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IMPROVING ON THE JAMES–STEIN POSITIVE-PART ESTIMATOR

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The purpose of this paper is to give an explicit estimator dominating the positive-part James–Stein rule.

The James–Stein estimator improves on the "usual" estimator X of a multivariate normal mean vector θ if the dimension p of the problem is at least 3. It has been known since at least 1964 that the positive-part version of this estimator improves on the James–Stein estimator. Brown's 1971 results imply that the positive-part version is itself inadmissible although this result was assumed to be true much earlier. Explicit improvements, however, have not previously been found; indeed, 1988 results of Bock and of Brown imply that no estimator dominating the positive-part estimator exists whose unbiased estimator of risk is uniformly smaller than that of the positive-part estimator.

1. Introduction. This paper gives explicit improvements to the James–Stein positive-part estimator of a multivariate normal mean vector.

Let $X \sim N_p(\theta, I)$ and $\delta(X)$ be an estimator of the mean vector θ with loss

$$L(\theta,\delta) = \|\delta - \theta\|^2. \tag{1.1}$$

The estimator $\delta_0(X) = X$ is the MLE, uniformly minimum variance unbiased estimator and is minimax. It is also admissible if $p = 1$ and 2. If $p \geq 3, X$ is inadmissible [Stein (1956)] and a well-known dominating estimator is the James–Stein estimator [James and Stein (1961)]

$$\delta^{\mathrm{JS}}(X) = \left(1 - \frac{p-2}{\|X\|^2}\right)X. \tag{1.2}$$

Baranchik (1964, 1970) showed that the James–Stein positive-part estimator

$$\delta_+^{\mathrm{JS}}(X) = \left(1 - \frac{p-2}{\|X\|^2}\right)_+ X \tag{1.3}$$

dominates (1.2), where

$$(a)_+ = \begin{cases} a, & \text{if } a \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It has long been known that the positive-part estimator (1.3) is itself inadmissible [see, e.g., Brown (1971)]. Bock (1988) and Brown (1988) showed that there

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does not exist an estimator whose unbiased estimator of risk [Stein (1981)] is everywhere less than that of the positive-part estimator. These results imply, in a technical sense at least, that improving on the positive-part estimator is difficult and that the now standard method in the normal theory case [the unbiased estimator of risk given by Stein (1981)] will not work in this case. The results, however, also help in directing attention to alternative approaches. Results of Chow (1987) and Chow and Hwang (1982) for the estimation of the noncentrality parameter of a noncentral chi-square also had an impact on our search for an improved estimator. Their results essentially indicated that an improving estimator had to "wiggle" sufficiently about the estimator to be improved. The close connections between their problem and the present one indicated that something similar would hold in our case as well.

Our results fall into two broad classes. The first set of results describe explicit estimators which change $\delta_+^{\mathrm{JS}}(X)$ only on the set $p-2 \le \|X\|^2 \le p$. These results depend on some properties of central chi-square distributions with degrees of freedom greater than or equal to p. These results and other preliminary results are given in Section 2. In Section 3, we will study alternative estimators of the form

$$\delta(a,g,X) = \delta_+^{\mathrm{JS}}(X) - \frac{ag(\|X\|^2)}{\|X\|^2} X I_{\{p-2 \le \|X\|^2 \le \|p\}}, \tag{1.4}$$

where $g(\cdot)$ is an even symmetric piecewise linear function about $\|X\|^2 = p-1$ with $g(p-2) = g(p) = 0$, $g'(p-2) < 0$ and $|g'(t)| \equiv 1$ a.e. on $[p-2, p]$. Hence, the simplest function $g(\cdot)$ is "W"-shaped on $[p-2, p]$. To specify $\delta(a,g,X)$ fully, it then suffices to specify the constant a and a value p^* in $(p-1, p)$ such that $g(p^*)$ attains its minimum value. Values of a and p^* are given such that $\delta(a,g,X)$ dominates $\delta_+^{\mathrm{JS}}(X)$. In Section 4, we study estimators of the form (1.4) for certain non-piecewise-linear g. It is easy to see that estimators of the form (1.4) cannot themselves be admissible. Therefore we investigate a more general class of estimators in Section 5.

This second class of estimators is of the form

$$\delta(X) = \delta_+^{\mathrm{JS}}(X) - a\left[\frac{g(\|X\|^2)}{\|X\|^2} I_{\{\|X\|^2 \ge q\}} + kh(\|X\|^2) I_{\{\|X\|^2 < q\}}\right] X. \tag{1.5}$$

We give conditions on a, k, q, g and h such that (1.5) dominates $\delta_+^{\mathrm{JS}}(X)$. In this case, $g(\cdot)$ will be "W"-shaped on $[q, \infty)$, and $h(\cdot)$ will be bounded continuous and nonpositive. We believe but have not proved that admissible improvements can be found in this class.

Some proofs of technical results are given in the Appendix.

2. Preliminaries. Our first result in this section is a lemma analogous to Stein's (1981) lemma for the evaluation of expectation of cross products appearing in risk functions. Its proof is straightforward and is omitted.

LEMMA 2.1. *Let $X \sim N_p(\theta, I)$; let $H(\cdot)$ be a continuous function on $[a,b]$; and let $H'(\cdot)$ have at most a finite number of discontinuities $0 \le a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$. If, for $i = 0, \ldots, k+1$, both $H'(a_i^+)$ and $H'(a_i^-)$ are finite, then*

$$\begin{aligned} &E(X-\theta)'XH\big(\|X\|^2\big)I_{\{a<\|X\|^2<b\}} \\ &\quad = E\Big[pH\big(\|X\|^2\big) + 2\|X\|^2H'\big(\|X\|^2\big)\Big]I_{\{a<\|X\|^2<b\}} + f(a) - f(b), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} g_1(c) = {}& \int_0^{\pi/2}\cdots\int_0^{\pi/2}(\cos\alpha)^2(\sin\alpha)^{p-2}\Big(\exp\big(\|\theta\|c\sin\alpha\cos\phi_1\big) \\ &+ \exp\big(-\|\theta\|c\sin\alpha\cos\phi_1\big)\Big)(\sin\phi_1)^{p-3}\cdots\sin\phi_{p-3}\,d\alpha\,d\phi_1\cdots d\phi_{p-2}, \\ g_2(c) = {}& \int_0^{\pi/2}\cdots\int_0^{\pi/2}(\cos\alpha)^2(\sin\alpha)^{p-2}\Big(\exp\big(\|\theta\|c\cos\alpha\big) + \exp\big(-\|\theta\|c\cos\alpha\big)\Big) \\ &\times(\sin\phi_1)^{p-3}\cdots\sin\phi_{p-3}\,d\alpha\,d\phi_1\cdots d\phi_{p-2}; \\ K_1 = {}& \left(\frac{1}{\sqrt{2\pi}}\right)^p 2^{p-1}H(a)(\sqrt{a})^p\exp\left(-\frac{\theta'\theta + a}{2}\right), \\ K_2 = {}& \left(\frac{1}{\sqrt{2\pi}}\right)^p 2^{p-1}H(b)(\sqrt{b})^p\exp\left(-\frac{\theta'\theta + b}{2}\right); \\ f(a) = {}& K_1g_1(\sqrt{a}) + (p-1)K_1g_2(\sqrt{a}), \\ f(b) = {}& K_2g_1(\sqrt{b}) + (p-1)K_2g_2(\sqrt{b}). \end{aligned}$$

Note that if $H(a) = H(b) = 0$, then $f(a) = f(b) = 0$ and the lemma essentially reduces to Stein's lemma for a spherically symmetric function $H(\cdot)$.

If X is as in Lemma 2.1, then $T = \|X\|^2$ has a noncentral chi-square distribution with p degrees of freedom and noncentrality parameter $\theta'\theta/2$. The density of T is well known [see, e.g., Stein (1956)] to be

$$\exp\left(-\frac{\theta'\theta}{2}\right)\sum_{j=0}^{\infty}\frac{\exp(-t/2)\,t^{p/2+j-1}\big(\theta'\theta/2\big)^j}{j!2^{p/2+j}\,\Gamma\big(p/2+j\big)}, \tag{2.2}$$

that is, a mixture of central chi-square distributions with $p + 2L$ degrees of freedom, where L has a Poisson$(\theta'\theta/2)$ distribution.

The next series of lemmas have to do with properties of functions $F_n(t)$ defined in Lemma 2.2. These properties play a crucial role in the development of Section 3 and may be of some independent interest. The functions $F_n(t)$ are essentially (modulo constants) $f_n(t) - f_n(2(p-1)-t)$ restricted to $t \in [p-1, p]$, where $f_n(t)$ is the density of a central chi-square with $p + 2n$ degrees of freedom.

Let $s(t) = 2(p-1) - t$. In this paper, if there is no possible confusion, we denote $s(t)$ by s.

LEMMA 2.2. *Let $p \geq 3$; let*

$$F_n(t) = \exp\left(-\frac{t}{2}\right)t^{p/2+n-1} - \exp\left(-\frac{s(t)}{2}\right)s(t)^{p/2+n-1}; \tag{2.3}$$

and let the domain of $F_n(t)$ be $[p-1, p]$. For $n = 0, 1, \ldots$, the following properties hold (they hold with strict inequalities if $t \in (p-1, p]$):

(i) $F_0(t) \leq 0, F_0'(t) < 0, F_0''(t) \geq 0$;

(ii) $F_n(t) \geq 0, F_n'(t) > 0, F_n(t)/F_0(t) < 0,$ *for all* $n \geq 1$;

(iii) $F_1''(t) \geq 0, F_2''(t) \leq 0$;

(iv) $[F_1(t)/F_0(t)]' \leq 0$.

See the Appendix for the proof.

LEMMA 2.3. *For any integer $p > 2$, there exists an integer $j(p)$ (given in the proof) such that, for all $n \geq j(p)$, $t \in [p-1, p]$,*

$$F_n''(t) \geq 0.$$

See the Appendix for the proof.

LEMMA 2.4. *If $0 < b < \frac{1}{2}$, and $n \geq p - 1$, then $e((p-2+b)/(p-b))^n$ is monotone decreasing in n, and*

$$e\left(\frac{p-2+b}{p-b}\right)^n < 1. \tag{2.4}$$

PROOF. Since $0 < (p-2+b)/(p-b) < 1$,

$$e\left(\frac{p-2+b}{p-b}\right)^n < e\left(\frac{p-1.5}{p-0.5}\right)^{p-1}.$$

Since

$$\exp\left(\frac{1}{p-1}\right)(p-1.5) = p - 0.5 + \sum_{i=2}^{\infty} \frac{\left(1/(p-1)\right)^i \cdot \left(1/(i+1) - (1/2)\right)}{i!} < p - 0.5,$$

hence (2.4) is true. □

Next we give some recursive formulas for F_n, for $n \geq 2$.

LEMMA 2.5.

$$F_{n+1}(t) = 2(p-1)F_n(t) - tsF_{n-1}(t); \tag{2.5}$$

$$F_{n+1}(t) - tF_n(t) = s\left(F_n(t) - tF_{n-1}(t)\right) = \cdots = s^n\left(F_1(t) - tF_0(t)\right); \tag{2.6}$$

$$\frac{F_{n+1}(t)}{F_0(t)} = \frac{t^{n+1} - s^{n+1}}{t-s}\frac{F_1(t)}{F_0(t)} - ts\frac{t^n - s^n}{t-s}. \tag{2.7}$$

See the Appendix for the proof.

LEMMA 2.6. *Let $t \in [p-1, p]$. Then $(t^m - s^m)/(t-s)$ are positive and monotone nondecreasing for all $m \geq 1$.*

See the Appendix for the proof.

The next two results show that $F_n(t)/F_0(t)$ is monotone decreasing in t for all $n \geq 2$. We already know this for $n = 1$ by Lemma 2.2(iv).

LEMMA 2.7. *Let $t \in [p-1, p]$. Then $F_2(t)/F_0(t)$ is a monotone decreasing function of t.*

See the Appendix for the proof.

A key result of this section is the following theorem.

THEOREM 2.1. *Let $t \in [p-1, p]$. For all $j > 1$, $F_j(t)/F_0(t)$ are monotone decreasing functions of t.*

See the Appendix for the proof.

The next lemmas use these properties of the $F_j(t)$'s to establish inequalities used in the remainder of the paper.

LEMMA 2.8. *There exists a point c_0 in $(p-1, p-1+\sqrt{2}/2)$ such that*

$$\int_{p-1}^{c_0} F_0(t)\,dt = \int_{c_0}^{p} F_0(t)\,dt.$$

PROOF. Since, by Lemma 2.2, $F_0(t)$ is convex, negative and decreasing with $F_0(p-1) = 0$, the result follows easily by comparison with a straight line through $(p-1, F_0(p-1))$ and $(p-1+\sqrt{2}/2, F_0(p-1+\sqrt{2}/2))$. $\square$

LEMMA 2.9. *Let $c_j \in (p-1, p)$ be such that $\int_{p-1}^{c_j} F_j(t)\,dt = \int_{c_j}^{p} F_j(t)\,dt$. Then $\min_{j \geq 1} c_j > c_0$ and $\min_{j \geq j(p)} c_j > p - 1 + \sqrt{2}/2$.*

PROOF. By Theorem 2.1, for all $j \geq 1$, $\alpha_j(t) = F_j(t)/F_0(t)$ is negative and strictly decreasing on $(p-1, p]$. Hence

$$-\int_{p-1}^{c_0} F_j(t)\,dt + \int_{c_0}^{p} F_j(t)\,dt > -\alpha_j(c_0)\int_{p-1}^{c_0} F_0(t)\,dt + \alpha_j(c_0)\int_{c_0}^{p} F_0(t)\,dt = 0. \tag{2.8}$$

Since $F_j(t) > 0$, so $c_j > c_0$ for all $j \geq 1$.

Since, by Lemma 2.3, $F_j(t)$ is convex, positive and increasing with $F_j(p-1) = 0$, the result for $j \geq j(p)$ follows readily by comparison with a straight line through $(p-1, F_j(p-1))$ and $(p-1+\sqrt{2}/2, F_j(p-1+\sqrt{2}/2))$. $\square$

3. A class of improved estimators. In this section we use the results of Section 2, particularly Theorem 2.1, to find classes of estimators dominating $\delta_+^{\mathrm{JS}}(X)$ for the loss (1.1). We consider estimators of the form

$$\delta(a,g,X) = \delta_+^{\mathrm{JS}}(X) - \frac{ag(\|X\|^2)}{\|X\|^2} X I_{\{p-2 \le \|X\|^2 \le p\}}, \tag{3.1}$$

where $g(t)$ is an even symmetric function around the point $t = p - 1$ such that $g(p-2) = g(p) = 0$.

In our first result, $g(\cdot)$ is taken to be piecewise linear.

THEOREM 3.1. *Let $p \ge 3$, let $j(p)$ be as in Lemma 2.3, let c_0 be as in Lemma 2.8 and let p^* be any value in $(c_0, \min(c_1, c_2, \dots, c_{j(p)-1}, p - 1 + \sqrt{2}/2))$ such that $b = 1 - \sqrt{2}(p^* - (p-1)) < \min[\frac{1}{2}, p - p^*]$. Define*

$$g(t) = \begin{cases} t - p, & \text{if } p^* \le t \le p, \\ 2p^* - p - t, & \text{if } p - 1 \le t < p^*, \end{cases}$$

and extend the definition of $g(t)$ to $[p-2, p-1)$ so that $g(t)$ is symmetric about $t = p - 1$.

Then $\delta(a,g,X)$ [given by (3.1)] dominates $\delta_+^{\mathrm{JS}}(X)$ provided $0 < a < \min\{B, 2(p-2)bA\}$, where

$$A = 1 - \exp(1-b)\left(\frac{p-2+b}{p-b}\right)^{p/2+j(p)-1} \tag{3.2}$$

and

$$B = \min\left\{ \frac{4\int_{p-2}^{p} g'(t)\exp(-t/2)\, t^{p/2+j-1}\,dt}{\int_{p-2}^{p} g^2(t)\exp(-t/2)\, t^{p/2+j-2}\,dt} \text{ for } j = 0, 1, \dots, j(p) - 1 \right\}. \tag{3.3}$$

Note that A and B are both positive, as will be shown in the proof.

PROOF OF THEOREM 3.1. The difference in risk between $\delta(a,g,X)$ and $\delta_+^{\mathrm{JS}}(X)$ is given by

$$\Delta R(\theta) = R(\delta(a,g,X), \theta) - R(\delta_+^{\mathrm{JS}}(X), \theta).$$

Let $t = \|X\|^2$. Since $g(t)$ is symmetric and continuous on $[p-2, p]$, $g'(t)$ is piecewise continuous on $[p-2, p]$ and $g(p-2) = g(p) = 0$, according to Lemma 2.1, we have

$$\Delta R(\theta) = E\frac{a^2 g^2(t) - 4atg'(t)}{t} I_{p-2 \le t \le p}. \tag{3.4}$$

A necessary condition that $\Delta R(\theta) \le 0$ is, for all θ,

$$Eg'(t) I_{\{p-2 \le t \le p\}} > 0. \tag{3.5}$$

By (2.2),

$$Eg'(t)I_{\{p-2\le t\le p\}}$$
$$= \sum_{j=0}^{\infty}\exp\left(\frac{-\theta'\theta}{2}\right)\frac{(\theta'\theta/2)^j}{j!2^{p/2+j}\Gamma(p/2+j)}\int_{p-2}^{p} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}dt.$$

One sufficient condition for (3.5) is that, for all $j\ge 0$,

$$\int_{p-2}^{p} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}dt > 0. \tag{3.6}$$

Since $g(t)$ is symmetric about $t=p-1$,

$$\int_{p-2}^{p} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}dt = \int_{p-1}^{p} g'(t)F_j(t)dt, \tag{3.7}$$

where

$$F_j(t) = \exp\left(-\frac{t}{2}\right)t^{p/2+j-1} - \exp\left(-\frac{2(p-1)-t}{2}\right)(2(p-1)-t)^{p/2+j-1}.$$

Also, since

$$g'(t) = \begin{cases} +1, & \text{if } p>t>p^*,\\ -1, & \text{if } p-1<t<p^*,\end{cases}$$

(3.6) is equivalent to

$$-\int_{p-1}^{p^*} F_j(t)\,dt + \int_{p^*}^{p} F_j(t)\,dt > 0$$

for all $j\ge 0$, which is guaranteed by Lemma 2.9.

We complete the proof by showing that

$$\int_{p-2}^{p} \frac{a^2g^2(t)-4atg'(t)}{t}\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}dt \le 0, \tag{3.8}$$

for all $j\ge 0$.

This holds for $0\le j<j(p)$ by the definition of B and the fact that $B>a>0$.

It remains to show (3.8) for $j\ge j(p)$. Note first that $\exp(-t/2)t^{p/2+j-1}$ is monotone increasing and $|g(t)|\le 1$ on $[p-2,p]$. Therefore, using the fact that $p-b>p^*$,

$$\begin{aligned}
&\int_{p-2}^{p} \frac{g^2(t)}{t}e^{-\frac{t}{2}}\,t^{p/2+j-1}dt\\
&\quad < \frac{1}{p-2}\int_{p-2}^{p} e^{-\frac{t}{2}}\,t^{p/2+j-1}dt\\
&\quad < \frac{2}{(p-2)b}\int_{p-b}^{p} e^{-\frac{t}{2}}\,t^{p/2+j-1}dt\\
&\quad = \frac{2}{(p-2)b}\int_{p-b}^{p} g'(t)e^{-\frac{t}{2}}\,t^{p/2+j-1}dt.
\end{aligned} \tag{3.9}$$

Now,

$$\int_{p-b}^{p} F_j(t)\,dt = \int_{p-b}^{p} \left\{e^{-\frac{t}{2}}\, t^{p/2+j-1} - e^{-\frac{2(p-1)-t}{2}}\left[2(p-1)-t\right]^{p/2+j-1}\right\}dt$$
$$= \int_{p-b}^{p} e^{-\frac{t}{2}}\, t^{p/2+j-1}\left\{1 - e^{1-p+t}\left[\frac{2(p-1)-t}{t}\right]^{p/2+j-1}\right\}dt.$$

Since, for all $j > 0$, $\exp(t-p+1)[(2(p-1)-t)/t]^{p/2+j-1}$ is monotone decreasing in t, for $b < 0.5, t \in [p-1, p]$ and by Lemma 2.4 and (3.2), we have

$$\begin{aligned}
&\int_{p-b}^{p} F_j(t)\,dt \\
&\quad > \int_{p-b}^{p} e^{-\frac{t}{2}}\, t^{p/2+j-1}\left\{1 - \exp(1-b)\left(\frac{p-2+b}{p-b}\right)^{p/2+j-1}\right\}dt \\
&\quad > \int_{p-b}^{p} e^{-\frac{t}{2}}\, t^{p/2+j-1}\left\{1 - \exp(1-b)\left(\frac{p-2+b}{p-b}\right)^{p/2+j(p)-1}\right\}dt \\
&\quad = A\int_{p-b}^{p} e^{-\frac{t}{2}}\, t^{p/2+j-1}\,dt.
\end{aligned} \tag{3.10}$$

Note also that $A > 0$, by Lemma 2.4, and $\exp(1-b)((p-2+b)/(p-b))^{p/2+j(p)-1}$ is monotone increasing in b. Hence, by (3.9) and (3.10),

$$\int_{p-2}^{p} \frac{g^2(t)}{t} e^{-\frac{t}{2}}\, t^{p/2+j-1}\,dt \le \frac{2}{(p-2)bA}\int_{p-b}^{p} g'(t)F_j(t)\,dt. \tag{3.11}$$

We will finally show

$$\int_{p-2}^{p-b} g'(t)F_j(t)\,dt = -\int_{p-1}^{p^*} F_j(t)\,dt + \int_{p^*}^{p-b} F_j(t)\,dt > 0, \tag{3.12}$$

for $j \ge j(p)$.

Hence, (3.11), (3.12) and (3.7) imply that, for $j \ge j(p)$,

$$\begin{aligned}
&\int_{p-2}^{p} \frac{g^2(t)}{t}\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}\,dt \\
&\quad < \frac{2}{(p-2)bA}\int_{p-1}^{p} g'(t)F_j(t)\,dt \\
&\quad = \frac{4}{2(p-2)bA}\int_{p-2}^{p} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}\,dt.
\end{aligned} \tag{3.13}$$

The fact that $0 < a < 2(p-2)bA$ implies that (3.8) holds for all $j \ge 0$ and hence the theorem holds.

It remains to show (3.12).

TABLE 1

p	p^*	A	B	a
3	2.695	0.9976	0.1627	0.0341
4	3.700	0.9779	0.3314	0.0393
5	4.700	0.9393	0.1968	0.0566
10	9.705	0.7071	0.0072	0.0337

By the definition of b,

$$\int_{p-1}^{p^*} \big(t-(p-1)\big)\,dt = \int_{p^*}^{p-b} \big(t-(p-1)\big)dt.$$

For $j \geq j(p)$, $F_j(p-1) = 0$, $F_j(t) \geq 0$, $F_j'(t) > 0$ and $F_j''(t) \geq 0$, that is, $F_j(t)$ are convex increasing by Lemmas 2.2 and 2.3. Hence

$$F_j(t)\begin{cases} < \dfrac{F_j(p^*)}{p^*-p+1}(t-p+1), & \text{if } p-1<t<p^*, \\ > \dfrac{F_j(p^*)}{p^*-p+1}(t-p+1), & \text{if } p^*<t\leq p. \end{cases}$$

Therefore

$$\begin{aligned} &-\int_{p-1}^{p^*} F_j(t)\,dt + \int_{p^*}^{p-b} F_j(t)\,dt \\ &\quad > \frac{F_j(p^*)}{p^*-(p-1)}\left\{-\int_{p-1}^{p^*}\big[t-(p-1)\big]dt + \int_{p^*}^{p-b}\big[t-(p-1)\big]dt\right\} = 0. \end{aligned}$$

This completes the proof. □

Table 1 gives some values of p^*, A, B and a for which the estimator given in the theorem improves on $\delta_+^{\mathrm{JS}}(X)$.

These examples are not intended to represent optical choices of p^* but to show the order of magnitude of the expected range of values of a.

4. Some further results. In the preceding section, we took the function $g(t)$ to be a piecewise linear function such that $g(p-2) = g(p) = 0$. We extend the previous result in two directions. In this section, we assume that $g(t)$ is a curve but is still symmetric about $t = p-1$ and $g(p) = 0$. In the next section, we do not assume that $g(t)$ is symmetric and let $g(t)$ be defined on $[0, \infty)$.

Let both $g_1(t)$ and $g_2(t)$ be continuously differentiable functions defined on $(p-1-\varepsilon, p+\varepsilon)$, for some $\varepsilon > 0$. Moreover, let $g_1(t)$ be a monotone decreasing function and let $g_2(t)$ be a monotone increasing function on the interval $[p-1, p]$. Assume $g_1(p) = g_2(p-1)$ and $g_2(p) = 0$; $F_j(t)$ is defined as in (2.3).

Let $w \in [0.1]$, and define

$$g(t,w) = wg_1\left(p-1+\frac{t-(p-1)}{w}\right)I_{\{p-1\le t\le p-1+w\}} \qquad (4.1)$$
$$+(1-w)g_2\left(p-1+\frac{t-(p-1)-w}{1-w}\right)I_{\{p-1+w<t\le p\}}.$$

If $w = 0$ or 1, we define $g(t,w) = g_2(t)$ or $g_1(t)$, respectively. If there is no risk of confusion, we write $g'(t,w)$ as $\partial g(t,w)/\partial t$. Now since $g'(t,0) = g_2'(t) > 0$ and $g'(t,1) = g_1'(t) < 0$, there exists w_0 such that

$$\int_{p-1}^{p} g'(t,w_0)F_0(t)\,dt = 0. \qquad (4.2)$$

If $g(t,w_0)$ satisfies

$$\int_{p-1}^{p} g'(t,w_0)\{t-(p-1)\}\,dt > 0, \qquad (4.3)$$

then we have the following result.

THEOREM 4.1. *Let $g_1(\cdot)$, $g_2(\cdot)$ and $g(\cdot,\cdot)$ be defined as in the previous paragraph. Extend the definition of $g(\cdot,\cdot)$ so that $g(t,w)$ is an even symmetric function (in t) around $t = p-1$, defined on $t \in [p-2, p]$ for each w. Then there exists $w^+ > w_0$ such that* (i) *and* (ii) *hold*:

(i) $\int_{p-1}^{p} g'(t,w^+)F_j(t)\,dt > 0$, *for all* $1 < j < j(p)$;
(ii) $\int_{p-1}^{p} g'(t,w^+)\{t-(p-1)\}\,dt > 0$.

Furthermore,

$$\delta(a,w^+,X) = \delta_+^{\mathrm{JS}}(X) - \frac{ag(\|X\|^2,w^+)}{\|X\|^2}XI_{\{p-2\le\|X\|^2\le p\}} \qquad (4.4)$$

dominates $\delta_+^{\mathrm{JS}}(X)$ *provided* (iii) *holds*:

(iii) $0 < a < \min\{B, 2(p-2)blA/L^2\}$, *where b satisfies the equation*

$$\int_{p-1}^{p-b} g'(t,w^+)\{t-(p-1)\}\,dt = 0, \qquad (4.5)$$

and where

$$L = \max\{|g(t,w^+)| \mid p-1 \le t \le p\},$$
$$l = \min\{g'(t,w^+) \mid p-b \le t \le p\}$$

and

$$B = \min\left\{ \frac{4\int_{p-2}^{p} g'(t, w^+)\exp(-t/2)t^{p/2+j-1}\,dt}{\int_{p-2}^{p} g^2(t, w^+)\exp(-t/2)t^{p/2+j-2}\,dt} \,\middle|\, j = 0, 1, \dots, j(p) - 1 \right\}.$$

PROOF. Note that the class of functions $g(\cdot, \cdot)$ satisfying (4.2) and (4.3) is nonempty, by Theorem 3.1. It suffices to show for all $j \geq 0$ that

$$\int_{p-1}^{p} \frac{a^2 g^2(t, w^+) - 4atg'(t, w^+)}{t} F_j(t)\,dt \leq 0. \tag{4.6}$$

Let

$$\alpha_j(t) = \frac{F_j(t)}{F_0(t)}.$$

When $t \in [p-1, p]$, by Lemmas 2.2 and 2.7 and Theorem 2.1, $\alpha_j(t)$ is monotone decreasing and nonpositive for all $j > 0$.

For $1 \leq j < j(p)$,

$$\int_{p-1}^{p} g'(t, w_0)F_j(t)\,dt > \alpha_j(p - 1 + w_0) \int_{p-1}^{p} g'(t, w_0)F_0(t)\,dt = 0.$$

Also, by (4.3) we have

$$\int_{p-1}^{p} g'(t, w_0)\{t - (p-1)\}\,dt > 0.$$

Therefore there exists $w^+ > w_0$, such that for $1 \leq j < j(p)$ the following inequalities still hold:

$$\begin{aligned} &\int_{p-1}^{p} g'(t, w^+)F_j(t)\,dt > 0, \\ &\int_{p-1}^{p} g'(t, w^+)\{t - (p-1)\}dt > 0. \end{aligned} \tag{4.7}$$

From (4.7), we also know that there exists $b > 0$ such that

$$\int_{p-1}^{p-b} g'(t, w^+)\{t - (p-1)\}dt = 0,$$

with $b < 1 - w^+$.

An argument similar to that in Theorem 3.1 shows

$$\int_{p-1}^{p} g'(t, w^{+}) F_j(t)\, dt > 0 \quad \text{for } j \geq j(p).$$

We complete the proof by showing that (4.6) holds for all $j \geq 0$. For $0 \leq j < j(p)$, this follows from assumption (iii).

For the case $j \geq j(p)$, using an argument similar to that leading to (3.13),

$$\int_{p-1}^{p} \frac{g^2(t, w^{+})}{t} F_j(t)\, dt \leq \frac{2L^2}{(p-2)blA} \int_{p-1}^{p} g'(t, w^{+}) F_j(t)\, dt.$$

This completes the proof. □

5. More general classes. The results of the previous two sections presented estimators which change the values of $\delta_{+}^{\mathrm{JS}}(X)$ only on the compact set $\{p-2 \leq \|X\|^2 \leq p\}$. While these estimators improve upon $\delta_{+}^{\mathrm{JS}}(X)$, they cannot themselves be generalized Bayes and hence are not admissible. In this section, we present a class of improved estimators which allow changes for $\|X\|^2$ in $(0, \infty)$ and which we believe (but have not proved) contains admissible improvements. A brief discussion of this is given in the comment after the statement of the theorem.

THEOREM 5.1. *Let $g(t)$ be a continuous and piecewise differentiable* W-*shaped function defined on $[q, \infty)$ with $g(q) = g(\infty) = 0$, where $p-2 < q < p-1$, that is, there exist $p-2 < q < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < \infty$, such that*

$$g(t_2) = g(t_4) = 0$$

and

$$g'(t) = \begin{cases} \geq 0, & \text{if } t_1 \leq t < t_3 \text{ or } t_5 \leq t < \infty, \\ \leq 0, & \text{if } q \leq t < t_1 \text{ or } t_3 \leq t < t_5. \end{cases}$$

Let $h(t)$ be a continuous nonpositive function defined on $[0, q]$ such that $|h(t)| < M < \infty$ and $h(q) = 0$. Let $b_j = 2^{p/2+j} j!\,\Gamma(p/2+j)$. Assume there exists a positive integer J such that (i) *and* (ii) *hold*:

(i) *We have*

$$\int_{q}^{t_6} g'(t) \exp\left(-\frac{t}{2}\right) t^{p/2+J+1}\, dt > 0.$$

(ii) *We have $\int_{t_3}^{t_6} g'(t) \exp(-t/2) t^{p/2+J-1}\, dt \geq 0$.*

(iii) *Assume, for all $t \geq t_6$, $g^2(t) \leq 4tg'(t)$.*

(iv) *Let $B_1 < B_2$ be two constants such that if $w \geq 2B_2$ or $0 \leq w \leq 2B_1$, for some integer $N > J$,*

$$\int_q^\infty \frac{g'(t)\exp(-t/2)t^{p/2-1}dt}{b_0} + 2\sum_{j=1}^{N} \frac{\int_q^{t_6} g'(t)\exp(-t/2)t^{p/2+j-1}dt}{b_j}\left(\frac{w}{2}\right)^j \geq 0.$$

(v) *Assume that*

$$k \geq \frac{4b_1\Sigma_{j=1}^{J-1} \mid \int_q^{t_6} g'(t)\exp(-t/2)\, t^{p/2+j-1}dt\left[(B_2)^j/b_j\right]}{B_1 \mid \int_0^{p-2} h(t)\exp(-t/2)t^{p/2}\,dt}.$$

(vi) *Assume that*

$$2\int_q^\infty g'(t)\exp(-t/2)t^{p/2-1}dt + k(q-p+2)\int_{p-2}^{q} h(t)t^{p/2-1}\exp(-t/2)\,dt > 0.$$

(vii) *Assume $0 < a < \min\{0.5, A_1, A_2,\ 1/(qkM),\ B_1/(kM)\}$, where*

$$A_1 = \frac{2\int_q^\infty g'(t)\exp(-t/2)t^{p/2-1}dt + k(q-p+2)\int_{p-2}^q h(t)\exp(-t/2)t^{p/2-1}\,dt}{\int_q^\infty g^2(t)\exp(-t/2)t^{p/2-2}\,dt + k^2\int_0^q h^2(t)\exp(-t/2)t^{p/2}\,dt},$$

$$A_2 = \frac{4\int_{t_6}^\infty g'(t)\exp(-t/2)t^{p/2-1}\,dt}{4\int_{t_6}^\infty g'(t)\exp(-t/2)t^{p/2-1}\,dt + \int_q^{t_6} g^2(t)\exp(-t/2)t^{p/2-1}\,dt}.$$

Then

$$\delta(X) = \delta_+^{\mathrm{JS}}(X) - a\frac{g(\|X\|^2)}{\|X\|^2}XI_{\{\|X\|^2>q\}} - akh(\|X\|^2)XI_{\{0<\|X\|^2<q\}} \tag{5.1}$$

dominates $\delta_+^{\mathrm{JS}}(X)$.

COMMENT. It is critical that $q > p-2$ in order that $\delta(X)$ have a chance to be admissible. To see this, note that generalized Bayes estimators are coordinatewise monotone. It follows that if $q \leq p-2$ and $\delta(X)$ is generalized Bayes, then $\delta(X) \equiv \mathbf{0}$ for $\|X\|^2 \leq q$ and hence by uniqueness must be identically $\mathbf{0}$.

The choice of $q > p-2$ allows choices of $g(\cdot)$ and $h(\cdot)$ such that the resulting $\delta(X)$ may be strictly monotone and sufficiently smooth to be possibly generalized Bayes. In particular it is important to note that $h(\cdot)$ is not assumed to be differentiable. This allows $h(\cdot)$ to be free to "correct" the discontinuity in the differential of the positive-part estimator in a neighborhood of $\|X\|^2 = p-2$. This is accomplished by having a similar discontinuity in the differential of $h(\|X\|^2)X$ but of opposite sign so that $\delta(X)$ is differentiable everywhere. If $g(\cdot)$ goes to 0 sufficiently quickly and $\delta(\cdot)$ is generalized Bayes, it follows from Strawderman and Cohen (1971) that $\delta(\cdot)$ is admissible.

PROOF OF THEOREM 5.1. If $X \sim N_p(\theta, I)$, for any fixed θ, there exists an orthogonal matrix Q, such that $Q\theta = (\mu, 0, \dots, 0)'$ and $Y = QX \sim N_p(Q\theta, I)$ with $\mu^2 = \|\theta\|^2$, $\|Y\|^2 = \|X\|^2$ and $\|X-\theta\|^2 = \|Y - Q\theta\|^2$. Therefore, we assume without loss of generality that $\theta = (\theta_1, 0, \dots, 0'.)$ We use the notation $t = \|X\|^2$ except in those expressions where $\theta_1 X_1$ and $\|X\|^2$ occur together;

$$\begin{aligned}\Delta R(\theta) &= R\big(\theta, \delta(X)\big) - R\big(\theta, \delta_+^{\mathrm{JS}}(X)\big)\\ &= E\left[\left(\frac{a^2g^2(t) - 4atg'(t)}{t}\right)\right] I_{\{t>q\}}\\ &\quad + Ea^2k^2th^2(t)I_{\{t<q\}} + 2akE\theta_1X_1h\big(\|X\|^2\big)I_{\{0<\|X\|^2<q\}}\\ &\quad + 2akE(p-2-t)h(t)I_{\{p-2<t<q\}}.\end{aligned}$$

Note that since $h(\cdot) < 0$ and $q - (p-2) < 1$,

$$\begin{aligned}&E\theta'Xh\big(\|X\|^2\big)I_{\{p-2<\|X\|^2<q\}} + 2E(p-2-t)h(t)I_{\{p-2<t<q\}}\\ &\quad < E\theta'Xh\big(\|X\|^2\big)I_{\{p-2<\|X\|^2<q\}} + 2(p-2-q)Eh(t)I_{\{p-2<t<q\}}\\ &\quad < \frac{2(q-p+2)}{(\sqrt{2\pi})^p}\int_{\{p-2<\|X\|^2<q\}} \big|h\big(\|X\|^2\big)\big|\\ &\quad\quad \times \exp\left(-\frac{\|X\|^2}{2}\right)\exp\left(-\frac{\theta_1^2}{2}\right)dx_1, \dots, dx_p.\end{aligned} \tag{5.2}$$

Similarly, since, by assumption (vii) and $akM < 1/q$,

$$\begin{aligned}&akEth^2(t)I_{\{0<t<q\}} + E\theta'Xh\big(\|X\|^2\big)I_{\{0<\|X\|^2<q\}}\\ &\quad < \frac{ak}{b_0}\int_0^q h^2(t)\exp\left(-\frac{t}{2}\right)t^{p/2}\,dt\,\exp\left(-\frac{\theta'\theta}{2}\right),\end{aligned} \tag{5.3}$$

then by (5.2) and (5.3),

$$
\begin{aligned}
\Delta R(\theta) < {} & E\frac{a^2g^2(t) - 4atg'(t)}{t} I_{\{t>q\}} \\
& + \frac{a^2k^2}{b_0}\int_0^q h^2(t)\exp\left(-\frac{t}{2}\right)t^{p/2}\,dt\exp\left(-\frac{\theta'\theta}{2}\right) \\
& - \frac{2ak(q-p+2)}{(\sqrt{2\pi})^p}\int_{\{p-2<\|X\|^2<q\}} h(\|X\|^2) \\
& \times \exp\left(-\frac{\|X\|^2}{2}\right)\exp\left(-\frac{\theta'\theta}{2}\right)dx_1\cdots dx_p \\
& + 2akE\theta' X h(\|X\|^2) I_{\{0<\|X\|^2<p-2\}}.
\end{aligned}
\tag{5.4}
$$

We separate the proof into two cases.

Case 1 ($\theta'\theta/2 \le B_1$ or $\theta'\theta/2 \ge B_2$). By assumptions (i) and (ii) we have, for $j \ge J$,

$$
\int_q^{t_6} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}\,dt > (t_1)^{j-J}\int_q^{t_6} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+J-1}\,dt > 0.
$$

By assumption (iv),

$$
\begin{aligned}
& 2\int_q^\infty \frac{g'(t)\exp(-t/2)t^{p/2-1}\,dt}{b_0} \\
& + 4\sum_{j=1}^\infty \frac{1}{b_j}\int_q^{t_6} g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}\,dt\left(\frac{\|\theta\|^2}{2}\right)^j > 0,
\end{aligned}
$$

and by the fact that $E\theta_1 X_1 h(\|X\|^2)I_{\{0<\|X\|^2<p-2\}} \le 0$ and by assumption (vi), we have

$$
\begin{aligned}
\exp\left(\frac{\theta'\theta}{2}\right)\Delta R(\theta) < {} & \left[\int_q^\infty a^2g^2(t)\exp\left(-\frac{t}{2}\right)t^{p/2-2}\,dt\right. \\
& - 2a\int_q^\infty g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2-1}\,dt \\
& \left. + a^2k^2\int_0^q h^2(t)\exp\left(-\frac{t}{2}\right)t^{p/2}\,dt\right]\frac{1}{b_0} \\
& + \sum_{j=1}^\infty \frac{(\theta'\theta/2)^j}{b_j}\left[a^2\int_q^\infty g^2(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-2}\,dt\right. \\
& \left. - 4a\int_{t_6}^\infty g'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-1}\,dt\right].
\end{aligned}
\tag{5.5}
$$

By assumptions (iii) and (vii), for all j,

$$
\begin{aligned}
& a \int_q^\infty g^2(t) \exp\left(-\frac{t}{2}\right) t^{p/2+j-2}\, dt \\
&\qquad - 4 \int_{t_6}^\infty g'(t) \exp\left(-\frac{t}{2}\right) t^{p/2+j-2}\, dt \\
&\le t_6^j \left\{ a \left[4 \int_{t_6}^\infty g'(t) \exp\left(-\frac{t}{2}\right) t^{p/2-1}\, dt \right.\right. \\
&\qquad \left. + \int_q^{t_6} g^2(t) \exp\left(-\frac{t}{2}\right) t^{p/2-2}\, dt \right] \\
&\qquad \left. - 4 \int_{t_6}^\infty g'(t) \exp\left(-\frac{t}{2}\right) t^{p/2-1}\, dt \right\} \\
&\le 0.
\end{aligned}
\tag{5.6}
$$

Therefore, $\Delta R(\theta) \le 0$ by (5.5), (5.6) and assumption (vii). This completes the proof for Case 1.

Case 2 ($B_1 < \theta'\theta/2 < B_2$). By (5.4) and the fact that $h(\cdot) < 0$,

$$
\begin{aligned}
& \exp\left(\frac{\theta'\theta}{2}\right) \Delta R(\theta) \\
&< \left[\int_q^\infty \left[a^2 g^2(t) \exp\left(-\frac{t}{2}\right) t^{p/2-2} \right] dt \right. \\
&\qquad - 4a \int_q^\infty g'(t) \exp\left(-\frac{t}{2}\right) t^{p/2-1}\, dt \\
&\qquad - 2ak(q-p+2) \int_{p-2}^q h(t) \exp\left(-\frac{t}{2}\right) t^{p/2-1}\, dt \\
&\qquad \left. + a^2 k^2 \int_0^q t h^2(t) \exp\left(-\frac{t}{2}\right) t^{p/2-1}\, dt \right] \frac{1}{b_0} \\
&\quad + \sum_{j=1}^{J-1} \left(\int_q^\infty [a^2 g^2(t) - 4atg'(t)] \exp\left(-\frac{t}{2}\right) t^{p/2+j-2}\, dt \right) \frac{(\theta'\theta/2)^j}{b_j} \\
&\quad + \sum_{j=J}^\infty \left(\int_q^\infty [a^2 g^2(t) - 4atg'(t)] \exp\left(-\frac{t}{2}\right) t^{p/2+j-2}\, dt \right) \frac{(\theta'\theta/2)^j}{b_j} \\
&\quad + ak \left(\int_{\{0 < \|X\|^2 < p-2\}} \left(\frac{1}{2\pi}\right)^{p/2} X_1^2 h(\|X\|^2) \right. \\
&\quad \left. \times \exp\left(-\frac{\|X\|^2}{2}\right) dx_1\, dx_2 \cdots dx_p \right) \frac{\theta'\theta}{2}.
\end{aligned}
\tag{5.7}
$$

By symmetry, the last term in (5.7) is equal to

$$\frac{ak}{b_1}\left[\int_0^{p-2} th(t)\exp\left(-\frac{t}{2}\right)t^{p/2-1}dt\right]\frac{\theta'\theta}{2}. \tag{5.8}$$

Additionally, the next-to-last term in (5.7) (i.e., $\Sigma_{j=J}^{\infty}\cdots$) is negative. Also note that the first term in brackets in (5.7) is negative by assumptions (vi) and (vii). Finally, by (5.6) and assumption (iii),

$$\begin{aligned}
&\sum_{j=1}^{J-1}\left(\int_q^{\infty}\left[a^2g^2(t)-4atg'(t)\right]\exp\left(-\frac{t}{2}\right)t^{p/2+j-2}dt\right)\frac{(\theta'\theta/2)^j}{b_j}\\
&\quad+\frac{ak}{b_1}\left(\int_0^{p-2} th(t)\exp\left(-\frac{t}{2}\right)t^{p/2-1}dt\right)\frac{\theta'\theta}{2}\\
&\quad<\sum_{j=1}^{J-1}\left|\int_q^{t_6} 4atg'(t)\exp\left(-\frac{t}{2}\right)t^{p/2+j-2}dt\right|\frac{(B_2)^j}{b_j}\\
&\quad-\frac{ak}{b_1}\left|\int_0^{p-2} th(t)\exp\left(-\frac{t}{2}\right)t^{p/2-1}dt\right|B_1.
\end{aligned}$$

This last expression is negative by assumption (v). Hence $\Delta R(\theta) < 0$. This completes the proof. □

APPENDIX

PROOF OF LEMMA 2.2. Since

$$\begin{aligned}
[F_n(t)]' &= (n-1)\Bigg[\exp\left(-\frac{t}{2}\right)t^{p/2+n-2}\\
&\qquad+\exp\left(\frac{t}{2}-p+1\right)(2(p-1)-t)^{p/2+n-2}\Bigg]\\
&\quad+\frac{p-t}{2}\exp\left(-\frac{t}{2}\right)t^{p/2+n-2}\\
&\quad+\frac{t-(p-2)}{2}\exp\left(\frac{t}{2}-(p-1)\right)(2(p-1)-t)^{p/2+n-2}
\end{aligned} \tag{A.1}$$

and, for any n,

$$F_n(p-1)\equiv 0, \tag{A.2}$$

by assumption $p-1\le t\le p$, from (A.1) and (A.2) we see $[F_0(t)]' < 0$ and $F_0(t)\le 0$. Also, for all $n\ge 1, F_n(t)\ge 0, [F_n(t)]' > 0$ and $F_n(t)/F_0(t) < 0$. As we assume $t = p - 1 + u$,

$$\text{(A.3)}\quad \begin{aligned} F_0''(t) &= \frac{1}{2}\exp\left(-\frac{p-1-u}{2}\right)(p-1-u)^{p/2-3}\left(p-2-\frac{(1-u)^2}{2}\right) \\ &\quad -\frac{1}{2}\exp\left(-\frac{p-1+u}{2}\right)(p-1+u)^{p/2-3}\left(p-2-\frac{(1+u)^2}{2}\right). \end{aligned}$$

Since

$$\begin{aligned} &\left[\exp\left(\frac{u}{2}\right)(p-1-u)^{p/2-3}\left(p-2-\frac{(1-u)^2}{2}\right)\right]' \\ &\quad = \exp\left(\frac{u}{2}\right)(p-1-u)^{p/2-4}\bigg[(p-1-u)(1-u)+(5-u) \\ &\qquad\qquad \times\left(\frac{p-2}{2}-\frac{(1-u)^2}{4}\right)\bigg], \end{aligned}$$

when $p > 3$ it is nonnegative for all $u \in [-1, 1]$. Hence the first term in (A.3) dominates the second.

When $p = 3$, it is easy to see $F_0''(t) \geq 0$. Therefore, for all $t \in [p-1, p]$,

$$\text{(A.4)}\quad F_0''(t) \geq 0.$$

Since

$$\begin{aligned} F_1''(t) &= \frac{1}{2}\bigg[\exp\left(-\frac{p-1-u}{2}\right)(p-1-u)^{p/2-1} \\ &\qquad - \exp\left(-\frac{p-1+u}{2}\right)(p-1+u)^{p/2-1}\bigg] \\ &\quad + \frac{(1-u)(1+u)}{4}\bigg[\exp\left(-\frac{p-1-u}{2}\right)(p-1-u)^{p/2-2} \\ &\qquad - \exp\left(-\frac{p-1+u}{2}\right)(p-1+u)^{p/2-2}\bigg], \end{aligned}$$

and since $\exp(-t/2)t^{p/2-1}$ and $\exp(-t/2)t^{p/2-2}$ are monotone decreasing,

$$\text{(A.5)}\quad F_1''(t) \geq 0.$$

Similarly, $\exp(-t/2)t^{p/2-1}(p-2t)$ and $\exp(-t/2)t^{p/2-1}(p-t)^2$ are monotone decreasing, so

$$\begin{aligned} F_2''(t) &= \exp\left(-\frac{t}{2}\right)t^{p/2-1}\frac{p-2t}{2} - \exp\left(-\frac{s}{2}\right)s^{p/2-1}\frac{p-2s}{2} \\ &\quad + \frac{(p-t)^2}{4}\exp\left(-\frac{t}{2}\right)t^{p/2-1} - \frac{(p-s)^2}{4}\exp\left(-\frac{s}{2}\right)s^{p/2-1} \\ &\leq 0. \end{aligned}$$

Since

$$\left[\frac{F_j(t)}{F_0(t)}\right]' = \frac{F_j'(t)F_0(t) - F_0'(t)F_j(t)}{\left(F_0(t)\right)^2},$$

let us define $G_j(t) = F_j'(t)F_0(t) - F_0'(t)F_j(t)$. Then, by (A.4) and (A.5),

$$G_1'(t) = F_1''(t)F_0(t) - F_0''(t)F_1(t) \le 0.$$

Because $G_1(p-1) = 0$, therefore

$$\left[\frac{F_1(t)}{F_0(t)}\right]' \le 0. \qquad \square$$

PROOF OF LEMMA 2.3. We have

$$\begin{aligned} F_n''(t) &= \exp\left(-\frac{t}{2}\right)t^{p/2+n-3}\left[(p-t)\left(n-\frac{3}{2}\right) + (n-1)(n-2) - \frac{t}{2} + \frac{(p-t)^2}{4}\right] \\ &\quad - \exp\left(-\frac{s}{2}\right)s^{p/2+n-3}\left[(p-s)\left(n-\frac{3}{2}\right) + (n-1)(n-2) - \frac{s}{2} + \frac{(p-s)^2}{4}\right]. \end{aligned}$$

Note that $F_n''(p-1) = 0$, and

$$\begin{aligned} &\left\{\exp\left(-\frac{t}{2}\right)t^{p/2+n-3}\left[-\frac{t}{2} + \frac{(p-t)^2}{4} + (n-1)(n-2) + \frac{p-t}{2}(2n-3)\right]\right\}' \\ \text{(A.6)}\qquad &\ge \exp\left(-\frac{t}{2}\right)t^{p/2+n-4}\left\{\frac{n-1}{2}\left[2(n-2)(n-3) - 3p\right]\right. \\ &\qquad \left. + \frac{p-t}{4}\left[6n^2 - 24n + 22 - 3p\right]\right\}. \end{aligned}$$

Take $j(p)$ to be the smallest integer bigger than $(5+\sqrt{1+6p})/2$. Then if $n \ge j(p)$, (A.6) is nonnegative for all $t \in [p-2, p]$ and, since $t > s$, for all $t \in [p-1, p]$. Hence, for $n \ge j(p), F_n''(t) \ge 0$. $\square$

PROOF OF LEMMA 2.5. Since $t + s = 2(p-1)$ and

$$(t+s)F_n(t) = F_{n+1}(t) + tsF_{n-1}(t),$$

we have

$$F_{n+1}(t) - tF_n(t) = s\left(F_n(t) - tF_{n-1}(t)\right).$$

By induction, $F_{n+1}(t) - tF_n(t) = s^n(F_1(t) - tF_0(t))$.

Let

$$T_{n+1}(t) = F_{n+1}(t) - tF_n(t).$$

Then

$$T_{n+1}(t) = s^n T_1(t);$$

also,

$$\begin{aligned} F_{n+1}(t) &= T_{n+1}(t) + tF_n(t) \\ &= T_{n+1}(t) + tT_n(t) + \cdots + t^n T_1(t) + t^{n+1} F_0(t). \end{aligned}$$

So we obtain

$$F_{n+1}(t) = T_1(t)\left(\frac{t^{n+1} - s^{n+1}}{t - s}\right) + t^{n+1} F_0(t)$$

and

$$\frac{F_{n+1}(t)}{F_0(t)} = \frac{t^{n+1} - s^{n+1}}{t - s} \frac{F_1(t)}{F_0(t)} - ts \frac{t^n - s^n}{t - s}. \qquad \square$$

PROOF OF LEMMA 2.6. Let $t = p - 1 + u$, so $s = p - 1 - u$, where $u \in [0, 1]$. For the case of $m = 1$ or $m = 2$, the ratios are 1 or $2(p - 1)$ and are nondecreasing. For the case $m \geq 3$, the derivative is given by

$$\begin{aligned} \left[\frac{t^m - s^m}{t - s}\right]' &= \frac{d}{du}\left[\frac{(p - 1 + u)^m - (p - 1 - u)^m}{2u}\right] \\ &= \frac{1}{2u} \sum_{i=0}^{m-1} \Big\{ \left[(p - 1 + u)^{m-1-i} - (p - 1 - u)^{m-1-i}\right] \\ &\qquad\qquad \times \left[(p - 1 + u)^i - (p - 1 - u)^i\right] \Big\} \\ &\geq 0. \end{aligned} \qquad \square$$

PROOF OF LEMMA 2.7. By Lemma 2.5,

$$\left[\frac{F_2(t)}{F_0(t)}\right]' = 2(p - 1)\left[\frac{F_1(t)}{F_0(t)}\right]' - \big(ts(t)\big)',$$

so it is sufficient by (A.1) to show for all $u \in [0, 1]$ that

$$\begin{aligned} &e^{-u}\left(1 + \frac{u}{p - 1}\right)^{p-1} - e^{u}\left(1 - \frac{u}{p - 1}\right)^{p-1} \\ &\qquad - 2\frac{u^3}{(p - 1)^2}\frac{p - 2}{p - 1}\left[1 - \frac{u^2}{(p - 1)^2}\right]^{p/2-2} \leq 0. \end{aligned}$$

Let

$$f(u) = e^{-u}\left(1 + \frac{u}{p-1}\right)^{p-1} - e^{u}\left(1 - \frac{u}{p-1}\right)^{p-1} - \frac{2u^3}{3(p-1)^2}.$$

It is straightforward to show that $f'(u)$ is nonpositive. Since $f(0) = 0$, $f(u) \leq 0$. Also, for all $p \geq 3$,

$$\frac{p-2}{p-1}\left[1 - \frac{u^2}{(p-1)^2}\right]^{p/2-2} > \frac{1}{e} > \frac{1}{3}.$$

Hence, $F_2(t)/F_0(t)$ is monotone decreasing on $[p-1, p]$. □

PROOF OF THEOREM 2.1. By Lemmas 2.2 and 2.7, we know for the cases $j = 1$ or $j = 2$ that the result is true.

For the case $j \geq 3$, by Lemma 2.5,

$$\left[\frac{F_j(t)}{F_0(t)}\right]' = \left(\frac{t^j - s^j}{t-s}\right)'\frac{F_1(t)}{F_0(t)} + \left(\frac{t^j - s^j}{t-s}\right)\left[\frac{F_1(t)}{F_0(t)}\right]' - \left(ts\frac{t^{j-1} - s^{s-1}}{t-s}\right)'.$$

By Lemma 2.2, we know that

$$\frac{F_1(t)}{F_0(t)} \leq 0$$

and

$$\left[\frac{F_1(t)}{F_0(t)}\right]' \leq 0.$$

Hence

$$\frac{F_1(t)}{F_0(t)} \leq \lim_{t \to p-1}\frac{F_1(t)}{F_0(t)} = \frac{\lim_{t\to p-1} F_1'(t)}{\lim_{t\to p-1} F_0'(t)} = -(p-1).$$

By Lemma 2.6,

$$\begin{aligned}\left[\frac{F_j(t)}{F_0(t)}\right]' &\leq -(p-1)\left(\frac{t^j - s^j}{t-s}\right)' - \left(ts\frac{t^{j-1} - s^{j-1}}{t-s}\right)' \\ &\leq 2u\big[-(p-1)(p-1+u)^{j-3} + (p-1+u)^{j-2} \\ &\qquad - (p-1-u)(p-1+u)^{j-3}\big] \\ &\quad + 2u\sum_{i=2}^{j-1}\big[-i(p-1)(p-1-u)^{i-1}(p-1+u)^{j-2-i} \\ &\qquad + (p-1+u)^{j-1-i}(p-1-u)^{i-1}\big].\end{aligned}$$

Since

$$-(p-1)+(p-1+u)-(p-1-u)\le -p+3\le 0,$$

the first term is nonpositive. The second term is also nonpositive, and the proof is complete. □

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