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ON THE EXISTENCE AND PATH PROPERTIES OF STOCHASTIC INTEGRALS[1]

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We study stochastic integrals of the form $Y(t) = \int_0^t V\,dX$, $t \geqq 0$, where X is a process with stationary independent increments while V is an adapted previsible process, thus continuing the work of Itô and Millar. In the case of vanishing Brownian component, we obtain conditions for existence which are considerably weaker than the classical requirement that V^2 be a.s. integrable. We also examine the asymptotic behavior of $Y(t)$ for large and small t, and we consider the variation with respect to suitable functions f. The latter leads us to investigate nonlinear integrals of the form $\int f(V\,dX)$.

The whole work is based on extensions of two general martingale-type inequalities, due to Esseen and von Bahr and to Dubins and Savage respectively, and on a super-martingale which was discovered and explored in a special case by Dubins and Freedman.

1. Introduction. The history of stochastic integrals goes back to the early work of Paley, Wiener, and Itô (see McKean (1969)), who defined the integral $\int_0^1 V\,dX$ in the case when X is a Brownian motion while V is an adapted [17] random process with paths a.s. in $L_2[0, 1]$, i.e. such that

$$\int_0^1 V^2 \equiv \int_0^1 (V(t))^2\,dt < \infty \quad \text{a.s.} \tag{1.1}$$

Later on, Doob, Courrège, Doléans-Dade and Meyer (1970), and Millar (1968), (1972) extended the definition to quasi-martingales X and previsible [20] processes V satisfying

$$\int_0^1 V^2\,dA \equiv \int_0^1 (V(t))^2\,dA(t) < \infty \quad \text{a.s.,} \tag{1.2}$$

where A is the natural increasing process associated with X. Millar (1972) also studied the path behavior of the process $\int_0^t V\,dX$, $t \geqq 0$, in the special case when V is a.s. bounded while X has stationary independent increments (in which case (1.1) and (1.2) become equivalent), and he showed in particular how several properties known for X itself carry over to $\int V\,dX$.

The investigation of the case when X has stationary independent increments is carried further in the present paper. In Section 3 we give conditions for the existence of $\int V\,dX$, which in particular cases are shown to be the best possible and which, in case of vanishing Brownian component, are substantially weaker

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than (1.1). In particular, it follows from our results that, if X has index $\beta < p$ in the sense of Blumenthal and Getoor (1961), then $\int V\,dX$ can be defined as a random process in $D[0, 1]$ for any process V whose paths lie a.s. in $L_p[0, 1]$. Section 4 is devoted to a study of the asymptotic behavior of $\int_0^t V\,dX$, both as $t \to 0$ and as $t \to \infty$ (without any further restrictions on V), and of the continuity for fixed X of the mapping $V \to \int V\,dX$. In the concluding Section 5, we give some variational results which lead us to introduce nonlinear stochastic integrals of the form $\int f(V\,dX)$. The whole work is based on extensions of two general inequalities, due to Esseen and von Bahr (1965) and to Dubins and Savage (1965) respectively, and on a super-martingale which was discovered and explored in a special case by Dubins and Freedman (1965). These extensions are given in Section 2.

The following function classes will be used throughout the paper:

$$\mathscr{F} = \{f : R \to R_+;\ f \not\equiv 0,\ f \text{ even and continuous with } f(0) = 0,\ f\uparrow \text{ on } R_+\},$$
$$\mathscr{F}_1 = \{f \in \mathscr{F} : f \text{ concave on } R_+\},$$
$$\mathscr{F}_2 = \{f \in \mathscr{F} : f \text{ absolutely continuous, } f' \text{ concave on } R_+ \text{ with } f'(0+) = 0\},$$
$$\mathscr{F}_3 = \{f \in \mathscr{F} : \textstyle\int_0^\infty (f(u))^{-1}\,du < \infty\}.$$

We define inverses of such functions f by $f^{-1}(x) = \sup\{y \geqq 0 : f(y) \leqq x\}$, $x \geqq 0$.

2. Some results on martingale-type sequences. In this section, let $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots$ be σ-algebras in the sample space, let the sequence $\{\xi_n\}$ of random variables be adapted [17] to $\{\mathscr{A}_n\}$, and put $X_n = \xi_1 + \cdots + \xi_n$, $n \in Z_+$. Our first lemma extends a result by Esseen and von Bahr (1965) (cf. [2]).

LEMMA 2.1. *Suppose that either* $f \in \mathscr{F}_1$, *or that* $f \in \mathscr{F}_2$ *and* $E[\xi_{n+1} \mid X_n] = 0$ a.s., $n \in N$. *Then*

$$Ef(X_n) \leqq Ef(\xi_1) + 2\textstyle\sum_{j=2}^n Ef(\xi_j), \qquad n \in N. \tag{2.1}$$

Note that the factor 2 in (2.1) may be replaced by 1 for $f \in \mathscr{F}_1$ and by 2^{2-p} for $f(x) \equiv |x|^p$, $p \in (1, 2]$ (cf. [2]). Similar improvements are possible in all subsequent formulae based on (2.1).

PROOF. The case $f \in \mathscr{F}_1$ being trivial, we need only consider the case when $f \in \mathscr{F}_2$ and $E[\xi_{n+1} \mid X_n] = 0$ a.s., $n \in N$. If we can prove the elementary inequality

$$f(a + x) \leqq f(a) + xf'(a) + 2f(x), \qquad a, x \in R, \tag{2.2}$$

then (2.1) will follow as in [2]. For $a, x \geqq 0$, we get by concavity

$$\begin{aligned} f(a + x) - f(a) - xf'(a) &= \textstyle\int_0^x [f'(a + u) - f'(a)]\,du \\ &\leqq \textstyle\int_0^x [f'(u) - f'(0)]\,du \leqq f(x), \end{aligned}$$

and (2.2) follows. Next define, for fixed $a > 0$,

$$F(x) = f(a) - xf'(a) + f(x) - f(a - x), \qquad x \geqq 0.$$

Then $F'(x) = -f'(a) + f'(x) + f'(a - x)$ is concave while $F'(0+) = F'(a-) \geqq 0$, so $F' \geqq 0$ on $(0, a)$, and hence F is nondecreasing on $[0, a]$. But $f(0) = 0$, so we get $F \geqq 0$ on $[0, a]$, proving (2.2) for $x \in [-a, 0]$. Finally note that f' must be nondecreasing on R_+, since otherwise $f(x) < 0$ for large x. Hence for $x > a$

$$\begin{aligned} F'(x) + f'(x) &= -f'(a) + 2f'(x) + f'(a - x) \\ &= [f'(x) - f'(a)] + [f'(x) - f'(x - a)] \geqq 0 . \end{aligned}$$

Since $F(a) + f(a) \geqq 0$, we get $F + f \geqq 0$ on $[a, \infty)$, proving (2.2) for $x \leqq -a$. This completes the proof of (2.2) for $a \geqq 0$. Its truth for $a \leqq 0$ follows by symmetry. □

Our next lemma extends a result by Dubins and Freedman (1965). Here and below, $\frac{0}{0}$ is to be interpreted as 0.

LEMMA 2.2. *Let $g \in \mathscr{F}_3$ and suppose that either $f \in \mathscr{F}_1$ or that $f \in \mathscr{F}_2$ and $\{X_n\}$ is local martingale* [17]. *Define*

$$Y_n = \textstyle\sum_{j=1}^n E[f(\xi_j) \mid \mathscr{A}_{j-1}] , \qquad n \in Z_+ , \tag{2.3}$$

$$Q(x, y) = \frac{f(x)}{g(y)} + c \textstyle\int_y^\infty \frac{du}{g(u)} , \qquad x \in R,\ y \geqq 0 , \tag{2.4}$$

where $c = 2$. Then $Q(x + X_n, y + Y_n)$, $n \in Z_+$, is a super-martingale for each $x \in R$ and $y > 0$, and also for $x = y = 0$.

PROOF. Let $x \in R$ and $y \in (0, \infty]$, let ξ be a random variable such that $E\xi = 0$ in the case $f \in \mathscr{F}_2$, and put $Ef(\xi) = a$. If $y + a = \infty$, then $Q(x + \xi, y + a) = 0$, and it follows trivially that $EQ(x + \xi, y + a) \leqq Q(x, y)$. But this inequality is also true for $y + a < \infty$, since by Lemma 2.1

$$\begin{aligned} EQ(x + \xi, y + a) &\leqq \frac{f(x)}{g(y + a)} + \frac{2a}{g(y + a)} + 2 \textstyle\int_{y+a}^\infty \frac{du}{g(u)} \\ &\leqq \frac{f(x)}{g(y)} + 2 \textstyle\int_y^\infty \frac{du}{g(u)} - 2 \left\{ \textstyle\int_y^{y+a} \frac{du}{g(u)} - \frac{a}{g(y + a)} \right\} \\ &\leqq Q(x, y) . \end{aligned}$$

If $x = 0$, it follows by monotone convergence that we may even take $y = 0$ here. Induction completes the proof. □

COROLLARY 2.1. *Let f, g, $\{X_n\}$ and $\{Y_n\}$ be such as in Lemma 2.2. Then $\{X_n\}$ is* a.s. *convergent on $\{Y_\infty < \infty\}$, while on $\{Y_\infty = \infty\}$, $X_n / f^{-1}g(Y_n) \to 0$* a.s.

This extends a result in Neveu (1972) page 150, (as was pointed out by the referee). An interesting choice of f and g might be $f(x) \equiv |x|^p$, $p \in (0, 2]$, and $g(y) \equiv y(\log y)^c$, $c > 1$.

PROOF. On $\{Y_\infty < \infty\}$ it follows from Lemma 2.2 that $\{f(x + X_n)\}$ converges a.s., so the same thing must be true for $\{X_n\}$ since x is arbitrary. On $\{Y_\infty = \infty\}$, it follows from the same lemma that $f(X_n)/g(Y_n) \to$ some $\rho < \infty$ a.s. For large

$n \in N$ we now define $t_n > 0$ by $\int_{t_n}^{\infty} (g(u))^{-1}\,du = 3^{-n}$. Putting $g_1(u) = (\frac{2}{3})^n g(u)$, $u \in [t_{n-1}, t_n)$, $n \in N$, and $h(y) = \sup\{g_1(u) : u \leqq y\}$, it is seen that even $h \in \mathscr{F}_3$, so we get $f(X_n)/h(Y_n) \to$ some $\rho' < \infty$ a.s., as above for g. But since $h(y)/g(y) \to 0$ as $y \to \infty$, this implies $\rho = 0$ a.s. Finally use the fact that $f(2x)/f(x) \leqq 4$, $x > 0$, to complete the proof. □

LEMMA 2.3. *Let f, g, $\{X_n\}$ and $\{Y_n\}$ be such as in Lemma* 2.2. *Then*

$$P\left\{\sup_n \frac{f(X_n)}{g(y + Y_n)} \geqq \varepsilon\right\} \leqq \frac{2}{\varepsilon}\int_y^{\infty} \frac{du}{g(u)}\,, \qquad \varepsilon > 0, y \geqq 0\,,$$

and in particular

$$P\left\{\sup_n [|f(X_n)|^{1/p} - aY_n] \geqq b\right\} \leqq \frac{2}{(p-1)ab^{p-1}}\,, \qquad a, b > 0, p > 1\,. \tag{2.5}$$

This is essentially an extension of a result by Dubins and Savage (1965) (see also [6], [17], [20]), who consider the special case $f(x) \equiv g(x) \equiv x^2$. The proof is similar to that of (29) in [6], once Lemma 2.2 is established. Contrary to the Dubins–Savage inequality, (2.5) has no content for small ab^{p-1}. However, it may be shown that, if $f(x) \equiv |x|^p$ in (2.3), $p \in (1, 2]$, then

$$P\{\sup_n (X_n - aY_n) \geqq b\} \leqq [1 + c_p a^{1/(p-1)} b]^{-p+1}\,, \qquad a, b > 0\,.$$

(Here and below, c_p denotes some positive constant depending on p only.) This follows as in [7], [17] from the following result of some independent interest, (cf. Doob (1973) for the case $p = 2$). We omit the elementary but tedious proof.

LEMMA 2.4. *Let ξ be an integrable random variable and define for $p \in (1, 2]$*

$$\begin{aligned} g_p(x) &= (1 - x)^{-p+1}\,, && x < 0\,, \\ &= 1\,, && x \geqq 0\,. \end{aligned}$$

Then

$$Eg_p(\xi) \leqq g_p(E\xi + c_p E|\xi|^p)\,.$$

3. Existence of stochastic integrals. For the remainder of the paper, let X be a random process in $D[0, \infty)$ with stationary independent increments and $X(0) \equiv 0$, and define the Lévy measure λ and the parameters σ and γ_ε, $\varepsilon > 0$ (even $\varepsilon = 0$ or ∞ if possible), by the formula

$$\begin{aligned} \log Ee^{iuX(1)} \equiv iu\gamma_\varepsilon - \tfrac{1}{2}u^2\sigma^2 &+ \int_{|x| \leqq \varepsilon} (e^{iux} - 1 - iux)\lambda(dx) \\ &+ \int_{|x| > \varepsilon} (e^{iux} - 1)\lambda(dx)\,. \end{aligned} \tag{3.1}$$

To any $f \in \mathscr{F}$, we associate the function $\tilde{f}$, defined by

$$\begin{aligned} \tilde{f}(u) &= \int_{-\infty}^{\infty} f(ux)\lambda(dx) + \sigma^2 f''(0)u^2/2 && \text{if } f'(0) = 0\,, \\ &= \int_{-\infty}^{\infty} f(ux)\lambda(dx) + |\gamma_0| f'(0+)|u| && \text{if } \sigma = 0\,, \end{aligned} \tag{3.2}$$

and the functions $\tilde{f}_\varepsilon$, defined in the same way w.r.t. the restrictions of λ to $[-\varepsilon, \varepsilon]$, $\varepsilon > 0$. Let us further suppose that $\mathscr{A}_t$, $t \geqq 0$, is an increasing family of σ-algebras in the sample space such that, for fixed $t \geqq 0$, $X(t)$ is $\mathscr{A}_t$-measurable while the process $X(t+s) - X(t)$, $s \geqq 0$, is independent of $\mathscr{A}_t$. By V with or without subscript we denote a random process on R_+ which is adapted to $\{\mathscr{A}_t\}$ and previsible [20]. We shall say that V is *simple* if it is a step process with all discontinuities belonging to some nonrandom finite set. For brevity, we shall use the short-hand notation indicated by (1.1) and (1.2), and we shall often write $X\{t\} = X(t) - X(t-)$ for the jump size of X at t.

Following Itô (see [16]) and Millar (1972), we shall define the integral $\int V\,dX$ by a limiting procedure, starting from simple V (for which $\int V\,dX$ is defined in the obvious way as a finite sum). For this purpose we need two lemmas, the first of which extends a result in McKean (1969) page 23.

LEMMA 3.1. *Let* $f \in \mathscr{F}$ *with* $\limsup_{x\to\infty} f(2x)/f(x) < \infty$, *and suppose that* $\int_0^\infty f \circ V < \infty$ a.s. *Then there exist some simple adapted processes* $V_1, V_2, \cdots$ *such that*

$$\int_0^\infty f \circ (V_n - V) \to 0 \quad \text{a.s.,} \tag{3.3}$$

$$\int_0^t f \circ V_n \to \int_0^t f \circ V \quad \text{a.s.,} \qquad t \in \bar{R}_+ . \tag{3.4}$$

If $E \int_0^\infty f \circ V < \infty$, *the* V_n *may even be chosen so as to satisfy* (3.3) *and* (3.4) *in* L_1.

PROOF. By an obvious truncation argument, we may restrict our attention to processes V with bounded support, say in $[0,1]$. Let us first assume that f is strictly increasing on R_+ with

$$\sup_{x>0} \frac{f(2x)}{f(x)} = c < \infty . \tag{3.5}$$

Then

$$f(x+y) \leqq cf(x) + cf(y) , \qquad x, y \in R , \tag{3.6}$$

since for $x, y \in R_+$

$$f(x+y) \leqq f(2(x \vee y)) \leqq cf(x \vee y) = c[f(x) \vee f(y)] \leqq c[f(x) + f(y)] .$$

Let us further suppose that V is nonrandom and nonnegative, put $V(t) = 0$ for $t < 0$, and define $V_x(t) \equiv V(t-x)$. By mean continuity ([11] page 199), $\int |f \circ V_x - f \circ V| \to 0$ as $x \to 0$, so we get

$$\limsup_{h\to 0} \int \left| \frac{1}{h} \int_0^h f \circ V_x\,dx - f \circ V \right| \leqq \limsup_{h\to 0} \frac{1}{h} \int_0^h dx \int |f \circ V_x - f \circ V| = 0 , \tag{3.7}$$

and in particular,

$$U_h \equiv f^{-1}\left(\frac{1}{h} \int_0^h f \circ V_x\,dx\right) \to V \tag{3.8}$$

in Lebesgue measure. Furthermore, (3.7) implies uniform integrability near

$h=0$ of $\{f\circ U_h\}$ (cf. [15] page 162), and hence by (3.6) of $\{f\circ(U_h-V)\}$, so (3.8) yields

$$\lim_{h\to 0}\sup_{t>0}\left|\int_0^t f\circ U_h-\int_0^t f\circ V\right|=0\,,\qquad \lim_{h\to 0}\int f\circ(U_h-V)=0\,. \tag{3.9}$$

For general nonrandom V, we may approximate $V^+=V\vee 0$ and $V^-=-V\wedge 0$ in the same way by processes U_h^+ and U_h^- and put $U_h=U_h^+-U_h^-$. By (3.6), $\{f\circ U_h\}$ is still uniformly integrable so (3.9) remains true. We next define

$$U_{hk}(t)=U_h(j/k)\,,\qquad t\in(j/k,(j+1)/k],\, j\in Z,\, k\in N\,,$$

and conclude from uniform continuity that

$$\begin{aligned}&\lim_{k\to\infty}\sup_{t>0}\left|\int_0^t f\circ U_{hk}-\int_0^t f\circ U_h\right|=0\,,\\ &\lim_{k\to\infty}\int f\circ(U_{hk}-U_h)=0\,,\qquad h>0\,.\end{aligned} \tag{3.10}$$

By (3.6), (3.9), and (3.10),

$$\begin{aligned}&\lim_{h\to 0}\limsup_{k\to\infty}\sup_{t>0}\left|\int_0^t f\circ U_{hk}-\int_0^t f\circ V\right|=0\,,\\ &\lim_{h\to 0}\limsup_{k\to\infty}\int f\circ(U_{hk}-V)=0\,,\end{aligned}$$

and hence, returning to random V and letting $n\in N$ be arbitrary, we may choose h and k such that $V_n\equiv U_{hk}$ satisfies

$$\begin{aligned}&P\left\{\sup_{t>0}\left|\int_0^t f\circ V_n-\int_0^t f\circ V\right|>2^{-n}\right\}\leqq 2^{-n}\,,\\ &P\left\{\int f\circ(V_n-V)>2^{-n}\right\}\leqq 2^{-n}\,.\end{aligned}$$

Thus (3.3) and (3.4) follow by the Borel–Cantelli lemma, while the last assertion follows by dominated convergence, since by (3.6), provided $hk\in N$,

$$\begin{aligned}\int f\circ U_{hk}&=\frac{1}{k}\sum_j f\circ U_h(j/k)\leqq\frac{c}{k}\sum_j[f\circ U_h^+(j/k)+f\circ U_h^-(j/k)]\\ &=\frac{c}{hk}\sum_j\int_0^h[f\circ V_x^+(j/k)+f\circ V_x^-(j/k)]\,dx\\ &=\frac{c}{hk}\sum_j\int_0^h f\circ V_x(j/k)\,dx=c\int f\circ V\,.\end{aligned}$$

To get rid of the assumptions on f, let $a>0$ be such that $f(a)>0$ and let $\hat f$ be the minimal even function $\geqq f$ which is concave on $[0,a]$. Then $\hat f$ is easily seen to be continuous at 0, and in particular $\hat f'(b)>0$ for some $b\in(0,a)$. Next define $g(x)\equiv\hat f(x)+(1-e^{b-|x|})\vee 0$, and note that g belongs to $\mathscr{F}$ and satisfies (3.5), and that g is strictly increasing on R_+. Since clearly $\int_0^1 f\circ V<\infty$ a.s. implies $\int_0^1 g\circ V<\infty$ a.s. and similarly for the corresponding expectations, the conclusions of the lemma hold with g in place of f. This proves the assertions involving (3.3), while those involving (3.4) carry over to f by uniform integrability. □

LEMMA 3.2. *Let $f\in\mathscr{F}$ be strictly positive outside* 0 *and satisfy* $\limsup_{x\to\infty}f(2x)/f(x)<\infty$. *Then $\int_0^1 f\circ V_n\to_P 0$ implies*

$$\int_0^1 f\circ(V_m-V_n)\to_P 0\,,\qquad m,n\to\infty\,. \tag{3.11}$$

PROOF. If f satisfies (3.5), then (3.11) follows by (3.6). For general f, let $\hat{f}$ be such as in the preceding proof. Since $\hat{f}$ is $\geqq f$ and satisfies (3.5), it is enough to prove that $\int_0^1 f \circ V_n \to_P 0$ implies $\int_0^1 \hat{f} \circ V_n \to_P 0$, and here it clearly suffices to consider nonrandom V_n. But in that case, the assertion follows easily by continuity and uniform integrability. □

We may now state the main result of this section.

THEOREM 3.1. *If* $\int_0^t \tilde{f}_1 \circ V < \infty$ a.s., $t > 0$, *where* $f(x) \equiv x^2 \wedge |x|$, *then there exists some* a.s. *unique process* $\int_0^t V\,dX$, $t \geqq 0$, *in* $D[0, \infty)$ *such that*

$$\sup_{s \leqq t} \left|\int_0^s V_n\,dX - \int_0^s V\,dX\right| \to_P 0\,, \qquad t > 0\,, \tag{3.12}$$

for any sequence $\{V_n\}$ *of simple processes satisfying* $\int_0^t \tilde{f}_1 \circ (V_n - V) \to_P 0$, $t > 0$, (*the existence of which is ensured by Lemma* 3.1). *If* $\sigma = 0$ *and* $\int_0^t \tilde{g} \circ V < \infty$ a.s., $t > 0$, $g(x) \equiv |x| \wedge 1$, *then the above conclusion remains true with f replaced by g, and furthermore, the Lebesgue–Stieltjes integral exists and equals* $\int V\,dX$ a.s.

The last assertion extends with essentially the same proof a result by Doléans-Dade and Meyer (1970), and by Millar (1972). As will be seen from the proof, this is the only part of the theorem that requires previsibility. For any fixed X with $\sigma = 0$, Theorem 3.1 yields a larger class of integrable processes than the one defined by (1.1) (cf. [20]). In fact, since $\tilde{f}_1(u) = o(u^2)$ as $u \to \infty$, it is easy to construct numerical functions v such that $\int_0^1 \tilde{f}_1 \circ v < \infty$ while $\int_0^1 v^2 = \infty$. Similarly, the condition $\int_0^t |V| < \infty$ a.s., $t > 0$, is too restrictive in the case when X has bounded variation and $\gamma_0 = 0$, since then $\tilde{g}(u) = o(u)$. If X is stable of index p, it follows from the theorem that $\int V\,dX$ exists for any V whose paths lie a.s. in L_p, and the same processes V are seen to be integrable whenever X is such that $\int_{-1}^1 |x|^p \lambda(dx) < \infty$ and $\sigma = 0$ (if $p < 2$) or $\sigma = \gamma_0 = 0$ (if $p < 1$). We finally point out the interesting fact that, for subordinators, the function $\tilde{g}$ above may be replaced by the exponent

$$-\log Ee^{-uX(1)} = u\gamma_0 + \int_0^\infty (1 - e^{-ux})\lambda(dx)\,, \qquad u \geqq 0\,,$$

occurring in the canonical representation of X (cf. (3.1)).

PROOF. We may clearly assume that V and all the V_n have support in $[0, 1]$. By a simple truncation argument [19], [20], we may further assume that λ has bounded support. In the proof of the first assertion, we may also take $\sigma = 0$ by [16], and since $u/\tilde{f}_1(u)$ is bounded as $u \to \infty$, we may finally assume that $EX(t) \equiv 0$. For convenience, let us replace the function f above by the equivalent function $f(x) \equiv x^2/(1 + |x|)$, which clearly belongs to $\mathscr{F}_2$ and is such that $f(\sqrt{\cdot})$ is concave on R_+. If V is a simple process of the form

$$V = \sum_{j=1}^k \alpha_j 1_{(t_{j-1}, t_j]}\,, \qquad 0 = t_0 < t_1 < \cdots < t_k = 1\,,$$

we get by Theorem 4.1 of Millar (1971), for any $j \in \{1, \cdots, k\}$,

$$\begin{aligned} \sum_{i=1}^j E\{f(\alpha_i[X(t_i) - X(t_{i-1})]) \mid \mathscr{A}_{t_{i-1}}\} &\leqq 2 \sum_{i=1}^j (t_i - t_{i-1}) \int_{-\infty}^\infty f(\alpha_i x)\lambda(dx) \\ &= 2 \sum_{i=1}^j (t_i - t_{i-1})\tilde{f}(\alpha_i) = 2 \int_0^{t_j} \tilde{f} \circ V\,, \end{aligned} \tag{3.13}$$

so by (2.5) with $p = 2$, we obtain for any finite subset $T \subset R_+$,

$$P\{\sup_{t\in T} |f(\textstyle\int_0^t V\,dX)|^{\frac{1}{2}} \geqq 2a \int_0^1 \tilde{f}\circ V + b\} \leqq \frac{2}{ab}\,, \qquad a, b > 0\,, \tag{3.14}$$

which extends to $T = R_+$ by monotone convergence and right continuity. Returning to the case of general V and approximating $\{V_n\}$, conclude from Lemma 3.2 that $\int_0^1 \tilde{f}\circ(V_m - V_n) \to_P 0$ as $m, n \to \infty$. For any $\varepsilon > 0$, we may therefore choose some $n_0 \in N$ such that $P\{\int_0^1 \tilde{f}\circ(V_m - V_n) > \varepsilon^3/32\} \leqq \varepsilon/2$ whenever $m, n > n_0$. By (3.14), we get for such m and n

$$\begin{aligned} &P\{\sup_t |f(\textstyle\int_0^t (V_m - V_n)\,dX)|^{\frac{1}{2}} \geqq \varepsilon\} \\ &\quad \leqq P\left\{\sup_t |f(\textstyle\int_0^t (V_m - V_n)\,dX|^{\frac{1}{2}} \geqq \frac{16}{\varepsilon^2} \int_0^1 \tilde{f}\circ(V_m - V_n) + \frac{\varepsilon}{2}\right\} + \frac{\varepsilon}{2} \leqq \varepsilon\,, \end{aligned}$$

proving that

$$\sup_t |\textstyle\int_0^t V_m\,dX - \int_0^t V_n\,dX| \to_P 0 \qquad \text{as } m, n \to \infty\,. \tag{3.15}$$

By restricting m and n to some suitable subsequence $N' \subset N$, it is even possible to obtain a.s. convergence in (3.15), and so there must exist some process Y in $D[0, \infty)$ satisfying

$$\sup_t |\textstyle\int_0^t V_m\,dX - Y(t)| \to 0 \quad \text{a.s.}, \qquad m \in N'\,. \tag{3.16}$$

The proof of (3.12) is now completed by combining (3.15) and (3.16). To see that any two approximating sequences $\{V_n\}$ and $\{V_n'\}$ lead to the same limit, it suffices to consider the mixed sequence $V_1, V_1', V_2, V_2', \cdots$. The proof of the second assertion is similar, except that Millar's Lemma 4.2 (1971) replaces his Theorem 4.1. □

The following simple property of $\int V\,dX$ will be needed here and in Section 5.

LEMMA 3.3. *With probability one,*

$$\textstyle\int_{t-}^t V\,dX = V(t)X\{t\}\,, \qquad t \geqq 0\,. \tag{3.17}$$

PROOF. By the uniform convergence in Theorem 3.1, $\int V\,dX$ has a.s. no jumps outside those of X. For fixed $\varepsilon > 0$, write X as a sum of independent processes X_ε and X_ε', where X_ε' corresponds to the last term in (3.1). Then clearly $\int V\,dX = \int V\,dX_\varepsilon + \int V\,dX_\varepsilon'$ a.s. (cf. Proposition 2.1 in [20]), and here the last term may be interpreted as a Lebesgue–Stieltjes integral, so with probability 1 we get for all $t \geqq 0$ with $|X\{t\}| > \varepsilon$

$$\textstyle\int_{t-}^t V\,dX = \int_{t-}^t V\,dX_\varepsilon' = V(t)X_\varepsilon'\{t\} = V(t)X\{t\}\,.$$

Since ε is arbitrary, this completes the proof. □

For fixed nonrandom V, the existence or nonexistence of $\int V\,dX$ as a random process in $D[0, \infty)$ satisfying (3.17) is an interesting sample path property of X. We shall consider a simple case when it is possible to obtain necessary and sufficient conditions for existence in this sense.

THEOREM 3.2. *Let $v : [0, 1] \to R_+$ be absolutely continuous on* $(0, 1]$ *with* $v' < 0$ *and* $v(1) \leqq 1$, *and suppose that either* $\sigma = 0$ *and*

$$-1 < \liminf_{t\to 0} \frac{tv'(t)}{v(t)} \leqq \limsup_{t\to 0} \frac{tv'(t)}{v(t)} < -\tfrac{1}{2}\,, \tag{3.18}$$

or that $\sigma = \gamma_0 = 0$ *and*

$$-\infty < \liminf_{t\to 0} \frac{tv'(t)}{v(t)} \leqq \limsup_{t\to 0} \frac{tv'(t)}{v(t)} < -1\,. \tag{3.19}$$

Then $\int v\,dX$ *exists as a random process in* $D[0, 1]$ *satisfying* (3.17) iff

$$\int_{-1}^{1} v^{-1}(|x|^{-1})\lambda(dx) < \infty\,, \tag{3.20}$$

and in that case it also exists in the sense of Theorem 3.1.

Thus, in this particular case, the sufficient conditions for existence given in Theorem 3.1 are also necessary. For Brownian motion, a proof of the corresponding statement may be found in [16]. Applying Theorem 3.2 to the product vw where $v(t) \equiv w(t) \equiv t^{-\frac{1}{2}}$, we obtain a counterexample to the claim made by Millar (1972) on the top of page 312.

PROOF. Suppose that $\sigma = 0$ and that (3.18) and (3.20) hold. If $f(x) \equiv x^2 \wedge |x|$, we get

$$\int_0^1 \tilde{f}_1 \circ v = \int_{-1}^{1} x^2\lambda(dx) \int_{v^{-1}(|x|^{-1})}^{1} v^2 + \int_{-1}^{1} |x|\lambda(dx) \int_0^{v^{-1}(|x|^{-1})} v\,,$$

and from (3.18) it is easily seen that

$$\int_u^1 v^2 = O(u[v(u)]^2)\,, \qquad \int_0^u v = O(uv(u))\,, \qquad u \to 0\,,$$

so by (3.20),

$$\int_0^1 \tilde{f}_1 \circ v = O\left(\int_{-1}^{1} v^{-1}(|x|^{-1})\lambda(dx)\right) < \infty\,,$$

proving the existence of $\int v\,dX$ in the sense of Theorem 3.1. A similar argument proves the existence under (3.19) when $\sigma = \gamma_0 = 0$. Conversely, when (3.20) is false, it follows by the argument of Fristedt (1971) page 180 (see also Theorem 2.3 in [12]) that with probability 1, $v(t)|X\{t\}| \geqq 1$ for infinitely many $t \in (0, 1]$, and so $\int v\,dX$ cannot exist. □

4. Asymptotic properties. We first extend Lemma 2.2 to stochastic integrals:

LEMMA 4.1. *Let* $g \in \mathscr{F}_3$, *and suppose that either* $\sigma = 0$ *and* $f \in \mathscr{F}_1$ *or that* $EX(t) \equiv 0$ *and* $f \in \mathscr{F}_2$. *Define* Q *by* (2.4) *with* $c = 4$. *Then*

$$Q\left(x + \int_0^t V\,dX, y + \int_0^t \tilde{f} \circ V\right), \qquad t \geqq 0\,, \tag{4.1}$$

is a super-martingale for every $x \in R$ *and* $y > 0$, *and also for* $x = y = 0$.

PROOF. Let $EX(t) \equiv 0$ and $f \in \mathscr{F}_2$. Since $f(x^{\frac{1}{2}})$ is clearly concave on R_+, the assertion follows for simple V by proceeding as in (3.13) and applying Lemma 2.2. In the general case, choose simple $V_1, V_2, \cdots$ such that (3.4) (with $\tilde{f}$ in place of f) and (3.12) hold a.s., and note that

$$(4.2)\qquad EQ(x + \textstyle\int_0^t V_n\,dX, y + \int_0^t \tilde{f} \circ V_n) \leqq Q(x, y)\,, \qquad x \in R,\ y, t > 0,\ n \in N\,.$$

Since f and g are continuous and $g(y) > 0$, we further have

$$\begin{aligned} &Q(x + \textstyle\int_0^t V_n\,dX, y + \int_0^t \tilde{f} \circ V_n) \\ &\qquad \to Q(x + \textstyle\int_0^t V\,dX, y + \int_0^t \tilde{f} \circ V) \quad \text{a.s.}, \qquad n \to \infty\,, \end{aligned}$$

so by Fatou's lemma, (4.2) remains true with V in place of V_n. By monotone convergence, we may even take $x = y = 0$. The proof for $\sigma = 0$ and $f \in \mathscr{F}_1$ is similar. □

We may now extend Corollary 2.1 as follows:

THEOREM 4.1. *Let $g \in \mathscr{F}_3$ and suppose that either $\sigma = 0$ and $f \in \mathscr{F}_1$ or that $f \in \mathscr{F}_2$. Then*

$$(4.3)\qquad \{f^{-1}g(\textstyle\int_0^t \tilde{f} \circ V)\}^{-1} \int_0^t V\,dX \to 0 \quad \text{a.s.}$$

as $t \to 0$, and this is also true with $\tilde{f}$ replaced by $\tilde{f}_\varepsilon$, $\varepsilon > 0$. If we further assume that $EX(t) \equiv 0$ in case $f \in \mathscr{F}_2$, then as $t \to \infty$, $\int_0^t V\,dX$ converges a.s. *on $\{\int_0^\infty \tilde{f} \circ V < \infty\}$ while* (4.3) *holds on $\{\int_0^\infty \tilde{f}_0 V = \infty\}$.*

PROOF. The assertions for $t \to \infty$ follow as in the proof of Corollary 2.1 by applying the a.s. continuity theorem for super-martingales ([15] page 526). The same theorem applies for $t \to 0$ provided $EX(t) \equiv 0$ when $f \in \mathscr{F}_2$, so it remains to consider the case when $f \in \mathscr{F}_2$ and X is arbitrarily centered. Let X' be the process obtained from X by deleting all jumps of sizes outside $[-a, b]$ for fixed $a, b > 0$, and put $c = EX'(1)$. Suppose that the corresponding truncation of λ changes $\tilde{f}$ to $\tilde{f}' \leqq \tilde{f}$. Applying (4.3) to the process $X'(t) - ct$, $t \geqq 0$, yields a.s.

$$\begin{aligned} (4.4)\qquad &\limsup_{t\to 0} \{f^{-1}g(\textstyle\int_0^t \tilde{f} \circ V)\}^{-1} |\int_0^t V\,dX - c\int_0^t V| \\ &\qquad \leqq \lim_{t\to 0} \{f^{-1}g(\textstyle\int_0^t \tilde{f}' \circ V)\}^{-1} |\int_0^t V\,dX' - c\int_0^t V| = 0\,, \end{aligned}$$

since the limiting behavior of X as $t \to 0$ is independent of the large jumps. If $\lambda \neq 0$, it is possible to choose a and b in two different ways leading to different values of c. Applying (4.4) to the corresponding processes, it follows that

$$(4.5)\qquad \{f^{-1}g(\textstyle\int_0^t \tilde{f} \circ V)\}^{-1} \int_0^t V \to 0 \quad \text{a.s.},$$

and (4.3) follows by combination with (4.4). We finally have to prove (4.5) in the case $\lambda = 0$. Since clearly $g(x)/x \to \infty$ as $x \to 0$ and $f(x) \leqq x^2 f''(0)/2$, it is enough to assume that $g(x) \equiv x$ and $f(x) \equiv x^2 f''(0)/2$. But in that case, (4.5) is a consequence of Schwarz's inequality. □

For $V \equiv 1$, Theorem 4.1 contains Millar's Theorem 4.3 (1971), which in turn improves some classical results of Blumenthal and Getoor (1961). It is interesting to observe that, for stable X and $V \equiv 1$, (4.3) is close to the results by Khinchine (1938). For arbitrary V and X, our result for $t \to 0$ extends Millar's Theorem 3.1 (1972). Note that the strongest possible conclusions from (4.3) are obtained by choosing $g(x)$ close to $x|\log x|$. If we content ourselves with convergence in probability, it is sometimes possible to get rid of this logarithm (cf. [19], [20]):

THEOREM 4.2. *Suppose that either* $\gamma_0 = \sigma = 0$ *and* $f \in \mathscr{F}_1$ *or that* $\sigma = 0$, $\int_{-1}^{1} |x|\lambda(dx) = \infty$ *and* $f \in \mathscr{F}_2$, *and let us further assume that*

(i) *either* $\limsup_{t\to 0} |V(t)| < \infty$ a.s. *or* $xf'(x)/f(x) \uparrow$ *on* R_+, *and*
(ii) *either* $\liminf_{t\to 0} |V(t)| > 0$ a.s. *or* $xf'(x)/f(x) \downarrow$ *on* R_+.

Then

$$\lim_{t\to 0} \sup_{c>0} P\{f(\sup_{s\leq t} |\textstyle\int_0^s V\,dX|) \geqq \delta c,\ \int_0^t \tilde{f}_\varepsilon \circ V \leqq c\} = 0\,, \qquad \varepsilon, \delta > 0\,, \tag{4.6}$$

and if X *and* V *are independent, then even*

$$\{f^{-1}(\textstyle\int_0^t \tilde{f}_\varepsilon \circ V)\}^{-1} \sup_{s\leq t} |\int_0^s V\,dX| \to_P 0 \qquad as\ t\to 0,\ \varepsilon > 0\,. \tag{4.7}$$

We need the following extension of Millar's Property 2.3 (1972), which may either be proved directly from Lemma 4.1 or by a limiting argument from Lemma 2.3.

LEMMA 4.2. *Let* X, f, *and* g *be such as in Lemma* 4.1. *Then*

$$P\left\{\sup_t \frac{f(\int_0^t V\,dX)}{g(y + \int_0^t \tilde{f} \circ V)} \geqq c\right\} \leqq \frac{4}{c} \int_y^\infty \frac{du}{g(u)}\,, \qquad c > 0, y \geqq 0\,, \tag{4.8}$$

and in particular,

$$P\{\sup\,[|f(\textstyle\int_0^t V\,dX)|^{1/p} - a\int_0^t \tilde{f} \circ V] \geqq b\} \leqq \dfrac{4}{(p-1)ab^{p-1}}\,, \qquad a, b > 0,\ p > 1\,. \tag{4.9}$$

PROOF OF THEOREM 4.2. Take $f \in \mathscr{F}_2$ and let us first assume that

$$\int_0^1 x\lambda(dx) = -\int_{-1}^0 x\lambda(dx) = \infty\,. \tag{4.10}$$

It is then possible to choose truncations $\lambda_1, \lambda_2, \cdots$ of λ such that $\int x^2\lambda_n(dx) \to 0$ and such that the corresponding processes $X_1, X_2, \cdots$ obtained from X by deleting large jumps have all mean 0. (If λ has point masses, a randomization may be needed to construct these X_n.) Denote the corresponding functions $\tilde{f}$ by $\tilde{f}_1, \tilde{f}_2, \cdots$. Let $\delta > 0$ be arbitrary. If

$$0 < \liminf_{t\to 0} |V(t)| \leqq \limsup_{t\to 0} |V(t)| < \infty \quad \text{a.s.},$$

we may choose m, M, and $t_1 > 0$ such that

$$PA \equiv P\{m \leqq |V(t)| \leqq M,\ t \leqq t_1\} \geqq 1 - \delta\,. \tag{4.11}$$

Furthermore, since $\tilde{f}$ and the $\tilde{f}_n$ are nondecreasing on R_+, we have

$$\tilde{f}_n(u)/\tilde{f}(u) \to 0 \qquad \text{as } n \to \infty\,, \tag{4.12}$$

uniformly in $[m, M]$, so for $n \in N$ large enough,

$$\sup_{t\leq t_1} \textstyle\int_0^t \tilde{f}_n \circ V / \int_0^t \tilde{f} \circ V \leqq \delta^2 \qquad \text{on } A\,. \tag{4.13}$$

Finally, choose $t_2 \in (0, t_1]$ such that

$$P\{X_n(t) = X(t),\, t \leqq t_2\} \geqq 1 - \delta\,. \tag{4.14}$$

Applying (4.8) with $y = 0$ and $g = 1/1_{[0,\delta^2 c]}$, (which is clearly a permissible choice although $g \notin \mathscr{F}_3$), and using (4.11), (4.13), and (4.14), we obtain for $t \leqq t_2$

$$\begin{aligned} &P\{f(\sup_{s\leq t} |\textstyle\int_0^s V\,dX|) \geqq \delta c,\ \textstyle\int_0^t \tilde{f}\circ V \leqq c\} \\ &\quad\leqq P\{f(\sup_{s\leq t} |\textstyle\int_0^s V\,dX_n|) \geqq \delta c,\ \textstyle\int_0^t \tilde{f}_n \circ V \leqq \delta^2 c\} + 2\delta \\ &\quad\leqq P\{\sup_{s\leq t} f(\textstyle\int_0^s V\,dX_n)/g(\textstyle\int_0^s \tilde{f}_n \circ V) \geqq \delta c\} + 2\delta \leqq \dfrac{4\delta^2 c}{\delta c} + 2\delta = 6\delta\,, \end{aligned}$$

uniformly in $c > 0$, proving (4.6) for $\varepsilon = \infty$. An obvious truncation argument yields the same result for arbitrary $\varepsilon > 0$.

If $xf'(x)/f(x)$ is nondecreasing in $x > 0$, it is easily verified that $f(ux)/f(uy)$ is nonincreasing in $u > 0$ for fixed x and y with $0 < x < y$. Assuming for simplicity of writing that $\lambda_n(dx) \equiv \lambda(dx)1_{[-a,b]}$, we have

$$\begin{aligned} \frac{\tilde{f}_n(u)}{\tilde{f}(u)} &\leqq \frac{\int_0^b f(ux)\lambda(dx)}{\int_b^\infty f(uy)\lambda(dy)} + \frac{\int_{-a}^0 f(ux)\lambda(dx)}{\int_{-\infty}^{-a} f(uy)\lambda(dy)} \\ &= \int_0^b \left\{\int_b^\infty \frac{f(uy)\lambda(dy)}{f(ux)}\right\}^{-1} \lambda(dx) + \int_{-a}^0 \left\{\int_{-\infty}^{-a} \frac{f(uy)\lambda(dy)}{f(ux)}\right\}^{-1} \lambda(dx)\,, \end{aligned}$$

and here the right-hand side is nonincreasing in $u > 0$ and tends to 0 as $a, b \to 0$ for any fixed $u > 0$. Therefore, the uniformity in (4.12) extends to any interval $[m, \infty)$, $m > 0$, and we may take $M = \infty$ in the above argument. The symmetric argument applies to the case of nonincreasing $xf'(x)/f(x)$. If exactly one of the integrals in (4.10) is finite, suppose that X, X', and X'' are independent and distributed as X, and apply (4.6) to the processes $X \pm (X' - X'')$. Since (3.6) holds with $c = 2$, it follows that (4.6) is also true for X. This completes the proof of (4.6) for $f \in \mathscr{F}_2$. The proof for $f \in \mathscr{F}_1$ is similar but much simpler, since no centering is needed in that case. Finally, when X and V are independent, (4.7) follows from (4.6) by Fubini's theorem and dominated convergence. □

When discussing the asymptotic behavior of X, or more generally of $\int V\,dX$, it is often useful to consider integrals of the form $\int V_n\,dX$ for variable V_n. This leads us to investigate the continuity of the mapping $V \to \int V\,dX$ for fixed X. By linearity, we may confine ourselves to continuity at 0.

THEOREM 4.3. *Suppose that either $\sigma = 0$ and $f \in \mathscr{F}_1$, or that $EX = 0$ and $f \in \mathscr{F}_2$. Then*

(i) $\int_0^\infty \tilde{f}\circ V_n \to_P 0$ *implies* $\sup_t |\int_0^t V_n\,dX| \to_P 0$;

(ii) *if $f \in \mathscr{F}_1$ or if $f \in \mathscr{F}_2$ with $\int_0^1 x^{-2}f(x)\,dx < \infty$ and* $\liminf_{x\to\infty} xf'(x)/f(x) > 1$, *then* $E\int_0^\infty \tilde{f}\circ V_n \to 0$ *implies* $Ef(\sup_t |\int_0^t V_n\,dX|) \to 0$;

(iii) *if g_n and h_n, $n \in N$, are strictly increasing functions on R_+ onto itself such* *that* $\sum_n h_n^{-1}(y)/g_n^{-1}(x) < \infty$, $0 < y < x$, *then*

$$\limsup_{n\to\infty} g_n \circ f(\sup_t |\textstyle\int_0^t V_n\, dX|) \leqq \limsup_{n\to\infty} h_n(\int_0^\infty \tilde{f} \circ V_n) \quad \text{a.s.} \tag{4.15}$$

Note in particular that (4.15) holds with $g_n(x) \equiv h_n(x) \equiv x^{1/n}$. A related result for Brownian motion may be found in [16] page 25. It should be observed that (i) does not hold in general with the convergence in probability replaced by a.s. convergence. For Brownian motion, a counterexample is easily constructed from the law of the iterated logarithm, and by using arguments from [14], it is even possible to obtain counterexamples with $\sigma = 0$.

PROOF. To prove (i), proceed as in the proof of Theorem 3.1, using (4.9) in place of (2.5). As for (ii), let $f \in \mathscr{F}_2$ and conclude from Lemma 2.1 and the argument in (3.13) that for simple V

$$Ef(\textstyle\int_0^\infty V\, dX) \leqq 4E \int_0^\infty \tilde{f} \circ V\,. \tag{4.16}$$

For general V with $E \int \tilde{f} \circ V < \infty$, choose simple $V_1, V_2, \cdots$ such that $\tilde{f}$ satisfies (3.4) in L_1 and (3.12) holds a.s., and conclude from Fatou's lemma that (4.16) is generally true. Our next step is to show that $\int V\, dX$ is a martingale. Again this is clear for simple V. In the general case, choose approximating simple $V_1, V_2, \cdots$ as above. Then $Ef(\int_0^\infty V_n\, dX)$ is bounded by (4.16), and assuming f to satisfy the assumptions under (ii) we have $f(x) \geqq x^{1+\varepsilon}$ for some $\varepsilon > 0$ and large x, so $\int_0^\infty V_n\, dX$ is uniformly integrable and we may proceed to the limit in the martingale defining relations. Let us now define the function

$$h(x) = |x| \int_0^{|x|} \frac{f'(u)\, du}{u}\,, \qquad x \in R\,,$$

which is finite and continuous whenever $\int_0^1 x^{-2} f(x)\, dx < \infty$. By the arguments in Meyer (1972) pages 28–29, we get

$$Ef(\sup_t |\textstyle\int_0^t V\, dX|) \leqq 4Ef(\frac{1}{2} \sup_t |\int_0^t V\, dX|) \leqq 4Eh(\int_0^\infty V dX)\,,$$

so by (4.16) it remains to prove that $Ef(\int V_n\, dX) \to 0$ implies $Eh(\int V_n\, dX) \to 0$. But this follows by continuity and uniform integrability, since $h(x) = 0(f(x))$ as $x \to \infty$. In fact, choosing $y > 0$ such that $xf'(x)/f(x) \geqq c > 1$, $x \geqq y$, and integrating by parts, we get for large x

$$\begin{aligned} h(x) &= f(x) + x \int_0^x \frac{f(u)\, du}{u^2} = 0(f(x)) + x \int_y^x \frac{f(u)\, du}{u^2} \\ &\leqq 0(f(x)) + c^{-1} x \int_y^x \frac{f'(u)\, du}{u} \leqq 0(f(x)) + c^{-1} h(x)\,. \end{aligned}$$

This completes the proof of (ii) for $f \in \mathscr{F}_2$. The case $f \in \mathscr{F}_1$ is similar but simpler.

To prove (iii), write

$$\xi_n = f(\sup_t |\textstyle\int_0^t V_n\, dX|)\,, \qquad \eta_n = \int_0^\infty \tilde{f} \circ V_n\,, \qquad n \in N\,,$$

and note that

$$P\{\limsup_{n\to\infty} g_n(\xi_n) > \limsup_{n\to\infty} h_n(\eta_n)\} \leqq \sum_{x,y,m} P\{\limsup g_n(\xi_n) > x > y > h_k(\eta_k),\ k \geqq m\}\,,$$

where the sum extends over all rational x and y with $0 < y < x$ and all $m \in N$. Since the sum is countable, it suffices to prove that each term equals 0, so let us consider fixed x, y, and m. Writing $A = \{h_k(\eta_k) < y,\ k \geqq m\}$ and applying (4.9) with $p = 2$, we get for $n \geqq m$

$$\begin{aligned} P(A \cap \{g_n(\xi_n) > x\}) &= P(A \cap \{\xi_n^{\frac{1}{2}} \geqq (g_n^{-1}(x))^{\frac{1}{2}}\}) \\ &\leqq P\{\xi_n^{\frac{1}{2}} \geqq \tfrac{1}{2}(g_n^{-1}(x))^{\frac{1}{2}}\}[1 + \eta_n/h_n^{-1}(y)] \leqq 16h_n^{-1}(y)/g_n^{-1}(x)\,, \end{aligned}$$

so by the Borel–Cantelli lemma,

$$P(A \cap \{\limsup_{n\to\infty} g_n(\xi_n) > x\}) \leqq P(A \cap \{g_n(\xi_n) > x \text{ i.o.}\}) = 0\,,$$

and the proof is complete. □

By the method for proving (ii), we may also obtain the following result of some independent interest, (cf. Property 2.4 of Millar (1972)):

COROLLARY 4.1. *Let $p \in (0.\ 2]$, and suppose that $\sigma = \gamma_\infty = 0$ when $p > 1$ and $\sigma = \gamma_0 = 0$ when $p \leqq 1$. Then*

$$E \sup_t |\textstyle\int_0^t V\,dX|^p \leqq c_p \int_{-\infty}^{\infty} |x|^p \lambda(dx) E \int_0^\infty |V|^p\,.$$

The modifications required in the cases $p = 2$, $\sigma \neq 0$, and $p = 1$, $\gamma_0 \neq 0$ are obvious.

5. Variation and nonlinear integrals. For $n \in N$, let $\Pi_n = \{0 = t_{n0} < t_{n1} < \cdots < t_{nk_n} = 1\}$ be a finite partition of $[0, 1]$, and define for any $f\colon [0, 1] \to R$ the measure

$$\Pi_n f = \sum_{j=1}^{k_n} \delta(f(t_{nj}) - f(t_{n,j-1}))\,,$$

where $\delta_x A \equiv \delta(x)A \equiv 1_A(x)$. We assume that $\max_j (t_{nj} - t_{n,j-1}) \to 0$ as $n \to \infty$. We further define, for any X and V, the point processes on $R\backslash\{0\}$

$$(V \cdot X)_t \equiv \sum_{0<s\leqq t} \delta(V(s)X\{s\})\,, \qquad t \geqq 0\,,$$

and we put in particular $(V \cdot X)_1 = V \cdot X$. We shall write $\to_w$ for weak convergence of finite measures on R, i.e. $\mu_n \to_w \mu$ means that $\mu_n f \to \mu f$ for any bounded continuous function $f\colon R \to R_+$. Here $\mu f = \int f(x)\mu(dx)$, and we further define $f\mu$ as the *measure* with $(f\mu)(dx) \equiv f(x)\mu(dx)$. For brevity, we also put $x^k\mu(dx) \equiv \mu^k(dx)$, $k = 1, 2$. If the μ_n and μ are random, then $\mu_n \to_{wP} \mu$ ($\mu_n \to_w \mu$ in probability) means that $\mu_n f \to_P \mu f$ with $\mu f < \infty$ a.s. for any bounded continuous f.

The following theorem extends results by Wong and Zakai (1965), and by Millar (1972), (see also [9], [18]). The idea to state variational results as limit theorems for random measures comes from [12].

THEOREM 5.1. *If* $\int_0^1 V^2 < \infty$ a.s., *then*

$$(\Pi_n \int V\,dX)^2 \to_{wP} \sigma^2\delta_0 \int_0^1 V^2 + (V \cdot X)^2\,, \tag{5.1}$$

while if $\sigma = 0$, $\int_{-1}^1 |x|\lambda(dx) < \infty$ *and* $\int_0^1 |V| < \infty$ a.s.,

$$|(\Pi_n \int V\,dX)^1| \to_w |\gamma_0|\delta_0 \int_0^1 |V| + |(V \cdot X)^1| \quad \text{a.s.} \tag{5.2}$$

If either $f \in \mathscr{F}_1$ *and* $\sigma = \gamma_0 = 0$ *or* $f \in \mathscr{F}_2$ *and* $\sigma = 0$, *and if moreover* $\int_0^1 \tilde{f}_1 \circ V < \infty$ a.s., *then*

$$f(\Pi_n \int V\,dX) \to_w f(V \cdot X) \tag{5.3}$$

holds a.s. *or in probability respectively.*

PROOF. By Lemma 3.3, we have in the space of measures on $\bar{R}\backslash\{0\}$

$$\Pi_n \int V\,dX \to_v \sum_{0<t\leq 1} \delta(\int_{t-}^t V\,dX) = V \cdot X \quad \text{a.s.} \tag{5.4}$$

(cf. [12]), where $\to_v$ stands for vague convergence. To prove (5.3) for $f \in \mathscr{F}_2$ and $\sigma = 0$, it therefore suffices to show that

$$(\Pi_n \int V\,dX)f \to_P (V \cdot X)f\,. \tag{5.5}$$

We need the fact that $f^{\frac{1}{2}}$ is concave on R_+. To see this, we have to verify that $(f')^2/f$ is nonincreasing, i.e. that $h = 2ff'' - (f')^2 \leqq 0$. But $f(x) \leqq xf'(x) - x^2(f''(x)/2)$ since f' is concave, and so

$$h(x) \leqq 2xf'(x)f''(x) - x^2(f''(x))^2 - (f'(x))^2 = -(xf''(x) - f'(x))^2 \leqq 0$$

as desired. As in [12], it now follows by Minkowski's inequality that, for any $x_j, y_j, j \in N$,

$$|(\sum f(x_j))^{\frac{1}{2}} - (\sum f(y_j))^{\frac{1}{2}}| \leqq (\sum f(x_j - y_j))^{\frac{1}{2}}\,, \tag{5.6}$$

and applying this to the decomposition $X = X_\varepsilon + X_\varepsilon'$ in the proof of Lemma 3.3, we get

$$|\{(\Pi_n \int V\,dX)f\}^{\frac{1}{2}} - \{(\Pi_n \int V\,dX_\varepsilon')f\}^{\frac{1}{2}}| \leqq \{(\Pi_n \int V\,dX_\varepsilon)f\}^{\frac{1}{2}}\,. \tag{5.7}$$

Now clearly

$$(\Pi_n \int V\,dX_\varepsilon')f \to (V \cdot X_\varepsilon')f \quad \text{a.s.,} \tag{5.8}$$

and it will be shown below that

$$(\Pi_n \int V\,dX_\varepsilon)f \to_P 0 \qquad \text{as } n \to \infty \text{ and then } \varepsilon \to 0\,. \tag{5.9}$$

Combining (5.6)—(5.8), it is seen that $(V \cdot X_\varepsilon')f$ is fundamental in probability as $\varepsilon \to 0$, which proves that $(V \cdot X)f < \infty$ a.s. Using this fact, (5.5) follows by another application of (5.6)—(5.8).

To prove (5.9), note first that V is a.s. integrable on [0, 1] since $\tilde{f}_1 \circ V$ is, and hence that the process $\int_0^t V$, $t \in [0, 1]$, is a.s. continuous and of bounded variation. Since $f'(0) = 0$, it follows that $(\Pi_n \int V)f \to 0$ a.s., so by (5.6) it suffices to prove (5.9) with X_ε replaced by the martingale $Y_\varepsilon(t) \equiv X_\varepsilon(t) - tEX_\varepsilon(t)$. By an obvious stopping time argument, (cf. Lemma 1.1 of Fisk (1966)), we may further assume that $E \int_0^1 \tilde{f}_1 \circ V < \infty$. Using (4.16), we then get

$$\sup_n E(\Pi_n \int V\, dY_\varepsilon)f \leqq 4E \int_0^1 \tilde{f}_\varepsilon \circ V \to 0 \,, \qquad \varepsilon \to 0 \,,$$

and (5.9) follows. For $f \in \mathscr{F}_1$ and $\sigma = \gamma_0 = 0$, it may be shown as above that $(V \cdot X)f < \infty$ a.s., and then the a.s. convergence in (5.3) follows by concavity (cf. [20], [12]).

To prove (5.1), it suffices by the above argument to consider the case when $\lambda[-\varepsilon, \varepsilon] = 0$ for some fixed $\varepsilon > 0$, and since $\int V\, dX$ has then only finitely many jumps, we may even assume that $\lambda = 0$. As before, we may further take $EX = 0$, so after a normalization it only remains to prove that, when X is a standard Brownian motion,

$$(5.10) \qquad (\Pi_n \int V\, dX)^2 R \to_P \int_0^1 V^2 \,.$$

(This was proved by Wong and Zakai (1965) under the assumption $E \int_0^1 V^4 < \infty$, but we prefer to start afresh.) Again we may assume that $E \int_0^1 V^2 < \infty$, and by Lemma 3.1, we may then choose simple processes $V_1, V_2, \cdots$ such that as $k \to \infty$

$$(5.11) \qquad \sup_n E(\Pi_n \int (V_k - V)\, dX)^2 R = E \int_0^1 (V_k - V)^2 \to 0$$

and $\int_0^1 V_k^{\,2} \to \int_0^1 V^2$ a.s. But by the classical result for Brownian motion,

$$(5.12) \qquad (\Pi_n \int V_k\, dX)^2 \to_P \int_0^1 V_k^{\,2} \to_P \int_0^1 V^2$$

as $n \to \infty$ and then $k \to \infty$. Using Minkowski's inequality, we obtain (5.10) from (5.11) and (5.12). To prove (5.2), we may use a similar argument with (5.10) replaced by the well-known relation

$$|(\Pi_n \int V)^1| = \sum_j |\int_{t_{n,j-1}}^{t_{nj}} V| \to \int_0^1 |V| \quad \text{a.s.,}$$

(which may e.g. be proved from Lemma 3.1). Our proof is now complete. □

Further assumptions must be added in order to obtain a.s. convergence in (5.1) and in (5.3) for $f \in \mathscr{F}_2$. Let us say that $\{\Pi_n\}$ is *nested* if it proceeds by successive refinements, and that (X, V) is *symmetric*, if for every fixed $t \geqq 0$, (X, V) has the same distributions as (Y, V) where

$$\begin{aligned} Y(s) &= X(s) \,, && s < t \,, \\ &= 2X(t) - X(s) \,, && s \geqq t \,. \end{aligned}$$

Note that the last notion reduces to symmetry of $X(1)$ when X and V are independent. We shall prove the following extension of results by Lévy, Cogburn and Tucker (1961), Millar (1971), and myself (1974).

THEOREM 5.2. *If in Theorem* 5.1 *we add the assumptions that* $\{\Pi_n\}$ *is nested and that either* (X, V) *is symmetric or* X *and* V *are independent*, *then* (5.1) *and* (5.3) *hold even in the sense of* a.s. *convergence*.

PROOF. For independent X and V, we may use Fubini's theorem to reduce the proof to the case of nonrandom V. But then $\int V\,dX$ has independent and infinitely divisible increments, and our assertions are contained in Theorems 4.1 and 4.2 of [12]. (Note that the assumptions of f in [12], Theorem 4.2, are fulfilled since for $f \in \mathscr{F}_2$, $f^{\frac{1}{2}}$ and $f(\cdot^{\frac{1}{2}})$ are both concave.) Next suppose that (X, V) is symmetric, and assume without loss that λ has bounded support and that $E\int_0^1 V^2 < \infty$ (or $E\int_0^1 \tilde{f} \circ V < \infty$ respectively). Proceeding as in [3] and [19], it is seen that $\{(\Pi_n \int V\,dX)^2 R\}$ (or $\{(\Pi_n \int V\,dX)f\}$) is an L_1-bounded reversed super-martingale and therefore converges a.s. This proves in particular that (5.5) and (5.10) remain true in the sense of a.s. convergence, and so the preceding proof yields the present stronger conclusions. □

It is suggestive to write $\int f(V\,dX)$ for the f-variation of $\int V\,dX$, whenever it exists in the sense of Theorem 5.1. More generally, we define for any X and V and for $f \in \mathscr{F}$

$$\begin{aligned} (5.13) \qquad \int_0^t f(V\,dX) &= (V \cdot X)_t f + \sigma^2 f''(0) \int_0^t V^2 && \text{if } f'(0) = 0\,, \\ &= (V \cdot X)_t f + |\gamma_0| f'(0+) \int_0^t |V| && \text{if } \sigma = 0\,. \end{aligned}$$

(Note the formal similarity with (3.2).) As a simple example, it may be seen from (5.13) that, for any $p > 0$,

$$\int_0^t |V\,dX|^p \equiv \int_0^t |V|^p |dX|^p \quad \text{a.s.},$$

where the integral on the right is the Lebesgue–Stieltjes integral of $|V|^p$ w.r.t. $\int_0^t |dX|^p$, $t \geqq 0$. Since the latter process has stationary independent increments, the above theory applies to this case.

Many previous results of this paper may be extended to integrals of the form $\int f(V\,dX)$. For brevity, we restrict our attention to the following partial extension of Theorems 3.1 and 4.1.

THEOREM 5.3. *If $f \in \mathscr{F}$ and either $f'(0) = 0$ or $\sigma = 0$, and if $\int_0^t (\widetilde{1 \wedge f})_1 \circ V < \infty$ a.s., $t > 0$, then $\int_0^t f(V\,dX) < \infty$ a.s., $t > 0$. If moreover $h \in \mathscr{F}_1$ and $g \in \mathscr{F}_3$, then*

$$(5.14) \qquad \{h^{-1} g(\textstyle\int_0^t (\widetilde{h \circ f}) \circ V)\}^{-1} \int_0^t f(V\,dX) \to 0 \quad \text{a.s.}$$

as $t \to 0$, and this is also true with $(\widetilde{h \circ f})$ replaced by $(\widetilde{h \circ f})_\varepsilon$, $\varepsilon > 0$. As $t \to \infty$, $\int_0^t f(V\,dX)$ converges a.s. *on $\{\int_0^\infty (\widetilde{h \circ f}) \circ V < \infty\}$, while* (5.14) *holds on $\{\int_0^\infty (\widetilde{h \circ f}) \circ V = \infty\}$.*

PROOF. We first extend Lemma 4.1 by showing that the process

$$(5.15) \qquad Q(x + \textstyle\int_0^t f(V\,dX), y + \int_0^t (\widetilde{h \circ f}) \circ V)\,, \qquad t \geqq 0\,,$$

with Q defined by (2.4) with h in place of f and with $c = 1$, is a super-martingale for every $x \in R$ and $y > 0$, and also for $x = y = 0$. For simple V, this follows from Lemma 2.2 and the easily verified fact that

$$Eh(\textstyle\int_0^t f(V\,dX)) \leqq E\int_0^t (\widetilde{h \circ f}) \circ V\,, \qquad t \geqq 0\,,$$

(cf. (3.13)). For general V satisfying $\int_0^t \widetilde{(h \circ f)} \circ V < \infty$ a.s., $t > 0$, consider fixed $t > 0$ and $n \in N$, write $D_n = \{s\colon |X\{s\}| > n^{-1}\}$, and choose $u > 0$ so large that the process $V_n' = u \wedge V$ satisfies

$$P\{|\textstyle\int_0^t \widetilde{(h \circ f)} \circ V_n' - \int_0^t \widetilde{(h \circ f)} \circ V| > n^{-1}\} \leqq 2^{-n}\,, \tag{5.16}$$

$$P\{V_n'(s) = V(s),\, s \in D_n\} = P\{|V(s)| \leqq u,\, s \in D_n\} \geqq 1 - 2^{-n}\,, \tag{5.17}$$

being possible by monotone convergence. According to Lemma 3.1 and Theorem 3.1, we may next choose some simple process V_n such that

$$P\{|\textstyle\int_0^t \widetilde{(h \circ f)} \circ V_n - \int_0^t \widetilde{(h \circ f)} \circ V_n'| > n^{-1}\} \leqq 2^{-n}\,, \tag{5.18}$$

$$P\{\sup_{s \leqq t} |\textstyle\int_0^s V_n\, dX - \int_0^s V_n'\, dX| > n^{-2}\} \leqq 2^{-n}\,. \tag{5.19}$$

(Note that the assumption $\limsup_{x \to \infty} f(2x)/f(x) < \infty$ in Lemma 3.1 is not needed when V is bounded, since in that case the construction in the proof leads to uniformly bounded processes V_n.) Combining (5.16)—(5.19), we get

$$P\{|\textstyle\int_0^t \widetilde{(h \circ f)} \circ V_n - \int_0^t \widetilde{(h \circ f)} \circ V| > 2n^{-1}\} \leqq 2^{-n+1}\,,$$
$$P\{|V_n(s) - V(s)| \leqq 2n^{-1},\, s \in D_n\} \geqq 1 - 2^{-n+1}\,,$$

and so, by the Borel–Cantelli lemma,

$$\textstyle\int_0^t \widetilde{(h \circ f)} \circ V_n \to \int_0^t \widetilde{(h \circ f)} \circ V\,, \tag{5.20}$$

$$V_n(s)X\{s\} \to V(s)X\{s\}\,, \qquad s \in [0, t]\,, \tag{5.21}$$

a.s. as $n \to \infty$. By Fatou's lemma, it follows from (5.21) that a.s.

$$(V \cdot X)_t f = \textstyle\sum_{0<s\leqq t} f(V(s)X\{s\}) \leqq \liminf_{n\to\infty} \sum_{0<s\leqq t} f(V_n(s)X\{s\})$$
$$= \liminf_{n\to\infty} (V_n \cdot X)_t f\,,$$

and by combination with (5.20),

$$\textstyle\int_0^t f(V dX) \leqq \liminf_{n\to\infty} \int_0^t f(V_n\, dX) \quad \text{a.s.} \tag{5.22}$$

Using Fatou's lemma once more, we get from (5.20) and (5.22) for $y > 0$

$$EQ(x + \textstyle\int_0^t f(V\, dX), y + \int_0^t \widetilde{(h \circ f)} \circ V)$$
$$\leqq E \liminf_{n\to\infty} Q(x + \textstyle\int_0^t f(V_n\, dX), y + \int_0^t \widetilde{(h \circ f)} \circ V_n)$$
$$\leqq \liminf_{n\to\infty} EQ(x + \textstyle\int_0^t f(V_n\, dX), y + \int_0^t \widetilde{(h \circ f)} \circ V_n) \leqq Q(x, y)\,,$$

where the last inequality follows from the fact that (5.15) is a super-martingale for simple V. This extends the super-martingale property to arbitrary V. The last two assertions may now be proved as were Corollary 2.1 and Theorem 4.1.

In particular, it follows that $\int_0^t f(V\, dX) < \infty$ a.s. when $\int_0^t \widetilde{(h \circ f)} \circ V < \infty$ a.s., or even provided $\int_0^t \widetilde{(h \circ f)}_\varepsilon \circ V < \infty$ a.s. for some $\varepsilon > 0$, since X has at most finitely many jumps in $[0, t]$ of modulus $> \varepsilon$. Choosing $h(x) \equiv |x| \wedge 1$ yields the first assertion and hence completes the proof. □

We may even go a step further and consider more complex integrals such as $\int Uf(V\,dX)$, $\int f(V;dX)$ etc. Though the formulae now become more complicated, the methods remain essentially the same.

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