$\begin{aligned}G(y, \lambda) = &\frac{T(y)}{n(1 + y)} \int_\lambda^\infty \pi(t)e^{-nt/y}(t/y)^{n+1}\,dt \\ &+ \left(\frac{(n + 1)T(y)}{n(1 + y)} - 1\right) \int_\lambda^\infty \pi(t) \int_{t/y}^\infty p_t(g, y)\,dg\,dt\,.\end{aligned}$$

Since π is non-increasing on $[0, \infty)$ and $T(y) \leqq n(1 + y)/(n + 1)$ for all $y > 0$,

$$G(y; \lambda) \leqq yT(y)\pi(\lambda) \int_{\lambda/y}^\infty e^{-nt}t^{n+1}\,dt\,. \tag{4.13}$$

But

$$\begin{aligned}c^*(y, \lambda) &= e^{n\lambda} \int_{\lambda/y}^\infty \exp[-ng(1 + y)]g^{n+1}\,dg \\ &= e^{-n\lambda/y} \int_0^\infty \exp[-ng(1 + y)](g + \lambda/y)^{n+1}\,dg\end{aligned}$$

so that

$$c^*(y, \lambda) \geqq e^{-n\lambda/y}(\lambda/y)^{n+1}[n(1+y)]^{-1}. \tag{4.14}$$

Together inequalities (4.13) and (4.14) imply

$$|M(y, \lambda)| \leqq yT(y) \int_{\lambda/y}^{\infty} e^{-nt}t^{n+1}\, dt\{e^{-n\lambda/y}(\lambda/y)^{n+1}\}^{-1}. \tag{4.15}$$

But by Lemma 4.1, T is bounded and hence, inequality (4.15) implies

$$|M(y, \lambda)| \leqq Ky, \qquad \lambda > 1,\ \lambda/y > 1. \tag{4.16}$$

Using identity (4.7) a bound for M will be obtained in the case where $\lambda/y \leqq 1$. From this identity we deduce that

$$G(y, \lambda) = \int_{-\infty}^{\lambda} \pi(t) \int_{H(t/y)}^{\infty} p_t(g, y)[1 - gT(y)]\, dg\, dt.$$

As a result, if $c' = n! \int_{-\infty}^{0} e^{nt}\pi(t)\, dt$ and $\omega = \lambda/y$,

$$G(y, \lambda) \leqq c'[n(1+y)]^{-(n+1)} + \frac{\lambda}{n(1+y)} \int_0^1 \pi(\lambda t)e^{-n\omega t} \int_0^{\infty} e^{-g}\left(\frac{g}{n(1+y)} + \omega t\right)^n dg\, dt.$$

Thus,

$$G(y, \lambda) \leqq c_1(\omega/\lambda)^{n+1} + \omega/n[c_2(\omega/\lambda)^n \int_0^1 \pi(\lambda t)e^{-n\omega t}\, dt + c_3\omega^n \int_0^1 \pi(\lambda t)e^{-n\omega t}t^n\, dt], \tag{4.17}$$

for some positive constants c_i, $i = 1, 2, 3$ which depend on n. Since π is bounded and, by A4, $\pi(\lambda t) \leqq c\pi(\lambda)t^{-\beta}$, $0 \leqq t \leqq 1$, inequality (4.17) implies

$$G(y, \lambda) \leqq K\omega^{n+1}[\lambda^{-(n+1)} + \lambda^{-n} + \pi(\lambda)].$$

But A4 also implies $\pi(1) \leqq c\pi(\lambda)\lambda^{\beta}$, $\lambda > 1$. Thus, $G(y, \lambda) \leqq K\omega^{n+1}\pi(\lambda)$. Combining the last inequality with (4.14) we obtain

$$|M(y, \lambda)| \leqq K(1+y), \qquad \lambda/y \leqq 1,\ \lambda > 1. \tag{4.18}$$

Together (4.16) and (4.18) imply (4.4).

THEOREM 4.1. *If T is defined by* (4.2) *and the hypotheses of Lemma* 4.2 *are satisfied, then $GT(Y)$ is admissible in D_I as an estimator of σ.*

PROOF. Suppose $GT(Y)$ is not admissible in D_I. By Lemma 3.1, T is not admissible in D^*. Therefore, there exists $T^* \in D^*$ such that $r^*(T^*, \lambda) \leqq r^*(T, \lambda)$, $\lambda \in \Theta^*$ with actual inequality for at least one point, say $\lambda = \lambda_0$. Let $B = \{y \in \mathscr{Y}^*: T^*(y) \neq T(y)\}$. It follows that $\int_B p(y \mid \lambda_0)\, dy > 0$.

Define an estimator ψ by $\psi = \frac{1}{2}T^* + \frac{1}{2}T$. Then $(\psi(y) - w(y, \lambda))^2 < \frac{1}{2}(T^*(y) - w(y, \lambda))^2 + \frac{1}{2}(T(y) - w(y, \lambda))^2$, $y \in B$, $\lambda \in \Theta^*$ and $(\psi(y) - w(y, \lambda))^2 = (T(y) - w(y, \lambda))^2$, $\lambda \in \Theta^*$, $y \in B^c$, the complement of B.

Now $r^*(T, \lambda) - r^*(\psi, \lambda) = \int_B c(y, \lambda)\{(T(y) - w(y, \lambda))^2 - (\psi(y) - w(y, \lambda))^2\} \times p(y \mid \lambda)\, dy$. Let

$$\Lambda = \{\lambda: \int_B p(y \mid \lambda)\, dy > 0\}.$$

Then if $\Pi(\Lambda) > 0$, the conclusion of Lemma 4.2 is contradicted. But, with $C_\lambda = \{(g, y) \in \mathscr{X} : gy > \lambda\}$,

$$\int_B p(y \mid \lambda_0)\, dy = Ke^{n\lambda_0} \int_{(\mathscr{G}\times B)\cap C_{\lambda_0}} \int g^{n-1} \exp[-ng(1+y)]\, dg\, dy > 0 ;$$

Since $C_{\lambda_0} \subset C_\lambda$, $\lambda \leqq \lambda_0$, $\int_B p(y \mid \lambda)\, dy > 0$, $\lambda \leqq \lambda_0$. This implies $\Pi(\Lambda) > 0$ and the proof is complete.

5. Numerical results. In this section we evaluate numerically, the risk function of the minimax estimator obtained in Section 2 under quadratic loss and of each of three members of the class of scale admissible equivariant estimators obtained in Section 4. This evaluation is carried out when $n = 4$ and $-2 \leqq \mu/\sigma \leqq 2$.

Consider the subclass of all possible choices of π given by $\{\pi_{a,b} : a > 0,\ b > 0\}$ where $\pi_{a,b}(t)$ is $(1 + a|t|)^{-1}$ or $(1 + b|t|)^{-1}$ according as $t \geqq 0$ or $t < 0$. Denote by $T_{n,a,b}$, the function obtained from the (4.3) by setting $\pi = \pi_{a,b}$.

In Figure 1 we represent $T_{n,a,b}$ for $n = 4$ and various values of a and b. Also represented there is $T_4{}^* = T^*$, the function obtained in Section 2 when $T(y) \equiv 1$. Figure 2 indicates the manner in which $T_{n,4,4}$ varies with n. Finally in Figure 3 the risk functions for $T_{4,a,b}$, $(a, b) = (4, 1)$, $(4, 4)$, $(20, 200)$, and $T_4{}^*$ are represented.

It is clear at a glance that use of the preliminary test estimator of Section 3 will not result in a large reduction in mean square error. At best the reduction is about 6% and this is achieved when μ/σ is about .2.

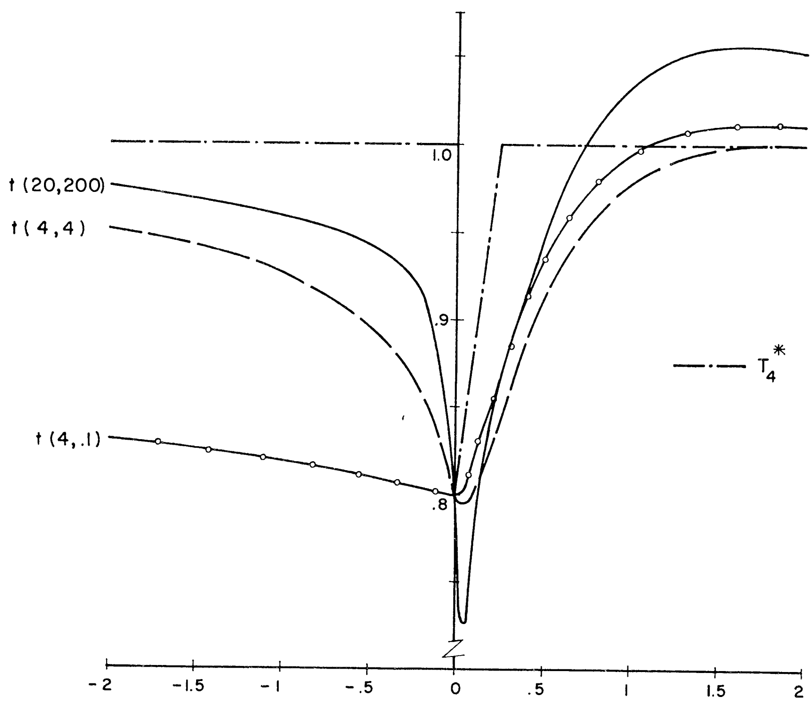

FIG. 1. $T_n{}^*$ and $T_{n,a,b} = t(a, b)$ for $n = 4$ and $(a, b) = (4, .1), (4, 4), (20, 200)$.

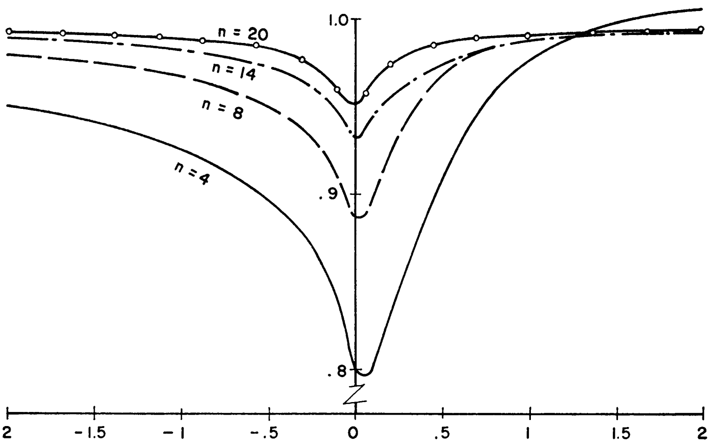

FIG. 2. $T_{n,4,4}$ for $n = 4, 8, 14, 20$.

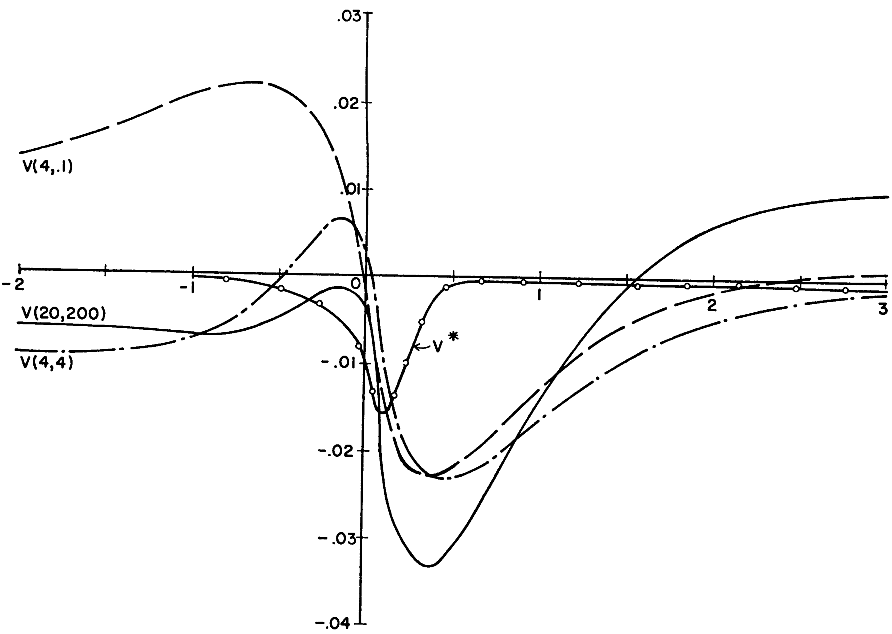

FIG. 3. $V^* = r(T_n^*, \lambda) - \frac{1}{4}$ and $v(a, b) = r(T'_{n,a,b}, \lambda) - \frac{1}{4}$ when $n = 4$ for various values of $(a, b) = (4, .1), (20, 200), (4, 4)$.

Arnold [2] is incorrect in asserting that for any n, the maximum improvement occurs when $\mu = 0$ as can readily be seen by examining his computed risk for the case $n = 2$. There the maximum relative reduction occurs when μ/σ is $\frac{5}{6}$. The maximum relative reduction itself is about 9%. Using the admissible equivariant estimator $T_{n,4,4}$, when $n = 4$, will result in a maximum relative reduction of about 9% when μ/σ is about .5. On the other hand this estimator is not minimax and there is a relative loss of about 2.5% when μ/σ is about $-.2$.

6. Computing $T_{n,a,b}$. We adopt the following notation:

$$V(x, r; \pi) = [(r-1)!]^{-1} \int_0^\infty \exp(-t) t^{r-1} \pi(xt)\, dt \qquad -\infty < x < \infty$$
$$V^*(y, r) = [(r-1)!]^{-1} \int_0^\infty \exp(-t) t^{r-1} (1+yt)^{-1}\, dt \qquad y \geqq 0\,.$$

Denote by $T_{n,\pi}$, the function given in (4.3).

Then for $-\infty < y < \infty$,

$$T_{n,\pi}(y) = n(1+y)(n+1)^{-1}[1 + y(1+y)^{n+1} \times V(y/n, n+2; \pi)\{\mathsf{U}(n, y; \pi)\}^{-1}]^{-1} \tag{6.1}$$

where

$$\mathsf{U}(n, y; \pi) = V(-n^{-1}, 1; \pi) + y \textstyle\sum_{k=0}^{n} (1+y)^k V(y/n, k+1; \pi)\,.$$

Moreover, if $-2 < y < -\frac{1}{2}$, $T_{n,\pi}(y)$ can be approximated, to any prescribed degree of accuracy by choosing $K = 0, 1, 2, \cdots$ large enough in

$$n(n+1)^{-1}[1 + y + V(y/n, n+2; \pi)/W_K(n, y; \pi)] \tag{6.2}$$

where

$$W_K(n, y; \pi) = \textstyle\sum_{k=n+2}^{n+2+K} (1+y)^{k-r-1} V(y/n, k+1; \pi)\,.$$

Now (6.1) and (6.2) may be applied for any choice of π satisfying the conditions determined in Section 4. But difficulty may be encountered in computing the function V which appears in both of these expressions. We describe below; methods which may be used when $\pi = \pi_{a,b}$ as defined in Section 5.

Observe that

$$\begin{aligned} V(x, r; \pi_{a,b}) &= V^*(ax, r)\,, & x \geqq 0 \\ &= V^*(b|x|, r)\,, & x < 0\,. \end{aligned} \tag{6.3}$$

Moreover

$$V^*(y, 1) = y^{-1} \exp(y^{-1}) Ei(y^{-1})$$

where $Ei(x)$ denotes the exponential integral, $\int_x^\infty \exp(-t) t^{-1}\, dt$, $x > 0$. Thus with the help of existing tables (for example, [1]), $V^*(y, 1)$ is easily computed. If $y > 1$ $V^*(y, r)$ can be computing recursively without roundoff or loss of significant figures due to cancellation from

$$V^*(y, r) = [(r-1)y]^{-1}[1 - V^*(y, r-1)]\,, \qquad r = 2, 3, \cdots, y > 1\,. \tag{6.4}$$

When $0 \leqq y \leqq 1$ the recursive method suggested above is unsatisfactory. An approximation, $V_m{}^*(y, r)$, to $V^*(y, r)$ based on a continued fractional expansion of V^* is preferable. It is

$$V^*(y, r) = \frac{1}{1+} \cdots \frac{(m-1)y}{1+} \frac{(r+m-1)}{1+} myf_m(y) \tag{6.5}$$

where $0 \leqq y \leqq 1$, and

$$f_m(y) = [2(m+1)y]^{-1}[1 - (r-1)y + \{[1-(r-1)y]^2 + 4(m+1)y\}^{\frac{1}{2}}]\,.$$

Numerical results indicate that $|V^*(y, r) - V_m{}^*(y, r)| < 10^{-6}$ if $m = 9, 0 \leqq y \leqq$

.5 and $r > 4$, if $m = 13$, $.5 < y \leqq 1$ and $r > 6$, or, if $m = 21$, $.5 < y \leqq 1$ and $r \leqq 6$.

In summary the computational procedure calls for the use of (6.1) if $y \notin (-2, -\frac{1}{2})$ and (6.2) if $y \in (-2, -\frac{1}{2})$. The required values of V are obtained using (6.3). This necessitates the computation of $V^*(z, r)$ for certain values of $z > 0$ and r. If $z > 1$, (6.4) may be employed to that end, while if $0 < z \leqq 1$, (6.5) may be used.

Our numerical results indicate that the procedure given above, while generally satisfactory, may encounter difficulties if $-2 < y < -\frac{3}{2}$ or $-\frac{3}{4} < y < 0$. There (6.1) may fail because of a loss of significant figures due to cancellation and (6.2) may be unsatisfactory because an unduly large number of terms is required.

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