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TRANSFORMATIONS OF THE EMPIRICAL MEASURE AND KOLMOGOROV–SMIRNOV TESTS

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The power of Kolmogorov–Smirnov tests can be increased by transforming the empirical process into a new process that converges to a Wiener process under the null hypothesis and by choosing the transformation in such a way that some families of local alternatives become as noticeable as possible.

1. Introduction. Given a random sample $X_1, X_2, \dots, X_n$ having distribution $\tilde{F}$ we consider the design of sequences of tests, one for each sample size, of the null hypothesis $\mathscr{H}_0$: $\tilde{F} = \tilde{F}_0$, $\tilde{F}_0$ absolutely continuous, with the following two main properties:

1. Consistency against any alternative, as in the classical Kolmogorov–Smirnov test.
2. A good asymptotic power when the tests are applied to a given sequence of contiguous alternatives converging to the null hypothesis.

In order to solve this problem, we introduce a large family of tests, indexed by a functional parameter in $L^2[0,1]$, all of them satisfying property 1, and give a criterion to select a member of the family that provides good discrimination of the alternatives [see also Cabaña (1993); the adjustment of Kolmogorov–Smirnov-type tests to certain local alternatives has also been studied in Janssen and Milbrodt (1993) and Drees and Milbrodt (1994)].

Transform the original sample to $U_i = \tilde{F}_0(X_i)$, $i = 1, \dots, n$, with distribution function $F = \tilde{F} \circ \tilde{F}_0^{-1}$. We may consider, without loss of generality the new problem of testing $\mathscr{H}_0$: $F(u) = u$.

The alternatives under consideration will be $\mathscr{H}_n$: $F^{(n)}(u) = u + (\delta/\sqrt{n})g_n(u)$, where g_n is a sequence of real functions that converges to a limit g in the following sense:

(A) g_n and g have derivatives $\dot{g}_n$ and $\dot{g}$ such that $\int_0^1 (\dot{g}_n(s) - \dot{g}(s))^2\, ds \to 0$ as $n \to \infty$.

(B) $\dot{g}_n/\sqrt{n} \to 0$ a.e. as $n \to \infty$.

In the terminology of Pfanzagl (1982), this family of alternatives converge to the null hypothesis following a path with tangent vector $\dot{g}$.

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Assumptions (A) and (B) imply that $(dF^{(n)}/du)^{1/2}$ can be written as $1 + \dot{h}_n \delta/\sqrt{n}$, where $\dot{h}_n$ converges in $L^2[0,1]$ to some limit $\dot{h}$, and so Theorem 1 in Oosterhoff and van Zwet (1979) implies that the sequence of alternatives $(\mathscr{H}_n)$ is contiguous to the uniform distribution in $[0,1]$.

Let us denote by $F_n(u) = (1/n)\sum_{i=1}^{n} 1_{\{U_i \le u\}}$ the empirical distribution function constructed from the sample (where 1_C is the indicator function of the set C) and denote by $b_n(u) = \sqrt{n}\,(F_n(u) - u)$ the empirical process. It is well known that under $\mathscr{H}_0$, b_n converges weakly in $D[0,1]$ to the standard Brownian bridge, a centered Gaussian process b characterized by $\mathbf{E}(b(u_1)b(u_2)) = (u_1 \wedge u_2) - u_1u_2$, and to $b + g$ under $\mathscr{H}_n$.

The stochastic integral

$$\mathscr{W}b(t) = \int_0^t db(s) + \int_0^t \frac{b(s)}{1-s}\,ds$$

maps a standard Brownian bridge b into a standard Wiener process $w = \mathscr{W}b$. It was introduced to goodness-of-fit theory in Khmaladze (1981).

We can replace the preceding expression by its differential form:

$$\frac{dw(u)}{1-u} = d\left(\frac{b(u)}{1-u}\right). \tag{1}$$

Define the *empirical martingale* w_n by means of the analogous equation

$$\frac{dw_n(u)}{1-u} = d\left(\frac{b_n(u)}{1-u}\right). \tag{2}$$

It is not hard to show by direct computation that w_n is actually a square integrable Martingale and that it has the same increasing process as the standard Wiener process w. Moreover, the weak limit in $L^2[0,1]$ of w_n is w under $\mathscr{H}_0$ and $w + \gamma$ under $(\mathscr{H}_n)$, where the drift γ is given by $\gamma(u) = \int_0^u (1-s)\,d(g(s)/(1-s))$ [see Khmaladze (1981)].

2. A family of transformations of the empirical process. For each measurable a: $[0,1] \to \mathbf{R}$, $\int_0^1 a^2(s)\,ds = 1$, define $V_a(u) = \int_0^u a^2(s)\,ds$ and introduce the sequence of processes

$$w_n^{V_a} = \mathscr{L}_a w_n = \int_0^{\cdot} a(s)\,dw_n(s). \tag{3}$$

If w is a standard Wiener process, the stochastic integral $w^{V_a} = \mathscr{L}_a w = \int_0^{\cdot} a(s)\,dw(s)$ is a V_a Wiener process, a centered Gaussian process characterized by $\mathbf{E}w^{V_a}(u_1)w^{V_a}(u_2) = V_a(u_1 \wedge u_2)$.

We will next show that $w_n^{V_a}$ converges weakly in $D[0,1]$ (equipped with Skorokhod's topology) to $w^{V_a} = \mathscr{L}_a w = \int_0^{\cdot} a(s)\,dw(s)$ under $\mathscr{H}_0$ and to $\mathscr{L}_a w + \mathscr{L}_a \gamma = w^{V_a} + \int_0^{\cdot} a(s)\,d\gamma(s)$ under $(\mathscr{H}_n)$ (Theorem 1). This allows the construction of a family of tests for the null hypothesis $F(u) = u$, consistent against any fixed alternative $F = G_0$, not uniform, by means of the rejection region $\{\sup_{0 \le u \le 1} |w_n^{V_a}(u)| > \text{const.}\}$, as described in Section 3.

Besides, if one is interested in detecting a specific family of contiguous alternatives, a particular $\hat{a}$ can be chosen so that the resulting test based on $w_n^{V_{\hat{a}}}$ is efficient, as will be seen in Section 3.2.

THEOREM 1. *Given* $a \in L^2[0,1]$, $\int_0^1 a^2(s)\,ds = 1$, *define* $A(u) = \int_0^u (|a(s)|/(1-s))\,ds$.

(i) (*Weak convergence of* $w_n^{V_a}$ *under* $\mathscr{H}_0$.) *When* $U_1,\dots,U_n$ *are i.i.d. uniformly in* $[0,1]$, $\mathscr{L}_a w_n$ *converges weakly in* $D[0,1]$ *to a* V_a*-Wiener process* w^{V_a}.

(ii) (*Consistency of the test against any fixed alternative.*) *When* $U_1,\dots,U_n$ *are i.i.d. with* $\mathbf{P}\{U_1 \le u\} = u + D(u)$, *where* $D(0) = D(1) = 0$, D *not identically zero and* V_a *is strictly increasing*, $\int_0^1 a^2(s)\,d(s+D(s)) < \infty$ *and* $\int_0^1 A^2(s)\,d(s+D(s)) < \infty$, *there exists* $u^* \in [0,1]$ *such that* $\lim_{n\to\infty} \mathbf{E} w_n^{V_a}(u^*) = \infty$ *and* $\mathbf{Var}\, w_n^{V_a}(u^*) < \infty$, *for every* n.

(iii) (*Asymptotic behavior under contiguous alternatives.*) *Assume that*: (a) *for each* n, $U_{n,1}, U_{n,2},\dots,U_{n,n}$ *are independent variables distributed according to* $F^{(n)}$ *with density* $f^{(n)}$ *such that* $f^{(n)}(u) = 1 + (\delta/\sqrt{n})\dot{g}_n(u)$, *where* g *and* g_n *are functions on* $[0,1]$ *with derivatives* $\dot{g}$ *and* $\dot{g}_n$ *satisfying* $g(0) = g(1) = 0$, $\lim_{n\to\infty}\int_0^1 (\dot{g}_n(s) - \dot{g}(s))^2\,ds = 0$ *and* $(\dot{g}_n)^2/\sqrt{n} \to 0$ *almost everywhere.*

If there exists a measure F^* *with density* f^* *such that* (b) *for all* n $f^{(n)} \le f^*$, $\int_0^1 a^4(s)\,df^*(s) < \infty$ *and* $\int_0^1 A^4(s)\,df^* < \infty$, *then* $\mathscr{L}_a w_n$ *converges weakly in* $D[0,1]$ *to* $w^{V_a} + \delta\gamma_a$, *where* $\gamma_a(u) = \int_0^u a(s)(1-s)\,d(g(s)/(1-s))$.

We state now some technical results to prepare the proof of Theorem 1.

LEMMA 1. *When the function* $g\colon [0,1] \to \mathbf{R}$ *has a square integrable derivative* $\dot{g}$ *and* $g(0) = g(1) = 0$, *then*:

(i) $\lim_{u\to 1}((g(u))^2/(1-u)) = 0$.
(ii) $(g(u))/(1-u)$ *is square integrable.*

LEMMA 2. *Let* U *be a random variable with distribution function* $F^{(n)}$ *and*

$$I(x,y,U) = \left[a(U) - \int_x^U \frac{a(s)}{1-s}\,ds\right]1_{\{x<U\le y\}} - \int_x^y \frac{a(s)}{1-s}\,ds\,1_{\{y<U\}}.$$

Then, under the assumptions of Theorem 1(iii), *there exist absolutely continuous finite measures* μ_1, μ_2, *on* $[0,1]$ *and* μ_3 *on* $[0,1]\times[0,1]$ *such that for* $x<y$, $|\mathbf{E}I(x,y,U)| \le \mu_1((x,y))$, $\mathbf{E}(I^2(x,y,U)) \le \mu_2((x,y))$ *and for* $x<y<z$, $\mathbf{E}(I^2(x,y,U)I^2(y,z,U)) \le \mu_3((x,y)\times(y,z))$ *for every* n.

The proofs of Lemmas 1 and 2 are given at the end of this section.

PROOF OF THEOREM 1. Part (i). Suppose first that a has an integrable derivative. Since the sequence w_n converges weakly to w in $D[0,1]$ [as a consequence of (iii) with $a(s) = 1$, $0 \le s \le 1$], there exist copies of w_n and w

with $\sup_{\{0\le u\le 1\}}|w_n(u)-w(u)|\to 0$ a.s. as $n\to\infty$. With these strongly convergent copies an integration by parts gives

$$\sup_{\{0\le u\le 1\}}|\mathscr{L}_a w_n(u)-\mathscr{L}_a w(u)|\le 2\sup_{\{0\le u\le 1\}}|a(u)|\sup_{\{0\le u\le 1\}}|w_n(u)-w(u)|\to 0.$$

For general a, it suffices to show that, for any uniformly continuous bounded functional Ψ in $D[0,1]$ with the Skorokhod distance ρ, $\mathbf{E}\Psi\mathscr{L}_a w_n\to\mathbf{E}\Psi\mathscr{L}_a w$. Let M be a bound for $|\Psi|$.

Given an arbitrary $\varepsilon>0$, choose δ such that $\rho(x,y)<\delta$ implies $|\Psi(x)-\Psi(y)|\le\varepsilon/4$ and choose an L^2 approximation a_ε of a with integrable derivative, such that the difference $\Delta_\varepsilon=a-a_\varepsilon$ has L^2-norm bounded by $\delta\sqrt{\varepsilon}/4\sqrt{M}$.

Then

$$\begin{aligned}&|\mathbf{E}\Psi(\mathscr{L}_a w_n)-\mathbf{E}\Psi(\mathscr{L}_a w)|\\&\quad\le|\mathbf{E}\Psi(\mathscr{L}_{a_\varepsilon}w_n)-\mathbf{E}\Psi(\mathscr{L}_{a_\varepsilon}w)|+|\mathbf{E}\Psi(\mathscr{L}_a w_n)-\mathbf{E}\Psi(\mathscr{L}_{a_\varepsilon}w_n)|\\&\qquad+|\mathbf{E}\Psi(\mathscr{L}_a w)-\mathbf{E}\Psi(\mathscr{L}_{a_\varepsilon}w)|.\end{aligned}$$

Because a_ε has an integrable derivative, the first term in the right-hand side is smaller than $\varepsilon/4$, for n sufficiently large.

From Doob's inequality applied to the Martingale $\mathscr{L}_{\Delta_\varepsilon}w_n$, we get

$$\mathbf{P}\left\{\left|\sup_{0\le u\le 1}\mathscr{L}_{\Delta_\varepsilon}w_n(u)\right|>\delta\right\}\le\frac{1}{\delta^2}\mathbf{E}\big(\mathscr{L}_{\Delta_\varepsilon}w_n(1)\big)^2=\frac{1}{\delta^2}\int_0^1\Delta_\varepsilon^2(s)\,ds\le\frac{\varepsilon}{16M}$$

and, therefore, $\mathbf{E}|\Psi(\mathscr{L}_a w_n)-\Psi(\mathscr{L}_{a_\varepsilon}w_n)|$ is bounded by $\varepsilon/4+2M\varepsilon/16M=3\varepsilon/8$.

The same argument applied to the Martingale $\mathscr{L}_{\Delta_\varepsilon}w$ leads to the estimate $\mathbf{E}|\Psi(\mathscr{L}_a w)-\Psi(\mathscr{L}_{a_\epsilon}w)|\le 3\varepsilon/8$.

Joining the previous results, the inequality $|\mathbf{E}\Psi(\mathscr{L}_a w_n)-\mathbf{E}\Psi(\mathscr{L}_a w)|\le\varepsilon$ follows for n sufficiently large, and this proves (i).

Part (iii). Suppose now that the assumptions in (iii) hold and $u<1$. Write

$$\begin{aligned}w_n^{V_a}(u)=\mathscr{L}_a w_n(u)&=\int_0^u a(s)(1-s)\,d\left(\frac{b_n(s)}{1-s}\right)\\&=\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(a(U_i)1_{\{U_i\le u\}}-\int_0^{u\wedge U_i}\frac{a(s)}{1-s}\,ds\right).\end{aligned}$$

The convergence of the finite-dimensional distributions of $w_n^{V_a}$ follows from the central limit theorem, after establishing

$$\lim_{n\to\infty}\mathbf{E}w_n^{V_a}(u)=\gamma_a(u)\tag{4}$$

and

$$\lim_{n\to\infty}\mathbf{Var}\,w_n^{V_a}(u)=V_a(u).\tag{5}$$

Compute

$$\mathbf{E}w_n^{V_a}(u) = \sqrt{n}\left[\int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right) f^{(n)}(s)\,ds - \int_0^u \frac{a(s)}{1-s}\,ds\,\mathbf{P}\{u < U\}\right],$$

where U has probability density $f^{(n)}$. This expectation becomes 0 when $f^{(n)}$ is replaced by the constant 1; hence, since $f^{(n)} = 1 + \delta(\dot{g}_n/\sqrt{n})$, then

$$\begin{aligned}\mathbf{E}w_n^{V_a}(u) &= \delta\int_0^u a(s)(1-s)\,d\left(\frac{g_n(s)}{1-s}\right)\\ &= \delta\int_0^u a(s)\dot{g}_n(s)\,ds + \delta\int_0^u a(s)\frac{g_n(s)}{1-s}\,ds.\end{aligned}$$

The first term converges to $\delta\int_0^u a(s)\dot{g}(s)\,ds$, because $\dot{g}_n$ converges to $\dot{g}$ in $L^2[0,1]$. Since $g_n(u) \to g(u)$, $\delta\int_0^u a(s)(g_n(s)/(1-s))\,ds \to \delta\int_0^u a(s)(g(s)/(1-s))\,ds$. This proves (4).

As for (5), compute

$$\begin{aligned}\mathbf{Var}\, w_n^{V_a}(u) &= \mathbf{Var}\left[\left(a(U) - \int_0^U \frac{a(s)}{1-s}\,ds\right)1_{\{U \le u\}} - \int_0^u \frac{a(s)}{1-s}\,ds\,1_{\{u<U\}}\right]\\ &= \int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right)^2 f^{(n)}(s)\,ds\\ &\quad + \left(\int_0^u \frac{a(s)}{1-s}\,ds\right)^2 \mathbf{P}\{u<U\} - \frac{1}{n}\mathbf{E}^2 w_n^{V_a}(u).\end{aligned}$$

Since

$$\int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right)^2 f^*(s)\,ds < \infty,$$

the dominated convergence theorem implies

$$\lim_{n\to\infty}\int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right)^2 f^{(n)}(s)\,ds = \int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right)^2 ds.$$

On the other hand, $\lim_{n\to\infty}\mathbf{P}\{u<U\} = 1-u$. Use finally that (4) implies $\lim_{n\to\infty}(\mathbf{E}^2 w_n^{V_a}(u)/n) = 0$ and conclude

$$\lim_{n\to\infty}\mathbf{Var}\, w_n^{V_a}(u) = \int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right)^2 ds + \left(\int_0^u \frac{a(s)}{1-s}\,ds\right)^2(1-u).$$

It is easily verified by differentiation that this last expression is $V_a(u)$. The case $u = 1$ is simpler, because $w_n^{V_a}(1)$ reduces to

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(a(U_i) - \int_0^{U_i}\frac{a(s)}{1-s}\,ds\right).$$

A minor modification of the moments criterion in Theorem 15.6 of Billingsley (1967) implies the tightness of $w_n^{V_a}$, provided there exists an absolutely continuous finite measure μ on $[0,1]\times[0,1]$ such that for $u_1 < u < u_2$, the inequality

$$\mathbf{E}\Big(\big(w_n^{V_a}(u) - w_n^{V_a}(u_1)\big)^2\big(w_n^{V_a}(u_2) - w_n^{V_a}(u)\big)^2\Big) \le \mu((u_1,u_2)\times(u_1,u_2)) \tag{6}$$

holds.

The increment $w_n^{V_a}(y) - w_n^{V_a}(x)$ is $(1/\sqrt{n})\sum_{i=1}^n I(x,y,U_i)$ [the notation $I(x,y,U)$ is introduced in Lemma 2]. Let $Y_i = I(u_1,u,U_i)$ and $Z_i = I(u,u_2,U_i)$, so that $w_n^{V_a}(u) - w_n^{V_a}(u_1) = (1/\sqrt{n})\sum_{i=1}^n Y_i$ and $w_n^{V_a}(u_2) - w_n^{V_a}(u) = (1/\sqrt{n})\sum_{i=1}^n Z_i$.

In order to establish (6), compute

$$\begin{aligned}&\mathbf{E}\Big(\big(w_n^{V_a}(u) - w_n^{V_a}(u_1)\big)^2\big(w_n^{V_a}(u_2) - w_n^{V_a}(u)\big)^2\Big)\\ &\quad = \frac{1}{n^2}\mathbf{E}\Big(\sum_{i,j,k,l} Y_iY_jZ_kZ_l\Big)\\ &\quad \le \mathbf{E}(Y_1^2)\mathbf{E}(Z_1^2) + 2(\mathbf{E}(Y_1Z_1))^2 + \mathbf{E}(Y_1^2Z_1^2)\end{aligned} \tag{7}$$

and apply the inequalities in Lemma 2 to get $\mu((u_1,u_2)\times(u_1,u_2)) = 3(\mu_2(u_1,u_2))^2 + \mu_3((u_1,u_2)\times(u_1,u_2))$. This ends the proof of (iii).

Part (ii). From $w_n^{V_a}(u) = \int_0^u a(s)(1-s)\,d((b_n(s)/(1-s)))$ and $\mathbf{E}b_n(u) = \sqrt{n}D(u)$, we get

$$\mathbf{E}w_n^{V_a}(u) = \sqrt{n}\int_0^u a(s)(1-s)\,d\left(\frac{D(s)}{1-s}\right). \tag{8}$$

Let us also denote by D the signed measure with distribution function D and denote by $|D|$ its total variation.

The expectation (8) is zero for all u, if and only if $|D|(\{s\colon a(s)\neq 0\}) = 0$. Observe that if C is any subset of $(0,1)$ with Lebesgue measure $\lambda(C) = 1$ and $D(C^c)\neq 0$, then $D(C^c)$ is necessarily positive and hence, $D(C)$ is negative and so the total variation $|D|(C)$ must be different from 0.

Under the assumption that V_a is strictly increasing, $\{s\colon a(s)\neq 0\}$ has Lebesgue measure 1, so that $|D|(\{s\colon a(s)\neq 0\})\neq 0$ and therefore there exists a u^* such that $\mathbf{E}w_n^{V_a}(u^*)\to\infty$ a.s.

Observe now that the variance of $w_n^{V_a}(u)$ is

$$\mathbf{Var}\left(a(U) - \int_0^U \frac{a(s)}{1-s}\,ds\,1_{\{U\le u\}} - \int_0^u \frac{a(s)}{1-s}\,ds\,1_{\{u<U\}}\right),$$

with $\mathbf{P}\{U\le u\} = u + D(u)$, so it is independent of n.

In order to show that it is finite, bound the second-order moment

$$\mathbf{E}\big(w_n^{V_a}(u)\big)^2 \le 2\int_0^u \left(a(s) - \int_0^s \frac{a(r)}{1-r}\,dr\right)^2 d(s + D(s)) + 2\left(\int_0^u \frac{a(s)}{1-s}\,ds\right)^2 \mathbf{P}\{u < U\}.$$

The first term is finite because a and A are in $L^2([0,1], \lambda + D)$. The remaining term is bounded by $\int_0^u \int_0^u (|a(s)|/(1-s))(|a(r)|/(1-r))\int_{r\vee s}^1 d(t + D(t))\,ds\,dr = \int_0^1 A^2(t)\,d(t + D(t)) < \infty$. This ends the proof of the theorem. □

PROOF OF LEMMA 1. Part (i) follows from $g^2(u) = (-\int_u^1 \dot g(s)\,ds)^2 \le (1-u)\int_u^1 (\dot g(s))^2\,ds$.

In order to prove (ii), define, for $0 \le u < 1$, $K(u) = \int_0^u (g^2(s)/(1-s)^2)\,ds$. This is an increasing function of u; therefore, it has a limit, finite or infinite, when $u \to 1$.

An integration by parts leads to

$$K(u) = \frac{g^2(u)}{1-u} - 2\int_0^u \frac{g(s)}{1-s}\dot g(s)\,ds;$$

hence

$$K^2(u) \le 2\left(\frac{g^2(u)}{(1-u)}\right)^2 + 4\|\dot g\|_{L^2} K(u).$$

Taking limits for $u \to 1$, it follows that $\lim_{u\to 1} K(u) < \infty$. □

PROOF OF LEMMA 2.

$$\mathbf{E}I(x,y,U) = \int_x^y \left(a(s) - \int_x^s \frac{a(r)}{1-r}\,dr\right) f^{(n)}(s)\,ds - \mathbf{P}\{y < U\}\int_x^y \frac{a(s)}{1-s}\,ds);$$

hence,

$$|\mathbf{E}I(x,y,U)| \le \int_x^y a^2(s) f^*(s)\,ds + \int_x^y A^2(s) f^*(s)\,ds + \mathbf{P}\{y < U\}\int_x^y \frac{|a(s)|}{1-s}\,ds.$$

Observe that

$$\mathbf{P}\{y < U\}\int_x^y \frac{|a(s)|}{1-s}\,ds \le \int_y^1 f^*(t)\,dt \int_x^y \frac{|a(s)|}{1-s}\,ds = \int_x^y \frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds.$$

This is a measure as a function of the interval (x, y), and it is finite because

$$\int_0^1 \frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds = \int_0^1 A(t)f^*(t)\,dt < \infty.$$

Then,

$$\mu_1((x,y)) = \int_x^y (a^2(s) + A^2(s))f^*(s)\,ds + \int_x^y \frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds.$$

In order to obtain a bound for the second-order moment of $I(x, y, U)$, write

$$\begin{aligned} I^2(x,y,U) \le{}& 3a^2(U)1_{\{x<U\le y\}} + 3A^2(U)1_{\{x<U\le y\}} \\ &+ 3(A(y) - A(x))^2 1_{\{y<U\}} \end{aligned}$$

and

$$\begin{aligned} \mathbf{P}\{y<U\}(A(y)-A(x))^2 &\le \int_y^1 f^*(t)\,dt \int_x^y 2(A(s)-A(x))\frac{|a(s)|}{1-s}\,ds \\ &\le 2\int_x^y A(s)\frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds. \end{aligned}$$

Joining these results, we obtain

$$\begin{aligned} &\mathbf{Var}\, I(x,y,U) \\ &\quad\le 3\int_x^y a^2(s)f^*(s)\,ds + 3\int_x^y A^2(s)f^*(s)\,ds \\ &\qquad + 3(A(y)-A(x))^2\mathbf{P}\{y<U\} \\ &\quad\le 3\int_x^y (a^2(s)+A^2(s))f^*(s)\,ds + 6\int_x^y A(s)\frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds. \end{aligned}$$

Each term in the right-hand side of this inequality is a finite measure as a function of the interval (x, y). For the first two, it is immediate from the assumptions on a and A. As for the third, it follows from the estimate

$$\begin{aligned} \int_0^1 A(s)\frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds &= \int_0^1 f^*(t)\int_0^t A(s)\frac{|a(s)|}{1-s}\,ds\,dt \\ &\le \int_0^1 f^*(t)A(t)\int_0^t \frac{|a(s)|}{1-s}\,ds\,dt \\ &= \int_0^1 f^*(t)A^2(t)\,dt < \infty. \end{aligned}$$

Suppose now that $x < y < z$. Since

$$I^2(x,y,U)I^2(y,z,U) = \left(\int_x^y \frac{a(s)}{1-s}\,ds\right)^2 I^2(y,z,U),$$

then

$$
\begin{aligned}
&\mathbf{E}I^2(x,y,U)I^2(y,z,U) \\
&\quad \le (A(y)-A(x))^2 \\
&\qquad \times\left[3\int_y^z (a^2(s)+A^2(s))f^*(s)\,ds + 3(A(z)-A(y))^2\mathbf{P}\{z<U\}\right].
\end{aligned}
$$

The expression $(A(y)-A(x))^2\int_y^z a^2(s)f^*(s)\,ds$ is bounded by the product of the finite measure $\int_y^z a^4(s)f^*(s)\,ds$ times $(A(y)-A(x))^2\int_y^z f^*(s)\,ds$, which is bounded by

$$
(A(y)-A(x))^2\int_y^1 f^*(s)\,ds \le 2\int_x^y A(s)\frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds,
$$

that is finite as was seen before. Thus, $(A(y)-A(x))^2\int_y^z a^2(s)f^*(s)\,ds$ is bounded by an absolutely continuous finite measure computed on $(x,y)\times(y,z)$.

A similar argument is used to bound

$$
\begin{aligned}
&(A(y)-A(x))^2\int_y^z A^2(s)f^*(s)\,ds \\
&\qquad \le 2\int_y^z A^4(s)f^*(s)\,ds\int_x^y A(s)\frac{|a(s)|}{1-s}\int_s^1 f^*(t)\,dt\,ds.
\end{aligned}
$$

Finally, observe that $(A(y)-A(x))^2(A(z)-A(y))^2\mathbf{P}\{z<U\}$ is bounded by

$$
4\int_x^y A(r)\frac{|a(r)|}{1-r}\int_y^z A(s)\frac{|a(s)|}{1-s}\int_{r\vee s}^1 f^*(t)\,dt\,ds\,dr.
$$

This is an absolutely continuous measure as a function of the Cartesian product $(x,y)\times(y,z)$. In order to show that it is finite, compute

$$
\begin{aligned}
&\int_0^1 A(r)\frac{|a(r)|}{1-r}\int_0^1 A(s)\frac{|a(s)|}{1-s}\int_{r\vee s}^1 f^*(t)\,dt\,ds\,dr \\
&\quad = \int_0^1 f^*(t)\int_0^t A(r)\frac{|a(r)|}{1-r}\,dr\int_0^t A(s)\frac{|a(s)|}{1-s}\,ds\,dt \\
&\quad \le \int_0^1 A^4(t)f^*(t)\,dt<\infty.
\end{aligned}
$$

□

NOTE ADDED IN PROOF. The hypotheses in Theorem 1(iii) can be weakened. In fact, (iii)(b) is not necessary: observe that the sequence $F^{(n)}$ is contiguous to the uniform distribution on $[0,1]$ [cf. Theorem 1 in Oosterhoff and van Zwet (1979)] and $\Lambda_n = \log(\prod_{i=1}^n f^{(n)}(U_i)/U_i)$, where the U_i's are i.i.d. uniform in $[0,1]$, is asymptotically Gaussian. On the other hand, $w_n^{V_a}$ is a sum of independent random variables, each of them depending on one of the U_i's. Then, the joint distribution of $w_n^{V_a}$ and Λ_n is asymptotically Gaussian. Therefore, Le Cam's third lemma [see Le Cam and Yang (1990), for instance] implies that, when replacing U_i by $U_{n,i}$ in $w_n^{V_a}$ and Λ_n, their joint distribution is still asymptotically Gaussian. The second order moments of $w_n^{V_a}$ do not change, and the bias is given by $\mathscr{L}_a\gamma = \int_0^{\cdot} a(u)\,d\gamma(u) = \int_0^{\cdot} a(s)(1-s)\,d(g(s)/1-s)$.

3. Goodness-of-fit tests

3.1. *The critical regions.* Given a random sample $X_1, X_2, \dots, X_n$ having continuous distribution $\tilde{F}$, to test the null hypothesis $\mathscr{H}_0$: $\tilde{F} = \tilde{F}_0$, transform the data by means of $U_i = \tilde{F}_0(X_i)$, so that, under $\mathscr{H}_0$, $U_1, U_2, \dots, U_n$ are i.i.d. uniform on $[0,1]$. For each $a \in L^2[0,1]$ construct $w_n^{V_a}(u) = \int_0^u a(s)(1-s)\,d((b_n(s)/(1-s)))$, define $K_n^{a,+} = \sup_{0\le u\le 1} w_n^{V_a}(u)$, $K_n^a = \sup_{0\le u\le 1}|w_n^{V_a}(u)|$ and use the critical regions $\{K_n^{a,+} > c_\alpha^+\}$ for a one-sided test for $\mathscr{H}_0$, or $\{K_n^a > c_\alpha\}$ for a two-sided test.

In view of Theorem 1(i), the tests with these critical regions have asymptotic level α, if c_α^+ and c_α are the well-known solutions of $\mathbf{P}\{\sup_{0\le u\le 1} w(u) > c_\alpha^+\} = \alpha$ and $\mathbf{P}\{\sup_{0\le u\le 1}|w(u)| > c_\alpha\} = \alpha$, where w is a standard Wiener process on$[0,1]$.

The test based on $\{K_n^a > c_\alpha\}$ is consistent under any alternative $F(u) = u + D(u)$, provided a and D satisfy the assumptions of Theorem 1(ii).

Suppose now that one is specially interested in detecting a specific sequence of contiguous alternatives $\mathscr{H}_n$: $F^{(n)}(u) = u + (\delta/\sqrt{n})g_n(u)$, where $F^{(n)}$ satisfies the assumptions of Theorem 1(iii)(b).

For any a with $\|a\|_{L^2} = 1$, the asymptotic distribution of $w_n^{V_a}(1)$ (and hence of K_n^a and $K_n^{a,+}$) under $\mathscr{H}_0$ is the same. If the assumptions in Theorem 1(iii)(b) were satisfied, the asymptotic drift under the alternative would be γ_a for each fixed a.

This suggests that one can look for an appropriate a for better discrimination of the alternatives of interest. As a heuristic criterion we propose to choose a in order to maximize the asymptotic drift in the point of maximum asymptotic variance, that is, to choose $\hat{a}$ such that $\gamma_{\hat{a}}(1) = \sup_{\{a:\ \|a\|_{L^2}=1\}} \gamma_a(1)$.

We verify in Section 3.2 that, if Theorem 1(iii)(b) is satisfied for the resulting $\hat{a}$, then the tests based on $K_n^{\hat{a}}$ and $K_n^{\hat{a},+}$ have high asymptotic efficiency, near optimal when the level and the power approach 0 and 1, respectively.

Denote $\gamma := \gamma_1$ and observe that $\gamma_a(1)$ is the inner product in $L^2[0,1]$ of $a(u)$ and $\dot{\gamma}(u) = \dot{g}(u) + (g(u)/(1-u))$; therefore, the optimum choice of a

under our heuristic criterion is

$$\hat{a} = \frac{\dot{\gamma}}{\|\dot{\gamma}\|_{L^2}}. \tag{9}$$

Notice that with this choice of $\hat{a}$,

$$\gamma_{\hat{a}}(u) = \int_0^u \hat{a}(s)\dot{\gamma}(s)\,ds = \int_0^u \|\dot{\gamma}\|_{L^2}\hat{a}^2(s)\,ds = \|\dot{\gamma}\|_{L^2} V_{\hat{a}}(u).$$

3.2. *On the efficiency of the test.* Assume that for $a = \hat{a}$, the assumptions in Theorem 1(iii)(b) hold. We shall compare now the efficiency of the modified Kolmogorov–Smirnov test (MKST) and the likelihood ratio test (LRT) by comparing their local asymptotic powers. We present in detail the case of the one-sided test to simplify the exposition. The two-sided case is similar.

Instead of comparing directly the MKST with the LRT, we compare both with a simple test (ST) with critical region $\{w_n^{V_{\hat{a}}}(1) > \text{const.}\}$. This ST is closely related to the MKST, and asymptotically equivalent to the LRT:

In fact, the LRT is based on an asymptotically Gaussian test variable with limit expectation and variance $(E(\delta), V_a)$, such that the efficacy $((\partial E(\delta)/\partial\delta)|_{\delta=0})^2 V_a^{-1}$ is $E_{\mathrm{LRT}} = \int_0^1 (\dot{g}(u))^2\,du = \|\dot{g}\|_{L^2}^2$, under our assumptions on the sequence g_n [Capon (1965)].

Moreover, the test variable of the ST is asymptotically Gaussian, with expectation $\delta\|\dot{\gamma}\|_{L^2}$ and variance 1. Hence the efficacy is

$$E_{\mathrm{ST}} = \|\dot{\gamma}\|_{L^2}^2 = \int_0^1 \left(\dot{g}(u) + \frac{g(u)}{1-u}\right)^2 du = \|\dot{g}\|_{L^2}^2 - \int_0^1 d\left(\frac{(g(u))^2}{1-u}\right) = \|\dot{g}\|_{L^2}^2$$

[Lemma 1(ii)]. Notice that γ is obtained from g via the isometry that maps $g \in L^2[0,1]$ onto $g(\cdot) - \int_0^{\cdot} (g(s)/(1-s))\,ds$. This isometry appears in Efron and Johnston (1990) and Ritov and Wellner (1988) in the context of hazard rates. See also Groneboom and Wellner (1992).

Let Φ be the standard Gaussian distribution function and let φ be the corresponding density.

The critical region for the ST with level α is $\{w_n^{V_{\hat{a}}}(1) > -\Phi^{-1}(\alpha)\}$ and its asymptotic local power is $1 - \beta_{\mathrm{ST}}(\alpha, \delta, \hat{a}) = 1 - \Phi(\delta\|\dot{\gamma}\|_{L^2} - \Phi^{-1}(\alpha))$.

The MKST with asymptotic level α has critical region $\{K_n^{\hat{a},+} > -\Phi^{-1}(\alpha/2)\}$, by the reflexion principle, and its asymptotic power is

$$\begin{aligned} 1 - \beta_{\mathrm{MKST}}(\alpha, \delta, \hat{a}) &= \mathbf{P}\left\{\sup_{0\le u\le 1} w^{V_{\hat{a}}}(u) + \delta\|\dot{\gamma}\|_{L^2} u > -\Phi^{-1}(\alpha/2)\right\} \\ &= \mathbf{P}\left\{\sup_{0\le z\le 1} w(z) + \delta\|\dot{\gamma}\|_{L^2} z > -\Phi^{-1}(\alpha/2)\right\}, \end{aligned} \tag{10}$$

where w is a standard Wiener process on $[0,1]$.

This probability can be computed exactly [see, e.g., Karatzas and Shreve (1991)]. Namely, if b denotes a standard Brownian bridge, with $c = -\Phi^{-1}(\alpha/2)$,

$$
\begin{aligned}
1 - \beta_{\mathrm{MKST}}(\alpha, \delta, \hat{a}) & \\
&= \int_{-\infty}^{\infty} \mathbf{P}\Big\{ \sup_{0 \le z \le 1} w(z) + \delta \|\dot{\gamma}\|_{L^2} z > c | w(1) = t \Big\} \varphi(t)\, dt \\
&= \int_{-\infty}^{c - \delta\|\dot{\gamma}\|_{L^2}} \mathbf{P}\Big\{ \sup_{0 \le z \le 1} b(z) + (\delta \|\dot{\gamma}\|_{L^2} + t) z > c | w(1) = t \Big\} \varphi(t)\, dt \\
&\qquad + 1 - \Phi(c - \delta \|\dot{\gamma}\|_{L^2}) \\
&= \int_{-\infty}^{c - \delta\|\dot{\gamma}\|_{L^2}} \mathbf{P}\big\{ \text{for some } z,\ b(z) > c - (\delta \|\dot{\gamma}\|_{L^2} + t) z \big\} \varphi(t)\, dt \\
&\qquad + 1 - \Phi(c - \delta \|\dot{\gamma}\|_{L^2}) \\
&= \int_{-\infty}^{c - \delta\|\dot{\gamma}\|_{L^2}} \exp\big(-2c(c - \delta \|\dot{\gamma}\|_{L^2})\big) \varphi(t)\, dt + 1 - \Phi(c - \delta \|\dot{\gamma}\|_{L^2}) \\
&= \Phi(-c - \delta \|\dot{\gamma}\|_{L^2}) \exp(2c\delta \|\dot{\gamma}\|_{L^2}) + 1 - \Phi(c - \delta \|\dot{\gamma}\|_{L^2}).
\end{aligned} \tag{11}
$$

In order to compare the performance of MKST and ST, we apply the ST to samples of size $[en]$ and the MKST to samples of size n, when the alternative is $F^{(n)}$. The value e for which both tests have the same local asymptotic power $1 - \beta$ for equal level α is a measure of the local asymptotic relative efficiency (LARE) and will depend, in general, on α and β.

The asymptotic distribution of the ST for the samples of size $[en]$ is $N(0, 1)$ under $\mathscr{H}_0$, and $N(\delta\sqrt{e}\|\dot{\gamma}\|_{L^2}, 1)$ under the alternative.

Fix α and β (and assume them both smaller than 1/2). In order to attain for both tests power $1 - \beta$ and level α, δ and e must satisfy

$$\beta = \Phi\big(-\Phi^{-1}(\alpha) - \delta\sqrt{e}\|\dot{\gamma}\|_{L^2}\big) \tag{12}$$

and

$$\beta = \beta_{\mathrm{MKST}}(\alpha, \delta, \hat{a}) \tag{13}$$

Using (11), a numerical computation gives the values of $e = e(\alpha, \beta)$ for given α and β, eliminating δ in (12) and (13).

Since $K_n^{\hat{a}, +} > w_n^{V_{\hat{a}}}$, $\beta_{\mathrm{MKST}}(\alpha, \delta, \hat{a}) \le \beta_{\mathrm{ST}}(\alpha/2, \delta, \hat{a})$; hence we can obtain a bound for e as follows: From (12),

$$\delta\sqrt{e}\|\dot{\gamma}\|_{L^2} = -\Phi^{-1}(\alpha) - \Phi^{-1}(\beta).$$

From $\beta = \beta_{\mathrm{MKST}}(\alpha, \delta, \hat{a}) \le \beta_{\mathrm{ST}}(\alpha/2, \delta, \hat{a}) = \Phi(-\Phi^{-1}(\alpha/2) - \delta\|\dot{\gamma}\|_{L^2})$,

$$\delta\|\dot{\gamma}\|_{L^2} \le -\Phi^{-1}(\alpha/2) - \Phi^{-1}(\beta).$$

Then

$$e \ge \left(\frac{\Phi^{-1}(\alpha) + \Phi^{-1}(\beta)}{\Phi^{-1}(\alpha/2) + \Phi^{-1}(\beta)} \right)^2$$

and the right-hand term of this inequality can be chosen as near to 1 as desired, for α and β sufficiently small.

By numerical computation it is readily verified that the lower bound is very close to the actual values of $e = e(\alpha, \beta)$.

3.3. *Some examples*. Alternatives having changes in location or scale can be written as $\mathscr{H}_n$: $\tilde{F}(t) = \tilde{F}^{(n)}(t)$ and, consequently, $\hat{a}$ and the corresponding test statistics can be derived easily for any specific null hypothesis.

Let us first study the change in location case, where the alternative hypothesis is $\mathscr{H}_n$: $\tilde{F}(t) = \tilde{F}_0(t - \delta/\sqrt{n})$.

From $\tilde{F}_0(t - \delta/\sqrt{n}) = \tilde{F}_0(t) - (\delta g_n \circ \tilde{F}_0(t)/\sqrt{n})$, we obtain

$$g_n\big(\tilde{F}_0(t)\big) = \frac{\sqrt{n}}{\delta}\left(\tilde{F}_0(t) - \tilde{F}_0\left(t - \frac{\delta}{\sqrt{n}}\right)\right) = \tilde{f}_0\left(t - \theta\frac{\delta}{\sqrt{n}}\right), \qquad 0 < \theta < 1,$$

and

$$\dot{g}_n\big(\tilde{F}_0(t)\big) = \frac{\tilde{f}_0'\big(t - \theta'(\delta/\sqrt{n})\big)}{\tilde{f}_0(t)}, \qquad 0 < \theta' < 1,$$

where $\tilde{f}_0$ is the density function of the distribution $\tilde{F}_0$ and so $g(\tilde{F}_0(t)) = \tilde{f}_0(t)$ and $\lim_{n\to\infty} \dot{g}_n(\tilde{F}_0(t)) = \tilde{f}_0'(t)/\tilde{f}_0(t)$.

We present some examples of change of location tests for symmetric unimodal distributions, with decreasing nonvanishing density $\tilde{f}_0$ on $[0, +\infty)$, and sectionally continuous derivative $\tilde{f}_0'$ bounded by a constant C, nondecreasing on $[K, \infty)$ for some positive constant K.

These assumptions lead to

$$|\dot{g}_n\big(\tilde{F}_0(t)\big)| \leq \begin{cases} \left|\dfrac{\tilde{f}_0'(|t| - |\delta|)}{\tilde{f}_0(t)}\right|, & \text{for } |t| > K + \delta, \\[2ex] \dfrac{C}{\tilde{f}_0(t)}, & \text{for } |t| \leq K + \delta, \end{cases}$$

that implies property (B) (Section 1). Moreover, the right-hand side is square integrable provided

$$\int_K^\infty \left(\frac{\tilde{f}_0'(t - |\delta|)}{\tilde{f}_0(t)}\right)^2 \tilde{f}_0(t)\, dt < \infty \tag{14}$$

and in this case property (A) holds.

On the other hand, the measures with distribution functions $\tilde{F}^{(n)}(t) = \tilde{F}_0(t - \delta/\sqrt{n})$ are dominated by the measure $\tilde{F}^*$ with density

$$\tilde{f}^*(t) = \begin{cases} \tilde{f}_0(t - |\delta|), & t > |\delta|, \\ \tilde{f}_0(0), & |t| \leq |\delta|, \\ \tilde{f}_0(t + |\delta|), & t < -\delta. \end{cases}$$

With the change of variables $u = \tilde{F}_0(t)$, $g(u) = \tilde{f}_0(\tilde{F}_0^{-1}(u)) = f_0(u)$ and according to (9), $\hat{a}$ has to be chosen proportional to

$$\dot{\gamma}(u) = \frac{f_0'(u)}{f_0(u)} + \frac{f_0(u)}{1-u}$$

and satisfying $\int_0^1 \hat{a}^2(s)\,ds = 1$.

The remaining assumptions needed in order to apply the results in Section 2 [hypothesis (iii)(b) of Theorem 1 concerning $\hat{a}$ and $f^* = (\tilde{f}^* \circ \tilde{F}_0^{-1} / \tilde{f}_0 \circ \tilde{F}_0^{-1})$ are verified for each of the following examples separately].

EXAMPLE 1. When $\tilde{F}_0$ is the logistic distribution, $\tilde{F}_0(t) = e^t/(1+e^t)$, $\tilde{f}_0(t) = e^t/(1+e^t)^2$, $g(u) = u(1-u)$, $\dot{\gamma}(u) = \dot{g}(u) + (g(u)/(1-u)) = 1-u$, $\|\dot{\gamma}\|^2 = 1/3$ and, hence, $\hat{a}(u) = \sqrt{3}(1-u)$.

The assumptions of Theorem 1 hold: $\tilde{f}_0'(t)$ is negative for $t > 0$ and decreases in absolute value for t large enough. The integrand in (14) is $O(e^{-t})$ for $t \to \infty$ and hence the integral is finite. Finally, since $\hat{a}(u)$ and the corresponding $A(u) = \sqrt{3}u$ are bounded and f^* is a finite measure, hypothesis (iii)(b) of Theorem 1 is in force.

EXAMPLE 2. Let F_0 be the standard Normal distribution, $F_0(t) = \Phi(t) = \int_{-\infty}^t \varphi(s)\,ds$, $\varphi(t) = (1/\sqrt{2\pi})e^{-t^2/2}$. Now $\dot{\gamma}(u) = (\varphi(\Phi^{-1}(u))/(1-u)) - \Phi^{-1}(u)$ and

$$\begin{aligned}\|\dot{\gamma}\|^2 &= \int_0^1 \left(\frac{\varphi(\Phi^{-1}(u))}{(1-u)} + \Phi^{-1}(u)\right)^2 du \\ &= \int_{-\infty}^{\infty} \left(\frac{\varphi(t)}{1-\Phi(t)} - t\right)^2 \varphi(t)\,dt \\ &= \int_{-\infty}^{\infty} d\left(\frac{\varphi^2(t)}{1-\Phi(t)}\right) + \int_{-\infty}^{\infty} t^2\varphi(t)\,dt = 1;\end{aligned}$$

hence, $\hat{a}(u) = \dot{\gamma}(u)$.

The function $\varphi'(t) = -t\varphi(t)$ is negative for $t > 0$ and nondecreasing for large t. From

$$1 - \Phi(t) = \frac{\varphi(t)}{t} - \frac{\varphi(t)}{t^3} + (3 + o(1))\frac{\varphi(t)}{t^5}, \tag{15}$$

the integrand in (14) is equivalent to $t^2\varphi(t)$ at $t = \infty$, and hence (14) holds.

From (15), $\lim_{u \to 1} \hat{a}(u) = 0$; hence $\int_{1/2}^1 \hat{a}^4(s) f^*(s)\,ds$ and $\int_{1/2}^1 A^4(s) f^*(s)\,ds$ are finite. For $u \to 0$, $\hat{a}(u)$ is equivalent to $-\Phi^{-1}(u)$ and $\int_0^{1/2} (\Phi^{-1}(s))^4 f^*(s)\,ds = \int_{-\infty}^0 t^4\varphi((t + |\delta|) \wedge 0)\,dt < \infty$. The finiteness of $\int_0^1 A^4(s) f^*(s)\,ds$ poses no additional problem, so the assumptions of Theorem 1 hold.

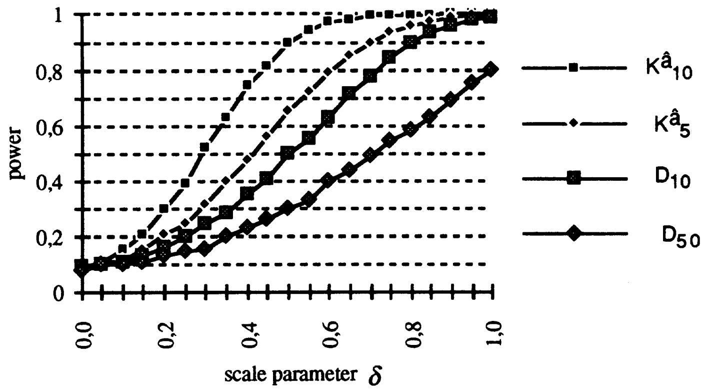

FIG. 1. *Estimated power of the tests based on $K_n^{\hat{a}}$ and the standard KS test based on D_n. Level $\alpha = 10\%$.*

EXAMPLE 3. Changes in scale are treated similarly: Write the alternative as $\mathscr{H}_n$: $\tilde{F}(t) = \tilde{F}_0(t - (\delta/\sqrt{n})(t-\mu)) = \tilde{F}_0(t) - (\delta/\sqrt{n})g_n \circ F_0(t)$, so that $g_n(\tilde{F}_0(t)) = (t-\mu)\tilde{f}_0(t - (\theta\delta/\sqrt{n})(t-\mu))$ and, consequently, $g(\tilde{F}_0(t)) = (t-\mu)f_0(t)$ and

$$\gamma(u) = \frac{\tilde{f}_0'\big(\tilde{F}_0^{-1}(u)\big)\big(\tilde{F}_0^{-1}(u) - \mu\big)}{\tilde{f}_0\big(\tilde{F}_0^{-1}(u)\big)} + \frac{\tilde{f}_0\big(\tilde{F}_0^{-1}(u)\big)\big(\tilde{F}_0^{-1}(u) - \mu\big)}{1-u} + 1,$$

and proceed as before.

We present now a numerical example of the proposed goodness-of-fit test. We have simulated samples of sizes $n = 50$ and $n = 100$ with laws $F^{(n)}(u) = u + \delta g(u)$, where $g(u) = 2u^2 - u$ for $0 \le u < 1/2$, and $g(u) = -2u^2 + 3u - 1$ for $1/2 \le u \le 1$, for different values of the scale parameter δ.

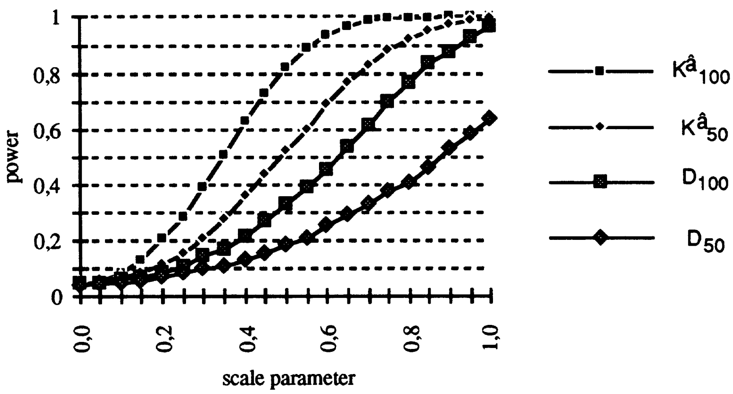

FIG. 2. *Estimated power of the tests based on $K_n^{\hat{a}}$ and the standard KS test based on D_n. Level $\alpha = 5\%$.*

The optimum choice of the score function $\hat{a}$ [see (9)] is $\hat{a}(u) = \sqrt{3}(2u - 2 + 1/(1-u))$ for $0 \le u < 1/2$ and $\hat{a}(u) = \sqrt{3}(2 - 2u)$ for $1/2 \le u \le 1$.

The power of the tests based on $K_n^{\hat{a}}$ and the standard Kolmogorov–Smirnov test based on $D_n(u) = \sup_{0 \le u \le 1} \sqrt{n}(F_n(u) - u)$ was calculated by simulation (5000 replications). The behavior of these tests is summarized in Figures 1 and 2.

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