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FUNCTIONAL LAWS OF THE ITERATED LOGARITHM FOR THE PARTIAL SUMS OF I.I.D. RANDOM VARIABLES IN THE DOMAIN OF ATTRACTION OF A COMPLETELY ASYMMETRIC STABLE LAW[1]

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Suppose X and X_n, $n \geqq 1$, are i.i.d. random variables whose common distribution lies in the domain of attraction of a completely asymmetric stable law of index α $(0 < \alpha < 2)$, so that (i) as $v \to \infty$, $v \to P\{X \geqq v\}$ varies regularly with exponent $-\alpha$, and (ii) $\lim_{v\to\infty} P\{X \leqq -v\}/P\{X \geqq v\} = 0$. Under a condition only slightly more stringent than (ii), we present Strassen-type functional laws of the iterated logarithm for the partial sums $S_n = \sum_{m \leqq n} X_m$, $n \geqq 1$. Our laws hold in particular when $X \geqq 0$; the proofs in this case utilize some new large deviation results for the S_n's.

1. Introduction. Suppose V_n, $n \geqq 1$, are i.i.d. random variables with mean 0 and variance 1. Put $S_n = V_1 + \cdots + V_n$, and define random functions H_n, $n \geqq 3$, on $[0, \infty)$ by

$$H_n(t) = (S_{[nt]} + (nt - [nt])V_{[nt]+1})/(2n \log(\log n))^{\frac{1}{2}}$$

$(t \geqq 0)$. H_n takes values in the space $C[0, \infty)$ of continuous functions on $[0, \infty)$. Strassen (1964) showed that with probability one (w.p. 1)

(1.1) the sequence (H_n) is relatively compact in the topology of uniform convergence on compact intervals, and its limit points coincide with $K_2 \equiv \{x \in C[0, \infty): x$ is absolutely continuous, with $\int \dot{x}^2(t)\, dt \leqq 1\}$

(actually Strassen worked with the restrictions of the H_n's to $[0, 1]$, but his argument yields (1.1)). Various improvements of (1.1) can be obtained. For example, (1.1) implies the Hartman–Wintner law $\limsup_n |H_n(1)| = 1$ w.p. 1 and this in turn implies $\lim_{s\to\infty} \limsup_n \sup_{t \geqq s} (|H_n(t)|/(t \log\log t)^{\frac{1}{2}}) = 0$ w.p. 1; it follows that

(1.2) (H_n) is relatively compact in the topology Φ_w

induced by the metric $d(x, y) = \sup_t |x(t) - y(t)|/(w(t))^{\frac{1}{2}}$, where $w(t) = (\max(t, 3) \log\log(\max(t, 3)))^{\frac{1}{2}}$, $t \geqq 0$.

Our purpose here is to describe similar results when V_1 lies in the domain of

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attraction of a completely asymmetric stable law of index α, $0 < \alpha < 2$. Under a mild condition that the negative tail of the distribution of V_1 decrease rapidly enough as compared to the positive tail, we obtain analogues of (1.1) for the "small value" behavior of the partial sums. We do not, however, have a satisfactory analogue of the more interesting (1.2). One can formulate functional laws of the iterated logarithm for the "large value" behavior of the partial sums; these laws are closer in spirit to those of Wichura (1974) than they are to those presented here, and we will give them elsewhere. In Section 2 we discuss the function space that plays the role of $C[0, \infty)$ in Strassen's work. The main results, including some large deviation theorems of independent interest, are presented in Section 3 for $\alpha \neq 1$, and in Section 4 for $\alpha = 1$; we limit attention to nonnegative random variables in these sections. Some complements and extensions are given in Section 5. Section 7 sets up some general machinery for proving limit theorems of the kind involved here. The proofs of the main results appear in Section 8, with some of the spade work having been done in Section 6. Proofs of some of the applications are sketched very briefly in Section 9.

In regard to the partial sums S_n, strong forms of the law of the iterated logarithm have been established under various conditions by Lipshutz (1956b) for $\alpha \neq 1$ and Mijnheer (1972) for $\alpha = 1$, and again by Kalinauskaĭte (1966) for $\alpha \neq 1$ and Breiman (1968) for $\alpha < 1$. These results imply weak laws, which have been dealt with further by, e.g., Zolotarev (1964) for $0 < \alpha < 2$, Miller (1967) for $\alpha \leqq 1$, and Fristedt and Pruitt (1971) for $\alpha < 1$, the latter under very weak conditions. The normalizations used in our functional laws were suggested by those of Fristedt and Pruitt.

In the context of completely asymmetric stable processes, Mijnheer (1973) has a result which implies that points outside the sets K_α introduced in Section 2 cannot be limit points of the random functions of Sections 3 and 4. Bingham (1973) has elaborated on the functional central limit theorem for stable processes; his results do not overlap with ours.

We shall make use of the terminology established in the last part of Section 1 of Wichura (1974); in particular $\psi^{\sim}$ denotes the inverse of a function ψ, and the notation $\psi \in \mathscr{R}\mathscr{V}_\infty(\rho)$ means that ψ is regularly varying at ∞ with exponent ρ. Given a number a, we write a^+ for $\max(a, 0)$, and a^- for $(-a)^+$. Given a real function x on the line and a finite interval A, we write $x(A)$ for $x(t) - x(s)$, where s and t are the endpoints of A, and $|A|$ for $t - s$.

2. The space Δ. Our goal in this section is to describe the function space, Δ, that plays the same role in our work as $C[0, \infty)$ does in Strassen's. We first introduce an auxiliary space, D, which we will later use in defining Δ.

For D, we take the space of all real-valued functions on $[0, \infty)$ which have finite left and right limits at each $t \geqq 0$, which are right-continuous at each $t > 0$, and which vanish at $t = 0$. It is to be emphasized that the functions in D need not be continuous at 0; the assumption that they all vanish at $t = 0$ is not particularly important, but is convenient for our purposes. We shall endow D with a topology analogous to the so-called M_1-topology introduced (on a different function space) by Skorokhod (1956), and treated further by Whitt (1974a and b) and Vervaat (1974). This topology is defined in terms of the closeness of the initial segments of the parametrized graphs of the functions of D, in the following manner.

Let $x \in D$. The *graph*, Γ_x, of x is the subset of $[0, \infty) \times (-\infty, \infty)$ consisting of all pairs (t, β) such that β belongs to the closed interval whose endpoints are $x(t-)$ and $x(t+)$; here $x(0-)$ is to be interpreted as $x(0)$, i.e. as 0. A *parametrization* of x is a one-to-one continuous mapping

$$s \to (\tau(s), \eta(s))$$

of $[0, \infty)$ onto Γ_x such that τ is nondecreasing. If χ and χ' are both parametrizations of x, then there exists a continuous, strictly increasing mapping, say ψ, of $[0, \infty)$ onto itself such that $\chi' = \chi\psi$. Given a parametrization $\chi = (\tau, \eta)$ of x, and positive numbers s and ε, we put

$$N_{x,\chi,s,\varepsilon} = \{y \in D\colon \text{ for some parametrization } \chi' = (\tau', \eta') \text{ of } y,\ |\chi'(r) - \chi(r)| < \varepsilon \text{ for } 0 \leqq r \leqq s\};$$

here

$$|\chi'(r) - \chi(r)| = |\tau'(r) - \tau(r)| + |\eta'(r) - \eta(r)|\,.$$

Roughly speaking, $N_{x,\chi,s,\varepsilon}$ consists of those y in D such that some initial segment of the graph of y can be "matched up" to within ε with the initial segment $\{\chi(r);\ r \leqq s\}$ of the graph of x. Although the individual sets $N_{x,\chi,s,\varepsilon}$ depend on the parametrization χ, the family $\{N_{x,\chi,s,\varepsilon}\colon s > 0, \varepsilon > 0\}$ does not.

Now put $\mathscr{B} = \{N_{x,\chi,s,\varepsilon}\colon x \in D,\ s > 0,\ \varepsilon > 0, \text{ and } \chi \text{ parametrizes } x\}$. It is easily checked that the class of unions of elements of $\mathscr{B}$ is a topology, which we will call the *M_1-topology on D*. Under this topology, $\mathscr{D}$ is second countable and regular, hence metrizable and separable. One may define the distance between x_1 and x_2 in D to be

$$\textstyle\sum_{1 \leqq m < \infty} 2^{-m} \inf \{\min (1, \sup (|\chi_2^{(m)}(s) - \chi_1^{(m)}(s)|\colon 0 \leqq s < \infty))\}$$

where the infimum in the mth summand is taken over all parametrizations $\chi_i^{(m)}$ of the functions $x_i^{(m)}$, $i = 1, 2$; here $x_i^{(m)}$ denotes the function in D whose value at $t \in [0, \infty)$ is $x_i(t) \min (1, \max [0, m + 1 - t])$. D is not complete under this metric; however an equivalent complete metric, of interest for weak convergence, has been constructed by Whitt (1973).

Given functions x_n, $n \geqq 1$, and x in D, we write $x_n \to x$ to mean that the x_n's converge to x in the M_1-topology. Necessary and sufficient for $x_n \to x$ is the existence of parametrizations χ_n of x_n, $n \geqq 1$, and χ of x, such that $\chi_n(s) \to \chi(s)$ uniformly for s in compact subsets of $[0, \infty)$. The following examples may help to give some feeling for M_1-convergence. Put

$$\begin{aligned} x_t &= I_{(0,\infty)}, & &\text{if } t = 0, \\ &= I_{[t,\infty)}, & &\text{if } 0 < t < \infty, \\ &= 0, & &\text{if } t = \infty. \end{aligned}$$

Then as $n \to \infty$,

(a) $x_{1/n} \to x_0$ and $x_n \to x_\infty$
(b) $x_{1-1/n}/2 + x_{1+1/n}/2 \to x_1$, whereas
(c) the sequence $(x_{1-1/n} - x_1 + x_{1+1/n})_{n \geq 1}$ does not converge.

Example (b) shows that the M_1-topology is weaker than the analogue of Skorokhod's J_1-topology (cf. Stone (1963) and Lindvall (1973)).

Here are some necessary and sufficient conditions for convergence and compactness in D. These conditions make use of the moduli $m_{\delta,v}$, $\delta > 0$, $v > 0$, which are defined on D through the formula

$$m_{\delta,v}(x) = \sup\{\Delta(x; s, t, u) : 0 \leqq s \leqq t \leqq u \leqq v,\ u - s < \delta\}, \tag{2.1}$$

where

$$\Delta(x; s, t, u) = (|x(u) - x(t)| + |x(t) - x(s)| - |x(u) - x(s)|)/2$$

is the Euclidean distance from $x(t)$ to the closed interval whose endpoints are $x(s)$ and $x(u)$. One has (cf. Skorokhod (1956) Section 2.4) $x_n \to x$ iff

$$\lim_{\delta \downarrow 0} \limsup_n m_{\delta,v}(x_n) = 0 \qquad \text{for each} \quad v > 0, \qquad \text{and} \tag{2.2}$$

$$x_n(t) \to x(t) \qquad \text{for each point} \quad t \quad \text{in a dense subset of } (0, \infty). \tag{2.3}$$

Moreover, $x_n \to x$ implies $x_n(t) \to x(t)$ at each continuity point t of x; if x is continuous at each point of some finite interval $[0, u]$, then the convergence is uniform for t in $[0, u]$. A sequence (x_n) of points of D is relatively compact in the M_1-topology iff (2.2) holds and

$$\limsup_n |x_n(t)| < \infty \qquad \text{for each point} \quad t \quad \text{in a dense subset of} \quad (0, \infty). \tag{2.4}$$

We are now in a position to introduce the function space Δ that figures prominently in our results. Roughly speaking, what we wish to do is allow $+\infty$ as a positive value for the functions in D; an expedient way to accomplish this is the following. Put $D_* = \{x \in D : x(t) \leqq 1 \text{ for all } t \geqq 0\}$, and endow D_* with the subspace M_1-topology. Define a mapping Φ on D_* by letting $\Phi(x)$ (for x in D_*) be the function on $[0, \infty)$ defined by

$$\Phi(x)(t) = \rho(x(t)) \tag{2.5}$$

where

$$\begin{aligned} \rho(a) &= a, & &\text{if } a < 0 \\ &= a/(1-a), & &\text{if } 0 \leqq a \leqq 1. \end{aligned} \tag{2.6}$$

We take

$$\Delta = \Phi(D_*) \tag{2.7}$$

to be the image of D_* under the mapping Φ, and define the *M_1-topology on* Δ to be that topology which makes Φ a homeomorphism. The function x in Δ for which $x(t) = \infty$ for all $t > 0$ has neighborhoods which are admittedly rather "coarse"; this is of no consequence here because our results are concerned with the "small," as opposed to "large," values of the random functions introduced in the following sections.

Throughout this paper, the convergence (or relative compactness) of a sequence of points in Δ is to be understood as convergence (or relative compactness) relative to the M_1-topology. Given a sequence (x_n) of points in Δ, we write, say,

$$x_n \to K \tag{2.8}$$

to mean that (x_n) is relatively compact and has K for its set of limit points.

Two subspaces of Δ will frequently be encountered in the sequel. The first is

$$\Delta_\uparrow = \{x \in \Delta : \ x \text{ is non-decreasing}\}\,; \tag{2.9}$$

the second is

$$\Delta_+ = \{x \in \Delta : \ x \text{ has no negative jumps}\}\,. \tag{2.10}$$

Each x in $\Delta_\uparrow$ is nonnegative, because $x(0) = 0$ by assumption. Every sequence (x_n) of points in $\Delta_\uparrow$ is relatively compact, and $x_n \to x$ iff $x_n(t) \to x(t)$ at each continuity point t of x. In other words, the M_1-topology relativized to $\Delta_\uparrow$ is just the usual topology of weak convergence. In relation to Δ_+, useful moduli are the $m_{\delta,v}^*$, $\delta > 0$, $v > 0$, defined on Δ by the formula

$$m_{\delta,v}^*(x) = \sup\{(x(u) - x(t))^- : 0 \leqq t < u \leqq v,\ u - t < \delta\}\,; \tag{2.11}$$

here $(\infty - \infty)^-$ is to be interpreted as 0. One has (cf. (2.1) and (2.5))

$$m_{\delta,v}(\Phi^\sim(x)) \leqq m_{\delta,v}^*(\Phi^\sim(x)) \leqq m_{\delta,v}^*(x) \tag{2.12}$$

for each $\delta > 0$, $v > 0$, and x in Δ. It follows that if (x_n) is a sequence of points in Δ for which

$$\lim_{\delta \downarrow 0} \limsup_n m_{\delta,v}^*(x_n) = 0 \qquad \text{for all } v > 0\,, \tag{2.13}$$

then (x_n) is relatively compact and all its limit points are in Δ_+.

The following mappings from Δ to $\Delta_\uparrow$ were studied by Whitt (1971) and (1975) in a slightly different context. The *supremum operator* $\mathscr{S}$ is the mapping which sends x in Δ into the function $\mathscr{S}x$ in $\Delta_\uparrow$, defined by

$$(\mathscr{S}x)(t) = \sup\{x(s) : 0 \leqq s \leqq t\}\,. \tag{2.14}$$

The *first passage time operator* $\mathscr{F}$ is the mapping which sends x in Δ into the $\mathscr{F}x$ in $\Delta_\uparrow$, defined by

$$\begin{aligned}(\mathscr{F}x)(t) &= 0\,. && \text{if } t = 0\\ &= \inf\{u : t < x(u)\}\,, && \text{if } t > 0\,;\end{aligned} \tag{2.15}$$

here the infimum of an empty set is taken to be ∞. One has

$$\mathscr{F} = \mathscr{F}\mathscr{S} \qquad \text{and} \qquad \mathscr{F}\mathscr{F} = \mathscr{S} \tag{2.16}$$

in particular, when restricted to $\Delta_\uparrow$, $\mathscr{T}$ is its own inverse. Both $\mathscr{T}$ and $\mathscr{S}$ are continuous (relative to the M_1-topology).

In the following paragraph we shall establish some terminology and notation concerning functions x belonging to Δ. Let $x \in \Delta$. For $t > 0$, set

(2.17) $\quad {}_xV_-(t) = \sup \sum_{A \in \mathscr{A}} (x(A))^- \quad$ and $\quad {}_xV_+(t) = \sup \sum_{A \in \mathscr{A}} (x(A))^+ ,$

where the suprema are taken over all finite partitions $\mathscr{A}$ of $[0, t]$ into disjoint intervals; in computing the increments $x(A)$, use the convention that $\infty - \infty = \infty$. Put

(2.18) $$ {}_xV_-(0) = 0 = {}_xV_+(0) .$$

We will call ${}_xV_-$ (resp. ${}_xV_+$) the *negative* (resp. *positive*) *variation* of x. We assume henceforth in this discussion that ${}_xV_-$ is absolutely continuous over $[0, t]$ for each $t < \infty$, as this is the only case of interest to us here. Then from the representation

(2.19) $$x(t) = {}_xV_+(t) - {}_xV_-(t) ,$$

holding for all $t \geqq 0$, it follows that there is a point t_x (possibly 0, possibly ∞) such that $x(t) < \infty$ for $t < t_x$ and $x(t) = \infty$ for $t > t_x$. Thus

(2.20) $$F_x = \{t : x(t) < \infty\}$$

is a (possibly degenerate) interval. Let ν be the measure on the Borel sets of F_x such that $\nu([0, t]) = x(t)$ for each $0 < t \in F_x$. Define the function x_a over F_x by setting $x_a(t) = \nu_a([0, t])$, where ν_a is the component of ν which is absolutely continuous with respect to Lebesgue measure on F_x. x_a is absolutely continuous on each subinterval of F_x over which it is bounded. We will call x_a the *absolutely continuous component* of x. The *derivative* of x_a is the almost surely unique function $\dot{x}_a$ on F_x such that

(2.21) $$x_a(t) = \int_0^t \dot{x}_a(s)\, ds$$

for all t in F_x. Frequently we will suppress the subscript a on $\dot{x}_a$, and just write $\dot{x}$. Making use of this convention, one has

(2.22) $\quad (\dot{x}(t))^- = {}_x\dot{V}_-(t) \quad$ and $\quad (\dot{x}(t))^+ = {}_x\dot{V}_+(t)$

for each t in F_x.

We are now going to define some compact subsets of Δ which will arise later on as the almost sure limit points of certain sequences of random functions. The reason for indexing these sets in the way we do will become clear in the next section.

Suppose first that α is a number in $(0, 1)$. Put $\lambda_\alpha = \alpha/(\alpha - 1)$, and note that $\lambda_\alpha < 0$. For each x in $\Delta_\uparrow$, set

(2.23) $$i_\alpha(x) = \int_{F_x} (\dot{x}_a(t))^{\lambda_\alpha}\, dt$$

where $\dot{x}_a$ and F_x are defined by (2.21) and (2.20) respectively. In Lemma 6.2(a) we show that

(2.24) $$i_\alpha(x) = j_\alpha(x) ,$$

where

$$j_\alpha(x) = \sup \textstyle\sum_{A \in \mathscr{A}} (x(A)/|A|)^{\lambda_\alpha}|A| \,, \tag{2.25}$$

with the supremum being taken over all finite collections $\mathscr{A}$ of disjoint bounded subintervals of F_x. Now put

$$K_\alpha = \{x \in \Delta_\uparrow : i_\alpha(x) \leqq 1\} \,. \tag{2.26}$$

The "singular" component of an x in K_α need not satisfy any condition, but (because $\lambda_\alpha < 0$) the integral condition in (2.26) forces x to be strictly increasing over F_x. In view of (2.24) and (2.25), K_α is closed in $\Delta_\uparrow$, and therefore compact in Δ.

Suppose next that $\alpha = 1$. For each x in Δ, set

$$j_\alpha(x) = \sup \textstyle\sum_{A \in \mathscr{A}} e^{-x(A)/|A|}|A| \tag{2.27}$$

where the supremum is taken over all finite collections $\mathscr{A}$ of disjoint bounded subintervals of $[0, \infty)$, and $\infty - \infty$ is to be interpreted as ∞. Then (cf. Lemma 6.2(b)) $j_\alpha(x) < \infty$ iff ${}_xV_-$ is absolutely continuous over $[0, \infty)$ and

$$i_\alpha(x) \equiv \textstyle\int_{F_x} e^{-\dot{x}_a(t)}\, dt < \infty \,; \tag{2.28}$$

moreover $j_\alpha(x) < \infty$ implies

$$i_\alpha(x) = j_\alpha(x) \,. \tag{2.29}$$

Set

$$K_\alpha = \{x \in \Delta : \ {}_xV_- \text{ is absolutely continuous and } i_\alpha(x) \leqq 1\} \,. \tag{2.30}$$

The positive variation of an x in K_α must have an absolutely continuous component which increases fast enough to guarantee $i_\alpha(x) \leqq 1$; however, there is no constraint on the singular component of ${}_xV_+$. By (2.29), (2.27), (2.13) and (6.2), K_α is a compact subset of Δ_+, hence also of Δ.

Next suppose $\alpha \in (1, \infty)$; only α's < 2 will be of interest in the sequel. Set $\lambda_\alpha = \alpha/(\alpha - 1)$, and note that now $\lambda_\alpha > 1$. For each x in Δ, put

$$j_\alpha(x) = \sup \textstyle\sum_{A \in \mathscr{A}} ((x(A))^-/|A|)^{\lambda_\alpha}|A| \tag{2.31}$$

where the supremum is taken over all finite collections $\mathscr{A}$ of disjoint bounded subintervals of $[0, \infty)$. Then (cf. Lemma 6.2(c)) $j_\alpha(x) < \infty$ iff ${}_xV_-$ is absolutely continuous over $[0, \infty)$ and

$$i_\alpha(x) \equiv \textstyle\int_0^\infty ({}_x\dot{V}_-(t))^{\lambda_\alpha}\, dt < \infty \,; \tag{2.32}$$

moreover $j_\alpha(x) < \infty$ implies

$$i_\alpha(x) = j_\alpha(x) \,. \tag{2.33}$$

Set

$$K_\alpha = \{x \in \Delta : \ {}_xV_- \text{ is absolutely continuous and } i_\alpha(x) \leqq 1\} \,. \tag{2.34}$$

There are no constraints on the positive variation of an x in K_α. In view of (2.33), (2.31), (2.13), and (6.1), K_α is a compact subset of Δ_+, hence also of Δ.

Finally, suppose $\beta > 1$. We shall have occasion to make use of the set

$$L_\beta = \{x \in \Delta_\uparrow : x \text{ is absolutely continuous and } \textstyle\int_0^\infty (\dot{x}(t))^{\lambda_\beta} dt \leqq 1\} . \tag{2.35}$$

A function x in $\Delta_\uparrow$ belongs to L_β iff

$$\sup \textstyle\sum_{A \in \mathscr{A}} (x(A)/|A|)^{\lambda_\beta} |A| \leqq 1 , \tag{2.36}$$

where the supremum is taken over all finite collections $\mathscr{A}$ of disjoint bounded subintervals of $[0, \infty)$.

There are a couple of relations among the above sets that will be used later on. Recalling the definition of the supremum operator $\mathscr{S}$ and the first passage time operator $\mathscr{F}$ (cf. (2.14) and (2.15)), we have

$$\mathscr{F}(K_\alpha) = L_\beta \quad \text{and} \quad K_\alpha = \mathscr{F}(L_\beta) \tag{2.37}$$

for $\alpha < 1$, $\beta > 1$, and $\alpha\beta = 1$, and

$$\mathscr{S}(-K_\alpha) = L_\alpha \tag{2.38}$$

for $1 < \alpha < 2$. These relations follow easily from the characterizations of the sets involved by means of the various "j" functions (cf. (2.25), (2.31), and (2.36)).

3. Statement of main results for $\alpha \neq 1$. Let $\alpha \in (0, 1) \cup (1, 2)$. We shall have occasion to make use of several quantities dependent on α, and for convenience we collect them together here. The first of these is

$$\lambda_\alpha = \alpha/(\alpha - 1) \tag{3.1}$$

which was introduced in the previous section (cf. (2.23) and (2.31)). Frequently we shall suppress the subscript α from λ_α. Notice that λ is conjugate to α in the sense that

$$1/\alpha + 1/\lambda = 1 ; \tag{3.2}$$

this fact will be used in the proofs in conjunction with Hölder's inequality. The remaining quantities are

$$\begin{aligned} \sigma_\alpha &= +1 , && \text{if} \quad \alpha < 1 \\ &= -1 , && \text{if} \quad \alpha > 1 \end{aligned} \tag{3.3}$$

and

$$\theta_\alpha = \alpha/|\alpha - 1|^{1/\lambda} = \alpha^{1/\alpha}\lambda^{1/\lambda} . \tag{3.4}$$

Throughout this section we let $V, V_1, V_2, \cdots$ be i.i.d. nonnegative random variables in the domain of attraction of a stable law with index α. There thus exists a function L_α slowly varying at ∞ such that for all $v > 0$,

$$\Pr\{V \geqq v\} = v^{-\alpha} L_\alpha(v)/|\Gamma(1 - \alpha)| ; \tag{3.5}$$

here Γ is the usual Gamma function. For $s \geqq 0$, set

$$\begin{aligned} g_\alpha(s) &= -\log E e^{-sV} , && \text{if} \quad \alpha < 1 \\ &= \log E e^{-s(V - EV)} , && \text{if} \quad \alpha > 1 . \end{aligned} \tag{3.6}$$

g_α is continuous and strictly increasing. By a standard Tauberian theorem (cf. Feller (1971) Section XIII. 5),

$$g_\alpha(s) \sim s^\alpha L_\alpha(1/s) \tag{3.7}$$

as $s \downarrow 0$.

For each $n \geqq 1$, put

$$S_n = V_1 + \cdots + V_n \tag{3.8}$$

and set

$$\begin{aligned} T_n &= S_n\,, && \text{if} \quad \alpha < 1 \\ &= S_n - ES_n\,, && \text{if} \quad \alpha > 1\,. \end{aligned} \tag{3.9}$$

The following probability estimate, which will be established in Section 6B, plays a crucial role in the proof of Theorem 3.1 below. Recall that g_α, λ, σ_α, and θ_α are defined by (3.6), (3.1), (3.3), and (3.4) respectively.

LEMMA 3.1. *Let (B_n) be a sequence of positive numbers such that*

$$B_n \to \infty \qquad \text{and} \qquad B_n/n \to 0\,. \tag{3.10}$$

Set

$$A_n = B_n/g_\alpha{}^{\sim}(B_n/n)\,. \tag{3.11}$$

Then for each $c > 0$,

$$\Pr\{T_n/A_n \leqq \sigma_\alpha c\} = \exp(-(1 + o(1))(c/\theta_\alpha)^\lambda B_n) \tag{3.12}$$

as $n \to \infty$.

To put this result in perspective, set

$$\gamma_n = 1/g_\alpha{}^{\sim}(1/n)\,. \tag{3.13}$$

Then from (3.7) it follows immediately that the random variables T_n/γ_n converge in distribution to the completely asymmetric stable law, $\mathscr{P}_\alpha$, having transform

$$\log\left(\textstyle\int_{-\infty}^{\infty} e^{-sv}\mathscr{P}_\alpha(dv)\right) = -\sigma_\alpha s^\alpha \tag{3.14}$$

($s \geqq 0$). The support of $\mathscr{P}_\alpha$ is $[0, \infty)$ when $\alpha < 1$, and $(-\infty, \infty)$ when $\alpha > 1$. Since $g_\alpha{}^{\sim} \in \mathscr{R}\mathscr{V}_0(1/\alpha)$, one has

$$A_n = B_n\gamma_n(g_\alpha{}^{\sim}(1/n)/g_\alpha{}^{\sim}(B_n/n)) = B_n{}^{1/\lambda+\varepsilon_n}\gamma_n\,, \tag{3.15}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Bearing in mind that $\lambda_\alpha < 0$ if $\alpha < 1$, and $\lambda_\alpha > 1$ if $\alpha > 1$, it is clear that the lemma is a statement about the probability with which T_n takes on very "small" values. Sharper results have been obtained by Lipschutz (1956a) under certain restrictions on the function L_α appearing in (3.5); when applicable, her conditions imply $B_n{}^{\varepsilon_n} \to 0$.

Here is the main result:

THEOREM 3.1. *Let T_n be defined by* (3.9). *Define random functions H_n, $n \geqq 1$,* *on* $[0, \infty)$ *by setting*

$$H_n(t) = T_{[nt]}/\theta_\alpha a_n \tag{3.16}$$

$(t \geqq 0)$, *where*

$$a_n = \log_2 n/g_\alpha^{\sim}(\log_2 n/n) \tag{3.17}$$

(*cf.* (3.4) *and* (3.6)). *Then* w.p. 1

$$H_n \to K_\alpha \tag{3.18}$$

in the M_1*-topology* (*cf.* (2.8)), *with* K_α *defined by* (2.26) *if* $\alpha < 1$ *and by* (2.34) *if* $\alpha > 1$.

Theorem 3.1 will be established in Section 8. A word or two is in order concerning the normalizing constants a_n. By (3.15) one has

$$a_n = \gamma_n(\log_2 n)^{1/\lambda + o(1)}.$$

In "nice" cases, the $o(1)$ term tends to 0 so rapidly that $(\log_2 n)^{o(1)} \to 1$; this is the situation when e.g., $L_\alpha(v) = (\log v)^c$ for some constant c. However, in general $(\log_2 n)^{o(1)}$ may diverge to ∞ (or 0); this is the case when, e.g., $g_\alpha^{\sim}(t) \sim \exp((\log t)(1 - 1/\log_4(1/t))/\alpha)$ as $t \downarrow 0$.

Next let $N = (N_t)_{t \geqq 0}$ be the counting process derived from the S_n's (cf. (3.8)):

$$\begin{aligned} N_t &= 0\,, && \text{if} \quad t = 0 \\ &= \textstyle\sum_{0 \leqq k < \infty} I_{[0,t]}(S_k)\,, && \text{if} \quad t > 0 \end{aligned} \tag{3.19}$$

$(S_0 = 0)$. The process N and the partial sum process $S \equiv (S_{[u]})_{u \geqq 0}$ are inverses in the sense that

$$N = \mathscr{F}(S) \qquad \text{and} \qquad S = \mathscr{F}(N) \tag{3.20}$$

where $\mathscr{F}$ is the first passage time operator of (2.15). According to Theorem 9.1 below, Theorem 3.1 is thus equivalent to the following Theorem 3.2. Recall that θ_α is defined by (3.4), g_α by (3.6).

THEOREM 3.2. *Let N be defined by* (3.19).

(a) *Suppose* $\alpha < 1$. *Define random functions* H_n, $n \geqq 1$, *on* $[0, \infty)$ *by setting*

$$H_n(t) = N_{[nt]}/(\theta_\alpha{}^\alpha \log_2 n/g_\alpha(\log_2 n/n))$$

$(t \geqq 0)$. *Then* w.p. 1

$$H_n \to L_{1/\alpha}$$

in the M_1*-topology, with* $L_{1/\alpha}$ *defined by* (2.35).

(b) *Suppose* $\alpha > 1$. *Let* a_n *be defined by* (3.17), *and put* $\mu = E(V)$. *Define random functions* H_n, $n \geqq 1$, *on* $[0, \infty)$ *by setting*

$$H_n(t) = -(N_{[nt]} - [nt]/\mu)/(a_n/\mu^{1+1/\alpha})$$

$(t \geqq 0)$. *Then* w.p. 1

$$H_n \to K_\alpha$$

in the M_1*-topology, with* K_α *defined by* (2.34).

Suppose now $(\mathscr{H}_n)$ is any sequence of random functions in Δ which w.p. 1 is relatively compact and has K_α (defined by (2.26) or (2.34)) as its set of limit points. Here are some consequences, which are proved by making use of the so-called mapping principle (cf. Strassen (1964) page 218 or Wichura (1974) Lemma 2.1).

(i) Let ν_n, $n \geqq 1$, and ν be nonnegative finite measures on the Borel sets of $[0, \infty)$, such that $\nu_n \to \nu$ weakly, in the sense that $\int f\, d\nu_n \to \int f\, d\nu$ for all continuous bounded real-valued functions f. Suppose that all the ν_n's and ν are concentrated in some finite interval; one consequence of this is that $\int x\, d\nu$ is defined for all x in Δ. To avoid trivialities, suppose that $\nu((0, \infty)) > 0$. Set $\hat{\nu}(t) = \nu([t, \infty))$ and put

$$c_{\alpha,\nu} = \left(\int_0^\infty (\hat{\nu}(t))^\alpha\, dt\right)^{1/\alpha} . \tag{3.21}$$

Then w.p. 1 the limit points of $(\int \mathscr{H}_n\, d\nu_n)$ coincide with $[\sigma_\alpha c_{\alpha,\nu}, \infty]$, and in particular

$$\liminf_n \int \mathscr{H}_n(t)\nu_n(dt) = \sigma_\alpha c_{\alpha,\nu} \tag{3.22}$$

(see (3.3) for the definition of σ_α). This result is true even if the ν_n's are random and converge weakly to ν w.p. 1.

There is some extra information available as to the shape of the function $\mathscr{H}_n$ when $\int \mathscr{H}_n\, d\nu_n$ is near $\sigma_\alpha c_{\alpha,\nu}$. For this, let t_ν be the largest point of support of ν. Set

$$x_{\alpha,\nu}(t) = \int_0^t (\hat{\nu}(s)/c_{\alpha,\nu})^{\alpha-1}\, ds \tag{3.23}$$

for $t < t_\nu$. Define a subset $\mathscr{S}_{\alpha,\nu}$ of Δ as follows. For $\alpha < 1$, $\mathscr{S}_{\alpha,\nu}$ consists of the single function in $\Delta_\uparrow$ which coincides with $x_{\alpha,\nu}$ over $[0, t_\nu)$ and equals ∞ over $[t_\nu, \infty)$. For $\alpha > 1$, $\mathscr{S}_{\alpha,\nu}$ consists of those functions in Δ_+ which agree with $-x_{\alpha,\nu}$ over $[0, t_\nu)$ and which are non-decreasing on $[t_\nu, \infty)$. The following statement is then true for almost all sample points ω in the underlying probability space on which the $\mathscr{H}_n$'s are defined: for each subsequence $(n_k)_{k\geqq 1}$ such that $(\int \mathscr{H}_{n_k}\, d\nu_{n_k})(\omega)$ converges to $\sigma_\alpha c_{\alpha,\nu}$, the distance in Δ between the function $\mathscr{H}_{n_k}(\omega)$ and the set $\mathscr{S}_{\alpha,\nu}$ tends to 0. Loosely speaking, for large n, $\int \mathscr{H}_n\, d\nu_n$ is near $\sigma_\alpha c_{\alpha,\nu}$ only if $\mathscr{H}_n$ is near $\mathscr{S}_{\alpha,\nu}$.

When ν is Lebesgue measure on [0, 1], one has

$$c_{\alpha,\nu} = 1/(1 + \alpha)^{1/\alpha} \tag{3.24}$$

and

$$x_{\alpha,\nu}(t) = (1 + \alpha)^{1/\lambda}\alpha^{-1}(1 - (1 - t^\alpha)) \tag{3.25}$$

for $0 \leqq t < 1$. When ν is a unit mass at the point $u \in (0, \infty)$,

$$c_{\alpha,\nu} = u^{1/\alpha} \qquad \text{and} \tag{3.26}$$

$$x_{\alpha,\nu}(t) = (t/u)c_{\alpha,\nu} \tag{3.27}$$

for $0 \leqq t < u$.

Applying the last result with $u = 1$ to the random functions H_n of Theorem 3.1, we get

$$\liminf_n T_n/a_n = \sigma_\alpha \theta_\alpha . \tag{3.28}$$

When $\alpha < 1$, (3.28), with the right-hand side replaced by an unspecified constant, is contained in Theorem 4 of Fristedt and Pruitt (1971). Under supplementary conditions on L_α, (3.28) (for $\alpha > 1$ as well as $\alpha < 1$) follows from Lipschutz's strong form of the law of the iterated logarithm.

(ii) Let $c \geqq 1$ and define the "cth-norm" of $\mathscr{H}_n$ to be

$$\begin{aligned} |\mathscr{H}_n|_c &= (\textstyle\int_0^1 |\mathscr{H}_n(t)|^c\,dt)^{1/c}, && \text{if } \alpha < 1 \\ &= -(\textstyle\int_0^1 ((\mathscr{H}_n(t))^-)^c\,dt)^{1/c}, && \text{if } \alpha > 1 . \end{aligned}$$

Let $\mathscr{B}(r, s) = \Gamma(r)\Gamma(s)/\Gamma(r + s)$ be the usual Beta coefficient. Then w.p. 1

$$\liminf_n |\mathscr{H}_n|_c = \sigma_\alpha u_{\alpha,c} \tag{3.29}$$

where

$$u_{\alpha,c} = (c + \alpha)(\alpha/(\alpha + c))^{1/c}(c/(\alpha + c))^{1/\alpha}/\mathscr{B}(1/\alpha, 1/c) . \tag{3.30}$$

Moreover for large n, $|\mathscr{H}_n|_c$ is near $\sigma_\alpha u_{\alpha,c}$ only if $\mathscr{H}_n$ is close to a certain subset $\mathscr{T}_{\alpha,c}$ of Δ which we shall now describe. Put

$$h_{\alpha,c}(u) = \textstyle\int_0^{f(u)} s^{1/c-1}(1 - s)^{1/\alpha-1}\,ds/\mathscr{B}(1/\alpha, 1/c)$$

where

$$f(u) = (u(c/(\alpha + c))^{1/\lambda}\mathscr{B}(1/\alpha, 1/c)/c)^c .$$

For $0 \leqq t < 1$, put

$$y_{\alpha,c}(t) = h_{\alpha,c}^{\sim}(t) . \tag{3.31}$$

When $\alpha < 1$, $\mathscr{T}_{\alpha,c}$ consists of the single function in $\Delta_\uparrow$ which agrees with $y_{\alpha,c}$ over $[0, 1)$ and equals ∞ over $[1, \infty)$. When $\alpha > 1$, $\mathscr{T}_{\alpha,c}$ consists of those functions in Δ_+ which coincide with $-y_{\alpha,c}$ over $[0, 1)$ and are non-decreasing over $[1, \infty)$.

(iii) For the final application, we suppose that the $\mathscr{H}_n$'s have a special structure, namely, that there exist a random process $\mathscr{X}$ and a fixed function $b \in \mathscr{RV}_\infty(1/\alpha)$, such that

$$\mathscr{H}_n(t) = \mathscr{X}(nt)/b(n) \tag{3.32}$$

for all t and n. This is the situation, e.g., in Theorems 3.1 and 3.2. We shall give a sharpening of (3.22) in the case that ν is a unit mass at the point $u = 1$ (cf. (3.24)). For each positive integer p, let $f_{p,c}$ be the random variable which records the number of integers $n \leqq p$ for which $\mathscr{H}_n(1) \leqq \sigma_\alpha c$; here $1 < c < \infty$ if $\alpha > 1$, and $0 < c < 1$ if $\alpha > 1$. Put

$$\tau_{\alpha,c} = e^{-\alpha\lambda(c-\lambda-1)} . \tag{3.33}$$

Then w.p. 1

$$\limsup_p f_{p,c} = 1 - \tau_{\alpha,c} . \tag{3.34}$$

Moreover for large p, $f_{p,c}$ is near $1 - \tau_{\alpha,c}$ only if $\mathscr{H}_p$ is near a certain subset $\mathscr{U}_{\alpha,c}$ of Δ which we shall now describe. Put

$$z_{\alpha,c}(t) = (t/\tau_{\alpha,c})c\tau_{\alpha,c}^{1/\alpha}\,, \qquad \text{if} \quad 0 \leqq t < \tau_{\alpha,c} \tag{3.35}$$
$$= ct^{1/\alpha}\,, \qquad \text{if} \quad \tau_{\alpha,c} \leqq t < 1\,.$$

If $\alpha < 1$, $\mathscr{U}_{\alpha,c}$ consists of the single function in $\Delta_\uparrow$ which coincides with $z_{\alpha,c}$ over $[0, 1)$ and equals ∞ over $[1, \infty)$. If $\alpha > 1$, then $\mathscr{U}_{\alpha,c}$ consists of those functions in Δ_+ which agree with $-z_{\alpha,c}$ over $[0, 1)$ and are non-decreasing over $[1, \infty)$.

REMARKS. (a) The similarity of these results to Strassen's for Brownian motion is striking. Indeed, if one formally sets $\alpha = 2$ in (3.21), (3.30), and (3.33), one gets the quantities that figure in Strassen's applications (ii), (iv), and (v) respectively.

(b) In view of part (a) of Theorem 3.2, it would be desirable to have results corresponding to (i), (ii), and (iii) above in the case that the sequence $(\mathscr{H}_n)$ is relatively compact and has as its limit points the set L_β, defined by (2.35). These results are easily obtained. In fact, using the same notation and conventions as above, one has in this case

$$\limsup_n \textstyle\int \mathscr{H}_n\, d\nu_n = c_{\beta,\nu} \tag{3.36}$$

$$\limsup_n \textstyle\int_0^1 (\mathscr{H}_n{}^+(t))^c\, dt = u_{\beta,c} \tag{3.37}$$

$$\limsup_p p^{-1} \operatorname{card}(n \leqq p : \mathscr{H}_n(1) > c) = 1 - \tau_{\beta,c}\,. \tag{3.38}$$

The sets corresponding to $\mathscr{S}_{\alpha,\nu}$, $\mathscr{T}_{\alpha,c}$, and $\mathscr{U}_{\alpha,c}$ above are respectively $\{x^*_{\beta,\nu}\}$, $\{y^*_{\beta,c}\}$, and $\{z^*_{\beta,c}\}$. Here, e.g., $x^*_{\beta,\nu}$ is the continuous function which coincides with $x_{\beta,\nu}$ over $[0, t_\nu)$ (cf. (3.23)) and is constant over $[t_\nu, \infty)$; $y^*_{\beta,c}$ and $z^*_{\beta,c}$ are obtained from $y_{\beta,c}$ and $z_{\beta,c}$ (cf. (3.31) and (3.35)) in a similar fashion.

4. Statement of main results for $\alpha = 1$. Throughout this section, we let V, V_1, V_2, $\cdots$ be i.i.d. nonnegative random variables in the domain of attraction of a stable law of index $\alpha = 1$, so

$$1 - F(v) \equiv \Pr\{V \geqq v\} = L(v)/v \tag{4.1}$$

($v > 0$) for some function L varying slowly at ∞. For $s \geqq 0$, set

$$g(s) = -\log(Ee^{-sV})\,. \tag{4.2}$$

In contrast to (3.6), one has here

$$g(s) = s(M(1/s) + O(sM^2(1/s))) \sim sM(1/s)\,, \tag{4.3}$$

as $s \downarrow 0$, with

$$M(t) = t \textstyle\int_0^\infty (1 - e^{-v/t})F(dv) = \int_0^\infty e^{-v/t}(1 - F(v))\, dv \tag{4.4a}$$

$$= t \textstyle\int_0^\infty \min(1, v/t)F(dv) - (\kappa + o(1))L(t) \tag{4.4b}$$
$$= \textstyle\int_0^t (1 - F(v))\, dv - (\kappa + o(1))L(t)$$

(4.4c) $$= t \int_0^t (v/t)F(dv) + (1 - \kappa + o(1))L(t)$$

(4.4d) $$= t \int_0^\infty \sin (v/t)F(dv) + o(1)L(t) ;$$

here $\kappa = 0.57721\cdots$ is Euler's constant. M varies slowly at ∞, but is asymptotically much larger than L: $\lim_{t\to\infty} M(t)/L(t) = \infty$ (cf. Feller (1971) Sections VIII. 9 and XIII. 5). For $t \geqq 0$, put

(4.5) $$\begin{aligned}\mathscr{M}(t) = M'(t) &= \int_0^\infty (1 - e^{-v/t} - (v/t)e^{-v/t})F(dv) \\ &= t^{-2} \int_0^\infty e^{-v/t}v(1 - F(v))\, dv .\end{aligned}$$

As t runs from 0 to ∞, $\mathscr{M}$ decreases strictly and continuously from 1 to 0; moreover (cf. again Feller (1971) Sections VIII. 9 and XIII. 5)

(4.6) $$\mathscr{M}(t) \sim L(t)/t$$

as $t \to \infty$, so $\mathscr{M} \in \mathscr{R}\mathscr{V}_\infty(-1)$. In other words, if we define $\mathscr{L}$ on $(0, \infty)$ by the relation

(4.7) $$\mathscr{M}(t) = \mathscr{L}(t)/t$$

$(t > 0)$, then $\mathscr{L}$ is slowly varying at ∞, and

(4.8) $$\mathscr{L}(t) \sim L(t)$$

as $t \to \infty$. From the identity

$$M(t) = \int_0^t \mathscr{L}(s)/s\, ds$$

we get the often used relation

(4.9) $$\lim_{t\to\infty} (M(ct) - M(t))/\mathscr{L}(t) = \log c$$

for each $c > 0$.

Put now

(4.10) $$S_n = V_1 + \cdots + V_n$$

and set

(4.11) $$\gamma_n = \mathscr{M}^{\sim}(1/n)$$

so that

$$\gamma_n/n = \mathscr{L}(\gamma_n) .$$

Then from (4.3) and (4.9) it follows immediately that as $n \to \infty$ the random variables $(S_n - nM(\gamma_n))/\gamma_n$ converge in distribution to the completely asymmetric stable law, $\mathscr{P}_1$, having transform

(4.12) $$\log (\int_{-\infty}^\infty e^{-sv}\mathscr{P}_1(dv)) = s \log s$$

$(s \geqq 0)$. Skorokhod (1961) showed that

$$\mathscr{P}_1\{v : v \leqq \xi\} = \exp(-(1 + o(1))e^{-1}e^{-\xi})$$

as $\xi \downarrow -\infty$. The corresponding result for the S_n's is given by the following analogue to Lemma 3.1.

LEMMA 4.1. *Let* (B_n) *be a sequence of positive numbers such that*

$$B_n \to \infty \quad \text{and} \quad B_n/n \to 0\,. \tag{4.13}$$

Set

$$A_n = B_n \mathscr{M}^{\sim}(B_n/n) \tag{4.14}$$

so that $A_n/n = \mathscr{L}(A_n/B_n)$. *Then for each* $c \in (-\infty, \infty)$

$$\Pr\{(S_n - nM(A_n/B_n))/A_n \leqq c\} = \exp(-(1+o(1))e^{-1}e^{-c}B_n) \tag{4.15}$$

as $n \to \infty$.

We note in passing that since $\mathscr{M}^{\sim} \in \mathscr{R}\mathscr{V}_0(-1)$, $A_n = \gamma_n B_n^{o(1)}$ (cf. (4.11)). Here is the main result.

THEOREM 4.1. *Let* S_n *be defined by* (4.10). *Define random functions* H_n, $n \geqq 1$, *on* $[0, \infty)$ *by setting*

$$H_n(t) = (S_{[nt]} - [nt]M(a_n/(e \log_2 n)))/a_n \tag{4.16}$$

where

$$a_n = \log_2 n \mathscr{M}^{\sim}(\log_2 n/n)\,. \tag{4.17}$$

Then w.p. 1

$$H_n \to K_1 \tag{4.18}$$

in the M_1*-topology, where* K_1 *is defined by* (2.30).

To illustrate the norming, we consider the case in which

$$\Pr\{V \geqq v\} = (\log v)^g/v \tag{4.19}$$

for $v \geqq v_0$, where $g \neq -1$, and v_0 satisfies $(\log v_0)^g/v_0 = 1$. After some simplification (cf. (4.4b)), (4.18) becomes

$$(S_{[n\cdot]} - [n\cdot]h_n)/(n(\log n)^g) \to K_1 \tag{4.20}$$

with

$$h_n = (\log n)^{g+1}/(g+1) + (\log n)^g(g \log_2 n - \log_3 n - \kappa - 1) + c \tag{4.21}$$

(recall κ denotes Euler's constant); here

$$\begin{aligned} c &= 0\,, && \text{if} \quad g \geqq 0 \\ &= -(\log v_0)^{g+1}/(g+1)\,, && \text{if} \quad -1 < g < 0 \\ &= E(V)\,, && \text{if} \quad g < -1\,. \end{aligned} \tag{4.22}$$

Suppose now $(\mathscr{H}_n)$ is any sequence of random functions in Δ which w.p. 1 is relatively compact and has K_1 (defined by (2.30)) as its set of limit points. Here are some consequences.

(i) As in Section 3(i), let ν_n, $n \geqq 1$, and ν be nonnegative finite measures on the Borel sets of $[0, \infty)$, all concentrated in some finite interval, and suppose that $\nu_n \to \nu$ weakly, and that $\nu((0, \infty)) > 0$. Put $\hat{\nu}(t) = \nu([t, \infty))$ and set

$$b_\nu = \int_0^\infty \hat{\nu}(t)\,dt\,. \tag{4.23}$$

Let t_ν be the largest point of support of ν, and set

$$\begin{aligned} x_{1,\nu}(t) &= -\int_0^t \log(\hat{\nu}(s)/b_\nu)\,ds\,, &&\text{if } 0 \leqq t < t_\nu \\ &= \infty\,, &&\text{if } t_\nu \leqq t < \infty\,. \end{aligned} \tag{4.24}$$

Put

$$c_{1,\nu} = -\int_0^\infty \hat{\nu}(t)\log(\hat{\nu}(t)/b_\nu)\,dt = \int_0^\infty x_{1,\nu}(t-)\nu(dt)\,. \tag{4.25}$$

Then w.p. 1 the limit points of $(\int \mathscr{H}_n\,d\nu_n)$ coincide with $[c_{1,\nu}, \infty]$, and in particular

$$\liminf_n \int \mathscr{H}_n(t)\nu_n(dt) = c_{1,\nu}\,; \tag{4.26}$$

moreover for large n, $\int \mathscr{H}_n\,d\nu_n$ is near $c_{1,\nu}$ only if $\mathscr{H}_n$ is close to $x_{1,\nu}$.

When ν is Lebesgue measure on $[0, 1]$,

$$c_{1,\nu} = -2^{-1}\log(2) + \tfrac{1}{4} \doteq -0.09657 \tag{4.27}$$

and

$$x_{1,\nu}(t) = (1-t)\log(1-t) + t(1-\log 2) \tag{4.28}$$

for $0 \leqq t < 1$. When ν is a unit mass at the point $u > 0$,

$$c_{1,\nu} = u\log(u) \tag{4.29}$$

and

$$x_{1,\nu}(t) = t\log(u) \tag{4.30}$$

for $0 \leqq t < u$. In particular

$$\liminf_n \mathscr{H}_n(1) = 0 \quad \text{a.s.} \tag{4.31}$$

The analogue of (4.31) for the completely asymmetric Cauchy process was established by Mijnheer (1972).

(ii) Suppose now that for all n and t,

$$\mathscr{H}_n(t) = (\mathscr{X}(nt) - [nt]g(n))/b(n)\,, \tag{4.32}$$

where $\mathscr{X}$ is some random process, $b \in \mathscr{R}\mathscr{V}_\infty(1)$, and g is a non-decreasing function on $(0, \infty)$ satisfying

$$\lim_{u\to\infty}(g(uc) - g(u))/(b(u)/u) = \log c \tag{4.33}$$

for all $c > 0$. This is the situation, e.g., in Theorem 4.1. For $c > 0$, let $f_{p,c}$ be the random variable which records the proportion of integers $n \leqq p$ for which $\mathscr{H}_n(1) \leqq c$. Set

$$\tau_{1,c} = e^{-e(e^c-1)} \tag{4.34}$$

and put

$$
\begin{aligned}
z_{1,c}(t) &= (t/\tau_{1,c})(\tau_{1,c}\log\tau_{1,c} + c\tau_{1,c})\,, && \text{if } 0 \leqq t < \tau_{1,c} \\
&= t\log t + ct\,, && \text{if } \tau_{1,c} \leqq t < 1 \qquad (4.35)\\
&= \infty\,, && \text{if } 1 \leqq t < \infty\,.
\end{aligned}
$$

Then w.p. 1

$$\limsup{}_p f_{p,c} = 1 - \tau_{1,c} \tag{4.36}$$

and for large p, $f_{p,c}$ is near $1 - \tau_{1,c}$ only if $\mathscr{H}_p$ is near the function $z_{1,c}$.

5. Extensions and complements. A. For the processes H_n of Theorems 3.1 and 4.1, one can relax the assumptions made in Sections 3(i) and 4(i), in the following manner. Assume as before that ν_n, $n \geqq 1$, and ν are nonnegative finite measures on $[0, \infty)$, with ν_n converging weakly to ν. Suppose now that there exists a number $\varepsilon > 0$ such that

$$\sup{}_{n\geqq 1} \textstyle\int s^{1/\alpha+\varepsilon}\nu_n(ds) < \infty\,; \tag{5.1}$$

automatically $\int_1^\infty s^{1/\alpha+\varepsilon}\nu(ds) < \infty$ also. If $\alpha \neq 1$, all the assertions of Section 3(i) hold, with $\mathscr{H}_n$ replaced by H_n of (3.16). If $\alpha = 1$, all the assertions of Section 4(i) hold, with $\mathscr{H}_n$ replaced by H_n of (4.16).

This result suggests that there should be an analogue of (1.2), but it is not clear as to just what it should be. The difficulty lies in the fact that the "large-value" behavior of the processes involved is strikingly different from the "small-value" behavior treated here (cf. Chover (1966)).

B. The condition that the random variables figuring in Theorem 3.1 and 4.1 be nonnegative can be weakened substantially. Suppose, for example, that $\alpha < 1$, and that $V, V_1, V_2, \cdots$ are i.i.d. random variables, not necessarily nonnegative, satisfying (3.5). Define θ_α by (3.4), g_α by (3.6) with V replaced by V^+, and a_n by (3.17). For $n \geqq 1$, $t \geqq 0$, set $H_n(t) = {}_+H_n(t) - {}_-H_n(t)$, where ${}_oH_n(t) = (V_1^o + \cdots + V_{[nt]}^o)/(\theta_\alpha a_n)$, for $o = +, -$. By Theorem 3.1 ${}_+H_n \to K_\alpha$ w.p. 1, and so to get

$$H_n \to K_\alpha \quad \text{w.p. } 1\,, \tag{5.2}$$

it suffices to impose a condition on the distribution of V^- which will ensure ${}_-H_n \to 0$ w.p. 1. Because $n \to a_n$ is increasing and regularly varying, it in turn suffices to guarantee that $(V_1^- + \cdots + V_n^-)/a_n \to 0$ w.p. 1, or, what is the same (cf. Feller (1946)), that

$$\textstyle\sum_n \Pr\{V^- \geqq a_n\} < \infty\,. \tag{5.3}$$

For this suppose that

$$\Pr\{V^- \geqq v\}/\Pr\{V^+ \geqq v\} = O(1/(\log v)^{1+\delta}) \tag{5.4}$$

for some $\delta > 0$. Then, with γ_n defined by (3.13),

$$
\begin{aligned}
\Pr\{V^- \geqq a_n\} &= (\Pr\{V^- \geqq a_n\}/\Pr\{V^+ \geqq a_n\}) \\
&\quad \times (\Pr\{V^+ \geqq a_n\}/\Pr\{V^+ \geqq \gamma_n\}) \Pr\{V^+ \geqq \gamma_n\} \\
&= O((1/(\log n)^{1+\delta})((\log_2 n)^{1-\alpha+o(1)})(1/n))\,,
\end{aligned}
$$

whence (5.3), and so also (5.2), holds. It would be of interest to know whether (5.2) holds when V satisfies (3.5) along with

$$\Pr\{V^- \geqq v\}/\Pr\{V^+ \geqq v\} = o(1)\,, \tag{5.5}$$

these being the necessary and sufficient conditions for V to lie in the domain of attraction of the stable law $\mathscr{P}_\alpha$ defined by (3.14).

C. Let $X = (X(t))_{t\geqq 0}$ be a homogeneous process with independent increments, and let Π be its Lévy jump measure: for each Borel set B of $(-\infty, \infty)$ disjoint from $\{0\}$, $t\Pi(B)$ is the expected number of jumps of X which occur prior to time t and which lie in B. Put $\Pi_+(\xi) = \Pi([\xi, \infty))$, $\Pi_-(\xi) = \Pi((-\infty, \xi])$. Under the assumption that $\Pi_+ \in \mathscr{R}\mathscr{V}_\infty(-\alpha)$ for some $\alpha \in (0, 2)$, and that $\Pi_-(\xi)/\Pi_+(\xi)$ tends to 0 sufficiently rapidly as $\xi \to \infty$, one has

$$(X(\tau\,\cdot) - (\tau\,\cdot)b(\tau))/a(\tau) \to K_\alpha \quad \text{w.p. } 1 \tag{5.6}$$

as $\tau \to \infty$ for suitable normalizing functions a and b.

The case in which X is a completely asymmetric stable process, so $\Pi_-(\xi) \equiv 0$ and $\Pi_+(d\xi) = m\,d\xi/\xi^{1+\alpha}$ for some $\alpha \in (0, 2)$, is of special interest. When $\alpha \neq 1$, and $E(X) = 0$ if $\alpha > 1$, one has

$$\log(Ee^{-sX_1}) = -(m\Gamma(1-\alpha)/\alpha)s^\alpha \tag{5.7a}$$

$$\log(Ee^{iuX_1}) = -(m\Gamma(1-\alpha)/\alpha)\cos(\alpha\pi/2)|u|^\alpha(1 - i(u/|u|)\tan(\alpha\pi/2)) \tag{5.7b}$$

and the normalizations in (5.6) are $b(\tau) \equiv 0$ and

$$a(\tau) = \theta_\alpha(m|\Gamma(1-\alpha)|/\alpha)^{1/\alpha}\tau^{1/\alpha}(\log_2 \tau)^{1/\lambda}\,.$$

When $\alpha = 1$ and the process is centered so that

$$\log(Ee^{-sX_1}) = ms\log s \tag{5.8a}$$

or equivalently

$$\log(Ee^{iuX_1}) = -(\pi/2)m|u|(1 + i(2/\pi)(u/|u|)\log|u|)\,, \tag{5.8b}$$

the normalizations in (5.6) are

$$a(\tau) = m\tau \qquad \text{and} \qquad b(\tau) = m(\log(\tau) - \log_3 \tau - \log(e/m))\,.$$

It is possible to deduce Theorems 3.1 and 4.1 from (5.6) by means of the following standard imbedding procedure. Suppose, for example, that $\alpha < 1$ and that $V, V_1, V_2, \cdots$ are nonnegative i.i.d. random variables satisfying (3.5). Define a process $X = (X(t))_{t\geqq 0}$ by setting

$$X(t) = \textstyle\sum_{n\leqq \mathscr{N}(t)} V_n\,,$$

where $\mathscr{N} = (\mathscr{N}(t))_{t\geqq 0}$ is a Poisson process of unit intensity, independent of the sequence (V_n). Then X is a homogeneous process with independent increments, whose Lévy measure is just the restriction of the distribution of V to $(0, \infty)$. Moreover, $V_1 + \cdots + V_n$ is the value of X at the time τ_n of the nth jump of $\mathscr{N}$. By the strong law of large numbers, $\tau_n/n \to 1$ w.p. 1. Using properties of the M_1-topology, one can easily go on from here to deduce the conclusion of Theorem 3.1 from (5.6).

It would be of interest to know whether Theorems 3.1 and 4.1 could be derived using imbedding schemes involving the completely asymmetric stable processes (5.7) and (5.8).

6. Proofs of auxiliary results. A. *Characterization of the sets K_α.* We will have need of the following "linearization" lemma.

LEMMA 6.1. *Let B be a finite interval, and let y be absolutely continuous over B with derivative $\dot{y}$. Then*

$$(y(B)/|B|)^\zeta |B| \leqq \int_B \dot{y}^\zeta(s)\,ds \tag{6.1}$$

$$e^{-y(B)/|B|}|B| \leqq \int_B e^{-\dot{y}(s)}\,ds \tag{6.2}$$

provided in (6.1) $\dot{y} \geqq 0$ *and either* $\zeta \leqq 0$ *or* $\zeta \geqq 1$.

The proof is an immediate consequence of Jensen's inequality. Roughly speaking, the lemma says that the linearization of y over B has no greater "variation" than does y itself.

We proceed now to characterize the sets K_α defined by (2.26), (2.30) and (2.34). Recall that ${}_xV_\pm$ is defined by (2.17)—(2.18).

LEMMA 6.2. (a) *Let $\alpha < 1$, and let i_α and j_α be defined by* (2.23) *and* (2.25) *respectively. Then $i_\alpha(x) = j_\alpha(x)$ for all x in $\Delta_\uparrow$.*

(b) *Let $\alpha = 1$, and let i_α and j_α be defined by* (2.28) *and* (2.27) *respectively. Then for each x in Δ, $j_\alpha(x) < \infty$ iff ${}_xV_-$ is absolutely continuous over* $[0, \infty)$ *and $i_\alpha(x) < \infty$; moreover when this is the case, $i_\alpha(x) = j_\alpha(x)$.*

(c) *Let $\alpha > 1$, and let i_α and j_α be defined by* (2.32) *and* (2.31) *respectively. Then for each x in Δ, $j_\alpha(x) < \infty$ iff ${}_xV_-$ is absolutely continuous over* $[0, \infty)$ *and $i_\alpha(x) < \infty$; moreover, when this is the case, $i_\alpha(x) = j_\alpha(x)$.*

PROOF. In the proof we shall make use of the partitions

$$\mathscr{A}_{t,n} = \{[0, t2^{-n}], (t2^{-n}, 2t2^{-n}], \cdots, ((2^n - 1)t2^{-n}, 2^n t2^{-n}]\}$$

of $[0, t]$, $t > 0$, $n \geqq 1$. Also we use μ_t to denote $1/t$ times Lebesgue measure on the Borel sets of $[0, t]$; μ_t is thus a probability measure for each t.

(a) First we show that for any x in $\Delta_\uparrow$, $i_\alpha(x) \leqq j_\alpha(x)$. For this, let t be an interior point of F_x (cf. (2.20)), and put

$$r_n = \sum_{A \in \mathscr{A}_{t,n}} (x(A)/|A|) I_A$$

($n \geqq 1$). The sequence (r_n) is a martingale with respect to μ_t, which by standard martingale theory (cf. Freedman (1971) (3.35)) converges almost everywhere on $[0, t]$ to the function $\dot{x}_a$ (cf. (2.21)). Fatou's lemma gives

$$\int_0^t (\dot{x}_a(s))^\lambda\,ds \leqq \liminf_n \int_0^t (r_n(s))^\lambda\,ds \leqq j_\alpha(x)\,.$$

As t is arbitrary, we get $i_\alpha(x) \leqq j_\alpha(x)$.

The opposite inequality follows from the linearization inequality (6.1), with $y = x_a$.

(b) Suppose first that $x \in \Delta$ with $j_\alpha(x) < \infty$. Let t be any point in $(0, \infty)$. We shall show that ${}_xV_-$ is absolutely continuous over $[0, t]$, and that

$$i_{\alpha,t}(x) \equiv \int_0^t e^{-\xi(s)}\, ds = \sup \sum_{A \in \mathscr{A}} e^{-x(A)/|A|}|A| \equiv j_{\alpha,t}(x)\,,$$

where the supremum is taken over all finite partitions $\mathscr{A}$ of $[0, t]$ into disjoint intervals, and where (cf. (2.20))

$$\begin{aligned} \xi(s) &= \dot{x}_a(s)\,, && \text{if} \quad s < t_x \\ &= \infty\,, && \text{if} \quad s \geqq t_x\,. \end{aligned} \tag{6.3}$$

Because

$$\int_0^\infty {}_x\dot{V}_-(s)\, ds \leqq \int_{\{{}_x\dot{V}_-(s)>0\}} e^{-\dot{x}_a(s)}\, ds$$

(cf. (2.22)), this will guarantee that ${}_xV_-$ is absolutely continuous over $[0, \infty)$, with $i_\alpha(x) = j_\alpha(x)$.

For $n \geqq 1$, set

$$r_n = \sum_{A \in \mathscr{A}_{t,n}} (x(A)/|A|)I_A$$

(use the convention that $\infty - \infty = \infty$). Since

$$e^{r_n^-} \leqq \max(e^{-r_n}, 1)\,,$$

one has

$$\sup_n \int_0^t e^{r_n^-(s)}\mu_t(ds) \leqq j_{\alpha,t}(x)/t + 1 < \infty\,.$$

It follows that the sequence (r_n^-) is a uniformly integrable submartingale with respect to μ_t, and thus converges in mean to an integrable random variable ρ, which closes the submartingale. Moreover, the integrable rv e^ρ closes the submartingale $(e^{r_n^-})$, and so the $e^{r_n^-}$'s are themselves uniformly integrable.

From the above mentioned L_1-convergence, one gets

$${}_xV_-(\tau) = \int_0^\tau \rho(s)\, ds$$

for each $\tau \leqq t$; consequently ${}_xV_-$ is absolutely continuous on $[0, t]$. By standard martingale theory, r_n converges almost everywhere to ξ (cf. (6.3)); remember our convention that $\infty - \infty = \infty$. Moreover, since $e^{-r_n} \leqq e^{r_n^-}$, the e^{-r_n}'s are uniformly integrable and we get

$$j_{\alpha,t}(x) = \lim_n \int_0^t e^{-r_n(s)}\, ds = \int_0^t e^{-\xi(s)}\, ds = i_{\alpha,t}(x)\,.$$

For the converse, suppose ${}_xV_-$ is absolutely continuous over $[0, \infty)$ with $i_\alpha(x) < \infty$. If A is a finite subinterval of F_x, then the absolute continuity of ${}_xV_-$ and (2.22) give

$$-x(A) = {}_xV_-(A) - {}_xV_+(A) \leqq {}_xV_-(A) - ({}_xV_+)_a(A) = -x_a(A)\,.$$

The linearization inequality (6.2), with $y = x_a$, thus implies that

$$\sup \sum_{A \in \mathscr{A}} e^{-x(A)/|A|}|A| \leqq \int_{F_x} e^{-\dot{x}_a(s)}\, ds = i_\alpha(x) \tag{6.4}$$

where the supremum is extended over all finite disjoint collections of subintervals of F_x. But since x is identically ∞ to the right of F_x, the left-hand side of (6.4) is in fact $j_\alpha(x)$.

(c) Suppose first that $x \in \Delta$ with $j_\alpha(x) < \infty$. Let t be any point in $(0, \infty)$. We shall show that ${}_xV_-$ is absolutely continuous over $[0, t]$, with

$$i_{\alpha,t}(x) \equiv \int_0^t {}_x\dot{V}_-^\lambda(s)\,ds = \sup \sum_{A\in\mathscr{A}} ((x(A))^-/|A|)^\lambda |A| \equiv j_{\alpha,t}(x) \tag{6.5}$$

where the supremum is taken over all partitions $\mathscr{A}$ of $[0, t]$ into disjoint intervals. This will give ${}_xV_-$ absolutely continuous over $[0, \infty)$ with $i_\alpha(x) = \int {}_x\dot{V}_-^\lambda(s)\,ds = j_\alpha(x)$.

Proceeding with the argument, put

$$r_n = \sum_{A\in\mathscr{A}_{t,n}} ((x(A))^-/|A|) I_A$$

(with the convention $\infty - \infty = \infty$). The sequence (r_n) is a nonnegative submartingale with respect to μ_t for which

$$\sup_n \int_0^t (r_n(s))^\lambda \mu_t(ds) = j_{\alpha,t}(x)/t\,.$$

There therefore exists a nonnegative function r on $[0, t]$ such that r_n converges to r in mean of order λ with respect to μ_t. But then

$$\int_0^\tau r(s)\,ds = {}_xV_-(\tau)$$

for each $\tau \leqq t$, and so ${}_xV_-$ is absolutely continuous over $[0, t]$, with ${}_x\dot{V}_- = r$. Moreover, since the λ-norm of r_n converges to the λ-norm of r, we have (6.5).

Next suppose ${}_xV_-$ is absolutely continuous with $i_\alpha(x) < \infty$. That $j_\alpha(x) < \infty$ follows from the inequality (cf. (2.19))

$$(x(A))^- \leqq {}_xV_-(A) \leqq \int_A {}_x\dot{V}_-(s)\,ds$$

and the linearization inequality (6.1), with $y = {}_xV_-$. □

B. *Proof of Lemmas* 3.1 *and* 4.1. We borrow the following result from Bahadur (1971), page 5, in a form convenient for our purposes.

LEMMA 6.3. *For each $n \geqq 1$, let Z_n be a random variable whose cumulant generating function* (cgf)

$$C_n(\zeta) \equiv \log(Ee^{\zeta Z_n}) \tag{6.6}$$

exists for $\zeta \in (0, \infty)$ and for which C_n' takes on both negative and positive values. Determine τ_n such that

$$C_n'(\tau_n) = 0\,. \tag{6.7}$$

Put

$$r_n = -C_n(\tau_n) \tag{6.8}$$

$$\sigma_n^2 = C_n''(\tau_n) \tag{6.9}$$

and let Q_n be the probability on $(-\infty, \infty)$ whose cgf *C_n^* is given by*

$$C_n^*(\zeta) = C_n(\tau_n + \zeta/\tau_n) - C_n(\tau_n)\,. \tag{6.10}$$

Suppose that as $n \to \infty$,

(6.11) $$r_n \to \infty$$

(6.12) $$\sigma_n \tau_n / r_n \to 0$$

and

(6.13) *no subsequence of (Q_n) converges weakly to the distribution degenerate at* 0.

Then as $n \to \infty$.

(6.14) $$\Pr\{Z_n \geqq 0\} = \exp(-(1 + o(1))r_n) .$$

PROOF OF LEMMA 3.1. We shall give the proof only for $\alpha > 1$; the case $\alpha < 1$ is similar. First consider the function g_α defined by (3.6). Put $\psi(s) = Ee^{-sV}$. Elementary considerations show that

$$\begin{aligned} g_\alpha(s) &= \psi(s) - 1 - s\psi'(0) + O(s^2) \\ g_\alpha'(s) &= \psi'(s) - \psi'(0) + O(s) \\ g_\alpha''(s) &= \psi''(s) + O(1) \end{aligned}$$

as $s \downarrow 0$. After expressing $\psi(s) - 1 - s\psi'(0)$, $\psi'(s) - \psi'(0)$, and $\psi''(s)$ as Laplace transforms of measures on $[0, \infty)$ and using the Tauberian theorems of Feller (1971) Section XIII. 5, one finds that as $s \downarrow 0$

$$\begin{aligned} g_\alpha(s) &\sim s^\alpha L_\alpha(1/s) \\ g_\alpha'(s) &\sim \alpha s^{\alpha-1} L_\alpha(1/s) \sim \alpha g_\alpha(s)/s \\ g_\alpha''(s) &\sim \alpha(\alpha - 1)s^{\alpha-2} L_\alpha(1/s) \sim \alpha(\alpha - 1) g_\alpha(s)/s^2 , \end{aligned}$$

where L_α is defined by (3.5).

To establish (3.12) we will apply Lemma 6.3 with

$$Z_n = -c - T_n/A_n .$$

Notice that (3.11) implies $g_\alpha(B_n/A_n) = B_n/n$, and so $B_n/A_n \to 0$ by (3.10). For ζ on the same order of magnitude as B_n, the regular variation of g_α gives (cf. (6.6))

$$\begin{aligned} C_n'(\zeta) &= -c + (n/A_n) g_\alpha'(\zeta/A_n) \\ &= -c + (1 + o(1))\alpha n \zeta^{-1} g_\alpha((\zeta/B_n)(B_n/A_n)) \\ &= -c + (1 + o(1))\alpha n \zeta^{-1} (\zeta/B_n)^\alpha g_\alpha(B_n/A_n) \\ &= -c + (1 + o(1))\alpha(\zeta/B_n)^{\alpha-1} . \end{aligned}$$

In the notation of Lemma 6.3 we thus have $\tau_n \sim B_n(c/\alpha)^{1/(\alpha-1)}$, $r_n \sim B_n c^\lambda \alpha^{-\lambda}(\alpha - 1)$, $\sigma_n \sim O(1/B_n^{\frac{1}{2}})$, while $C_n^{*\prime\prime}(\zeta) \to 1$ uniformly for ζ in bounded intervals, so Q_n converges to the standard normal distribution. Consequently (3.12) follows from (6.14). □

PROOF OF LEMMA 4.1. Consider first the function g defined by (4.2):

$$g(s) = -\log \psi(s) \qquad \text{with} \quad \psi(s) = Ee^{-sV} .$$

Several applications of the Tauberian theorems of Feller (1971) Section XIII. 5 show that as $s \downarrow 0$

$$\begin{aligned}\psi(s) - 1 &\equiv -sM(1/s)\\ \psi'(s) &\sim -M(1/s)\\ \psi''(s) &\sim \mathscr{M}(1/s)/s^2 = \mathscr{L}(1/s)/s\,,\end{aligned}$$

where M, $\mathscr{M}$, and $\mathscr{L}$ are defined by (4.4), (4.5), and (4.7) respectively. Since M and $\mathscr{L}$ are slowly varying, it follows that as $s \downarrow 0$

$$\begin{aligned}g(s) &= sM(1/s) + s^{2+o(1)}\\ g'(s) &= M(1/s) - \mathscr{L}(1/s) + s^{1+o(1)}\\ g''(s) &\sim -\mathscr{L}(1/s)/s\,.\end{aligned}$$

To establish (4.15) we will apply Lemma 6.3 with

$$Z_n = c - (S_n - nM(A_n/B_n))/A_n\,.$$

By (4.13) and (4.14), $A_n/B_n = \mathscr{M}^{\sim}(B_n/n) \to \infty$ as $n \to \infty$. Thus for ζ of the same order of magnitude as B_n, (4.9) and the slow variation of $\mathscr{L}$ give

$$\begin{aligned}C_n'(\zeta) &= c - (n/A_n)g'(\zeta/A_n) + (n/A_n)M(A_n/B_n)\\ &= c + (M(A_n/B_n) - M((A_n/B_n)(B_n/\zeta)))/\mathscr{L}(A_n/B_n)\\ &\quad + \mathscr{L}((A_n/B_n)(B_n/\zeta))/\mathscr{L}(A_n/B_n) + 1/(A_n/B_n)^{1+o(1)}\\ &= c + (1 + o(1))\log(\zeta/B_n) + (1 + o(1)) + o(1)\,.\end{aligned}$$

Thus in the notation of Lemma 6.3 we have $\tau_n \sim e^{-(c+1)}B_n$, $r_n \sim \tau_n$, $\sigma_n \sim 1/\tau_n^{\frac{1}{2}}$, while $C_n^{*\prime\prime}(\zeta) \to 1$ uniformly for ζ in compact sets, so Q_n converges weakly to the standard normal distribuiion. And (4.15) follows from (6.14). □

C. *Some large deviation results.* The preceeding subsection established estimates of the left tails of the distributions involved. The following lemma is concerned with estimates for the right tails. The first assertion is essentially due to Heyde (1968).

LEMMA 6.4. (a) *Suppose* $\alpha \neq 1$, *and let* T_n *be defined by* (3.9), γ_n *by* (3.13). *Then for any sequence* (c_n) *tending to* ∞,

$$\Pr\{T_n \geqq c_n\gamma_n\} \sim n\Pr\{T_1 \geqq c_n\gamma_n\}\,. \tag{6.15}$$

(b) *Suppose* $\alpha = 1$, *and let* S_n *be defined by* (4.10), γ_n *by* (4.11), *and* M *by* (4.4). *Then for any sequence* (c_n) *tending to* ∞,

$$\Pr\{S_n - nM(\gamma_n) \geqq c_n\gamma_n\} \sim n\Pr\{S_1 \geqq c_n\gamma_n\}\,. \tag{6.16}$$

Moreover, (6.15) *and* (6.16) *continue to hold when the random variables figuring in them are replaced by their absolute values.*

PROOF. (a) Simple modifications of arguments due to Heyde (1967) and (1968) yield (6.15); we will briefly indicate the changes that need to be made. Put $r_n = \Pr\{T_n \geqq c_n\gamma_n\}/(n\Pr\{T_1 \geqq c_n\gamma_n\})$. For the proof of $\liminf_n r_n \geqq 1$, replace the events A_i and B_i in Heyde (1967), page 1576 by

$$\{X_i \geqq (1+\varepsilon)x_n\} \quad \text{and} \quad \{\textstyle\sum_{j\neq i} X_j \geqq -\varepsilon x_n\}$$

respectively (Heyde's notation). For the proof of $\limsup_n r_n \leqq 1$, replace the event E_n in Heyde (1968), page 255, by

$$E_n^* = \{X_k \geqq (1-\varepsilon)x_n B_n \ \text{ for at least one } \ k \leqq n\}$$

and observe

$$\{S_n \geqq x_n B_n\} \subset E_n^* \cup F_n \cup G_n$$

(Heyde's notation).

(b) The proof is similar. In verifying the analogue of (11)—(12) in Heyde (1968), one uses (4.4c) and (8.9) below. □

D. *A fluctuation inequality*. The following lemma is well known (cf. Kiefer (1969) page 325 for the method of proof)

LEMMA 6.5. *Let $Y_1, Y_2, \cdots, Y_n$ be* i.i.d. *random variables with partial sums $Z_0 = 0$, $Z_m = Y_1 + \cdots + Y_m$, $1 \leqq m \leqq n$. Then for each number c,*

$$\Pr\{\min_{0\leqq i<j\leqq n}(Z_j - Z_i) \leqq c\} \leqq \Pr\{Z_n \leqq c\}/(\min_{1\leqq j\leqq n}\Pr\{Z_j \leqq 0\})^2 . \tag{6.17}$$

7. **How to identify limit points.** The lemma below is a useful aid for identifying the limit points of a random sequence. In order to state the result, we need to introduce the notion of a *limit determining system* (LDS). Let x be a point in a metric space S, and let $(N_j)_{j\geqq 1}$ be a decreasing sequence of neighborhoods of x in S; we'll say that $(N_j)_{j\geqq 1}$ is a LDS at x if x is the limit of every convergent sequence (x_n) of points of S such that, for each j, $x_n \in N_j$ for all large n. For example, suppose $(t_i)_{i\geqq 1}$ is a dense subset of $(0, \infty)$, and for $j \geqq 1$, put

$$N_j = \{y \in \Delta : |y(t_i)| \leqq 1/j,\ 1 \leqq i \leqq j\} . \tag{7.1}$$

Then $(N_j)_{j\geqq 1}$ is a LDS at $x = 0$ in the M_1-topology on Δ. As this example shows, a LDS at x need not be a neighborhood base at x.

LEMMA 7.1. *Let $(\Omega, \mathscr{A}, P)$ be a probability space. Let S be a separable metric space and let $\mathscr{H}_n$, $n \geqq 1$, be mappings from Ω to S.*

(a) *Let C be a subset of S. Suppose there exists a sequence (n_j) tending to infinity such that*

$$\begin{aligned} &P\{\textit{the sequence } (\mathscr{H}_n)_{n\geqq 1} \textit{ has a limit point in } C\} \\ &\qquad \leqq P\{\textit{the sequence } (\mathscr{H}_{n_j})_{j\geqq 1} \textit{ has a limit point in } C\} \end{aligned} \tag{7.2}$$

and such that each x in C has a neighborhood N_x for which

$$P\{\mathscr{H}_{n_j} \in N_x \ \textit{ for infinitely many } \ j\} = 0 . \tag{7.3}$$

Then

$$P\{\textit{the sequence } (\mathscr{H}_n) \textit{ has no limit points in } C\} = 1 . \tag{7.4}$$

(b) *Let B be a subset of S, B_0 a dense subset of B. Suppose that for each $x \in B_0$*

there is a LDS $(N_j)_{j\geq 1}$ *at* x *such that*

$$P\{\mathscr{H}_n \in N_j \text{ for infinitely many } j\} = 1 \tag{7.5}$$

for each j. *Suppose also that*

$$P\{\text{the sequence } (\mathscr{H}_n) \text{ is relatively compact in } S\} = 1\,. \tag{7.6}$$

Then

$$P\{\text{every point of } B \text{ is a limit point of } (\mathscr{H}_n)\} = 1\,. \tag{7.7}$$

PROOF. Most of this was proved in Lemma 5.1 of Wichura (1974). The only thing left to do here is to show that for each x in B_0, one has

$$\liminf_n d(\mathscr{H}_n, x) = 0$$

w.p. 1, where d is a metric for S. For this, fix x and let (N_j) be a LDS at x for which (7.5) holds. Then for almost all sample points ω in Ω, one can inductively construct an ω-dependent sequence (n_j) such that $n_j \uparrow \infty$ and such that $\mathscr{H}_{n_j}(\omega) \in N_j$ for each j. Because the N_j's decrease with j, one then has $\mathscr{H}_{n_k}(\omega) \in N_j$ for $k \geqq j$. But for almost all ω, the sequence $(\mathscr{H}_n(\omega))$ is itself relatively compact, by (7.6). So for almost all ω, there exists an ω-dependent subsequence $(n_{j_l})_{l\geq 1}$ such that $\mathscr{H}_{n_{j_l}}(\omega)$ converges to some point of S as $l \to \infty$; the limit must be x because (N_j) is a LDS at x. □

Typically, one would verify (7.3) and (7.5) using some form of the first and second Borel–Cantelli lemmas. It should be noted that (7.3) is more stringent than the condition $P\{x \text{ is a limit point of } (\mathscr{H}_{n_j})\} = 0$ for each x in C; take, e.g., $\Omega = S = C = [0, 1]$, $P =$ Lebesgue measure, and set $\mathscr{H}_n(\omega) = \omega$ for all ω and n.

The following remqrks are of use in verifying condition (7.2). Say that a sequence (x_n) of points of S is *cohesive* if, whenever (m_j) and (n_j) are sequences tending to ∞ such that

$$n_j/m_j \to 1 \quad \text{and} \quad \lim_j x_{m_j} \text{ exists,} \tag{7.8}$$

then

$$\lim_j x_{n_j} \text{ exists and equals } \lim_j x_{m_j}\,. \tag{7.9}$$

For example, in the context of $S = \Delta$ endowed with the M_1-topology, a sequence (x_n) is cohesive if there exist numbers $a_{m,n}$, $b_{m,n}$, and $c_{m,n}$ such that

$$x_n(t) = a_{m,n} x_m(b_{m,n} t) + c_{m,n} t \tag{7.10}$$

for all t, and such that

$$a_{m,n} \to 1\,, \qquad b_{m,n} \to 1\,, \qquad c_{m,n} \to 0 \tag{7.11}$$

as m and n tend to ∞ in such a way that $n/m \to 1$.

LEMMA 7.2. *Condition* (7.2) *in Lemma* 7.1 *holds for any sequence* (n_j) *satisfying*

$$n_j \to \infty \quad \text{and} \quad n_j/n_{j-1} \to 1 \tag{7.12}$$

provided

(7.13) $$P\{\text{the sequence } (\mathscr{H}_n) \text{ is cohesive}\} = 1 .$$

8. Proofs of Theorems 3.1 and 4.1. We shall first give the proof of Theorem 4.1, which is a little more delicate than Theorem 3.1. Here are some preliminary remarks. For notational convenience, set

$$b_n = \log_2 n \qquad \text{and} \qquad c_n = a_n/(b_n e)$$

where a_n is defined by (4.17). Notice that

(8.1) $$n \to a_n \text{ is 1-varying,} \qquad n \to b_n \text{ is 0-varying,}$$
$$n \to c_n \text{ is 1-varying.}$$

For any m and n, one has (cf. (4.16))

(8.2) $$\begin{aligned} H_n(t) &= (S_{[nt]} - [nt]M(c_n))/a_n \\ &= (a_m/a_n)H_m((n/m)t) + ([nt]/n)(M(c_m) - M(c_n))/\mathscr{L}(ec_n) . \end{aligned}$$

It follows from (4.9) and Lemma 4.1 that as $n \to \infty$

(8.3) $$\Pr\{H_n(t) \leqq c\} = \exp(-(1 + o(1))te^{-c/t}b_n)$$

for each t and c.

We proceed now with the proof, which will be divided into three lemmas.

LEMMA 8.1. W.p. 1, *the sequence* (H_n) *defined by* (4.16) *is relatively compact and all its limit points lie in* Δ_+.

PROOF. By (2.13) we need to check that for each $v > 0$

(8.4) $$\lim_{\delta \downarrow 0} \limsup_n m^{m}_{\delta,v}(H_n) = 0 \quad \text{w.p. } 1 ,$$

where $m^*_{\delta,v}$ is defined by (2.11). For this fix v and let $\varepsilon > 0$ be given. For any integer $k \geqq 1$, one can find approximately $2kv$ intervals of length $1/k$ such that every subinterval of $[0, v]$ of length $1/(2k)$ is contained in some one of them; it follows from this and the fluctuation inequality (6.17) that

(8.5) $$\begin{aligned} &\Pr\{m^*_{1/(2k),v}(H_n) \geqq \varepsilon\} \\ &\quad \leqq O(1)\Pr\{\inf_{0 \leqq s < t \leqq 1/k}(H_n(t) - H_n(s)) \leqq -\varepsilon\} \\ &\quad \leqq O(1)\Pr\{H_n(1/k) \leqq -\varepsilon\}/(\inf_{t \leqq 1/k}\Pr\{H_n(t) \leqq 0\})^2 . \end{aligned}$$

Now for $t \leqq 1/k$,

$$\begin{aligned} \Pr\{H_n(t) \leqq 0\} &= \Pr\{S_{[nt]} - [nt]M(c_n) \leqq 0\} \\ &\geqq \Pr\{S_{[nt]} - [nt]M(c_{[nt]}) \leqq 0\} = \Pr\{H_{[nt]}(1) \leqq 0\} , \end{aligned}$$

and (8.3) implies that $\Pr\{H_m(1) \leqq 0\} = 1/(\log m)^{1+o(1)}$ as $m \to \infty$; thus

$$\inf_{t \leqq 1/k}\Pr\{H_n(t) \leqq 0\} \geqq 1/(\log n)^{1+o(1)} .$$

Combining this with (8.3) we find that the right-hand side of (8.5) is dominated by

$$1/(\log n)^{(1+o(1))(e^{k\varepsilon}/k-2)} .$$

So if we choose k sufficiently large and take $n_j = [e^{j/\log j}]$, the first Borel–Cantelli lemma gives

$$\limsup_j m^*_{1/(2k),v}(H_{n_j}) \leqq \varepsilon$$

w.p. 1. As ε is arbitrary, we have

$$\lim_{\delta \downarrow 0} \limsup_j m^*_{\delta,v}(H_{n_j}) = 0$$

w.p. 1, and together with (8.1) and (8.2), this gives (8.4). □

LEMMA 8.2. W.p. 1, *the sequence* (H_n) *has no limit points outside of the set* K_1 *defined by* (2.30).

PROOF. We shall verify the conditions of part (a) of Lemma 7.1. Since (8.1), (8.2), and (4.9) imply that (H_n) is cohesive in the sense of (7.8)—(7.9), Lemma 7.2 implies that (7.2) holds with $C = \Delta_+ - K_1$ and $n_j = [e^{j/\log j}]$. So by (7.3), it suffices to check that each x in $\Delta_+ - K_1$ has a neighborhood N such that

$$\Pr\{H_{n_j} \in N \text{ for infinitely many } j\} = 0 . \tag{8.6}$$

Suppose then that x has no negative jumps and (cf. (2.27))

$$j_1(x) = \sup \textstyle\sum_{A \in \mathscr{A}} |A| e^{-x(A)/|A|} > 1 .$$

One can then choose continuity points $0 < t_0 < t_1 < \cdots < t_q < \infty$ of x such that $x(t_p) < \infty$ for each p and such that

$$\textstyle\sum_{1 \leqq p \leqq q} \delta t_p e^{-\delta x_p/\delta t_p} > 1 ;$$

here

$$\delta t_p = t_p - t_{p-1} \qquad \text{and} \qquad \delta x_p = x(t_p) - x(t_{p-1}) .$$

Choose g_p, h_p, $1 \leqq p \leqq q$ such that

$$g_p < x(t_p) < h_p$$

for each p, and

$$\zeta \equiv \textstyle\sum_{1 \leqq p \leqq q} \delta t_p e^{-(h_p - g_{p-1})/\delta t_p} > 1 .$$

Put

$$N = \{y \in \Delta : g_p < y(t_p) < h_p\} .$$

Then by (8.3)

$$\begin{aligned}\Pr\{H_n \in N\} &\leqq \textstyle\prod_p \Pr\{h_{p-1} - g_p < H_n(t_p) - H_n(t_{p-1}) < h_p - g_{p-1}\} \\ &= 1/(\log n)^{(1+o(1))\zeta}\end{aligned}$$

and so (8.6) follows from the first Borel–Cantelli lemma. □

LEMMA 8.3. W.p. 1, *each point of* K_1 *is a limit point of* (H_n).

PROOF. The set

$$K_1{}^* = \{x \in \Delta : j_1(x) < 1 \text{ and } x(t) < \infty \text{ for all } t > 0\}$$

is dense in K_1. So by Lemma 7.1(b), it is enough to show that each x in $K_1{}^*$ has a LDS satisfying (7.5).

Suppose then that $x \in K_1{}^*$. In view of example (7.1), it suffices to show that for any system $t_1 < t_2 < \cdots < t_q$ of points in $(0, \infty)$ and any sufficiently small $\varepsilon > 0$, one has

$$\Pr(\limsup_k F_k) = 1 = \Pr(\liminf_k G_k) , \tag{8.7}$$

where

$$G_k = \{|H_{n_k}(t_0)| < \varepsilon\}$$
$$F_k = \bigcap_{1\le p\le q} \{g_p < H_{n_k}(t_p) - H_{n_k}(t_{p-1}) < h_p\}$$

with

$$n_k = [\exp(k^{1+\varepsilon})] , \qquad t_0 = t_0(k) = t_q n_{k-1}/n_k$$

and

$$g_p = \delta x_p - \varepsilon , \qquad h_p = \delta x_p + \varepsilon .$$

By (8.3),

$$\Pr(F_k) = 1/(\log n_k)^{(1+o(1))\sum_{1\le p\le q} \delta t_p \exp(-h_p/\delta t_p)} .$$

Because $j_1(x) < 1$, we have only to take ε sufficiently small to get $\sum_k \Pr(F_k) = \infty$ and thus $\Pr(\limsup_k F_k) = 1$ by virtue of the second Borel–Cantelli lemma.

Next we show $\Pr(\liminf_k G_k) = 1$ regardless of the value of ε, thereby completing the proof of (8.7). Put $m_k = [t_q n_{k-1}]$. If $|H_{n_k}(t_q n_{k-1}/n_k)|$ exceeds ε, then

$$|S_{m_k} - m_k M(\gamma_{m_k})| \geqq \varepsilon a_{n_k} - |m_k(M(c_{n_k}) - M(\gamma_{m_k}))| , \tag{8.8}$$

where γ_m is defined by (4.11). Because $\mathscr{L}$ is slowly varying

$$(M(ct) - M(t))/\mathscr{L}(t) = \textstyle\int_t^{ct} (\mathscr{L}(u)/\mathscr{L}(t))u^{-1}\,du = c^{o(1)} \log c = c^{o(1)} \tag{8.9}$$

if $c = c(t)$ and t both tend to ∞ (compare (4.9)). Consequently the rightmost term in (8.8) is majorized by

$$\gamma_{m_k}(M(c_{n_k}) - M(\gamma_{m_k}))/\mathscr{L}(\gamma_{m_k}) = o(a_{n_k})$$

because $c_n < a_n$ and $a_n = b_n{}^{o(1)}\gamma_n$, whence

$$\begin{aligned}(\gamma_{m_k}/a_{n_k})^{1+o(1)} &= b_{n_k}^{o(1)}(\gamma_{m_k}/\gamma_{n_k}) = b_{n_k}^{o(1)}(m_k/n_k)^{1+o(1)} \\ &= (\log k)^{o(1)} \exp(-(1 + o(1))(1 + \varepsilon)k^{\varepsilon}) = o(1) .\end{aligned} \tag{8.10}$$

Thus the right-hand side of (8.8) is $(1 + o(1))\varepsilon a_{n_k}$. By Lemma 6.4(b),

$$\begin{aligned}\Pr\{|S_{m_k} - m_k M(\gamma_{m_k})| \geqq \varepsilon a_{n_k}\} &\sim m_k \mathscr{M}(\varepsilon a_{n_k}) = m_k \mathscr{M}(b_{n_k}^{o(1)}\gamma_{n_k}) \\ &= b_{n_k}^{o(1)} \mathscr{M}(\gamma_{n_k})/\mathscr{M}(\gamma_{m_k}) = b_{n_k}^{o(1)}(\gamma_{m_k}/\gamma_{n_k})^{1+o(1)} .\end{aligned}$$

By (8.10) the sum of these terms over k is finite, and so the first Borel–Cantelli lemma implies the desired result. □

The proof of Theorem 3.1 follows a similar pattern and will be omitted.

9. Proofs of applications of the main theorems. A. *On the equivalence of Theorems* 3.1 *and* 3.2. The equivalence of Theorems 3.1 and 3.2 is a consequence of the following result, which is modelled after Vervaat (1972) and Whitt (1975). It is to be noted that although the process U is assumed to be finite-valued, its limit points can take on infinite values. Let I be the identity map on $[0, \infty)$, put

$$\Delta_{\uparrow}^{\infty} = \{x \in \Delta_{\uparrow} : x(t) < \infty \text{ for all } t \text{ and } \lim_{t\to\infty} x(t) = \infty\}$$

(cf. (2.9)), and recall that $\mathscr{F}$ is defined by (2.15).

THEOREM 9.1. *Let $U = (U(\tau))_{\tau>0}$ be a $\Delta_{\uparrow}^{\infty}$-valued random process. Let C be a subset of Δ, and let ψ be a mapping from $(0, \infty)$ into itself.*

(a) *Suppose ψ is ultimately continuous and strictly increasing, with $\lim_{\tau\to\infty} \psi(\tau) = \infty$. The following two statements are then equivalent (cf. (2.8)):*

(i) W.p. 1, $U(\tau I)/\psi(\tau) \to C$ *as* $\tau \to \infty$.
(ii) W.p. 1, $(\mathscr{F}U)(\tau I)/\psi^{\sim}(\tau) \to \mathscr{F}(C)$ *as* $\tau \to \infty$.

(b) *Suppose $\psi(\tau) = o(\tau)$ as $\tau \to \infty$, and let μ be a strictly positive number. The following two statements are then equivalent:*

(i) W.p. 1, $(U(\tau I) - \tau\mu I)/\psi(\tau) \to C$ *as* $\tau \to \infty$.
(ii) W.p. 1, $-((\mathscr{F}U)(\tau I) - \tau I/\mu)/\psi(\tau) \to \{\mu^{-1}y(I/\mu) : y \in C\}$ *as* $\tau \to \infty$.

B. *Proofs of* (i), (ii), *and* (iii) *in Section* 3, *and* (i) *and* (ii) *in Section* 4. The proofs of these results follow pretty much along the lines of Strassen (1964), and will be omitted (see also (Wichura (1974)). We will point out, however, two of the tools which, in addition to the linearization Lemma 6.1, are used in the proofs.

The first part of the following lemma is just a special case of the Hölder inequalities (cf. Hardy, Littlewood, and Pólya (1934) pages 24–25):

LEMMA 9.1. (a) *Let $\alpha \in (0, 1) \cup (1, 2)$, and let $\lambda = \lambda_\alpha$ be defined by* (3.1), *so that $1/\alpha + 1/\lambda = 1$. Let E be a Borel measurable subset of $[0, \infty)$ and let x and y be two Borel measurable mappings of E into itself. Put*

$$\|x\|_\alpha = (\textstyle\int_E x^\alpha(t)\,dt)^{1/\alpha} \quad \text{and} \quad \|y\|_\lambda = (\textstyle\int_E y^\lambda(t)\,dt)^{1/\lambda}. \tag{9.1}$$

Then, using the convention that $0 \cdot \infty = 0$, one has

$$\textstyle\int_E x(t)y(t)\,dt \gtreqqless \|x\|_\alpha\|y\|_\lambda \tag{9.2}$$

according to whether $\alpha \lesseqgtr 1$. When the right-hand side of (9.2) *is finite and strictly positive, equality holds in* (9.2) iff $(x(t)/\|x\|_\alpha)^\alpha = (y(t)/\|y\|_\lambda)^\lambda$ *for almost all t in E.*

(b) *Let E be as in* (a), *and let x and y be two Borel measurable mappings of E into $[-\infty, \infty]$, with $y \geqq 0$. Then*

$$\textstyle\int_E x(t)y(t)\,dt \geqq -\textstyle\int_E (e^{-x(t)} + y(t)(\log(y(t)) - 1))\,dt. \tag{9.3}$$

When the right-hand side of (9.3) *is finite, equality holds in* (9.3) iff $x(t) = -\log(y(t))$ *for almost all t in E.*

PROOF OF (b): For any a in $[-\infty, \infty]$ and b in $[0, \infty]$, one has

$$ab \geqq -e^{-a} - b(\log(b) - 1). \tag{9.4}$$

Moreover, when equality holds in (9.4), $a = -\log b$ unless $b = \infty$ and $-\infty < a < 0$, or $0 < b < \infty$ and $a = -\infty$. (9.3) follows easily. $\square$

LEMMA 9.2. *Suppose ν_n, $n \geqq 1$, and ν are finite nonnegative measures on the Borel sets of $[0, \infty)$, and that ν_n converges weakly to ν. Suppose further that these measures are all concentrated on some finite closed interval F. Let $y_n \to y$ in the M_1-topology on Δ.*

(a) *If $y \in \Delta_+$, then*

$$\liminf_n \int y_n \, d\nu_n \geqq \int y(s-)\nu(ds) . \tag{9.5}$$

(b) *If y is finite and continuous at each point of F, then*

$$\int y_n \, d\nu_n \to \int y \, d\nu . \tag{9.6}$$

PROOF. (a) From the M_1-convergence of y_n to y, it follows that the y_n's and y are uniformly bounded away from $-\infty$ over finite intervals. Moreover because y has no negative jumps, $\liminf_n y_n(t_n) \geqq y(t-)$ whenever $t_n \to t$ in $[0, \infty)$. The weak convergence version of Fatou's lemma implies (9.5).

(b) From the continuity of y, one has $y_n \to y$ uniformly over F.

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