The Annals of Probability
1975, Vol. 3, No. 2, 247–261

SAMPLE PATH BEHAVIOR FOR BROWNIAN MOTION IN BANACH SPACES[1]

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We establish integral tests for upper and lower functions for Brownian motion in Banach spaces of infinite-dimension as well as some related results for Brownian motion in $\mathbb{R}^d$.

1. Introduction. Throughout the paper B is a real separable Banach space with norm $\|\cdot\|$, and all measures on B are assumed to be defined on the Borel subsets of B generated by the norm open sets. We denote the topological dual of B by B^*.

A measure μ on B is called a mean zero Gaussian measure if every continuous linear function f on B has a mean zero Gaussian distribution with variance $\int_B [f(x)]^2\mu(dx)$. The bilinear function T defined on B^* by

$$T(f, g) = \int_B f(x)g(x)\mu(dx) \qquad (f, g \in B^*)$$

is called the covariance function of μ. It is well known that a mean zero Gaussian measure on B is uniquely determined by its covariance function. This is so because T uniquely determines μ on the Borel subsets of B generated by the weakly open sets, and since B is separable the Borel sets generated by the weakly open sets are the same as those generated by the norm open sets.

However, a mean zero Gaussian measure μ on B is also determined by a unique subspace H_μ of B which has a Hilbert space structure. The norm on H_μ will be denoted by $\|\cdot\|_\mu$ and it is well known that the B norm $\|\cdot\|$ is weaker than $\|\cdot\|_\mu$ on H_μ. In fact, $\|\cdot\|$ is a measurable norm on H_μ in the sense of [3]. Since $\|\cdot\|$ is weaker than $\|\cdot\|_\mu$ it follows that B^* can be linearly embedded (by the restriction map) into the dual of H_μ, call it $H_\mu{}^*$, and identifying H_μ with $H_\mu{}^*$ in the usual way we have $B^* \subseteqq H_\mu{}^* \subseteqq B$. Then by the main result in [3] the measure μ is the extension of the canonical normal distribution on H_μ to B. We describe this relationship by saying μ *is generated by* H_μ. For details and additional references see [3], [5].

Let Ω denote the space of continuous functions ω from $[0, \infty)$ into B such that $\omega(0) = 0$, and let $\mathscr{T}$ be the sigma-algebra of Ω generated by the functions $\omega \to \omega(t)$. Let μ be a mean zero Gaussian measure on B generated by H_μ, and

Received January 24, 1974; revised August 9, 1974.

[1] Supported in part by NSF Grant 18759.

AMS 1970 *subject classifications.* Primary 60G15, 60G17; Secondary 60B05.

Key words and phrases. Gaussian measures, law of the iterated logarithm, Brownian motion in a Banach space, measurable norm, integral tests, upper and lower functions.

suppose $\{\mu_t : t \geqq 0\}$ is the family of Gaussian measures on B given by

$$\begin{aligned} \mu_t(A) &= \delta_0(A) && t = 0 \\ &= \mu(A/t^{\frac{1}{2}}) && t > 0 . \end{aligned}$$

Then $\mu_{s+t} = \mu_s * \mu_t$ for $s, t \geqq 0$, where * denotes convolution, and there is a unique probability measure P on $\mathscr{T}$ such that, if $0 = t_0 < t_1 < \cdots < t_n$, then the B-valued random variables $\omega(t_j) - \omega(t_{j-1})$ $(j = 1, \cdots, n)$ are independent and $\omega(t_j) - \omega(t_{j-1})$ has distribution $\mu_{t_j - t_{j-1}}$ on B. The stochastic process $\{W(t) : t \geqq 0\}$ defined on $(\Omega, \mathscr{T}, P)$ by $W(t, \omega) = \omega(t)$ has stationary independent mean zero Gaussian increments, and we call it *μ-Brownian motion in B*. The existence of μ-Brownian motion in B is discussed in [2], and a number of its important properties are examined in [5].

In [6] we proved the analogue of Strassen's version of the law of the iterated logarithm for μ-Brownian motion in B, and as a corollary we obtain

$$P\left(\limsup_{t\to\infty} \frac{||W(t)||}{(2t \log\log t)^{\frac{1}{2}}} = \Gamma\right) = 1 \tag{1.1}$$

where

$$\Gamma = \sup_{x \in K} ||x|| \tag{1.2}$$

and K is the unit ball of H_μ in the norm $||\cdot||_\mu$.

In view of (1.1) and the great body of literature regarding asymptotic properties of stochastic processes (see, for example, [8], [9], and [10]) it is natural to ask about upper and lower functions for μ-Brownian motion in B and to seek integral tests regarding these functions. First, however, we need some terminology.

A nonnegative, non-decreasing, continuous function $\phi(t)$ defined for large values of t is called a *lower function for $\{W(t) : t \geqq 0\}$ with respect to the norm $||\cdot||$* if

$$P(||W(t)|| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ for an unbounded set of } t\text{'s}) = 1 , \tag{1.3}$$

and an *upper function for $\{W(t) : t \geqq 0\}$ with respect to the norm $||\cdot||$* if

$$P(||W(t)|| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ for only a bounded set of } t\text{'s}) = 1 . \tag{1.4}$$

We write $\phi \in \mathscr{L}$ and $\phi \in \mathscr{U}$, respectively. Here, of course, Γ is given by (1.2).

Upper and lower functions for μ-Brownian motion in B are given in Theorems 2.3 and 2.4. These results, then, motivate Theorem 3.1, Theorem 4.1, and Corollary 4.2 which deal with upper and lower functions for standard Brownian motion in $\mathbb{R}^d$ $(d \geqq 2)$ when norms other than the usual Euclidean norm are used in $\mathbb{R}^d$. As can be seen from these latter results, changing the norm greatly changes the upper and lower functions for standard Brownian motion even in $\mathbb{R}^2$.

Throughout the paper C stands for an unimportant positive constant which may change from line to line.

It is a pleasure to acknowledge that the results of Section 2 were motivated by a question of Professor G. Kallianpur.

2. Upper and lower functions for Brownian motion in a Banach space. Let μ denote a mean-zero Gaussian measure on B and assume K is the unit ball of the generating Hilbert space H_μ. Viewing $B^* \subseteqq H_\mu^* \equiv H_\mu \subseteqq B$ as in the introduction we define the *index of* μ to be

$$(2.1) \qquad n_1 = \sup\{k : \exists f_1, \cdots, f_k \in B^*; f_1, \cdots, f_k \text{ orthogonal in } H_\mu; \\ \|f_j\|_{B^*} = 1 \text{ and } \|f_j\|_\mu = \sup_{x \in K} \|x\| \ (1 \leqq j \leqq k)\}.$$

LEMMA 2.1. *The index of a mean zero Gaussian measure μ on B exists and is finite.*

PROOF. If $i : H_\mu \to B$ is the identity map then the embedding of B^* into $H_\mu^* \equiv H_\mu$ via restriction is given by the adjoint map i^*. Now i is compact [2] and hence i^* is also compact. Further, since the unit ball of B^* is weak star compact and metrizable it is easy to see that $U = i^*\{f \in B^* : \|f\|_{B^*} \leqq 1\}$ is closed in H_μ so U is actually compact in $H_\mu^* \equiv H_\mu$. We also have

$$\sup_{\|f\|_{B^*} \leq 1} \|f\|_\mu = \sup_{x \in K} \|x\|$$

since

$$\sup_{\|f\|_{B^*} \leq 1} \|f\|_\mu = \sup_{\|f\|_{B^*} \leq 1} \sup_{x \in K} |f(x)| \\ = \sup_{x \in K} \sup_{\|f\|_{B^*} \leq 1} |f(x)| = \sup_{x \in K} \|x\| .$$

Thus n_1 exists and is finite since no compact subset in H_μ contains an infinite sequence of orthonormal elements.

LEMMA 2.2. *If B is a Hilbert space, then the index of a mean zero Gaussian measure μ on B equals the multiplicity of the maximal eigenvalue of the covariance operator for μ.*

PROOF. In case B is a Hilbert space it is well known that the covariance function for μ determines an operator of the form $Tx = \sum_k \lambda_k(x, e_k)e_k$ where $(\cdot, \cdot)$ is the inner product on B, $\lambda_k > 0$, $\sum_k \lambda_k < \infty$, and $\{e_k\}$ is an orthonormal set in B. In fact, if M denotes the closed subspace of B generated by $\{e_k\}$, then from [4] we know that the generating Hilbert space for μ is

$$H_\mu = \{x \in M : \textstyle\sum_k (x, e_k)^2/\lambda_k < \infty\} .$$

Further, the inner product for H_μ is given by

$$(x, y)_\mu = \sum_k \frac{(x, e_k)(y, e_k)}{\lambda_k} ,$$

and hence $K = \{x \in M : \sum_k (x, e_k)^2/\lambda_k \leqq 1\}$. Thus for $x \in K$

$$\|x\|^2 = \sum_k (x, e_k)^2 = \sum_k \lambda_k \frac{(x, e_k)^2}{\lambda_k} \leqq \sup_k \lambda_k \sum_k \frac{(x, e_k)^2}{\lambda_k} \\ \leqq \sup_k \lambda_k ,$$

and hence

$$\sup_{x \in K} \|x\| \leqq (\sup_k \lambda_k)^{\frac{1}{2}} = \lambda^{\frac{1}{2}}$$

where λ is the largest eigenvalue of T. Now each of the eigenvectors e_j corresponding to λ has B norm one and hence the linear function $f_j(x) = (x, e_j)$ has B^* norm one. Now f_j restricted to H_μ can be written as

$$f_j(x) = (x, \gamma_j)_\mu \qquad (x \in H_\mu)$$

where $\gamma_j \in H_\mu$. Hence we have $(x, e_j) = (x, \gamma_j)_\mu$ for each $x \in H_\mu$ and since $(x, \gamma_j)_\mu = \sum_k (x, e_k)(\gamma_j, e_k)/\lambda_k = (x, e_j)$ for $x \in H_\mu$ it follows easily that $\gamma_j = \lambda_j e_j = \lambda e_j$. That is, if i^* is defined as in the previous lemma, then $i^*(f_j) = \gamma_j = \lambda_j e_j = \lambda e_j$ so $||f_j||_\mu = ||\gamma_j||_\mu = ||\lambda e_j||_\mu = \lambda^{\frac{1}{2}}$. Thus the index of μ, call it n_1, is greater than or equal to the multiplicity of λ.

By relabeling (if necessary) we can assume $\lambda_1 = \cdots = \lambda_r = \lambda$ and $\lambda_j < \lambda$ for $j \geqq r + 1$. Now assume $f \in B^*$, $||f||_{B^*} = 1$, and $i^*(f) = \gamma \in H_\mu^* \equiv H_\mu$ where γ is orthogonal to $\gamma_j = \lambda e_j$ $(j = 1, \cdots, r)$. Hence $f(x) = (x, e)$ where $e = \sum_{k \geqq r+1} (e, e_k)e_k + h$, h is orthogonal to M, and e has B norm one. Since h is orthogonal to M we have $i^*h = 0$ and hence

$$\begin{aligned} ||f||_\mu = ||\gamma||_\mu = ||i^*e||_\mu &= ||\textstyle\sum_{k \geqq r+1} (e, e_k) i^*(e_k)||_\mu \\ &= ||\textstyle\sum_{k \geqq r+1} (e, e_k)\lambda_k e_k||_\mu \\ &= \left\{\sum_{k \geqq r+1} \frac{\lambda_k^2 (e, e_k)^2}{\lambda_k}\right\}^{\frac{1}{2}} \\ &\leqq \sup_{k \geqq r+1} \lambda_k^{\frac{1}{2}} \textstyle\sum_{k \geqq r+1} (e, e_k)^2 \\ &\leqq \sup_{k \geqq r+1} \lambda_k^{\frac{1}{2}} < \lambda^{\frac{1}{2}} . \end{aligned}$$

Thus $n_1 \leqq r$ so the lemma is proved.

THEOREM 2.3. *Let $\{W(t) : 0 \leqq t < \infty\}$ be μ-Brownian motion in a real separable Banach space B having norm $||\cdot||$, and assume $\phi(t)$ is a nonnegative, non-decreasing, continuous function defined for large values of t. Let n_1 denote the index of μ. Then there is an equivalent norm $||\cdot||_1$ on B such that* $\sup_{x \in K} ||x||_1 = \sup_{x \in K} ||x|| = \Gamma$ *and ϕ is in $\mathscr{U}(\mathscr{L})$ with respect to $||\cdot||_1$* iff

$$\int^\infty \frac{[\phi(t)]^{n_1}}{t} e^{-\phi^2(t)/2} dt < \infty \; (= \infty) . \tag{2.2}$$

PROOF. Assume μ is generated by $H_\mu \subseteqq B$. Since the index of μ is n_1 there exists $f_1, \cdots, f_{n_1} \in B^*$ such that $f_1, \cdots, f_{n_1}$ are orthogonal when viewed as elements of H_μ and satisfying

$$||f_j||_{B^*} = 1 \qquad \text{and} \qquad ||f_j||_\mu = \sup_{x \in K} ||x|| \qquad (j = 1, \cdots, n_1) .$$

Let $\pi(x) = \sum_{j=1}^{n_1} e_j(x) e_j$ $(x \in B)$ and $Q(x) = x - \pi(x)$ where $e_j(\cdot)$ denotes the linear functional $f_j(\cdot)/\Gamma$ and Γ is as in (1.2). Thus $||e_j||_{B^*} = \Gamma^{-1}$ and $||e_j||_\mu = 1$. Let

$$||x||_1 = \max \{\Gamma ||\pi x||_\mu, ||Qx||\} . \tag{2.3}$$

Then

$$\sup_{x \in K} ||x||_1 = \sup_{x \in K} ||x|| = \Gamma \tag{2.4}$$

since it follows easily from (2.3) that

$$\sup_{x\in K}\|x\|_1 = \max\{\sup_{x\in K}\Gamma\|\pi x\|_\mu, \sup_{x\in K}\|Qx\|\}$$

and since $Q: K\to K$ we have (2.4). Further, $\|\cdot\|_1$ is a norm on B which is equivalent to the given norm $\|\cdot\|$.

Using the argument of ([10] page 151) it suffices to prove the theorem when ϕ satisfies

$$(\log\log t)^{\frac{1}{2}} \leqq \phi(t) \leqq 2(\log\log t)^{\frac{1}{2}} \tag{2.5}$$

for all sufficiently large t. Hence we make this assumption on ϕ throughout the proof.

If (2.2) diverges then by (2.3) and (2.4) we have

$$\begin{aligned} &P\{\|W(t)\|_1 > t^{\frac{1}{2}}\phi(t)\sup_{x\in K}\|x\|_1 \text{ for an unbounded set of } t\text{'s}\}\\ &\quad\geqq P\{\|\pi W(t)\|_\mu > t^{\frac{1}{2}}\phi(t) \text{ for an unbounded set of } t\text{'s}\}\\ &\quad= 1\end{aligned} \tag{2.6}$$

by ([9] Theorem 2.2) since $\pi W(t)$ is standard n_1 dimensional Brownian motion in $\pi B = \pi H_\mu$. Thus divergence in (2.2) implies $\phi\in\mathscr{L}$.

If (2.2) converges then since ϕ is non-decreasing we have

$$\begin{aligned} &P(\|W(t)\|_1 > t^{\frac{1}{2}}\phi(t)\Gamma \text{ for an unbounded set of } t\text{'s})\\ &\quad\geqq P(\sup_{t_k\leqq t\leqq t_{k+1}}\|W(t)\|_1 > t_k^{\frac{1}{2}}\phi(t_k)\Gamma \text{ for infinitely many } k)\end{aligned} \tag{2.7}$$

where $\{t_k\}$ is any sequence increasing to infinity. For our purposes we choose $\{t_k\}$ such that

$$t_{k+1} = t_k\left(1+\frac{1}{\phi^2(t_k)}\right) \tag{2.8}$$

where $t_1 > 3$ is sufficiently large so that (2.5) holds for $t > t_1$ and hence $\lim_k t_k = \infty$.

Now observe that

$$\begin{aligned} &P(\sup_{t_k\leqq t\leqq t_{k+1}}\|W(t)\|_1 > t_k^{\frac{1}{2}}\phi(t_k)\Gamma)\\ &\quad\leqq P(\sup_{0\leqq t\leqq t_{k+1}}\|\pi W(t)\|_\mu > t_k^{\frac{1}{2}}\phi(t_k))\\ &\qquad+ P(\sup_{0\leqq t\leqq t_{k+1}}\|QW(t)\| > t_k^{\frac{1}{2}}\phi(t_k)\Gamma)\\ &\quad\leqq 2\{P(\|\pi W(t_{k+1})\|_\mu > t_k^{\frac{1}{2}}\phi(t_k) + P(\|QW(t_{k+1})\| > t_k^{\frac{1}{2}}\phi(t_k)\Gamma)\}\\ &\quad= 2\left\{P\left(\|\pi W(1)\|_\mu > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}}\phi(t_k)\right)\right.\\ &\qquad\left.+ P\left(\|QW(1)\| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}}\phi(t_k)\Gamma\right)\right\}\end{aligned} \tag{2.9}$$

since the second inequality follows in the usual manner.

Using the fact that (2.2) is finite we will show that

$$\textstyle\sum_k P\left(\|\pi W(1)\|_\mu > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}}\phi(t_k)\right) < \infty \tag{2.10}$$

and

$$\sum_k P\left(\|QW(1)\| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\Gamma\right) < \infty . \tag{2.11}$$

Hence by the Borel–Cantelli lemma and combining (2.9), (2.7), and (2.4) we have $\phi \in \mathscr{U}$.

We now turn to the proof of (2.10) and (2.11). To show (2.10) we first observe that $\pi W(1)$ is a standard n_1-dimensional Gaussian random variable with mean zero and identity covariance matrix. Hence by standard estimates we have

$$\begin{aligned} P\left(\|\pi W(1)\|_\mu > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\right) &\leqq C\left[\left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\right]^{n_1-2} \exp\left\{-\frac{t_k\phi^2(t_k)}{2t_{k+1}}\right\} \\ &\leqq C[\phi(t_k)]^{n_1-2} \exp\left\{-\frac{\phi^2(t_k)}{2}\right\} \end{aligned} \tag{2.12}$$

since $t_k/t_{k+1} < 1$ and $t_k/t_{k+1} = (1 + 1/\phi^2(t_k))^{-1} > 1 - 1/\phi^2(t_k) > \frac{1}{2}$ for all k sufficently large. Applying (2.5) and (2.8) we find for $k \geqq 2$ that

$$\begin{aligned} &P\left(\|\pi W(1)\|_\mu > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\right) \\ &\leqq C \int_{t_{k-1}}^{t_k} [\phi(t_k)]^{n_1-2} \exp\{-\tfrac{1}{2}\phi^2(t_k)\} \frac{t_k}{s(t_k - t_{k-1})}\, ds \\ &\leqq C \int_{t_{k-1}}^{t_k} [\phi(t_k)]^{n_1-2} \exp\{-\tfrac{1}{2}\phi^2(t_k)\} \frac{\phi^2(t_{k-1})t_k}{st_{k-1}}\, ds \\ &\leqq C \int_{t_{k-1}}^{t_k} \frac{[\phi(s)]^{n_1}}{s} \exp\{-\tfrac{1}{2}\phi^2(s)\}\, ds \end{aligned} \tag{2.13}$$

since $[\phi(s)]^{n_1} \exp\{-\frac{1}{2}\phi^2(s)\}$ eventually decreases as a function of s. Therefore (2.2) converging implies (2.10).

To verify (2.11) we first note that

$$\sup_{x \in K} \|Qx\| = \rho < \Gamma . \tag{2.14}$$

This follows since $\rho \leqq \Gamma$ is clear. Further, if $\rho = \Gamma$, then there exists a point $x_0 \in QK$ such that $\|x_0\| = \Gamma$ (QK is compact in B) and by the Hahn–Banach theorem a linear functional f such that $\|f\|_{B^*} = 1$, $f(x_0) = \Gamma$, and f vanishes on the subspace $\pi B = \pi H_\mu$. Thus f viewed as an element in H_μ is orthogonal to the elements $f_1, \cdots, f_{n_1}$ defined at the beginning of the proof, $\|f\|_{B^*} = 1$, and $\|f\|_\mu = \Gamma$. This contradicts the fact that the index of μ is n_1, and hence we must have $\rho < \Gamma$.

Since B is a separable Banach space we have a countable sequence of $\{\Lambda_j\}$ in B^* such that $\|\Lambda_j\|_{B^*} = 1$ and $\|x\| = \sup_j |\Lambda_j(x)|$ for every $x \in B$. Further, $QW(1)$ is a B-valued random variable whose distribution on B is the mean zero Gaussian measure μ^Q defined by $\mu^Q(A) = \mu(Q^{-1}(A))$ for Borel sets A. In fact, $\mu^Q(QB) = 1$ and if $\varepsilon < 1/2 \sup \sigma_n^2$ where $\sigma_n^2 = \int_B \Lambda_n^2(x)\mu^Q(dx)$ then

$$
\begin{aligned}
(2.15)\qquad & P\left(\|QW(1)\| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\Gamma\right) \\
&= \mu^Q\left(x \in QB : \|x\| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\Gamma\right) \\
&= \mu^Q\left(x \in QB : \sup_j |\Lambda_j(x)| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\Gamma\right) \\
&\leqq C \exp\left\{-\varepsilon \frac{t_k}{t_{k+1}} \phi^2(t_k)\Gamma^2\right\}
\end{aligned}
$$

where the last inequality follows from Theorem 1.11 of [7]. Now $\sigma_n^2 = \int_B \Lambda_n^2(x)\mu^Q(dx) = \|\Lambda_n \circ Q\|_\mu^2$ so

$$
\begin{aligned}
(2.16)\qquad \sup_n \sigma_n^2 &= \sup_n \|\Lambda_n \circ Q\|_\mu^2 \leqq \sup_{\|f\|_{B^*}\leq 1} \|f \circ Q\|_\mu^2 \\
&= \sup_{\|f\|_{B^*}\leq 1} \sup_{x\in K} |f(Q(x))|^2 \\
&= \sup_{x\in K} \sup_{\|f\|_{B^*}\leq 1} |f(Qx)|^2 \\
&= \sup_{x\in K} \|Q(x)\|^2 = \rho^2 < \Gamma^2 .
\end{aligned}
$$

Thus with $\varepsilon = (1 + \delta)/2\Gamma^2$ (where $\delta > 0$ is such that $\varepsilon < 1/2\rho^2$) we have that

$$
(2.17)\qquad P\left(\|QW(1)\| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\Gamma\right) \leqq C \exp\left\{-\frac{\phi^2(t_k)}{2}\right\} \exp\left\{-\frac{\delta}{2}\phi^2(t_k)\right\}
$$

since $t_k/t_{k+1} = (1 + 1/\phi^2(t_k))^{-1} > \frac{1}{2}$ for k sufficiently large. Arguing as above in (2.13) we see (2.2) converging and (2.17) implies (2.11). This completes the proof.

The next result provides an integral test for upper and lower functions for Brownian motion in a Hilbert space H. Of course, Theorem 2.3 applies to the Hilbert space case, but in Theorem 2.4 we work directly with the given norm on H and do not introduce an equivalent norm. As will be seen later in this paper, norms other than Euclidean norms yield considerably different upper and lower functions even for Brownian motion in $\mathbb{R}^d$. In view of these results, then, Theorem 2.3 is somewhat natural.

THEOREM 2.4. *Let $\{W(t) : 0 \leqq t < \infty\}$ be μ-Brownian motion in a real separable Hilbert space H with norm $\|\cdot\|$, and suppose $\phi(t)$ is a nonnegative, non-decreasing, continuous function defined for large values of t. Then ϕ is in $\mathscr{U}(\mathscr{L})$ with respect to the given norm $\|\cdot\|$* iff

$$
(2.18)\qquad \int^\infty \frac{[\phi(t)]^{n_1}}{t} e^{-\phi^2(t)/2}\, dt < \infty \ (= \infty)
$$

where n_1 denotes the multiplicity of the maximal eigenvalue of the covariance operator for μ.

PROOF. By Lemma 2.2 the index of μ is also n_1. Hence if (2.18) diverges the proof that $\phi \in \mathscr{L}$ follows exactly as in Theorem 2.3.

In case (2.18) converges we proceed as in (2.7) and (2.8) assuming as before.

(2.5). Then with $\Gamma = \sup_{x \in K} ||x||$ we easily obtain

$$(2.19) \qquad P(\sup_{t_k \leq t \leq t_{k+1}} ||W(t)|| > t_k^{\frac{1}{2}}\phi(t_k)\Gamma) \leq 2P\left(||W(1)|| > \left(\frac{t_k}{t_{k+1}}\right)^{\frac{1}{2}} \phi(t_k)\Gamma\right)$$
$$\leq C[\phi(t_k)]^{n_1-2} \exp\left\{-\frac{t_k \phi^2(t_k)\Gamma^2}{2\lambda t_{k+1}}\right\}$$

where λ equals the maximal eigenvalue of the covariance of μ and the last inequality follows by applying [11]. Using (2.19) and arguing as in (2.13) we then have

$$(2.20) \qquad \sum_k P(\sup_{t_k \leq t \leq t_{k+1}} ||W(t)|| > t_k^{\frac{1}{2}}\phi(t_k)\Gamma) < \infty$$

since (2.18) converges and $\Gamma^2 = \lambda$ as proved in Lemma 2.2. Hence $\phi \in \mathscr{U}$ and the proof is complete.

3. A result for Brownian motion in $\mathbb{R}^d$. The introduction of the equivalent norm in Theorem 2.3 seems somewhat unnatural, but here we show that norms other than Euclidean norms yield considerably different upper and lower functions even for Brownian motion in $\mathbb{R}^d$. In fact, Theorem 3.1 provides a complete description of the upper and lower functions when $j^* = d$ (the definition of j^* appears below), but when $j^* < d$ it is less complete. It seems reasonable to conjecture that $j^* < d$ and the integral in (3.3) finite implies that the function ϕ is an upper function for the norm under consideration, but this is not the case. For example, the polygonal norms appearing in Theorem 4.1 provide examples when it is possible to have $j^* < d$ and the integral in (3.3) finite, yet still have the function ϕ a lower function.

Let $\{W(t) : 0 \leq t < \infty\}$ be Brownian motion in $\mathbb{R}^d$ and assume $||\cdot||_2$ denotes the usual Euclidean norm on $\mathbb{R}^d$. Let $||\cdot||$ denote a second norm on $\mathbb{R}^d$ and define

$$(3.1) \qquad I = S_{d-1} \cap \{x \in \mathbb{R}^d : ||x|| = \Gamma\}$$

where $S_{d-1} = \{x \in \mathbb{R}^d : ||x||_2 = 1\}$ and $\Gamma = \sup_{x \in S_{d-1}} ||x||$. Let $\mathscr{S}_j$ denote the collection of all j dimensional spheres in S_{d-1} for $j = 0, 1, \cdots, d-1$, and let $m_{j,S}$ denote the measure on any j-dimensional sphere S corresponding to surface area on S. In case $j = 0$ the sphere consists of two points and we assume mass one at each point. Let

$$(3.2) \qquad j^* = 1 + \max\{j : m_{j,S}(I \cap S) > 0 \text{ for some } S \in \mathscr{S}_j\} .$$

Since $||\cdot||$ is a norm on $\mathbb{R}^d$ and all norms on $\mathbb{R}^d$ are equivalent I is a nonempty closed subset of S_{d-1}. Thus $m_{j,S}(I \cap S) > 0$ for some j $(0 \leq j \leq d-1)$ and $S \in \mathscr{S}$, and hence j^* satisfies $1 \leq j^* \leq d$. We call j^* *the dimension of the norm* $||\cdot||$.

THEOREM 3.1. *Let $\{W(t) : 0 \leq t < \infty\}$ be standard Brownian motion in $\mathbb{R}^d$ and assume $||\cdot||$ is a norm on $\mathbb{R}^d$ with dimension j^*. Let $\phi(t)$ be a nonnegative, nondecreasing, continuous function defined for large values of t. Then $\phi \in \mathscr{L}$ relative* *to* $\|\cdot\|$ *if*

$$\int^{\infty} \frac{[\phi(t)]^{j^*}}{t} e^{-\phi^2(t)/2}\, dt = \infty . \tag{3.3}$$

In the case $j^* = d$, *then* ϕ *is in* $\mathscr{U}(\mathscr{L})$ *relative to* $\|\cdot\|$ iff

$$\int^{\infty} \frac{[\phi(t)]^{d}}{t} e^{-\phi^2(t)/2}\, dt < \infty \; (= \infty) . \tag{3.4}$$

PROOF. Let I be defined as in (3.1) with $\Gamma = \sup_{x \in S_{d-1}} \|x\|$. Using the argument of [10] page 151 it suffices to prove the theorem when $\phi(t)$ satisfies

$$(\log\log t)^{\frac{1}{2}} \leqq \phi(t) \leqq 2(\log\log t)^{\frac{1}{2}} \tag{3.5}$$

for all sufficiently large t. Hence we make this assumption on ϕ throughout the proof.

Let $j^* = d$ and assume (3.4) converges. Since $U = \{x \in \mathbb{R}^d : \|x\| \leqq \Gamma\} \supseteqq \{x : \|x\|_2 \leqq 1\}$ we have

$$\begin{aligned} &P(\|W(t)\| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ for an unbounded set of } t\text{'s}) \\ &\qquad \leqq P(\|W(t)\|_2 > t^{\frac{1}{2}}\phi(t) \text{ for an unbounded set of } t\text{'s}) \\ &\qquad = 0 \end{aligned} \tag{3.6}$$

where the last equality follows from Theorem 2.4 or by ([9] Theorem 2.2). Thus $\phi \in \mathscr{U}$ with respect to $\|\cdot\|$.

Now assume $j^* \leqq d$ and (3.3) diverges. Let I be defined as in (3.1) and assume $S \in S_{j^*-1}$ is such that $m_{j^*-1,S}(I \cap S) > 0$. Let π_S denote the projection onto the subspace generated by S. Then

$$\begin{aligned} &P(\|W(t)\| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ for an unbounded set of } t\text{'s}) \\ &\qquad \geqq P\Big(\|W(t)\| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ and } \frac{\pi_S(W(t))}{\|\pi_S W(t)\|_2} \in I \cap S \\ &\qquad\qquad \text{for an unbounded set of } t\text{'s}\Big) \\ &\qquad \geqq P\Big(\|\pi_S W(t)\|_2 > t^{\frac{1}{2}}\phi(t) \text{ and } \frac{\pi_S(W(t))}{\|\pi_S W(t)\|_2} \in I \cap S \\ &\qquad\qquad \text{for an unbounded set of } t\text{'s}\Big) \end{aligned} \tag{3.7}$$

since $\{x \in \mathbb{R}^d : \|x\| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ and } \pi_S x/\|\pi_S x\|_2 \in I \cap S\} \supseteqq \{x \in \mathbb{R}^d : \|\pi_S x\|_2 > t^{\frac{1}{2}}\phi(t)$ and $\pi_S x/\|\pi_S x\|_2 \in I \cap S\}$.

Since $\{\pi_S W(t) : 0 \leqq t < \infty\}$ is j^*-dimensional Brownian motion we need only prove that (3.3) implies the last member in (3.7) equals one. To do this we can, without loss of generality, assume $j^* = d$, $S = S_{d-1}$, and that I has positive surface measure in S_{d-1}. Under these assumptions π_S is the identity map so it suffices to prove that

$$\int^{\infty} \frac{[\phi(t)]^{d}}{t} e^{-\phi^2(t)/2}\, dt = \infty \tag{3.8}$$

implies

$$(3.9)\qquad P\left(\|W(t)\|_2 > t^{\frac{1}{2}}\phi(t) \text{ and } \frac{W(t)}{\|W(t)\|_2} \in I \text{ for an unbounded set of } t\text{'s}\right) = 1$$

when I has positive surface measure on S_{d-1}.

Let $t_{p,k} = 2^p + 2^p k/[\log p]$ $(p = 3, 4, \cdots)$ where $[\log p]$ denotes the greatest integer in $\log p$ and $[\log p/2] \leqq k \leqq [\log p]$. Let

$$(3.10)\qquad E_{p,k} = \left\{\|W(t_{p,k})\|_2 > (t_{p,k})^{\frac{1}{2}}\phi(t_{p,k}) \text{ and } \frac{W(t_{p,k})}{\|W(t_{p,k})\|_2} \in I\right\}.$$

We order the $E_{p,k}$'s by saying

$$(p', k') < (p, k)$$

if

(a) $p' < p$ or
(b) $p' = p$ and $k' < k$.

We enumerate these events with respect to this ordering and denote the resulting sequence by $\{E_n\}$.

The theorem will be proved if we show that (3.8) implies that $P(E_n \text{ i.o.}) = 1$. This can be shown in the usual manner by applying the Chung–Erdös lemma [1] to the events $\{E_n\}$. The reader may consult [9] Theorem 2.2 as a guide to the analysis, so we omit further details here. We do, however, emphasize that Lemma 1.5 and Lemma 1.6 of [9] must be modified slightly for the current situation. For example, the form of Lemma 1.6 needed in this setup is that if U and V are Gaussian random variables in $\mathbb{R}^d$ with mean zero, identity covariance matrix, and such that

$$(3.11)\qquad E(U_i V_j) = \rho\delta_{ij},$$

then there is a positive constant C, independent of ρ, such that for all $a \geqq 0$

$$(3.12)\qquad P\left(\|U\|_2 > a, \frac{U}{\|U\|_2} \in I, \|V\|_2 > a, \frac{V}{\|V\|_2} \in I\right)$$
$$\leqq C\exp[-(1-\rho^2)a^2/8]P\left\{\|U\|_2 > a, \frac{U}{\|U\|_2} \in I\right\}.$$

The proof of (3.12) follows by the argument given for Lemma 1.6 in [9] so we omit the details.

4. Generalized polygonal norms on $\mathbb{R}^2$. The results of this section provide examples of norms other than Euclidean norms for which precise integral tests can be obtained. Furthermore, they show that if $j^* < d$ and the integral in (3.3) is finite, then ϕ need not be an upper function.

Recall that a closed half-space in $\mathbb{R}^2$ is a set of the form $\{x \in \mathbb{R}^2 : f(x) \leqq c\}$ where c is a real number and f is some linear functional on $\mathbb{R}^2$.

We now turn our attention to those norms $\|\cdot\|$ on $\mathbb{R}^2$ such that if I is given as in (3.1), then

$$U = \{x \in \mathbb{R}^2 : ||x|| \leqq \Gamma\} = \textstyle\bigcap_{i \in I} H_i \tag{4.1}$$

where H_i is a closed half-space of the form $\{x : f_i(x) \leqq c_i\}$, and the line $\{x : f_i(x) = c_i\}$ is tangent to S_1 at the point $i \in I$. Such norms will be called *generalized polygonal norms*, and we will denote the class of all such norms with $\sigma(I) = 0$ (σ is the measure corresponding to arc length on S_1) by $\mathscr{P}$. The restriction of our investigation to the case $\sigma(I) = 0$ is natural since $\sigma(I) > 0$ implies Theorem 3.1 applies with $j^* = 2$.

The set I is a closed symmetric subset of S_1 and represents the points of tangency of U and S_1. We will say that I consists of *the points of tangency of the norm* $||\cdot||$.

On the other hand, it is not difficult to see that if I is any closed symmetric subset of S_1, then we can construct a tangent line at each of the points of I and the intersection of the related closed half-spaces containing S_1 will be a closed, symmetric, absorbing, convex set, call it U_I. Now if I generates a subspace equal to $\mathbb{R}^2$ then U_I is the unit ball of the norm

$$||x||_I = \min\{\lambda > 0 : x \in \lambda U_I\}, \tag{4.2}$$

and $||\cdot||_I \in \mathscr{P}$ provided $\sigma(I) = 0$. Further, $||\cdot||_I$ has as its points of tangency the set I. It might be well to point out that if I is a subset of a one dimensional subspace of $\mathbb{R}^2$ (iff I contains exactly two points) then the right-hand side of (4.2) defines a semi-norm but not a norm on $\mathbb{R}^2$.

Suppose $||\cdot|| \in \mathscr{P}$ has the closed set I as its points of tangency and U is as in (4.1). Then

$$E_i = U \cap \{x : f_i(x) = c_i\} \qquad (i \in I) \tag{4.3}$$

is called a *face of the ball* U, and its *contraction to* S_1 is the set

$$\hat{E}_i = \{x \in S_1 : \lambda x \in E_i \quad \text{for some} \quad \lambda \geqq 1\}. \tag{4.4}$$

Now $S_1 - I$ is the union of countably many disjoint connected open sets which, in view of the natural mapping between S_1 and $[0, 2\pi)$, we will call intervals. Each of these intervals is a subset of the contraction of two faces of U, and hence $S_1 - I = \bigcup_{k \geqq 1} J_k$ where each interval J_k is the contraction of a subset of a *single face* of U. If J_k is denoted by $\{\theta_k, \theta_k + \delta_k\}$ where $0 \leqq \theta_k < \theta_k + \delta_k \leqq 2\pi$ then either θ_k or $\theta_k + \delta_k$ is in I (but not both), and the sequence of pairs $\theta(\delta) = \{(\theta_k, \delta_k)\}$ is called the *partition of* S_1 *induced by* $||\cdot||$.

THEOREM 4.1. *Let $||\cdot||$ be a generalized polygonal norm in $\mathscr{P}$ and suppose $\theta(\delta) = \{(\theta_k, \delta_k)\}$ is the partition of S_1 induced by $||\cdot||$. If ϕ is a non-decreasing, nonnegative, continuous function defined for large t, then $\phi \in \mathscr{U}(\mathscr{L})$ with respect to $||\cdot||$* iff

$$\int^{\infty} \frac{\phi(t)}{t} g(t) e^{-\phi^2(t)/2} \, dt < \infty \; (= \infty) \tag{4.5}$$

where

$$g(t) = \sum_{k\geq 1} \int_0^{\delta_k \phi(t)} e^{-y^2/2}\, dy\,. \tag{4.6}$$

We write $f(t) \approx h(t)$ if there are positive constants c_1 and c_2 such that $c_1 h(t) \leqq f(t) \leqq c_2 h(t)$ for all t sufficiently large. A similar notation applies to sequences.

The following Corollary is now immediate from Theorem 4.1.

COROLLARY 4.2. *Assume the setup in Theorem* 4.1. *Then*

(a) *If the points of tangency of the norm* $\|\cdot\|$ *consist of a finite set in* S_1, *then* $g(t) \sim c > 0$ *and* $\phi \in \mathscr{U}(\mathscr{L})$ *with respect to* $\|\cdot\|$ iff

$$\int^\infty \frac{\phi(t)}{t} e^{-\phi^2(t)/2}\, dt < \infty\ (= \infty)\,. \tag{4.7}$$

(b) *If* $\delta_k \approx x^k$ *for some* $x \in (0, 1)$ *then* $g(t) \approx \log \phi(t)$ *and* $\phi \in \mathscr{U}(\mathscr{L})$ *with respect to* $\|\cdot\|$ iff

$$\int^\infty \frac{\phi(t)}{t} \log \phi(t) e^{-\phi^2(t)/2}\, dt < \infty\ (= \infty)\,. \tag{4.8}$$

PROOF OF THEOREM 4.1. Using the argument of [10] page 151 it suffices to prove the theorem when $\phi(t)$ satisfies

$$(\log \log t)^{\frac{1}{2}} \leqq \phi(t) \leqq 2(\log\log t)^{\frac{1}{2}} \tag{4.9}$$

for all sufficiently large t. Hence we make this assumption on ϕ throughout the proof.

First assume (4.5) converges where $g(t)$ is as given in (4.6). Since ϕ is non-decreasing we have

$$\begin{aligned} &P(\|W(t)\| > t^{\frac{1}{2}}\phi(t)\Gamma \text{ for an unbounded set of } t\text{'s}) \\ &\qquad \leqq P(\sup_{t_j \leq t \leq t_{j+1}} \|W(t)\| > t_j^{\frac{1}{2}}\phi(t_j)\Gamma \text{ for infinitely many } j) \end{aligned} \tag{4.10}$$

where $\{t_j\}$ is any increasing sequence converging to infinity. For our purpose we choose $\{t_j\}$ such that

$$t_{j+1} = t_j(1 + 1/\phi^2(t_j)) \tag{4.11}$$

where $t_1 > 3$ is sufficiently large so that (4.9) holds for $t > t_1$ and hence $\lim_j t_j = \infty$.

Now observe that by an extension of Lévy's inequality and the scaling property

$$P(\sup_{t_j \leq t \leq t_{j+1}} \|W(t)\| > t_j^{\frac{1}{2}}\phi(t_j)\Gamma) \leqq 2P\left(\|W(1)\| > \left(\frac{t_j}{t_{j+1}}\right)^{\frac{1}{2}} \phi(t_j)\Gamma\right), \tag{4.12}$$

and using the fact that $\|\cdot\| \in \mathscr{P}$ and that (4.5) converges we next prove

$$\sum_j P\left(\|W(1)\| > \left(\frac{t_j}{t_{j+1}}\right)^{\frac{1}{2}} \phi(t_j)\Gamma\right) < \infty\,. \tag{4.13}$$

Fix $\lambda > 0$. Let $\theta(\delta) = \{(\theta_k, \delta_k)\}$ denote the partition of S_1 induced by $\|\cdot\|$ and assume $\{J_k : k \geqq 1\}$ are the corresponding intervals as described following (4.4).

For each $k \geqq 1$ let E_k' denote the subset of a single face of U such that J_k is the contraction of E_k' to S_1. Then, since $\sigma(I) = 0$, we have

$$P(\|W(1)\| > \lambda\Gamma) = \textstyle\sum_{k\geqq 1} P(W(1) \in E_k'(\lambda)) = \sum_{k\geqq 1} P(W(1) \in F_k'(\lambda)) \tag{4.14}$$

where for any subset A of R^2 we write $A(\lambda)$ to denote $\bigcup_{r\geqq\lambda} rA$ and F_k' is the image of E_k' under a rotation of R^2 which takes $J_k = \{\theta_k, \theta_k + \delta_k\}$ to $\{0, \delta_k\}$. Now

$$\begin{aligned} P(W(1) \in F_k'(\lambda)) &= \frac{1}{2\pi} \textstyle\int_0^{\delta_k} \int_{\lambda \sec\theta}^{\infty} re^{-r^2/2}\, dr\, d\theta \\ &= \frac{1}{2\pi} \textstyle\int_0^{\delta_k} e^{-(\lambda^2/2)\sec^2\theta}\, d\theta \\ &= \frac{e^{-\lambda^2/2}}{\lambda 2\pi} \textstyle\int_0^{\delta_k} \lambda e^{-\lambda^2 \tan^2\theta/2}\, d\theta \\ &\sim \frac{e^{-\lambda^2/2}}{2\pi\lambda} \textstyle\int_0^{\delta_k} \lambda e^{-\lambda^2\theta^2/2}\, d\theta \qquad \text{as } k \to \infty \text{ since } \delta_k \to 0 \\ &= \frac{e^{-\lambda^2/2}}{2\pi\lambda} \textstyle\int_0^{\lambda\delta_k} e^{-y^2/2}\, dy\,. \end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15) we see

$$P(\|W(1)\| > \lambda\Gamma) \approx \frac{e^{-\lambda^2/2}}{\lambda} \textstyle\sum_{k\geqq 1} \int_0^{\lambda\delta_k} e^{-y^2/2}\, dy\,. \tag{4.16}$$

Hence from (4.16) we see

$$\begin{aligned} &P\left(\|W(1)\| > \left(\frac{t_j}{t_{j+1}}\right)^{\frac{1}{2}} \phi(t_j)\Gamma\right) \\ &\quad\approx \frac{e^{-(t_j/t_{j+1})(\phi^2(t_j)/2)}}{(t_j/t_{j+1})\phi(t_j)} \textstyle\sum_{k\geqq 1} \int_0^{(t_j/t_{j+1})^{\frac{1}{2}}\phi(t_j)\delta_k} e^{-y^2/2}\, dy \\ &\quad\approx \frac{e^{-\phi^2(t_j)/2}}{\phi(t_j)} g(t_j) \quad \text{since} \quad 1 > \left(\frac{t_j}{t_{j+1}}\right)^{\frac{1}{2}} > \left(1 - \frac{1}{\phi^2(t_j)}\right)^{\frac{1}{2}} > 1 - \frac{1}{\phi(t_j)} \\ &\quad\leqq \textstyle\int_{t_{j-1}}^{t_j} \frac{e^{-\phi^2(t_j)/2}}{\phi(t_j)} g(t_j) \frac{t_j}{s(t_j - t_{j-1})}\, ds \\ &\quad\leqq C \textstyle\int_{t_{j-1}}^{t_j} \frac{e^{-\phi^2(t_j)/2}}{\phi(t_j)} g(t_j) \frac{\phi^2(t_{j-1})}{s} \frac{t_j}{t_{j-1}}\, ds \\ &\quad\leqq C \textstyle\int_{t_{j-1}}^{t_j} \frac{\phi(s)}{s} g(s) e^{-\phi^2(s)/2}\, ds \end{aligned} \tag{4.17}$$

by using (4.9), (4.11), and that $g(s)e^{-\phi^2(s)/2}$ is eventually decreasing in s. Therefore (4.5) converging and (4.17) imply (4.13) and hence $\phi \in \mathscr{U}$.

If (4.5) diverges we define events

$$E_{p,k} = \{\|W(t_{p,k})\| > (t_{p,k})^{\frac{1}{2}}\phi(t_{p,k})\Gamma\} \tag{4.18}$$

where $t_{p,k} = 2^p(1 + k/[\log p])$ $(p = 3, 4, \cdots)$ and $[\log p/2] \leqq k \leqq [\log p]$. Ordering the $E_{p,k}$'s as in the proof of Theorem 3 we obtain a sequence of events $\{E_n\}$ to which we apply the Chung-Erdös lemma [1] obtaining

$$P(E_n \text{ i.o.}) = 1 . \tag{4.19}$$

The proof of (4.19) uses the estimates of (4.17) and follows the pattern of Theorem 2.2 in [9]. However, here one needs the estimates obtained in Lemmas 1.5 and 1.6 of [9] as they apply to the polygonal norm $||\cdot||$.

Now Lemma 1.5 can be established for the norm $||\cdot||$ by exactly the same argument used in [9], but Lemma 1.6 requires a slightly different argument and it is this we turn to now. After this fact is established, however, the proof is exactly as before so we omit the additional details. Hence (4.5) diverging implies $\phi \in \mathscr{L}$ and the proof is complete.

The analogue of Lemma 1.6 of [9] is:

LEMMA 4.3. *Let $||\cdot||$ be a generalized polygonal norm in $\mathscr{P}$ and Γ be as in* (1.2). *If U and V are Gaussian random variables in $\mathbb{R}^2$ with mean zero, identity covariance matrix, and such that*

$$E(U_i V_j) = \rho\delta_{ij} \qquad (i, j = 1, 2),$$

then there is a positive constant c, independent of ρ, such that for all $a \geqq 0$

$$P(||U|| > a, ||V|| > a) \leqq c \exp\left\{\frac{-(1-\rho^2)a^2}{8\Gamma^2}\right\} P(||U|| > a) . \tag{4.20}$$

PROOF. First of all (4.20) is obvious if $(1 - \rho^2)a^2$ is small so assume the contrary. Next observe that

$$P(||U|| > a, ||V|| > a) = 2P(a < ||U|| < ||V||) . \tag{4.21}$$

Now

$$\begin{aligned} J &= P(a < ||U|| < ||V||) \\ &= \frac{c}{1-\rho^2} \iint_{a<||u||<||v||} \exp\left\{-\frac{1}{2(1-\rho^2)}[|u|^2 - 2\rho(u, v) + |v|^2]\right\} dv\, du \end{aligned}$$

where $|u^2| = u_1^2 + u_2^2$ and $(u, v) = u_1v_1 + u_2v_2$. Hence

$$\begin{aligned} J = &\frac{c}{1-\rho^2} \int_{a\leqq||u||} \exp\{-\tfrac{1}{2}|u^2|\} \\ &\times \iint_{||u||\leqq||(s+\rho u_1, t+\rho u_2)||} \exp\left\{\frac{-1}{2(1-\rho^2)}(s^2 + t^2)\right\} ds\, dt\, du \end{aligned}$$

where $s = v_1 - \rho u_1$, $t = v_2 - \rho u_2$

$$\begin{aligned} \leqq &\frac{C}{1-\rho^2} \int_{a<||u||} \exp\{-\tfrac{1}{2}|u|^2\} \\ &\times \iint_{\{(s,t):\, ||(s,t)||>(1-|\rho|)||u||\}} \exp\left\{-\frac{1}{2(1-\rho^2)}(s^2 + t^2)\right\} ds\, dt\, du \end{aligned}$$

$$\leqq C \int_{a<\|u\|} \exp\{-\tfrac{1}{2}|u|^2\} \times \iint_{\{(\omega_1,\omega_2):\, \|(\omega_1,\omega_2)\|>(1-|\rho|)\|u\|/(1-\rho^2)^{\frac{1}{2}}\}} \exp\{-\tfrac{1}{2}(\omega_1^2+\omega_2^2)\}\, d\omega_1\, d\omega_2\, du$$

$$\leqq C \int_{a<\|u\|} \exp\{-\tfrac{1}{2}|u|^2\} \exp\left\{-\frac{(1-|\rho|)^2\|u\|^2}{2(1-\rho^2)\Gamma^2}\right\} du$$

by arguments used in (4.17) since $(1-\rho^2)a^2$ large implies $(1-|\rho|)\|u\| > (1-|\rho|)a$ is also large

$$\leqq C \exp\left\{-\frac{(1-\rho^2)a^2}{2(1+|\rho|)^2\Gamma^2}\right\} P(\|U\| > a)\,.$$

REMARK 4.4. It is obvious, of course, that if the partition $\theta(\delta) = \{(\theta_k, \delta_k)\}$ of S_1 induced by a norm $\|\cdot\|$ in $\mathscr{P}$ satisfies other regular rates of decay such as $\delta_k = 1/k^2$ or $\delta_k = 1/k^p$ $(p > 1)$ etc., then other nice functions $g(t)$ arise in describing the upper and lower functions relative to $\|\cdot\|$. We can also pursue the concept of generalized polygonal norms in $\mathbb{R}^d$ and obtain results similar to those in Theorem 4.1 under some additional conditions on the faces generated by the polygonal norm.

REFERENCES

[1] CHUNG, K. L. and ERDÖS, P. (1952). On the application of the Borel-Cantelli lemma. *Trans. Amer. Math. Soc.* **72** 179-186.

[2] GROSS, L. (1967). Potential theory, on Hilbert space. *J. Functional Analysis* **1** 123-181.

[3] GROSS, L. (1970). Lectures in modern analysis and applications II **140**, *Lecture Notes in Mathematics*, Springer-Verlag, New York.

[4] KUELBS, J. (1970). Gaussian measures on Banach spaces. *J. Functional Analysis* **5** 354-367.

[5] KUELBS, J. (1973). Some results for probability measures on linear topological vector spaces with an application to Strassen's log log law. *J. Functional Analysis* **14** 28-43.

[6] KUELBS, J. and LEPAGE, R. (1973). The law of the iterated logarithm for Brownian motion in a Banach space. *Trans. Amer. Math. Soc.* **185** 253-264.

[7] MARCUS, M. B. and SHEPP, L. A. (1972). Sample behavior of Gaussian processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press, Berkeley, 423-439.

[8] NISIDA, T. and SIRAO, T. (1952). On some asymptotic properties concerning Brownian motion. *Nagoya Math. J.* **4** 97-101.

[9] OREY, S. and PRUITT, W. E. (1973). Sample functions of the N-parameter Wiener process. *Ann. Probability* **1** 138-163.

[10] SIRAO, T. (1960). On the continuity of Brownian motion with a multidimensional parameter. *Nagoya Math. J.* **16** 135-156.

[11] ZOLOTAREV, V. M. (1961). Concerning a certain probability problem. *Theor. Probability Appl.* **6** 201-204.

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