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A NEW DECOMPOSITION OF INFINITELY DIVISIBLE DISTRIBUTIONS

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In this paper, it is shown that a random variable Z having an infinitely divisible distribution with a finite left extremity can be decomposed in distribution as the sum of two random variables such that, given one, the other has an infinitely divisible distribution of prescribed type. Some extensions of this result when the left extremity is infinite are also given, and a new decomposition of a normal variable is exhibited.

1. Introduction. Let Z be a random variable (rv) distributed as the sum of two rv's X and Y, not necessarily independent. We shall, in this paper, state that Z is decomposable in distribution as the sum of X and Y; this terminology generalizes the traditional one which requires that X and Y be independently distributed nondegenerate rv's.

In a recent paper, Gani and Shanbhag (1974) considered the decomposition in distribution of a Poisson rv with parameter λ. It was shown that this could always be decomposed into the sum of 2 rv's $X, Y \geqq 0$ such that, given a set of parameters $\lambda > \mu_0 \geqq \mu_1 \cdots \geqq 0$, the conditional distribution $P^{Y|X=i}$ was Poisson with parameter μ_i, and X had a unique distribution $\{a_i > 0\}$ dependent on the μ_i. In a private communication to the authors, R. Pyke gave an alternative derivation of this result, which suggested that the problem could be viewed somewhat differently and generalized as follows.

If a proper counting process $\{N(t), t \geqq 0\}$ with independent increments is defined, then assuming the same relations between the parameters λ, μ_i as above, an rv Z with the distribution the same as that of $N(\lambda)$ is decomposable as the sum of two nonnegative integer-valued rv's X, Y such that $P^{Y|X=i}$ is the same as the distribution of $N(\mu_i)$ almost surely (a.s.).

In the case of a counting process with stationary independent increments this result implies that the probability generating function (pgf) of $N(1)$ is compound Poisson of the type

$$\psi(z) = e^{-\eta+\eta f(z)} \quad |z| \leqq 1$$

where $f(z)$ is the pgf of a positive integer valued rv, and satisfies

$$\sum_{i=0}^{\infty} a_i z^i [\psi(z)]^{\mu_i} = [\psi(z)]^{\lambda} ,$$

with $\{a_i\}$ a probability distribution. This, in turn, implies that any discrete rv

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with pgf ψ^λ can be decomposed in distribution as the sum of two rv's X, Y such that given $X = i$ the rv Y is infinitely divisible (i.d.) with pgf ψ^{μ_i} a.s.

The purpose of this paper is to extend this result to any arbitrary i.d. distribution with a finite left extremity. Let ϕ be the characteristic function (ch.f.) of an i.d. distribution and F_α the distribution function (df) where ch.f. is ϕ^α; further, let μ be a continuous monotonic decreasing function on the real line taking values in the interval $[0, \lambda]$, where $\lambda < \infty$. Then, the rv Z having the df F_λ with finite left extremity is decomposable in distribution as the sum of rv's X, Y such that given $X = x$, the conditional df of Y is $F_{\mu(x)}$ a.s. We shall use Helly's selection theorems (Lukacs (1970) pages 44–47) to provide limiting arguments to prove this result. We shall also establish that our decomposition theorem remains valid for an i.d. distribution with infinite left extremity provided it is the weak limit of a sequence of i.d. distributions with finite left extremities. Using this, we prove the existence of the mixtures of normal distributions which are normal. This result contrasts with that given earlier by Kelker (1971). Before discussing our main results a lemma will be required, which is proved in the next section.

2. A lemma for bivariate characteristic functions.

LEMMA. *Let X, Y have the bivariate* ch.f. *ψ given by*

$$\psi(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1x}\psi_x(t_2)\, dG(x)\,, \tag{2.1}$$

where G is a df. *For each x the function $\psi_x(t_2)$ $(-\infty < t_2 < \infty)$ is a* ch.f. *and for each t_2, $\psi_x(t_2)$ is a complex-valued Borel measurable function of x. Then the conditional* df *of Y given $X = x$ is $H_x(y)$* a.s., *where H_x is the* df *whose* ch.f. *is ψ_x.*

PROOF. Denote the joint distribution function of X and Y by $G^*(x, y)$.
Then it is well known that

$$G^*(x, y) = \int_{-\infty}^{x} H(y \mid z)\, dG(z) \tag{2.2}$$

where for fixed z the function $H(y \mid z)$ denotes a df, and for fixed y it is a Borel measurable function of z (cf. Burrill (1972) page 396). This implies that $\phi_x(t_2)$ the ch.f. of $H(y \mid x)$ is a Borel measurable function of x for every fixed t_2 and

$$\psi(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1x}\phi_x(t_2)\, dG(x)\,. \tag{2.3}$$

From (2.1), using the result (6.15) of Lukacs and Laha (1964), it can be seen after some minor manipulation that for every Borel set B,

$$\int_B \phi_x(t_2)\, d\nu = \int_B \psi_x(t_2)\, d\nu\,, \tag{2.4}$$

where ν is the measure induced by G. We have that

$$\begin{aligned} &\{x : \phi_x(t_2) \neq \psi_x(t_2) \ \text{ for some } \ t_2\} \\ &\qquad = \{x : \phi_x(t_2) \neq \psi_x(t_2) \ \text{ for some rational } \ t_2\} \\ &\qquad = \textstyle\bigcup_r \{x : \phi_x(r) \neq \psi_x(r)\} \end{aligned} \tag{2.5}$$

where r are rationals. From (2.4) it follows that $\{x: \phi_x(r) \neq \psi_x(r)\}$ is a subset of a ν-null set and hence it follows that $\bigcup_r \{x: \phi_x(r) \neq \psi_x(r)\}$ is also a subset of a ν-null set. From this, it is seen that the set of values x for which $H(y \mid x)$ does not have the ch.f. $\psi_x(t)$ is a subset of a ν-null set. Hence we see that the conditional df of Y is H_x a.s.

3. The main theorem for i.d. distributions. We now proceed to prove the main decomposition theorem for i.d. distributions. We recall that ϕ is the ch.f. of an i.d. distribution, and F_α the df whose ch.f. is ϕ^α. The function μ is continuous and monotonic decreasing on the real line, taking values in $[0, \lambda]$ where $\lambda < \infty$. We now require the following Theorem 1 before we proceed to the main Theorem 2.

THEOREM 1. *Let the nonnegative* rv Z *have the* df F_λ; *then* Z *is decomposable in distribution as a sum of two, nonnegative random variables* X *and* Y *such that given* $X = x$ *the conditional* df *of* Y *is* $F_{\mu(x)}$ a.s.

PROOF. It is sufficient to prove that the result is valid for $\lambda = 1$. The result is trivially true for the case $\mu(0) = \lambda$; hence in what follows we shall restrict ourselves to the case $\mu(0) < \lambda$, $\lambda = 1$.

We shall first establish that there exists a df G such that

$$\int_{[0,\infty)} e^{itx}(\phi(t))^{\mu(x)}\, dG(x) = \phi(t)\,. \tag{3.1}$$

Let A_{rj} ($j = 1, 2, \cdots, k_r$; $r = 1, 2, \cdots$) be disjoint intervals such that A_{rj_1} lies to the left of A_{rj_2} for $j_1 < j_2$, $\bigcup_{j=1}^{k_r} A_{rj} = [0, \infty)$, and

$$\sup\{\mu(x): x \in A_{rj}\} - \inf\{\mu(x): x \in A_{rj}\} < \frac{1}{r}\,,$$
$$j = 1, 2, \cdots, k_r;\ r = 1, 2, \cdots\,.$$

Further, let $c_{rj} = \inf\{\mu(x): x \in A_{rj}\}$ and $\mu_r: R \to [0, 1]$ be such that it equals c_{rj} on A_{rj} ($j = 1, \cdots, k_r$). We have

$$\begin{aligned} \phi(t) &= [\phi(t)]^{c_{r1}}[\phi(t)]^{1-c_{r1}} \\ &= \int_{[0,\infty)} e^{itx}[\phi(t)]^{c_{r1}}\, dG_{r1}(x)\,, \end{aligned} \tag{3.2}$$

where $G_{r1} = F_{1-c_{r1}}$. Proceeding inductively it can be seen that there exist df's G_{rj} such that

$$\begin{aligned} \phi(t) &= \int_{A_{r1}} e^{itx}[\phi(t)]^{c_{r1}}\, dG_{r1}(x) + \int_{R^+-A_{r1}} e^{itx}[\phi(t)]^{c_{r1}-c_{r2}}[\phi(t)]^{c_{r2}}\, dG_{r1}(x) \\ &= \sum_{j=1}^{2} \int_{A_{rj}} e^{itx}[\phi(t)]^{c_{rj}}\, dG_{rj}(x) \\ &\quad + \int_{R^+-A_{r1}\cup A_{r2}} e^{itx}[\phi(t)]^{c_{r2}-c_{r3}}[\phi(t)]^{c_{r3}}\, dG_{r2}(x) \\ &= \sum_{j=1}^{k_r} \int_{A_{rj}} e^{itx}[\phi(t)]^{c_{rj}}\, dG_{rj}(x)\,. \end{aligned} \tag{3.3}$$

From this it follows that there exists a df G_r of a nonnegative rv such that

$$\phi(t) = \int_{[0,\infty)} e^{itx}[\phi(t)]^{\mu_r(x)}\, dG_r(x)\,. \tag{3.4}$$

It then follows that we can view F_1 as the df of a sum of two nonnegative rv's one of which has df G_r. From this, the inequality

$$F_1(x) \leqq G_r(x) \tag{3.5}$$

is immediately obvious.

Now, according to Helly's first theorem (cf. Lukacs (1970) page 44) there exists a subsequence $\{G_{n_r}\}$ of $\{G_r\}$ converging weakly to a non-decreasing right continuous function G. From (3.5) it follows that G is the df of a nonnegative rv; we now wish to express ϕ in a form analogous to (3.4) with μ and G replacing μ_r and G_r. From (3.4), it is readily seen that

$$\begin{aligned} &|\phi(t) - \textstyle\int_{[0,\infty)} e^{itx}(\phi(t))^{\mu(x)}\, dG(x)| \\ &\quad = |\textstyle\int_{[0,\infty)} e^{itx}[\phi(t)]^{\mu_{n_r}(x)}\, dG_{n_r}(x) - \int_{[0,\infty)} e^{itx}(\phi(t))^{\mu(x)}\, dG(x)| \\ &\quad \leqq |\textstyle\int_{[0,\infty)} e^{itx}[\phi(t)]^{\mu(x)}\, dG_{n_r}(x) - \int_{[0,\infty)} e^{itx}[\phi(t)]^{\mu(x)}\, dG(x)| \\ &\qquad + \textstyle\int_{[0,\infty)} |1 - (\phi(t))^{\mu(x)-\mu_{n_r}(x)}|\, dG_{n_r}(x)\,. \end{aligned} \tag{3.6}$$

Given an $\varepsilon > 0$, we can find $n_0{}^*$ such that for all $n_r \geqq n_0{}^*$ and x in $[0, \infty)$

$$|1 - (\phi(t))^{\mu(x)-\mu_{n_r}(x)}| \leqq \varepsilon\,. \tag{3.7}$$

Hence using the extension of Helly's second theorem (cf. Lukacs, page 47), we can see that the right-hand side of (3.6) has the limit zero as $n_r \to \infty$. This implies that (3.1) is valid. Since the ch.f. ψ given by

$$\psi(t_1, t_2) = \textstyle\int_{[0,\infty)} e^{it_1x}[\phi(t_2)]^{\mu(x)}\, dG(x)\,, \tag{3.8}$$

has the form (2.1) of the Lemma, the proof of Theorem 1 follows.

We may now extend Theorem 1 to establish our main

THEOREM 2. *Let the* rv *Z have the* df F_λ *with finite left extremity; then Z is decomposable in distribution as a sum of two* rv'*s X and Y such that, given* $X = x$, *the conditional* df *of Y is* $F_{\mu(x)}$ a.s. *This is equivalent to stating that the joint* ch.f. *of X, Y is*

$$\psi(t_1, t_2) = \textstyle\int_{-\infty}^{\infty} e^{it_1x}[\phi(t_2)]^{\mu(x)}\, dG(x)\,, \tag{3.9}$$

where G is a df.

PROOF. We can write that

$$\phi(t) = e^{-itk}\psi(t) \tag{3.10}$$

where ψ is the infinitely divisible ch.f. of a nonnegative random variable. From (3.1) it follows that there exists a df G_1 such that

$$[\psi(t)]^\lambda = \textstyle\int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu^*(x)}\, dG_1(x)\,, \tag{3.11}$$

where μ^* is defined as having the same properties as μ. Let $\nu\colon R \to R$ be such that

$$\nu(x) = x + k(\lambda - \mu(x))\,. \tag{3.12}$$

We shall now show that ϕ satisfies an equation of type (3.11) with G_1 replaced by some other df. Since the result is already known to be valid for nonnegative rv's, we may now assume $k > 0$ to establish the result. In that case the mapping ν is one to one from R onto itself, and both ν and its inverse ν^{-1} are strictly monotone increasing and continuous. Defining

$$\mu^*(x) = \mu(\nu^{-1}(x))$$

we see that μ^* is monotone decreasing and continuous and is bounded by 0 and λ. Substituting this in (3.11), we have

$$[\psi(t)]^\lambda = E\{e^{itX_1}(\psi(t))^{\mu(\nu^{-1}(X_1))}\} = E\{e^{it\nu(X)}(\psi(t))^{\mu(X)}\} . \tag{3.13}$$

In this, X_1 denotes an rv with df G_1 and $X = \nu^{-1}(X_1)$. As a result of (3.12), we have

$$(\psi(t))^\lambda = E\{\exp[it\{X + k(\lambda - \mu(X))\}](\psi(t))^{\mu(X)}\} \tag{3.14}$$

from which we obtain

$$(\phi(t))^\lambda = E\{e^{itX}(\phi(t))^{\mu(X)}\} = \textstyle\int_{-\infty}^{\infty} e^{itx}(\phi(t))^{\mu(x)}\, dG(x) , \tag{3.15}$$

where G denotes the df of X. Hence by the use of our Lemma (2.1) the theorem follows.

4. Some examples, and an extension of the theorem. We now give examples to establish that the decomposition given in Theorem 2 is not valid in general when the finiteness condition for the left extremity of the distribution is relaxed.

EXAMPLE 1. Let Z be a random variable having a conjugate Poisson distribution with mean -1. Then its moment generating function is given by

$$M(t) = e^{-1+e^{-t}} . \tag{4.1}$$

Take $\lambda = 1$ and μ a monotone decreasing nonnegative continuous function such that its value on $(-\infty, -1]$ is μ_1 and at zero is μ_0 where $1 > \mu_1 > \mu_0 > 0$. Suppose now that our decomposition is valid for this Z; then we have a probability distribution $\{a_i\}$ satisfying the identity

$$a_0 e^{-\mu_0+\mu_0 s} + e^{-\mu_1+\mu_1 s} \textstyle\sum_{i=1}^{\infty} a_i s^i = e^{-1+s} \tag{4.2}$$

for all s in $(-\infty, \infty)$. This implies that for all s in $(-\infty, \infty)$, we have

$$\begin{aligned} &G_2(s) + G_1(s) \\ &\qquad = \exp[(\mu_1 - 1) - s(\mu_1 - 1)] - a_0 \exp[(\mu_1 - \mu_0) - (\mu_1 - \mu_0)s] \\ &G_2(s) - G_1(s) \\ &\qquad = \exp[(\mu_1 - 1) + s(\mu_1 - 1)] - a_0 \exp[(\mu_1 - \mu_0) + (\mu_1 - \mu_0)s] \end{aligned} \tag{4.3}$$

where

$$G_1(s) = \textstyle\sum_{i=0}^{\infty} a_{2i+1} s^{2i+1} \qquad \text{and} \qquad G_2(s) = \sum_{i=1}^{\infty} a_{2i} s^{2i} . \tag{4.4}$$

We may select μ_1 and μ_0 such that $1 - \mu_1 < \mu_1 - \mu_0$; we should then find that

$$\lim_{s\uparrow\infty} \frac{G_1(s) - G_2(s)}{G_1(s) + G_2(s)} = \infty , \tag{4.5}$$

which is definitely not true since, for $s > 0$,

(4.6) $$|G_1(s) - G_2(s)|/[G_1(s) + G_2(s)] < 1 .$$

Hence the required decomposition will not hold for the present Z, λ and μ.

EXAMPLE 2. Let Z be an rv with moment generating function

(4.7) $$M(t) = \frac{\frac{1}{4}}{(1 - \frac{1}{2}e^{-t})^2} .$$

Take $\lambda = 1$ and μ to be a monotonic decreasing nonnegative continuous function such that it takes the value 1 on $(-\infty, -1]$ and $\frac{1}{2}$ at zero. Then

(4.8) $$\begin{aligned} M(t) &= \tfrac{1}{2}e^{-t}(M(t)) + \tfrac{1}{2}(M(t))^{\frac{1}{2}} \\ &= \textstyle\int_{-\infty}^{\infty} e^{tx}(M(t))^{\mu(x)}\, dG(x) \end{aligned}$$

where G is a df whose support is a doublet. Hence it is obvious that in the present case, for our choice, the decomposition is valid.

We shall now attempt to extend our result of Section 3 to the case where the left extremity is infinite, but with some additional restrictions on the df of Z.

Let $\{\phi_n\}$ be a sequence of i.d. ch.f.'s such that the corresponding df's possess finite left extremities, and the sequence converges to a ch.f. Define

(4.9) $$\psi(t) = \lim_{n\to\infty} \phi_n(t) .$$

It is well known that the ch.f. ψ is i.d. (cf. Lukacs page 110). We shall now show that if the ϕ of Section 3 is replaced by this new ch.f., Theorem 2 will hold without the restriction of a finite left extremity.

From Theorem 2 it follows that there exists a sequence of df's $\{G_n\}$ such that

(4.10) $$[\phi_n(t)]^\lambda = \textstyle\int_{-\infty}^{\infty} e^{itx}[\phi_n(t)]^{\mu(x)}\, dG_n(x) .$$

Given an $\varepsilon > 0$, there exists $n_0 > 0$ such that

(4.11) $$\begin{aligned} |[\psi(t)]^\lambda - [\phi_n(t)]^\lambda| &\leqq \varepsilon/2 \\ |[\phi_n(t)]^{\mu(x)} - [\psi(t)]^{\mu(x)}| &\leqq \varepsilon/2 \end{aligned}$$

for all $n \geqq n_0$ and t, x satisfying $-T \leqq t \leqq T$, $-\infty < x < \infty$. Then for all $n \geqq n_0$ and t in $[-T, T]$

(4.12) $$\begin{aligned} &|\textstyle\int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG_n(x) - [\psi(t)]^\lambda| \\ &\quad \leqq |\textstyle\int_{-\infty}^{\infty} e^{itx}\{[\psi(t)]^{\mu(x)} - [\phi_n(t)]^{\mu(x)}\}\, dG_n(x)| + |[\phi_n(t)]^\lambda - [\psi(t)]^\lambda| \\ &\quad \leqq \textstyle\int_{-\infty}^{\infty} |[\psi(t)]^{\mu(x)} - [\phi_n(t)]^{\mu(x)}|\, dG_n(x) + \varepsilon/2 \leqq \varepsilon . \end{aligned}$$

This implies that $\int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG_n(x)$ has the limit $[\psi(t)]^\lambda$ as $n \to \infty$ uniformly in every finite t-interval $[-T, T]$.

Given an $\varepsilon > 0$, we can find a $\delta > 0$ such that for all t in $[-\delta, \delta]$ we have

(4.13) $$\begin{aligned} |\textstyle\int_{-\infty}^{\infty} e^{itx}\, dG_n(x) - \int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG_n(x)| &\leqq \textstyle\int_{-\infty}^{\infty} |1 - [\psi(t)]^{\mu(x)}|\, dG_n(x) \\ &\leqq \varepsilon/2 \end{aligned}$$

for all n.

We can find $n_0 > 0$ such that for all $n \geqq n_0$ and t in $[-\delta, \delta]$ we have

$$|[\psi(t)]^\lambda - \int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG_n(x)| \leqq \varepsilon/2$$

and hence

$$|[\psi(t)]^\lambda - \int_{-\infty}^{\infty} e^{itx}\, dG_n(x)| \leqq \varepsilon\,. \tag{4.14}$$

We can choose δ such that we also have

$$|1 - [\psi(t)]^\lambda| \leqq \varepsilon$$

for all t in $[-\delta, \delta]$. Hence for all $n \geqq n_0$ and t in $[-\delta, \delta]$ we obtain that

$$|1 - \int_{-\infty}^{\infty} e^{itx}\, dG_n(x)| \leqq 2\varepsilon\,. \tag{4.15}$$

This implies that given an $\varepsilon > 0$, we can find $n_0 > 0$ and $\delta > 0$ such that for all $n \geqq n_0$ and t in $[-\delta, \delta]$ we have

$$|1 - g_n(t)| \leqq 2\varepsilon\,,$$

where g_n is the ch.f. of G_n.

From a lemma of Chung ((1968) page 149) it follows that

$$\begin{aligned} G_n(2\delta^{-1}) - G_n(-2\delta^{-1}-) &\geqq \delta^{-1}|\textstyle\int_{-\delta}^{\delta} g_n(t)\, dt| - 1 \\ &\geqq 1 - \delta^{-1} \textstyle\int_{-\delta}^{\delta} |1 - g_n(t)|\, dt \\ &\geqq 1 - 4\varepsilon \qquad\qquad \text{for all } n \geqq n_0\,. \end{aligned} \tag{4.16}$$

From Helly's first theorem it follows that there is a subsequence $\{G_{n_r}\}$ of $\{G_r\}$ converging weakly to a non-decreasing right continuous function G. From the above inequality we see that $G(-\infty) = 0$ and $G(+\infty) = 1$. Hence from the extension of Helly's second theorem it follows that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG_n(x) = \int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG(x)\,. \tag{4.17}$$

This implies that

$$[\psi(t)]^\lambda = \int_{-\infty}^{\infty} e^{itx}[\psi(t)]^{\mu(x)}\, dG(x)\,, \tag{4.18}$$

which, in turn, shows that any random variable having the ch.f. ψ^λ can be decomposed in distribution as a sum of two rv's X and Y such that their joint ch.f. is given by

$$\phi(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1x}[\psi(t_2)]^{\mu(x)}\, dG(x)\,. \tag{4.19}$$

Hence our extended result follows for this class of distributions.

5. Some remarks on the extended result. It is interesting to note that this last result has the immediate consequence that if Z follows the normal distribution $N(\lambda m, \lambda\sigma^2)$ then

$$Z =_d X + Y\,,$$

where the rv's X and Y are such that the conditional distribution of Y given $X = x$ is $N(\mu(x)m, \mu(x)\sigma^2)$ a.s. This is seen from the fact that if Z_n is a Poisson rv with expectation n then

$$Z_n^* = \sigma \frac{Z_n - n}{n^{\frac{1}{2}}} + m$$

has an i.d. distribution with finite left extremity such that the sequence of distributions tends to $N(m, \sigma^2)$ as $n \to \infty$. From the present decomposition of the normal distribution it follows that there exist mixtures of normal distributions $N(x + m\mu(x), \mu(x)\sigma^2)$ $(-\infty < x < \infty)$ which are also normal. As was briefly mentioned in the introduction, this contrasts with Kelker's (1971) result that no mixture of normal distributions $N(0, \mu(x)\sigma^2)$ $(-\infty < x < \infty)$ for the case $\mu(x) \not\equiv c$ can be i.d.

In conclusion, it should be noted that the results of the present paper remain valid when the right extremities of the distributions are finite provided μ is taken to be increasing instead of decreasing. This implies that the above decomposition of the normal distribution is valid even when μ is taken as increasing.

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