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A TWO-DIMENSIONAL FUNCTIONAL PERMUTATIONAL CENTRAL LIMIT THEOREM FOR LINEAR RANK STATISTICS[1]

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Some two-dimensional time-parameter stochastic processes are constructed from a sequence of linear rank statistics by considering their projections on the spaces generated by the (double) sequence of anti-ranks. Under appropriate regularity conditions, it is shown that these processes weakly converge to Brownian sheets in the Skorokhod J_1-topology on the $D^2[0, 1]$ space. This unifies and extends earlier one-dimensional invariance principles for linear rank statistics to the two-dimensional case. The case of contiguous alternatives is treated briefly.

1. Introduction. Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d. rv) with a continuous distribution function (df) F on $(-\infty, \infty)$. Consider the *linear rank statistics*

$$T_N = \sum_{i=1}^N (c_i - \bar{c}_N)a_N(R_{Ni}), \qquad \bar{c}_N = N^{-1} \sum_{i=1}^N c_i \qquad N \geqq 1;\ T_0 = 0, \tag{1.1}$$

where $\{c_i, i \geqq 1\}$ is a sequence of (known) constants, $\{[a_N(1), \cdots, a_N(N)], N \geqq 1\}$ is a (triangular) sequence of real scores, and the ranks $\{R_{Ni}\}$ are defined by

$$R_{Ni} = \sum_{j=1}^N u(X_i - X_j), \quad 1 \leqq i \leqq N, \qquad \text{where} \quad u(t) = 1, \quad t \geqq 0; \quad = 0, \quad t < 0. \tag{1.2}$$

Note that $\mathbf{R}_N = (R_{N1}, \cdots, R_{NN})$ assumes all possible permutations of $(1, \cdots, N)$ with the common probability $(N!)^{-1}$, and by the classical Wald–Wolfowitz–Noether–Hoeffding–Hajék permutational central limit theorem (PCLT) [viz, Theorems 6.1 and 6.2 of Hajék (1961)], under suitable regularity conditions,

$$\mathscr{L}(A_N^{-1}C_N^{-1}T_N) \to \mathscr{N}(0, 1) \qquad \text{as} \quad N \to \infty, \tag{1.3}$$

where $C_1 = A_1 = 0$, and for $N \geqq 2$,

$$C_N^2 = \sum_{i=1}^N (c_i - \bar{c}_N)^2, \qquad A_N^2 = \frac{1}{N-1} \sum_{i=1}^N [a_N(i) - \bar{a}_N]^2 \quad \text{and} \quad \bar{a}_N = \frac{1}{N} \sum_{i=1}^N a_N(i). \tag{1.4}$$

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For $N \geqq 2$, consider a stochastic process $W_N^{(1)} = \{W_N^{(1)}(t) : t \in I\}$, $I = [0, 1]$, where

$$(1.5) \qquad \begin{aligned} W_N^{(1)}(t) &= A_N^{-1}C_N^{-1}T_{n(t)} \quad \text{and} \\ n(t) &= \max\{k : C_k^2 \leqq tC_N^2\} \qquad t \in I. \end{aligned}$$

From Sen and Ghosh (1972), it follows that as $N \to \infty$, $W_N^{(1)}$ converges in law to a standard Brownian motion on I; this provides the first functional PCLT for linear rank statistics. In the context of progressively censored nonparametric tests for the life-testing problem, Chatterjee and Sen (1973) have formulated a second functional PCLT which may be posed as follows. For every $N \geqq 1$, let $T_{N,0} = 0$, and

$$(1.6) \qquad \begin{aligned} T_{N,r} &= \textstyle\sum_{i=1}^{r} (c_{S_{Ni}} - \bar{c}_N)[a_N(i) - a_N^*(r)], \quad 1 \leqq r \leqq N-2, \quad \text{and} \\ &= T_N, \quad r\ N-1, N, \end{aligned}$$

where $\mathbf{S}_N = (S_{N1}, \cdots, S_{NN})$ is the vector of antiranks, that is

$$(1.7) \qquad X_{S_{Ni}} = X_{N,i}, \quad 1 \leqq i \leqq N \quad \text{or} \quad R_{NS_{Ni}} = S_{NR_{Ni}}, \quad 1 \leqq i \leqq N,$$

where $X_{N,1} < \cdots < X_{N,N}$ are the order statistics corresponding to $X_1, \cdots, X_N$, and

$$(1.8) \qquad \begin{aligned} a_N^*(r) &= (N-r)^{-1} \textstyle\sum_{j=r+1}^{N} a_N(j), \quad 1 \leqq r \leqq N-1; \\ &= 0 \quad \text{for} \quad r = N. \end{aligned}$$

Thus, in the usual terminology, $T_{N,r}$ is a *censored linear rank statistic* where the censoring occurs at the rth order statistic $X_{N,r}$, $r = 1, \cdots, N$. For $N \geqq 1$, let $V_{N,0} = 0$, and define

$$(1.9) \qquad \begin{aligned} V_{N,r} &= A_N^2 - (N-1)^{-1} \textstyle\sum_{j=r+1}^{N} [a_N(j) - a_N^*(r)]^2, \quad 1 \leqq r \leqq N-2; \\ &= A_N^2, \quad r = N-1, N. \end{aligned}$$

On denoting by $\mathbf{S}_N(r) = (S_{N1}, \cdots, S_{Nr})$, $1 \leqq r \leqq N$, and by $\mathscr{B}_N^{(r)}$, the σ-field generated by $\mathbf{S}_N(r)$, we observe that $E[T_N \mid \mathscr{B}_N^{(r)}] = T_{N,r}$ and $V(T_{N,r}) = V_{N,r}$, $1 \leqq r \leqq N$. Consider the process $W_N^{(2)} = \{W_N^{(2)}(t) : t \in I\}$, where

$$(1.10) \qquad \begin{aligned} W_N^{(2)}(t) &= A_N^{-1}C_N^{-1}T_{N,r(t)} \quad \text{and} \\ r(t) &= \max\{r : V_{N,r} \leqq tA_N^2\} \qquad t \in I. \end{aligned}$$

Then, $W_N^{(2)}$ converges in distribution to a standard Brownian motion on I, and this provides the second functional PCLT for linear rank statistics. [Actually, in Sen et al. (1972, 1973), these processes were defined on the $C[0, 1]$ space with slightly different definitions; however, their results remain good on the $D[0, 1]$ space with the current definitions.]

In the current paper, a two-dimensional time-parameter process is constructed on the $D^2[0, 1]$ space, such that its vertical and horizontal cross sections are $W_N^{(1)}$ and $W_N^{(2)}$. Under suitable regularity conditions, it is shown that the proposed process converges weakly to a standard Brownian sheet on the unit square; this extends the earlier results of Sen et al. (1972, 1973) to the two-dimensional case. The theorem is formulated in Section 2 and its proof is presented in Section 3. The last section includes an extension of the theorem for contiguous alternatives.

2. The basic theorem. Let $I^2 = \{\mathbf{t} = (t_1, t_2) : \mathbf{0} \leqq \mathbf{t} \leqq \mathbf{1}\}$ be the unit square, where $\mathbf{a} \leqq \mathbf{b}$ means that $a_i \leqq b_i$, $i = 1, 2$. For every $N (\geqq 2)$, consider a two-parameter stochastic process $W_N = \{W_N(\mathbf{t}) : \mathbf{t} \in I^2\}$, where

$$W_N(\mathbf{t}) = A_N^{-1} C_N^{-1} T_{n(t_1), r(t_2)} \qquad \mathbf{t} \in I^2, \tag{2.1}$$

$$n(t_1) = \max\{n : C_N^{-2} C_n^2 \leqq t_1\} \quad \text{and} \tag{2.2}$$

$$r(t_2) = \max\{r : V^2_{n(t_1), r} \leqq t_2 A^2_{n(t_1)}\},$$

and A_N^2, C_N^2 and $V_{N,r}$ are defined by (1.4) and (1.9). Let $D^2[0, 1]$ be the space of all real functions on I^2 with no discontinuity of the second kind, and associate with it the Skorokhod J_1-topology as in Neuhaus (1971). Then $W_N \in D^2[0, 1]$, $\forall N \geqq 2$. Also, note that the space $C^2[0, 1]$ of all real, continuous functions on I_2 is a subspace of $D^2[0, 1]$. We introduce the following assumptions needed for our subsequent results:

ASSUMPTION 1(a). $\{c_i, i \geqq 1\}$ satisfy the Noether-condition, viz,

$$\lim_{N\to\infty} \{C_N^{-2}[\max_{1\leqq i\leqq N} (c_i - \bar{c}_N)^2]\} = 0 ; \tag{2.3}$$

in fact, we will need the more restrictive condition:

ASSUMPTION 1(b). $\{c_i, i \geqq 1\}$ satisfy the Hájek (1968) condition, i.e.,

$$\{[\max_{1\leqq i\leqq N} N(c_i - \bar{c}_N)^2]/C_N^2\} = O(1) . \tag{2.4}$$

ASSUMPTION 2(a). For every $N (\geqq 1)$, the scores are defined by

$$a_N(i) = E\phi(U_{Ni}) , \quad 1 \leqq i \leqq N , \qquad \phi(u) = \phi_1(u) - \phi_2(u) \qquad u \in I , \tag{2.5}$$

where $U_{N1} < \cdots < U_{NN}$ are the ordered random variables of a sample of size N from the rectangular (0, 1) df, and both ϕ_1, ϕ_2 are nondecreasing and square-integrable inside I.

ASSUMPTION 2(b). [Hoeffding (1973).] For some $r > 1$,

$$\int_0^1 [\phi_j(u)]^2 [\log(1 + |\phi_j(u)|)]^r \, du < \infty \qquad j = 1, 2 . \tag{2.6}$$

Note that (2.6) is less restrictive than $\phi_j \in L_r$, $r > 2$, for $j = 1, 2$.

Note that if $\mathscr{F}_n^*$ be the σ-field generated by $\mathbf{R}_n$, $n \geqq 1$, then by (1.7), $\mathscr{F}_n^* = \mathscr{B}_n^{(n)}$. Now, by Lemma 2.1 of Sen and Ghosh (1972), $E(T_{n+1} | \mathscr{F}_n^*) = T_n$, while by Lemma 4.1 of Chatterjee and Sen (1973), $E(T_n | \mathscr{B}_n^{(r)}) = T_{n,r}$, $1 \leqq r \leqq n$. As $\mathscr{B}_n^{(r)}$ is $\nearrow$ in r: $1 \leqq r \leqq n$, from the above two identities, we conclude that under (2.5),

$$E[T_N | \mathscr{B}_n^{(r)}] = T_{n,r} \qquad \forall\, 1 \leqq r \leqq n \leqq N . \tag{2.7}$$

Thus, the process W_N in (2.1) is fabricated by the σ-field generated by $\{\mathbf{S}_n(r) : 1 \leqq r \leqq n \leqq N\}$.

Let $W = \{W(\mathbf{t}) : \mathbf{t} \in I^2\}$ be a standard Brownian sheet on I^2, so that $P\{W \in C^2[0, 1]\} = 1$, all finite dimensional distributions of W are Gaussian, $EW(\mathbf{t}) = 0$, $\forall\, \mathbf{t} \in I^2$, and

$$E\{W(\mathbf{s})W(\mathbf{t})\} = \mathbf{s} \wedge \mathbf{t} = [\min(s_1, t_1)][\min(s_2, t_2)] \qquad \forall\, \mathbf{s}, \mathbf{t} \in I^2 . \tag{2.8}$$

Then, the main result of the paper is the following.

THEOREM 1. *Under Assumptions* 1(b) *and* 2(b), *as* $N \to \infty$,

$$W_N \to_{\mathscr{D}} W , \quad \textit{in the } J_1\textit{-topology on the } D^2[0, 1] \textit{ space.} \tag{2.9}$$

The proof of the theorem along with certain related results is presented in Section 3.

3. The proof of the theorem. For simplicity of manipulations, we may standardize (without any loss of generality) the score function $\phi(u)$, $u \in I$, such that

$$\textstyle\int_0^1 \phi(u)\, du = \bar{\phi} = 0 \qquad \text{and} \qquad A^2 = \int_0^1 \phi^2(u)\, du = 1 . \tag{3.1}$$

Note that, by definition, $C_{k+1}^2 - C_k^2 = [(k+1)/k][c_{k+1} - \bar{c}_{k+1}]^2$, $k \geqq 1$, so that by (2.3),

$$\lim_{N\to\infty} \{C_N^{-2}[\max_{1\leqq k\leqq N-1} (C_{k+1}^2 - C_k^2)]\} = 0 . \tag{3.2}$$

Also, by Assumption 2(a) and Lemma 4.2 of Chatterjee and Sen (1973),

$$\lim_{N\to\infty} \{A_N^{-2}[\max_{1\leqq r\leqq N} (V_{N,r} - V_{N,r-1})]\} = 0 . \tag{3.3}$$

For every $p \in I$, let us define

$$\begin{aligned} \tilde{\phi}_p(u) &= \phi(u) , & 0 < u \leqq p , \\ &= \phi_p^* = (1-p)^{-1} \textstyle\int_p^1 \phi(t)\, dt , & p < u \leqq 1 ; \end{aligned} \tag{3.4}$$

$$\begin{aligned} \nu(p) &= (\textstyle\int_0^1 \tilde{\phi}_p^2(u)\, du - [\int_0^1 \tilde{\phi}_p(u)\, du]^2) = \int_0^1 \tilde{\phi}_p^2(u)\, du - (\int_0^1 \phi(u)\, du)^2 \\ &= \textstyle\int_0^p \phi^2(u)\, du + (1-p)(\phi_p^*)^2 , \quad 0 < p \leqq 1 ; \\ \nu(1) &= A^2 = 1 . \end{aligned} \tag{3.5}$$

It may be noted that $\nu(p)$ is $\uparrow$ in $p\ (\in I)$, and as in Hájek (1961), we have

$$[\lim_{N\to\infty} r/N = p] \Rightarrow [\lim_{N\to\infty} A_N^{-2} V_{N,r} = \nu(p)] \qquad \forall\, p \in I . \tag{3.6}$$

First, consider the convergence of finite dimensional distributions of $\{W_N\}$ to those of W. Define $\tilde{\varphi}_p$ as in (3.4) and let

$$\tilde{T}_n(p) = \textstyle\sum_{i=1}^n (c_i - \bar{c}_n)\tilde{a}_n(R_{ni}; p) , \tag{3.7}$$

$$T_n^*(p) = \textstyle\sum_{i=1}^n (c_i - \bar{c}_n)\tilde{\varphi}_p(U_i) \qquad n \geqq 1,\ p \in I , \tag{3.8}$$

where $\{U_i, i \geqq 1\}$ are i.i.d. rv's with the rectangular (0, 1) df and

$$\tilde{a}_n(i, p) = E\tilde{\varphi}_p(U_{ni}) \qquad 1 \leqq i \leqq n,\ p \in I . \tag{3.9}$$

Then, by Theorem 6.2 of Hájek (1961), for every $p \in I$,

$$E[\tilde{T}_n(p) - T_n^*(p)]^2/C_n^2 \to 0 \qquad \text{as } n \to \infty . \tag{3.10}$$

Also, note that for every r: $1 \leqq r \leqq n-1$,

$$(3.11)\qquad C_n^{-2}E[T_{n,r} - \tilde{T}_n(r/n)]^2 = (n-1)^{-1}\{\textstyle\sum_{i=1}^{r}[a_n(i) - \tilde{a}_n(i, p)]^2 + \sum_{i=r+1}^{n}[a_n^*(r) - \tilde{a}_n(i, p)]^2\},$$

where by (1.8), (2.5), (3.4), (3.9) and Assumption 2(a) along with Theorem a (on page 157) of Hájek and Šidák (1967), it can be easily shown that if $r/n \to p$ (as $n \to \infty$), $p \in I$, then

$$(3.12)\qquad \begin{aligned} &n^{-1}\textstyle\sum_{i=1}^{n}[\tilde{a}_n(i; r/n) - \tilde{\varphi}_{r/n}(i/(n+1))]^2 \to 0,\\ &n^{-1}\{\textstyle\sum_{i=1}^{r}[a_n(i) - \varphi(i/(n+1))]^2 + (n-r)[\varphi_{r/n}^* - a_n^*(r)]^2\} \to 0,\end{aligned}$$

so that

$$(3.13)\qquad [r/n \to p] \Rightarrow C_n^{-1}|T_{n,r} - \tilde{T}_n(r/n)| \to_p 0.$$

Finally, for every $p \in I$, $p' \in I$, as $p' \to p$,

$$(3.14)\qquad E[T_n^*(p) - T_n^*(p')]^2/C_n^2 = \textstyle\int_0^1[\tilde{\varphi}_p(u) - \tilde{\varphi}_{p'}(u)]^2\,du \to 0.$$

Hence, from (3.6) and (3.10) through (3.14), it follows that

$$(3.15)\qquad W_N(t) - W_N^*(t) \to_p 0, \qquad \text{for every } \mathbf{t} \in I,$$

where

$$(3.16)\qquad W_N^*(\mathbf{t}) = C_N^{-1}A_N^{-1}T_{n(t_1)}^*(\nu^{-1}(t_2)) \qquad t \in I^2.$$

Consequently, it suffices to show that for every $m\ (\geqq 1)$ and $\mathbf{t}_1, \cdots, \mathbf{t}_m \in I^2$, as $n \to \infty$,

$$(3.17)\qquad \mathbf{W}_{N,n}^* = [W_N^*(\mathbf{t}_1), \cdots, W_N^*(\mathbf{t}_m)] \to_{\mathscr{D}} [W(\mathbf{t}_1), \cdots, W(\mathbf{t}_m)].$$

Since $\int_0^1 \tilde{\varphi}_p(u)\,du = 0$, $p \in I$ and $\int_0^1 \tilde{\varphi}_p(u)\tilde{\varphi}_{p'}(u)\,du = \int_0^p \varphi^2(u)\,du + (1-p)(\varphi_p^*)^2 = \nu(p)$ for $0 \leqq p \leqq p' \leqq 1$, $EW_N^*(\mathbf{t}) = 0$, $\mathbf{t} \in I^2$ and

$$(3.18)\qquad \begin{aligned} &\mathrm{Cov}[W_N^*(\mathbf{s}), W_N^*(\mathbf{t})]\\ &= E[W_N^*(\mathbf{s})W_N^*(\mathbf{t})]\\ &= C_N^{-2}A_N^{-2}\{\textstyle\sum_{i=1}^{n^*}(c_i - \bar{c}_{n(s_1)})(c_i - \bar{c}_{n(t_1)})E[\tilde{\phi}_{\nu^{-1}(s_2)}(U_i)\tilde{\phi}_{\nu^{-1}(t_2)}(U_i)]\}\\ &\qquad\qquad (n^* = n(s_1) \wedge n(t_1))\\ &= (s_2 \wedge t_2)A_N^{-2}C_N^{-2}\textstyle\sum_{i=1}^{n^*}(c_i - \bar{c}_{n^*})^2\\ &= (s_2 \wedge t_2)A_N^{-2}(C_{n^*}^2C_N^{-2}) \to (s_2 \wedge t_2)(s_1 \wedge t_1) = \mathbf{s} \wedge \mathbf{t}.\end{aligned}$$

Consequently, to prove (3.17), it suffices to show that $\mathbf{W}_{N,m}^*$ has asymptotically a multinormal distribution. For this, note that

$$(3.19)\qquad \begin{aligned}\mathbf{W}_{N,m}^* &= A_N^{-1}C_N^{-1}(\textstyle\sum_{i=1}^{n(t_{j1})}(c_i - \bar{c}_{n(t_{j1})})\tilde{\phi}_{\nu^{-1}(t_{j2})}(U_i),\ 1 \leqq j \leqq m)\\ &= \textstyle\sum_{i=1}^{N}(d_{ij}^{(N)}\tilde{\phi}_{\nu^{-1}(t_{j2})}(U_i),\ 1 \leqq j \leqq m) = \sum_{i=1}^{N}\mathbf{Z}_{N,i}, \qquad \text{say,}\end{aligned}$$

where the $\mathbf{Z}_{N,i}$ are mutually independent vectors, and $d_{ij}^{(N)} = A_N^{-1}C_N^{-1}(c_i - \bar{c}_{n(t_{j1})})$, if $i \leqq n\ (t_{j1})$, and is 0, otherwise. Also, by definition, $E\mathbf{Z}_{N,i} = 0$, $\forall\ 1 \leqq i \leqq N$, and

$$(3.20)\qquad Ch_1[E(\mathbf{Z}_{N,i}\mathbf{Z}_{N,i}')] \leqq \nu(1)Ch_1[((d_{ij}^{(N)}d_{ij'}^{(N)}))_{j,j'=1,\cdots,m}],$$

so that by (2.3) and (3.19), as $N \to \infty$,

$$(3.21) \quad \max_{1\le i\le N}\{\sup_{\lambda\ne 0}[V(\mathbf{Z}_{N,i}\boldsymbol{\lambda})/\boldsymbol{\lambda}'\boldsymbol{\lambda}]\} \leqq \nu(1)\{\max_{1\le i\le N}[Ch_1[((d_{ij}^{(N)}d_{ij'}^{(N)}))_{j,j'=1,\cdots,m}]]\} \to 0 .$$

Hence, as $N \to \infty$, $\mathbf{Z}_{N,i}$, $1 \leqq i \leqq N$, have all infinitesimal elements. So, by virtue of (3.18) and (3.21), to prove (3.17), we only need to show that for every $\varepsilon > 0$,

$$(3.22) \quad \sum_{i=1}^N E\{||\mathbf{Z}_{N,i}||^2 I(||\mathbf{Z}_{N,i}|| > \varepsilon)\} \to 0 \qquad \text{as } N \to \infty ,$$

where $||\mathbf{u}|| = (\mathbf{uu}')^{\frac{1}{2}}$. By (3.19),

$$(3.23) \quad ||\mathbf{Z}_{N,i}||^2 = \sum_{j=1}^m [d_{ij}^{(N)}]^2 \tilde{\phi}^2_{\nu^{-1}(t_{j2})}(U_i) \leqq \{\max_{1\le j\le m}[d_{ij}^{(N)}]^2\}\{\sum_{j=1}^m \tilde{\phi}^2_{\nu^{-1}(t_{j2})}(U_i)\} \qquad 1 \leqq i \leqq N .$$

Let us denote by

$$(3.24) \quad \alpha_N = \max_{1\le i\le N}\max_{1\le j\le m}|d_{ij}^{(N)}| \qquad \text{and} \qquad U_N{}^* = \sum_{j=1}^m \tilde{\phi}^2_{\nu^{-1}(t_{j2})}(U_1) ,$$

so that by (2.3) and (3.18), $\alpha_N \to 0$ as $N \to \infty$, and by (3.22)—(3.23), for every $\varepsilon > 0$,

$$(3.25) \quad \sum_{i=1}^N E\{||\mathbf{Z}_{N,i}||^2 I(|\mathbf{Z}_{N,i}| > \varepsilon)\} \leqq (\sum_{i=1}^N \sum_{j=1}^m [d_{ij}^{(N)}]^2)(E[U^* I(U^* > \varepsilon^2/\alpha_N{}^2)]) \qquad \varepsilon/\alpha_N \to \infty \text{ as } N \to \infty .$$

Also, $\sum_{i=1}^N \sum_{j=1}^m [d_{ij}^{(N)}]^2 \leqq m$, and by Assumption 2, (3.4) and (3.24), $\lim_{K\to\infty} E[U^* I(U^* > K)] = 0$. Consequently, (3.25) implies (3.22), and the proof of the convergence of f.d.d. of $\{W_N{}^0\}$ is complete.

For $q \geqq 1$, for the tightness of D^q-valued processes, Wichura (1969) has considered some interesting inequalities for a multidimensional array of independent random summands. Unfortunately, our $W_N{}^0$ (or W_N) does not involve a two-dimensional array of independent random variables, so that his inequalities are not usable for our purpose. Usually, for D^q-valued processes, some prabability (or moment) inequalities [viz, Billingsley (1968, pages 128–130)] insure tightness. In our case, these require the evaluation of absolute moments (of order greater than 2) of $[W_N{}^0(t_1, t_2) - W_N{}^0(t_1, s_2) - W_N{}^0(s_1, t_2) + W_N{}^0(s_1, s_2)]$ (where $\mathbf{s} \leqq \mathbf{t}$), and these, in turn, require additional regularity conditions on the score function ϕ in (2.5) in order that the moments conform to the desired pattern. In the following approach, we establish *tightness* under Assumptions 1 (b) and 2 (b). Note that W_N vanishes at the lower boundary of I^2. For every $\mathbf{s} \in I^2$, let $B_\delta(s) = \{\mathbf{t} : \mathbf{s} \leqq \mathbf{t} \leqq (\mathbf{s} + \delta\mathbf{1}) \wedge \mathbf{1}\}$, $0 < \delta < 1$. Then, to prove the tightness of $\{W_N\}$, it suffices to show that for every $\mathbf{s} \in I^2$ and $\varepsilon > 0$, $\eta > 0$, there exist a δ (> 0) and an integer N_0, such that

$$(3.26) \quad P\{\sup[|W_N(\mathbf{t}) - W_N(\mathbf{s})| : \mathbf{t} \in B_\delta(\mathbf{s})] > \varepsilon\} < \eta\delta^2 \qquad N \geqq N_0 .$$

Towards this, note that $|W_N(\mathbf{t}) - W_N(\mathbf{s})| \leqq |W_N(\mathbf{t}) - W_N(s_1, t_2)| + |W_N(s_1, t_2) - W_N(\mathbf{s})|$, $\forall\, \mathbf{t} \in B_\delta(\mathbf{s})$ where

$$(3.27) \quad \sup_{\mathbf{t}\in B_\delta(\mathbf{s})} |W_N(s_1, t_2) - W_N(\mathbf{s})| = C_N{}^{-1}A_N{}^{-1}\{\max_{r_1\le r\le r_2}|T_{n_\cdot,r} - T_{n_1,r_1}|\} ,$$

$n_1 = n(s_1)$, $r_1 = r(s_2)$ and $r_2 = r((s_2 + \delta) \wedge 1)$ are defined by (2.2). Now, by Lemma 4.1 of Chatterjee and Sen (1973), for every $n\ (\geqq 1)$, $\{T_{n,r}, \mathscr{B}_n^{(r)};\ 1 \leqq r \leqq n\}$ is a martingale and by our (3.15)—(3.17), $C_N^{-1}A_N^{-1}(T_{n_1,r_2} - T_{n_1,r_1}) \to_{\mathscr{D}} N(0, s_1\delta)$. Consequently, on using Lemma 4 of Brown (1971), it follows by some standard steps that for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta\ (> 0)$ and an N_0, such that for $N \geqq N_0$,

$$(3.28)\qquad P\{\sup[|W_N(s_1, t_2) - W_N(\mathbf{s})| : s_2 \leqq t_2 \leqq (s_2 + \delta) \wedge 1] > \tfrac{1}{2}\varepsilon\} < \tfrac{1}{2}\eta\delta^2 .$$

Hence, to prove (3.26), it remains to show that

$$(3.29)\qquad P\{\sup[|W_N(\mathbf{t}) - W_N(s_1, t_2)| : \mathbf{t} \in B_\delta(\mathbf{s})] > \tfrac{1}{2}\varepsilon\} < \tfrac{1}{2}\eta\delta^2 .$$

Since, for a fixed t_2, $\{W_N(t_1, t_2) - W_N(s_1, t_2),\ s_1 \leqq t_1 \leqq s_1 + \delta\}$ is not necessarily a martingale sequence, the usual proof [as for (3.27)] is not directly applicable here. We need some back and forth replacement of W_N by some related processes. For this reason, we define first $\tilde{T}_n(p)$ as in (3.7) and consider a process $\tilde{W}_N^0 = \{\tilde{W}_N^0(\mathbf{t}) : \mathbf{t} \in I^2\}$, where

$$(3.30)\qquad \tilde{W}_N^0(\mathbf{t}) = A_N^{-1}C_N^{-1}\tilde{T}_{n(t_1)}(\nu^{-1}(t_2)) \qquad \mathbf{t} \in I^2$$

and $n(t_1)$ and ν are defined by (2.2) and (3.5). Note then

$$(3.31)\qquad T_{n,r} - \tilde{T}_n(r/n) = \textstyle\sum_{i=1}^n (c_i - \bar{c}_n) b_n(R_{ni}; r/n)$$

where

$$(3.32)\qquad \begin{aligned} b_n(i; r/n) &= a_n(i) - \tilde{a}_n(i; r/n)\,, \quad 1 \leqq i \leqq r\,, \\ &= a_n^*(r) - \tilde{a}_n(i; r/n)\,, \quad r < i \leqq n\,. \end{aligned}$$

Since, by Assumption 2, $\varphi = \varphi_1 - \varphi_2$ with $\varphi_j \nearrow$, $j = 1, 2$, the $b_n(i, r)$ can be expressed as the difference of two terms where for each term, φ_j is $\nearrow$. Thus, for convenience of manipulation, we assume in the sequel that φ is $\nearrow$. Then, by (2.5), (3.4) and (3.9), for $i \leqq r$, $|b_n(i; r/n)| = |n\binom{n-1}{i-1} \int_{r/n}^1 [\varphi(u) - \varphi_{r/n}^*]u^{i-1}(1 - u)^{n-1}\,du|$ and noting that $u^{i-1}(1-u)^{n-i}$ attains a unique maximum at $u = (i-1)/(n-1)$ $(< r/n)$ and exponentially converges to 0 as $u \to 0$ or to 1, we claim on letting $k_n = [r(n-r)/n]^{\frac{1}{2}} \log n$ that

$$(3.33)\qquad \begin{aligned} |b_n(i; r)| &= O(n^{-1})\,, & i &\leqq r - k_n\,, \\ &= d_i|\varphi(r/n) - \varphi_{r/n}^*|\,, & d_i &\leqq 1\,, \quad r - k_n \leqq i \leqq r\,. \end{aligned}$$

In a similar manner, it can be shown that

$$(3.34)\qquad \begin{aligned} |b_n(i; r)| &= O(n^{-1}k_n)\,, & i &\geqq r + k_n\,, \\ &= O(1)|\varphi(r/n) - \varphi_{r/n}^*|\,, & r &< i < r + k_n\,. \end{aligned}$$

Further, by (2.5), (3.4) and (3.9), $\sum_{i=1}^n b_n(i; r/n) = 0$ and by Assumption 2(a), $[r(n-r)/n^2][\varphi(t/n) - \varphi_{r/n}^*]^2$ is bounded for all $1 \leqq r \leqq n$ (and it goes to 0 as $r/n \to 0$ or 1). Hence using Assumption 1(b) and proceeding as in the Wald–Wolfowitz (1944) proof of the PCLT, we obtain that for every $m\ (\geqq 1)$,

$$(3.35)\qquad \begin{aligned} C_n^{-2m}E(T_{n,r} - \tilde{T}_n(r/n))^{2m} &= O(([n^{-1}k_n][\varphi(r/n) - \varphi_{r/n}^*]^2)^m) \\ &= O((\log n)^{-m}[r(n-r)/n]^{-m/2})\,. \end{aligned}$$

Hence, using the Markov and the Bonferroni inequalities (and choosing m adequately large) we obtain that for every $\eta > 0$,

$$\max_{n \leqq N} \max_{n\eta \leqq r \leqq n - n\eta} C_N^{-1}|T_{n,r} - \tilde{T}_n(r/n)| \rightarrow_p 0 \quad \text{as } N \rightarrow \infty . \tag{3.36}$$

We shall comment on the two tails ($r < n^\eta$ and $r > n - n^\eta$) later on. Again, by (3.4), for $q \leqq p$, $\tilde{\varphi}_p(u) - \tilde{\varphi}_q(u) = 0$, if $u \leqq q$, $\leqq \varphi(q) - \varphi_q{}^*$ if $q < u \leqq p$ and is $\varphi_p{}^* - \varphi_q{}^*$ if $u > p$. Hence, under Assumption 1 (b),

$$\begin{aligned}
&C_n^{-1}|\tilde{T}_n(p) - \tilde{T}_n(q)| \\
&\quad \leqq O(n^{-\frac{1}{2}}) \textstyle\sum_{i=1}^n |\tilde{a}_n(i, p) - \tilde{a}_n(i; q)| \\
&\quad \leqq O(n^{-\frac{1}{2}})[|\varphi(q) - \varphi_q{}^*| \textstyle\sum_{i=1}^n P\{U_{ni} \in [q, p]\} + |\varphi_p{}^* - \varphi_q{}^*| \sum_{i=1}^n P\{U_{ni} > p\}] \\
&\quad = O(n^{-\frac{1}{2}})[|\varphi(q) - \varphi_q{}^*|n(p - q) + |\varphi_p{}^* - \varphi_q{}^*| \\
&\qquad\quad \times \{n(1 - p) + (np(1 - o))^{\frac{1}{2}} \log n + o(1)\}] \\
&\quad \leqq O(n^{-\frac{1}{2}})\{n(p - q)(1 - q)^{-\frac{1}{2}}[\textstyle\int_q^1 \varphi^2(u)\, du]^{\frac{1}{2}} \\
&\qquad\quad + [n(1 - p) + (np(1 - p))^{\frac{1}{2}} \log n + o(1)][o(1)[(p - q)^{\frac{1}{2}}(1 - p)^{-1}]]\} \\
&\quad = O([n(1 - q)]^{-\frac{1}{2}}[n(p - q)][\textstyle\int_q^1 \varphi^2(u)\, du]^{\frac{1}{2}}) \\
&\qquad\quad + o([n(p - q)]^{\frac{1}{2}} + (\log n)[p(p - q)]^{\frac{1}{2}}) = o(1)
\end{aligned}$$

for every $p - n^{-1} \leqq q \leqq p \in I$. Thus, by (3.6), (3.36) and the above discussion, it follows that to prove (3.29) when $0 < s_2 < 1 - 2\delta$, it suffices to show that for $N \geqq N_0$,

$$\begin{aligned}
P\{\sup [|\tilde{W}_N{}^0(\mathbf{t}) - \tilde{W}_N{}^0(s_1, t_2)| :& \\
s_1 \leqq t_1 \leqq s_1 + \delta,\ s_2 \leqq t_2 \leqq s_2 + 2\delta] &> \tfrac{1}{4}\varepsilon\} < \tfrac{1}{4}\eta\delta^2
\end{aligned} \tag{3.37}$$

(later on, we shall comment on the case where $s_2 = 0$ or $s_2 + 2\delta = 1$).

Let us consider the family of one-parameter processes

$$\tilde{W}_N{}^0(t; \cdot) = \{\tilde{W}_N{}^0(t, s) : 0 \leqq s \leqq 1\} \qquad t \in I . \tag{3.38}$$

Let $\mathscr{F}_n{}^*$ be the σ-field generated by $\mathbf{R}_n = (R_{n1}, \cdots, R_{nn})$, for $n \geqq 1$; $\mathscr{F}_n{}^*$ is $\nearrow$ in n. Also for every $N (\geqq 1)$, on defining $n(t_1)$ as in (2.2), we denote by

$$\mathscr{F}_N{}^*(t_1) = \mathscr{F}^*_{n(t_1)} \qquad t_1 \in I , \tag{3.39}$$

so that $\mathscr{F}_N{}^*(t)$ is $\nearrow$ in $t \in I$.

LEMMA 3.1. *For every* $N (\geqq 1)$, $\{\tilde{W}_N{}^0(t; \cdot), \mathscr{F}_N{}^*(t); t \in I\}$ *is a martingale.*

PROOF. By (3.30), (3.38) and (3.39) it suffices to show that

$$E[\{\tilde{T}_{n+1}(p) : p \in I\} \,|\, \mathscr{F}_n{}^*] = \{\tilde{T}_n(p) : p \in I\} \quad \text{almost surely.} \tag{3.40}$$

Since by (3.4) and (3.9), $\sum_{i=1}^n \tilde{a}_n(i; p) = n \int_0^1 \tilde{\phi}_p(u)\, du = n \int_0^1 \phi(u)\, du = 0$, $\forall\, p \in I$, $n \geqq 1$, we may rewrite $\tilde{T}_n(p)$ as $\sum_{i=1}^n c_i \tilde{a}_n(R_{ni}; p)$, $p \in I$, $n \geqq 1$. Hence, for every $p \in I$,

$$\begin{aligned}
E[\tilde{T}_{n+1}(p) \,|\, \mathscr{F}_n{}^*] = \textstyle\sum_{i=1}^n c_i E\{\tilde{a}_{n+1}(R_{n+1i}; p) \,|\, \mathscr{F}_n{}^*\}& \\
+ c_{n+1} E\{\tilde{a}_{n+1}(R_{n+1n+1}; p) \,|\, \mathscr{F}_n{}^*\}& .
\end{aligned} \tag{3.41}$$

Now, given $\mathscr{F}_n^*$, R_{n+1n+1} can assume the values $1, \cdots, (n+1)$ with the common probability $(n+1)^{-1}$, so that $E\{\tilde{a}_{n+1}(R_{n+1n+1}; p) \mid \mathscr{F}_N^*\} = (n+1)^{-1} \sum_{i=1}^{n+1} \tilde{a}_{n+1}(i; p) = 0$. Thus, the second term on the rhs (right hand side) of (3.41) vanishes. Also, for $1 \leqq i \leqq n$, given $\mathscr{F}_n^*$, R_{n+1i} can be either R_{ni} or $(R_{ni}+1)$ with respective conditional probabilities $(n+1)^{-1}(n+1-R_{ni})$ and $(n+1)^{-1}R_{ni}$, so that

$$\begin{aligned} &E[\tilde{a}_{n+1}(R_{n+1i}; p) \mid \mathscr{F}_n^*] \\ &\quad = (n+1)^{-1}[(n+1-R_{ni})\tilde{a}_{n+1}(R_{ni}; p) + R_{ni}\tilde{a}_{n+1}(R_{ni}+1; p)] \\ &\quad = \tilde{a}_n(R_{ni}; p) \qquad 1 \leqq i \leqq n\,, \end{aligned} \tag{3.42}$$

where the last identity follows from a well-known recursion relation among the expected order statistics. From the above, we have $E[\tilde{T}_{n+1}(p) \mid \mathscr{F}_n^*] = \tilde{T}_n(p)$ for every $p \in I$, $n \geqq 1$, and this implies (3.40). □

Let us now denote by $\tilde{W}_N^0(t_1, s_2; \delta) = \{\tilde{W}_N^0(t_1, t_2) - \tilde{W}_N^0(t_1, s_2) : s_2 \leqq t_2 \leqq s_2 + \delta\}$, $t_1 \in I$, $0 \leqq s_2 \leqq 1 - \delta$, and let $\|\tilde{W}_N^0(t_1, s_2; \delta)\| = \sup_{s_2 \leqq t_2 \leqq s_2+\delta} |\tilde{W}_N^0(t_1, t_2) - \tilde{W}_N^0(t_1, s_2)|$, $t_1 \in I$, be the corresponding process of "sup" norms. Since $\|\cdot\|$ is a convex function, we obtain from Lemma 3.1 that for each $s_2 \in I$,

$$\{\|\tilde{W}_N^0(t_1, s_2; \delta)\|, \mathscr{F}_N^*(t_1); t_1 \in I\} \quad \text{is a nonnegative sub-martingale.} \tag{3.43}$$

Therefore, using Theorem 3.3 of Doob (1967, page 316), proceeding as in the proof of Lemma 4 of Brown (1971), and finally, using the Schwarz inequality, we obtain the following.

LEMMA 3.2. *For every* $s_1 \in I$, $0 \leqq s_2 < s_2 + \delta' \leqq 1$, $\varepsilon > 0$ *and* $N \geqq 1$,

$$\begin{aligned} &P\{\sup_{s_1 \leqq t_1 \leqq s_1+\delta} \sup_{s_2 \leqq t_2 \leqq s_2+\delta'} |\tilde{W}_N^0(t_1, t_2) - \tilde{W}_N^0(t_1, s_2)| > \varepsilon\} \\ &\quad = P\{\sup_{s_1 \leqq t_1 \leqq s_1+\delta} \|\tilde{W}_N^0(t_1, s_2; \delta)\| > \varepsilon\} \\ &\quad < (2/\varepsilon)[P\{\|\tilde{W}_N^0(s_1+\delta, s_2; \delta') - \tilde{W}_N^0(s_1, s_2; \delta')\| > \varepsilon/2\} \\ &\qquad \times E\{\|\tilde{W}_N^0(s_1+\delta, s_2; \delta') - \tilde{W}_N^0(s_1, s_2; \delta')\|^2\}]^{\frac{1}{2}}\,. \end{aligned} \tag{3.44}$$

Returning now to the proof of (3.39), we observe that by virtue of Lemmas 3.1 and 3.2, it suffices to show that for every $\varepsilon'' > 0$ and $\eta'' > 0$, there exist a $\delta' : 0 < \delta' < 1$ and an N_0, such that for $N > N_0$,

$$P\{\|\tilde{W}_N^0(s_1+\delta, s_2; \delta') - \tilde{W}_N^0(s_1, s_2; \delta')\| > \varepsilon''\} < \eta''(\delta')^3\,; \tag{3.45}$$

$$E\{\|\tilde{W}_N^0(s_1+\delta, s_2; \delta') - \tilde{W}_N^0(s_1, s_2; \delta')\|^2\} < C\delta' \qquad C < \infty\,. \tag{3.46}$$

Unfortunately, $\{\tilde{W}_N^0(s, t), t \in I\}$ does not possess a martingale property. So, to prove (3.45)—(3.46), we again make use of (3.36). First, note that by (3.6) and (3.36), to prove (3.45), it suffices to replace $\tilde{W}_N^0$ by W_N and δ' by $\delta'' = 2\delta'$. Also, $\|W_N(s_1+\delta, s_2; \delta'') - W_N(s_1, s_2, \delta'')\| < \|W_N(s_1+\delta, s_2; \delta'')\| + \|W_N(s_1, s_2; \delta'')\|$, where

$$\|W_N(s, s_2; \delta'')\| = \max\{A_N^{-1}C_N^{-1}|T_{n,r} - T_{n,r_1}|; r_1 \leqq r \leqq r_2\}\,, \tag{3.47}$$

with $n = n(s)$, $r_1 = r(s_2)$ and $r_2 = r(s_2 + \delta'')$ being defined by (2.2). Third, we recall that by Lemma 4.1 of Chatterjee and Sen (1973), $\{T_{n,r}, \mathscr{B}_n^{(r)}; 1 \leqq r \leqq n\}$ is a martingale, for each $n \geqq 1$. As such, proceeding as in (3.27), we obtain that for every $\varepsilon'' > 0$,

$$(3.48) \qquad \begin{aligned} &P\{\max [|T_{n,r} - T_{n,r_1}| : r_1 \leqq r \leqq r_2] > \varepsilon'' A_N C_N\} \\ &\quad \leqq (2/\varepsilon'')[EZ_N^2(\delta'')P\{|Z_N(\delta'')| > \varepsilon''/2\}]^{\frac{1}{2}} , \end{aligned}$$

where $Z_N(\delta'') = A_N^{-1}C_N^{-1}(T_{n,r_2} - T_{n,r_1})$, $EZ_N^2(\delta'') = C_n^2 C_N^{-2} A_N^{-2}(V_{n,r_2} - V_{n,r_1}) \rightarrow s\delta''$ and $Z_N(\delta'')$ converges in law to a normal df with 0 mean and variance $s\delta''$. Since, for $x > 1$, $P\{|\mathscr{N}(0, 1)| \geqq x\}$ exponentially converges to 0, we conclude that the rhs of (3.48) can be made smaller than $\eta''(\delta'')^3/2$ when δ'' (or δ) is chosen small. Hence, the proof of (3.45) is complete.

To prove (3.46), it is enough to show that for every $s \in I$, $E\{\|\tilde{W}_N^0(s, s_2; \delta')\|^2) < (C/2)\delta'$. For this, note that by (3.6) and (3.36), for N sufficiently large,

$$(3.49) \qquad \|\tilde{W}_N^0(s, s_2; \delta')\| \leqq \|W_N(s', s_2; \delta'')\| + Q_N$$

where $s' = (s - \delta'/2) \vee 0$, $\delta'' = 2\delta'$ and $Q_N = Q_N^* + o(1)$ with

$$(3.50) \qquad Q_N^* = \max_{r_1 \leqq r \leqq r_2} \{A_N^{-1}C_N^{-1}|\tilde{T}_{n'}(r/n') - \tilde{T}_{n'}(r_1/n') - T_{n',r} + T_{n',r_1}|\}$$

and $n' = n(s')$, $r_1 = r(s_2)$ and $r_2 = r(s_2 + \delta'')$ are defined by (2.2). Thus, it suffices to show that for every $\delta'' > 0$, there exists an N_0 such that for $N \geqq N_0$,

$$(3.51) \qquad E\{\|W_N(s', s_2; \delta'')\|^2\} < (C/4)\delta'' ;$$

$$(3.52) \qquad E(Q_N^{*2}) < (C/4)\delta'' .$$

Now, (3.51) follows directly by using (3.43), (3.47) and Theorem 3.3 of Doob (1967, page 316), while to prove (3.52), note that

$$(3.53) \qquad \begin{aligned} Q_N^* &\leqq 2\{\max_{r_1 \leqq r \leqq r_2} |\tilde{T}_{n'}(r/n') - T_{n',r}|/A_N C_N\} \\ &= 2\{\max_{r_1 \leqq r \leqq r_2} Q_{Nr}^*\} , \qquad \text{say,} \end{aligned}$$

where, by (3.35),

$$(3.54) \qquad \begin{aligned} [E(Q_N^{*2})]^2 &\leqq 2^4[E(\max_{r_1 < r < r_2} Q_{Nr}^{*2})]^2 \\ &\leqq 2^4 \textstyle\sum_{r_1 \leqq r \leqq r_2} E(Q_{Nr}^{*4}) \leqq 2^4(r_2 - r_1)[O(N^{-1}(\log N)^{-2})] \\ &= [O(1)][O((\log N)^{-2})] \rightarrow 0 \qquad \text{as } N \rightarrow \infty . \end{aligned}$$

Hence, $E(Q_N^{*2})$ can be made arbitrarily small by choosing N adequately large. This completes the proof of the theorem when $0 < s_2 < 1$.

To complete the proof of the theorem, we need to show now that (3.37) holds for $s_2 = 0$ and $s_2 \geqq 1 - 2\delta$. We consider only the case of $s_2 \geqq 1 - 2\delta$ as the other case follows on parallel lines. For this, first, we need to extend in (3.36) the domain of r from $n - n^\gamma$ to n. Note that under Assumption 1(b)

$$(3.55) \qquad \begin{aligned} C_n^{-1}|T_{n,r} - T_n| &= |\textstyle\sum_{i=r+1}^n [(c_{S_{ni}} - \bar{c}_n)/C_n][a_n(i) - a_n^*(r)]| \\ &\leqq \{[\textstyle\sum_{i=r+1}^n (c_{S_{ni}} - \bar{c}_n)^2/C_n^2][\sum_{i=r+1}^n (a_n(i) - a_n^*(r))^2]\}^{\frac{1}{2}} \\ &= \{[O((n - r)/n)][n(A_n^2 - V_{n,r})]\}^{\frac{1}{2}} \\ &= O([(n - r)(A_n^2 - V_{n,r})]^{\frac{1}{2}}) , \end{aligned}$$

where by Assumption 2(b), $(n-r)(A_n^2 - V_{n,r}) \to 0$ as $n \to \infty$ whenever $r > n - K\log n$. [Note that $\varphi^2(u)\{\log(1+|\varphi(u)|)\}^r = o((1-u)^{-1})$ as $u \to 1$.] Hence, as $n \to \infty$

$$\max_{n-K\log n \leq r \leq n} C_n^{-1}|T_{n,r} - T_n| \to 0 \quad \text{with probability 1.} \tag{3.56}$$

Similarly, under Assumptions 1(b) and 2(b), as $n \to \infty$,

$$\max_{n-K\log n \leq r \leq n} C_n^{-1}|\tilde{T}_n(r/n) - T_n| \to 0 \quad \text{with probability 1.} \tag{3.57}$$

Thus, we need only to prove a result parallel to (3.36) for $n - n^\eta \leqq r \leqq n - K\log n$ where $\eta > 0$. For this, we proceed as in (3.31) through (3.34) where we let $k_n = (n-r)$ and instead of evaluating the $2m$th moment, we bound the moment generating function of $C_n^{-1}(\tilde{T}_n(r/n) - T_{n,r})$; the latter exists for all such r and $t = O(\log n)$. Using then the Bernstein and the Bonferroni inequalities, we obtain that as $N \to \infty$,

$$\max_{n \leq N} \max_{n-n^\eta \leq r \leq n-K\log n} C_N^{-1}|T_{n,r} - \tilde{T}_n(r/n)| \to_p 0\,. \tag{3.58}$$

The proof of (3.52) for $s_2 \geqq 1 - 2\delta$ also follows as in (3.53)—(3.54) where in (3.54) we use the inequality that for $\vartheta > 0$, $\exp(\vartheta E Q_N^*) \leqq E\{\exp(\vartheta Q_N^*)\} \leqq \sum_{r=r_1}^{r_2} E[\exp(\vartheta Q_{Nr}^*)] = \sum_{r=r_1}^{r_2} O(N^{-1-\gamma}) = O(N^{-\gamma}) \to 0$ as $N \to \infty$ (where $\gamma > 0$). □

Note that by (2.1), (3.4) and (3.9), $W_N{}^0(t,1) = \tilde{W}_N{}^0(t,1)$ for every $t \in I$ and hence, the lengthy steps in (3.30) through (3.37) are not needed. Further, by Lemma 2.1 of Sen and Ghosh (1972), $\{W_N(t,1);\ t \in I\}$ is a martingale, and hence, by using Lemma 4 of Brown (1971), it follows that under assumptions 1(a) and 2(a),

$$\sup\{|W_N(t,1) - W_N(s,1)| : |t-s| < \delta\} \to_P 0 \qquad \text{as } N \to \infty\,. \tag{3.59}$$

Hence, $W_N{}^{(1)}$ in (1.5) converges weakly to a Brownian motion under Assumptions 1(a) and 2(a). Similarly, $W_N(1,t) = \tilde{W}_N{}^0(1,t)$ for every $t \in I$, and by Lemma 4.1 of Chatterjee and Sen (1973), $\{W_N(1,t);\ t \in I\}$ is a martingale, so that $W_N{}^{(2)}$ in (1.10) weakly converges to a standard Brownian motion under Assumptions 1(a) and 2(a). Third, if we let for every $\varepsilon: 0 < \varepsilon < \frac{1}{2}$, $W_{N,\varepsilon} = \{W_N(\mathbf{t}) : t_1 \in I,\ \varepsilon \leqq t_2 < 1 - \varepsilon\}$ and $W_\varepsilon = \{W(\mathbf{t}) : t_1 \in I,\ \varepsilon \leqq t_2 \leqq 1 - \varepsilon\}$, then for the tightness of $W_{N,\varepsilon}$, we do not need Assumption 2(b) and hence, $W_{N,\varepsilon} \to_{\mathscr{D}} W_\varepsilon$ under 1(b) and 2(a). However, for (2.9) to hold, we are unable to replace (2.4) by (2.3) and Assumption 2(b) by 2(a).

In general, the scores $\{a_N(i)\}$ need not be specified by $a_N(i) = E\phi(U_{Ni})$ $1 \leqq i \leqq N$, as in (2.5), e.g., $a_N(i) = \phi(i/(N+1))$, $1 \leqq i \leqq N$. Thus, the question arises whether in (2.5) one may replace $\{a_N(i)\}$ by more general scores. For arbitrary scores, the martingale result in Lemma 3.1 does not hold, and this causes complications in the proof. However, if $\{a_N(i),\ 1 \leqq i \leqq N;\ N \geqq 1\}$ be any arbitrary sequence for which

$$N^{-1} \sum_{i=1}^N |a_N(i) - E\phi(U_{Ni})| = o(N^{-\frac{1}{2}})\,, \tag{3.60}$$

where ϕ satisfies Assumption 2, then the results hold without any difficulty by noting that in (1.6), the difference of the two linear rank statistics (for the two sets of scores) is $o(C_N)$, and hence, in (2.1), their difference is asymptotically negligible. In fact, (2.6) insures (3.60) for $a_N(i) = \phi(i/(N+1))$, $i = 1, \cdots, N$, whenever ϕ is absolutely continuous inside I.

4. Weak convergence to a drifted Brownian sheet. When the X_i are not identically distributed, the PCLT does not hold. Nevertheless, the asymptotic normality of the standardized form of T_N can be established under certain additional regularity conditions (viz, [6, 7, 10, 11]). The convergence of the f.d.d.'s of $\{W_N\}$ to those of some appropriate Gaussian functions can be established by similar techniques. But, Lemma 2.1 of Sen and Ghosh (1972) and Lemma 4.1 of Chatterjee and Sen (1973) do not hold when the X_i do not have the common df and, for this reason, the current proof of the tightness of $\{W_N\}$ will not work in such a case. If one intends to use the basic projection approach of Hájek (1968) or the weak convergence approach of Pyke and Shorack (1968), one needs to extend their basic results to the entire sequence of sample sizes $\{n \leqq N\}$ and this, in turn, demands more restrictive regularity conditions. We intend to show here that for contiguous alternatives, weak convergence of $\{W_N\}$ to a drifted Brownian sheet follows quite easily from our Theorem 1.

Consider a triangular array $\{X_{Ni},\ 1 \leqq i \leqq N;\ N \geqq 1\}$ of row-wise independent rv's where

$$F_{Ni}(x) = P(X_{Ni} \leqq x) = F(x - d_{Ni}) \qquad 1 \leqq i \leqq N, \tag{4.1}$$

and $\{d_{Ni},\ 1 \leqq i \leqq N;\ N \geqq 1\}$ is a triangular array of constants, such that

$$\sup_N \textstyle\sum_{i=1}^N d_{Ni}^2 < \infty \qquad \text{and} \qquad \max_{1\leqq i\leqq N} d_{Ni}^2 \to 0 \qquad \text{as } N \to \infty. \tag{4.2}$$

Further, assume that F is absolutely continuous with a continuous density function f and a finite Fisher information

$$I(f) = \textstyle\int_{-\infty}^{\infty} [f'(x)/f(x)]^2\, dF(x). \tag{4.3}$$

We denote by $P_{N,k}$ the joint df of $(X_{N1}, \cdots, X_{Nk})$ when the d_{Ni} are all equal to 0 and by $Q_{N,k}$ when (4.1)—(4.3) hold, for $k = 1, \cdots, N$. Then, from the basic results of Chapter 6 of Hájek and Šidák (1967), we conclude that under (4.1)—(4.3), Q_N is contiguous to P_N and this implies that under (4.1)—(4.3),

$$Q_{N,k} \quad \text{is contiguous to} \quad P_{N,k} \qquad \forall\, k \leqq N. \tag{4.4}$$

Let us now denote by

$$\psi(u) = -f'(F^{-1}(u))/f(F^{-1}(u)) \qquad 0 < u < 1 \tag{4.5}$$

and defining $\tilde{\phi}_p$ as in (3.4), we let

$$\rho(p) = \left(\textstyle\int_0^1 \tilde{\phi}_p(u)\psi(u)\, du\right)/[AI^{\frac{1}{2}}(F)] \qquad p \in I. \tag{4.6}$$

Also, we assume that on defining $n(t_1)$ as in (2.2),

$$\lambda(t_1) = \lim_{N\to\infty} C_N^{-1} \textstyle\sum_{i=1}^{n(t_1)} (c_i - \bar{c}_{n(t_1)}) d_{Ni} \qquad \text{exists for all} \quad t_1 \in I. \tag{4.7}$$

Then let ν be defined as in (3.5) and let

$$\mu = \{\mu(\mathbf{t}) = I^{\frac{1}{2}}(f)\lambda(t_1)\rho(\nu^{-1}(t_2));\ \mathbf{t} \in I^2\}\,. \tag{4.8}$$

Finally we construct the $T_{n,k}$, $1 \leqq k \leqq n \leqq N$ as well as W_N from the sequence $\{X_{N1}, \cdots, X_{NN}\}$, for $N \geqq 1$. Then, we have the following.

THEOREM 2. *Under Assumptions* 1(b) *and* 2(b) *and* (4.1)—(4.3), (4.7),

$$W_N - \mu \rightarrow_{\mathscr{D}} W \qquad \textit{as } N \rightarrow \infty\,. \tag{4.9}$$

OUTLINE OF THE PROOF. (3.17) and (4.4) really imply that the f.d.d.'s of $W_N - \mu$ converge to those of W; since the proof follows by the same technique as in Hájek and Šidák (1967, pages 216–219), the details are omitted. For the proof of tightness of $\{W_N\}$ under $\{Q_N\}$, we denote the *modulus of continuity* by $\omega_\delta(x) = \sup\{|x(\mathbf{t}) - x(\mathbf{s})| : |\mathbf{t} - \mathbf{s}| < \delta,\ \mathbf{s}, \mathbf{t} \in I^2\}$, $0 < \delta < 1$ and let

$$B_{N,\varepsilon}^{(\delta)} = \{(X_{N1}, \cdots, X_{NN}) : \omega_\delta(W_N) > \varepsilon\} \qquad \text{where } \varepsilon > 0\,. \tag{4.10}$$

Then, in the course of the proof of the tightness part of Theorem 1, we have really shown that

$$\limsup_N P\{B_{N,\varepsilon}^{(\delta)} \mid P_N\} \rightarrow 0 \qquad \text{as } \delta \rightarrow 0\,. \tag{4.11}$$

Therefore, by the contiguity of Q_N to P_N and (4.11),

$$\limsup_N P\{B_{N,\varepsilon}^{(\delta)} \mid Q_N\} \rightarrow 0 \qquad \text{as } \delta \rightarrow 0\,. \tag{4.12}$$

Hence, $\{W_N\}$ is also tight under $\{Q_N\}$. □

In the context of nonparametric tests under progressive censoring, Chatterjee and Sen (1973) have utilized the process $W_N^{(2)}$ in (1.10) for constructing suitable sequential tests. In many practical problems, the N observations do not enter the scheme at the same time and this introduces additional complications necessitating a two-time parameter process to describe the phenomenon. Both Theorems 1 and 2 are quite useful in this context and the details will be studied in a separate paper.

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