The Annals of Probability
1976, Vol. 4, No. 5, 744-771

A STRONG CONVERGENCE THEOREM FOR BANACH SPACE VALUED RANDOM VARIABLES[1]

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We prove a strong convergence theorem for Banach space valued random variables. One corollary of this result establishes necessary and sufficient conditions for the law of the iterated logarithm (LIL) in the Banach space setting. We also prove an exact generalization of the Hartman-Wintner law of the iterated logarithm provided the random variables involved take values in a real separable Hilbert space or some other Banach space with smooth norm.

1. Introduction. Let B denote a real separable Banach space with norm $\|\cdot\|$, and throughout assume $X_1, X_2, \cdots$ are i.i.d. B-valued random variables such that $EX_k = 0$ and $E\|X_k\|^2 < \infty$. As usual $S_n = X_1 + \cdots + X_n$ for $n \geqq 1$, and we write $\mathrm{L}\, x$ to denote $\log x$ for $x \geqq e$ and 1 otherwise. The function $\mathrm{L}\,(\mathrm{L}\, x)$ is written $\mathrm{LL}\, x$, and B^* denotes the topological dual of B with norm $\|\cdot\|_{B^*}$.

If (M, d) is a metric space and $A \subseteq M$ we define the distance from $x \in M$ to A by $d(x, A) = \inf_{y\in A} d(x, y)$. If $\{x_n\}$ is a sequence of points in M, then $C(\{x_n\})$ denotes the cluster set of $\{x_n\}$. That is, $C(\{x_n\})$ is all possible limit points of the sequence $\{x_n\}$. We also will sometimes use the notation $\{x_n\} \to\to A$ if *both* $\lim_n d(x_n, A) = 0$ and $C(\{x_n\}) = A$.

We prove a strong convergence theorem for B-valued random variables which is related to the law of the iterated logarithm. To motivate such a result we recall the Hartman-Wintner LIL [10] as given by Strassen [27].

THEOREM A (Hartman-Wintner, Strassen). *If* $X_1, X_2, \cdots$ *are* i.i.d. *real valued random variables such that* $EX_k = 0$ *and* $EX_k^2 = \sigma^2 > 0$, *then*

$$P\left\{\omega : \lim_n d\left(\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}}, [-\sigma, \sigma]\right) = 0\right\} = 1 \tag{1.1}$$

and

$$P\left\{\omega : C\left(\left\{\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}}\right\}\right) = [-\sigma, \sigma]\right\} = 1\,. \tag{1.2}$$

Now let $X_1, X_2, \cdots$ be i.i.d. B-valued random variables such that $E\|X_1\|^2 < \infty$ and $E(X_1) = 0$. In view of (1.1) and (1.2) one might expect that there is a fixed bounded symmetric set K in B such that

$$P\left\{\omega : \lim_n d\left(\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}}, K\right) = 0\right\} = 1\,, \tag{1.3}$$

Received August 17, 1974; revised January 20, 1976.
[1] Supported in part by NSF Grant GP 18759.
AMS 1970 *subject classifications.* Primary 60B05, 60B10, 60F10; Secondary 28A40.
Key words and phrases. Measurable norm, Gaussian measure, law of the iterated logarithm, differentiable norm, submartingale, Berry-Esseen estimates.

and

$$P\left\{\omega: C\left(\left\{\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}}\right\}\right) = K\right\} = 1\,. \tag{1.4}$$

A result of this generality is, however, not true as can easily be seen from an example of R. Dudley and V. Strassen [5]. Their example was constructed to show the classical form of the central limit theorem was not valid in the Banach space $C[0, 1]$, but applies equally well to the law of the iterated logarithm. More precisely, they show that there exist i.i.d. $C[0, 1]$-valued random variables $X_1, X_2, \cdots$ satisfying $\|X_1\|_\infty \leqq 1$, $E(X_1) = 0$, and such that

$$\liminf_n P\left\{\omega: \max_{0\leqq t\leqq 1}\frac{S_n(t, \omega)}{n^{\frac{1}{2}}} > n^{\frac{1}{4}}\right\} \geqq \tfrac{1}{2}\,. \tag{1.5}$$

Now (1.5) implies $S_n/(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}$ is unbounded in probability and hence (1.3) is impossible for any bounded subset K of $C[0, 1]$.

The limit set K in (1.3) and (1.4) is uniquely determined by the covariance function

$$T(f, g) = E(f(X_1)g(X_1)) \quad f, g \in B^*\,,$$

and, in fact, is always the unit ball of a Hilbert space determined by T.

In Section 2 we examine K in detail, and one thing we prove is that K is necessarily compact in B whenever $E\|X_1\|^2 < \infty$. In fact, K is compact even if the covariance function $T(f, g)$ is only weak-star sequentially continuous on $B^* \times B^*$, but we do not use that fact here. It is important, however, to realize that the compactness of K is forced on us even though our original formulation in (1.3) and (1.4) only anticipated that K be bounded in B.

Since the random variables in the example of Dudley and Strassen mentioned above are uniformly bounded with probability one necessary and sufficient conditions for the LIL in the Banach space setting must involve conditions other than the classical moment conditions. The precise formulation of these conditions is given in the following corollary whose proof will follow easily from the general convergence result obtained in Theorem 3.1. Before stating the corollary we point out that the limit set K constructed from the covariance function in Section 2 satisfies the corollary. Some explicit examples of K will also be given in Section 2.

COROLLARY 3.1 (N.A.S.C. for the LIL in the Banach space setting). *Let* $X_1, X_2, \cdots$ *be* i.i.d. *B-valued such that* $E(X_k) = 0$ *and* $E\|X_k\|^2 < \infty$. *Then*

I. *There exists a compact, symmetric, convex* $K \subseteqq B$ *such that*

$$P\left\{\omega: C\left(\left\{\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}} : n \geqq 1\right\}\right) \not\subseteqq K\right\} = 0\,. \tag{1.6}$$

II. *In addition, there exists a compact, symmetric, convex set K satisfying* (1.6) *such that*

$$P\left\{\omega: \lim_n d\left(\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}}, K\right) = 0\right\} = 1 \tag{1.7}$$

and

$$P\left\{\omega : C\left(\left\{\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}} : n \geqq 1\right\}\right) = K\right\} = 1\,, \tag{1.8}$$

iff

$$P\left\{\omega : \left\{\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}} : n \geqq 1\right\} \text{ is conditionally compact in } B\right\} = 1\,. \tag{1.9}$$

REMARK. The event in (1.9) is a tail event for the sequence $X_1, X_2, \cdots$ so it has probability zero or one. Hence the LIL holds with limit set K or not at all. Furthermore, as mentioned previously, the limit set K can be uniquely determined by the covariance function of the common distribution, and since K is compact (1.7) obviously implies (1.9). The point of the corollary is that (1.7) and (1.9) are actually equivalent in the present situation. The additional interesting fact is that (1.8) always holds in this setup provided (1.7) or (1.9) hold. This is of interest because it is (1.8) which is usually the most cumbersome step to verify in proving this type of result.

Since the LIL in an arbitrary Banach space involves something other than the moment conditions of Theorem A it is of interest to see if there are infinite dimensional Banach spaces in which the classical moment assumptions alone are sufficient. Theorem 4.1 is in this direction, and has as a corollary an exact generalization of Theorem A provided the $\{X_k\}$ take values in a real separable Hilbert space or some other Banach space with smooth norm.

As usual $D[0, 1]$ denotes the space of real valued functions on $[0, 1]$ which are right continuous on $[0, 1]$ and have left limits on $(0, 1]$. The cylinder sets of $D[0, 1]$ induced by the maps $x \to x(t)$ induce a sigma algebra which we denote by $\mathscr{D}$ and it follows in a straightforward manner that the map $(x, y) \to x + y$ from $(D[0, 1] \times D[0, 1], \mathscr{D} \times \mathscr{D})$ into $(D[0, 1], \mathscr{D})$ is measurable.

For each $x \in D[0, 1]$ we define the norm

$$||x||_\infty = \sup_{0 \leq t \leq 1} |x(t)|\,, \tag{1.10}$$

and use $C[0, 1]$ to denote the subset of $D[0, 1]$ consisting of continuous functions on $[0, 1]$. It is obvious that $||x||_\infty$ is finite for each $x \in D[0, 1]$.

The next result deals with the law of the iterated logarithm for sums of i.i.d. $(D[0, 1], \mathscr{D})$ valued random variables with convergence and clustering computed in the sup-norm. Originally the proof of this result involved an application of Corollary 3.1 to a related sequence of random variables with values in $C[0, 1]$, but at the suggestion of Professor M. J. Wichura and J. Crawford we now proceed in a slightly different manner. First we establish some terminology.

A sequence $\{x_n\}$ of elements of $D[0, 1]$ is said to be asymptotically equicontinuous if

$$\lim_{\delta \downarrow 0} \limsup_n \omega_\delta(x_n) = 0\,,$$

where $\omega_\delta(x) \equiv \sup_{|s-t| \leq \delta} |x(t) - x(s)|$ is the usual modulus of continuity.

If $T = \{t_0, \cdots, t_r\}$ is a finite subset of $[0, 1]$ containing both zero and one and $x \in D[0, 1]$, then we define $\Lambda_T(x)$ to be the continuous polygonal function such that

$$\Lambda_T(x)(t) = x(t_j) \qquad \text{if} \quad t = t_j \qquad\qquad j = 0, \cdots, r$$

and linear elsewhere.

The following lemma is easily proved so the details are omitted. A similar result also appears in [23].

LEMMA 1.1. *Let $\{Y_n : n \geqq 1\}$ be $(D[0, 1], \mathscr{D})$ valued random variables and suppose $\{T_m : m \geqq 1\}$ are increasing finite subsets of $[0, 1]$ each of which contain zero and one, and such that $\bigcup_{m \geqq 1} T_m$ is dense in $[0, 1]$. If K is a compact subset of $C[0, 1]$, then*

$$P(\omega : Y_n(\omega) \rightarrow\!\!\rightarrow K) = 1 \tag{1.11}$$

iff

$$\text{(i)} \quad P(\omega : (Y_n(\omega)(t))_{t \in T_m} \rightarrow\!\!\rightarrow K(T_m) \equiv \{(x(t))_{t \in T_m} : x \in K\}) = 1$$

for each m, and any (or all) of the following hold:

$$\begin{aligned} &\text{(iia)} \quad P(\omega ; \{Y_n(\omega)\} \text{ is uniformly bounded and asymptotically} \\ &\qquad\qquad \text{equicontinuous}) = 1\,, \\ &\text{(iib)} \quad P(\omega : \{Y_n(\omega)\} \text{ is asymptotically equicontinuous}) = 1\,, \\ &\text{(iic)} \quad P(\omega : \limsup_m \limsup_n \|Y_n(\omega) - \Lambda_{T_m} Y_n(\omega)\|_\infty = 0) = 1\,. \end{aligned} \tag{1.12}$$

In the LIL for $D[0, 1]$ valued random variables the limit set is uniquely determined by the covariance function. To be precise let $\{X(t) : 0 \leqq t \leqq 1\}$ be a stochastic process with mean function identically zero and continuous covariance function $R(s, t) = E(X(s)X(t))$ defined on $[0, 1] \times [0, 1]$. Then, since $R(s, t)$ is symmetric, continuous, and nonnegative definite, by Mercer's theorem ([25], page 245) it has the eigenfunction expansion $\sum_n \lambda_n \phi_n(s)\phi_n(t)$ which converges uniformly on $[0, 1] \times [0, 1]$, the eigenfunctions $\{\phi_n(t)\}$ are continuous orthonormal elements of $L^2[0, 1]$, and the eigenvalues λ_n are positive numbers such that $\sum_n \lambda_n < \infty$.

Let H_R denote the set of elements in $L^2[0, 1]$ which are in the closure of the span of $\{\phi_n : n \geqq 1\}$ and such that

$$\sum_n \frac{(x, \phi_n)^2}{\lambda_n} < \infty\,, \tag{1.13}$$

where, of course, $(x, y) = \int_0^1 x(t)y(t)\,dt$. H_R is a Hilbert space in the inner product

$$(x, y)_{H_R} = \sum_n \frac{(x, \phi_n)(y, \phi_n)}{\lambda_n}\,,$$

and $\alpha_n = \lambda_n^{\frac{1}{2}}\phi_n$ $(n \geqq 1)$ is a complete orthonormal set in H_R.

If K_R is the unit ball of H_R (in the H_R norm), then since $R(s, t)$ is continuous it is fairly easy to see that K_R is a compact subset of $C[0, 1]$ in the sup-norm, and we shall see that K_R is the limit set of interest. Here, of course, we identify equivalence classes of H_R with their continuous representative.

The Hilbert space H_R is commonly called the reproducing kernel Hilbert space (RKHS) of the kernel R and in [22], pages 84–85, H_R is identified with its more usual definition as given, for example, in [1], page 344, and [22], page 84.

The LIL for $D[0, 1]$ valued random variables is the following:

THEOREM 4.2. *Let $X_1, X_2, \cdots$ be* i.i.d. *$(D[0, 1], \mathscr{D})$ valued random variables with $EX_k(t) = 0$ $(0 \leqq t \leqq 1)$ and such that the covariance function*

$$R(s, t) = E(X_k(s)X_k(t))$$

is continuous on $[0, 1] \times [0, 1]$. *If K_R is the unit ball of the* RKHS H_R, *then*

$$P\left(\omega:\ H_n(\omega) \equiv \frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}} \rightarrow\!\!\rightarrow K_R\right) = 1 \tag{1.14}$$

iff *one (or both) of the following hold*:

$$\begin{aligned} &\text{(i)}\quad P(\omega : \{H_n(\omega)\} \textit{ is asymptotically equicontinuous}) = 1\ , \\ &\text{(ii)}\quad P(\omega : \limsup_m \limsup_n \|H_n(\omega) - \Lambda_{T_m}(H_n(\omega))\|_\infty = 0) = 1\ . \end{aligned} \tag{1.15}$$

As fairly immediate corollaries of Theorem 4.2 we have:

COROLLARY 4.1. *Let $X_1, X_2, \cdots$ be independent identically distributed $(D[0, 1], \mathscr{D})$ valued random variables such that each $\{X_k(t) : 0 \leqq t \leqq 1\}$ is a martingale. Further, assume there exists a $\delta > 0$ such that*

$$E(X_k(t)) = 0 \qquad \textit{and} \qquad E|X_k(t)|^{2+\delta} < \infty \qquad\qquad 0 \leqq t \leqq 1\ , \tag{1.16}$$

and the covariance function

$$R(s, t) = E(X_k(s)X_k(t))$$

is continuous on $[0, 1] \times [0, 1]$. *If K_R denotes the unit ball of the* RKHS H_R, *then*

$$P\left\{\omega : \lim_n d\left(\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}}, K_R\right) = 0\right\} = 1\ , \tag{1.17}$$

and, in fact,

$$P\left\{\omega : C\left(\frac{S_n(\omega)}{(2n\,\mathrm{LL}\,n)^{\frac{1}{2}}} : n \geqq 1\right) = K_R\right\} = 1\ , \tag{1.18}$$

where convergence and clustering are computed with respect to the sup-norm.

COROLLARY 4.2. *If the processes $X_k(t)$ are independent increment processes, then* (1.17) *and* (1.18) *hold with only a second moment condition in* (1.16) *rather than the* $(2 + \delta)$th *moment.*

COROLLARY 4.3 (H. Finkelstein, [7]). *Let $U_1, U_2, \cdots$ be independent random variables uniformly distributed on* $[0, 1]$, *and let $F_n(t)$ be the empirical distribution function at stage n. If K_R denotes the unit ball of the* RKHS H_R *where*

$$R(s, t) = \min(s, t) - st\ , \tag{1.19}$$

then

$$P\left(\omega : \frac{n(F_n(t, \omega) - t)}{(2n \operatorname{LL} n)^{\frac{1}{2}}} \to\to K_R\right) = 1 . \tag{1.20}$$

The proof of Corollary 4.3 given here is due to M. J. Wichura, and I am also indebted to J. Crawford for comments in this area. Crawford's formulation of such results consists of part of [3].

Another application of Theorem 3.1 is Corollary 3.2 which is related to the work of [20]. Theorem 4.3 and Corollary 4.4 are applications of Corollary 3.2 and contain the main result of [20].

In Theorem 4.4 we provide another example of a situation where Corollary 3.1 can be applied. This result is a generalization of Theorem 2.1 in [19], and since the details of its proof are in the same spirit as those of the proof of Theorem 2.1 they are omitted. Theorem 4.4 is also interesting in that its hypotheses are exactly those used to establish the central limit theorem in this setting.

It is a pleasure to acknowledge a number of constructive remarks provided by the referee as well as the previously mentioned comments of M. J. Wichura and J. Crawford.

The paper [28] is another direction of possible generalization of Strassen's fundamental results.

2. Construction of the limit set K. The limit set K in our limit theorems for B-valued random variables depends on the covariance function of the random variables involved, and is intimately related to the mean-zero Gaussian measure on B with the given covariance function provided this measure exists.

A measure μ on B is called a mean-zero Gaussian measure if every $f \in B^*$ has a mean-zero Gaussian distribution with variance $\int_B [f(x)]^2 \, d\mu(x)$.

If μ is a measure on B (not necessarily Gaussian) such that $\int_B x \, d\mu(x) = 0$ and $\int_B ||x||^2 \, d\mu(x) < \infty$, then the bilinear function T defined on $B^* \times B^*$ by

$$T(f, g) = \int_B f(x)g(x) \, d\mu(x) \qquad f, g \in B^*$$

is called the covariance function of μ.

If μ is a mean-zero Gaussian measure then it is well known that $\int_B ||x||^2 \, d\mu(x) < \infty$, and that μ is uniquely determined by its covariance function. However, a mean-zero Gaussian measure μ is determined by a unique subspace H_μ of B which has a Hilbert space structure. We describe this relationship by saying μ is generated by H_μ, and mention that the pair (B, H_μ) is an abstract Wiener space in the sense of [9].

One method of finding this Hilbert space is given in the next lemma which applies to non-Gaussian measures as well. It also provides a construction of the limit set K used in our results, and the relationship to Gaussian measures is given in part (vi) of the lemma. As we shall see, the limit set K is always the unit ball of this Hilbert space. Finally, I emphasize that most of Lemma 2.1 is known in one form or another, but to avoid sending the reader to various references the crucial facts regarding K are collected here.

LEMMA 2.1. *Let μ denote a Borel probability measure on B (not necessarily Gaussian) such that $\int_B \|x\|^2 \, d\mu(x) < \infty$ and $\int_B x \, d\mu(x) = 0$. Let S denote the linear operator from B^* to B defined by the Bochner integral*

$$Sf = \int_B x f(x) \, d\mu(x) \qquad f \in B^* . \tag{2.1}$$

Let H_μ denote the completion of the range of S with respect to the norm obtained from the inner product

$$(Sf, Sg)_\mu = \int_B f(x) g(x) \, d\mu(x) . \tag{2.2}$$

Then: (i) *H_μ can be realized as a subset of B and the identity map $i \colon H_\mu \to B$ is continuous. In fact, for $x \in H_\mu$*

$$\|x\| \leqq \left(\int_B \|y\|^2 \, d\mu(y)\right)^{\frac{1}{2}} \|x\|_\mu . \tag{2.3}$$

(ii) *If $e \colon B^* \to H_\mu{}^*$ is the linear map obtained by restricting an element in B^* to the subspace H_μ of B and if we identify $H_\mu{}^*$ and H_μ in the usual way then*

$$e = S .$$

(iii) *Let $\{f_k \colon k \geqq 1\}$ be a weak-star dense subset of the unit ball of B^*. Let $\{\alpha_k \colon k \geqq 1\}$ be an orthonormal sequence obtained from the sequence $\{f_k\}$ by the usual Gram–Schmidt orthogonalization method with respect to the inner product given by the right side of* (2.2). *The each $\alpha_k \in B^*$, and $\{S\alpha_k \colon k \geqq 1\}$ is a* C.O.N.S. *in $H_\mu \subseteqq B$. Further, the linear operators*

$$\Pi_N(x) = \sum_{k=1}^N \alpha_k(x) S\alpha_k \qquad \text{and} \qquad Q_N(x) = x - \Pi_N(x) \qquad N \geqq 1 \tag{2.4}$$

are continuous from B into B where by $\alpha_k(x)$ we mean the linear functional α_k applied to x. Π_N and Q_N, when restricted to H_μ, are orthogonal projections onto their ranges.

(iv) *If K is the unit ball of H_μ, then K is a compact symmetric convex set in B. Further, for each $f \in B^*$ we have*

$$\sup_{x \in K} f(x) = \left\{\int_B [f(y)]^2 \, d\mu(y)\right\}^{\frac{1}{2}} . \tag{2.5}$$

(v) *If μ and ν are two measures on B satisfying the basic hypothesis of the lemma and having common covariance function, then $H_\mu = H_\nu$.*

(vi) *If μ is a mean-zero Gaussian measure on B, then $\int_B \|x\|^2 \, d\mu(x) < \infty$ and H_μ is the generating Hilbert space for μ.*

PROOF. Take $f \in B^*$. Then $\int_B \|y\|^2 \, d\mu(y) < \infty$ implies the Bochner integral defining $Sf = \int_B y f(y) \, d\mu(y)$ exists and $Sf \in B$. Further,

$$\|Sf\| \leqq \left(\int_B \|y\|^2 \, d\mu(y)\right)^{\frac{1}{2}} \|Sf\|_\mu . \tag{2.6}$$

and hence the map $i \colon S(B^*) \to B$ is continuous. Now (2.6) also implies the completion of $S(B^*)$ with respect to the norm given by the inner product in (2.2) can be realized as a subspace of B, and that the map $i \colon H_\mu \to B$ is continuous as indicated. Further, (2.3) follows from (2.6) since $S(B^*)$ is dense in H_μ with respect to the norm $\|\cdot\|_\mu$. Hence (i) holds.

Let $e: B^* \to H_\mu^* \equiv H_\mu$ as in (ii). Take $f \in B^*$. Then for $g \in B^*$ we have

$$f(Sg) = \int_B f(x)g(x)\,d\mu(x) = (Sf, Sg)_\mu\,,$$

and hence $e(f) = Sf$ when acting on the elements in SB^*. Since SB^* is dense in H_μ we have $e(f) = Sf$ provided we identify H_μ and H_μ^* in the canonical way.

The assertions of (iii) are obvious since each α_k is a finite linear combination of the f_j's. To see that $\{S\alpha_k : k \geqq 1\}$ is complete in H_μ, simply observe that the f_j's separate points of B (and hence in H_μ). That is, if $\alpha_k(y) = 0$ for every k and some $y \in H_\mu$, then by undoing the Gram–Schmidt procedure we thus have $f_j(y) = 0$ for every j. Since the f_j's separate points we have $y = 0$ as required. Perhaps it should be pointed out that when we undo the Gram–Schmidt procedure we omit all f_j's which are linear combinations of previous f_i $(i < j)$ and those such that $\int_B [f_j(x)]^2\,d\mu(x) = 0$. However, if f_j is a finite linear combination of f_i $(i < j)$ and $f_i(y) = 0$ for $i < j$ then $f_j(y) = 0$ as asserted. On the other hand, if $\int_B [f_j(x)]^2\,d\mu(x) = 0$, then $S(f_j) \equiv e(f_j) = 0$ and hence $f_j(y) = 0$ again.

To verify (2.5) note that

$$\begin{aligned}\sup_{x \in K} f(x) = \sup_{Sg \in K} f(Sg) &= \sup_{Sg \in K} \int_B f(x)g(x)\,d\mu(x)\\ &\leqq \left(\int_B f^2(x)\,d\mu(x)\right)^{\frac{1}{2}}\end{aligned}$$

since $Sg \in K$ implies $(\int_B g^2(x)\,d\mu(x))^{\frac{1}{2}} \leqq 1$. Now set $g = f/(\int_B f^2(x)\,d\mu(x))^{\frac{1}{2}}$ and (2.5) holds.

To finish the proof of (iv) we show K is compact in B by first showing K is closed in B and then verifying that every subsequence $\{y_n\} \subseteqq K$ has a convergent subsequence in B.

Take $\{y_n\} \subseteqq B$ and assume $\|y_n - y\| \to 0$ for $y \in B$. Since K is compact in the weak topology induced by H_μ^* we have a subsequence $\{y_{n_j}\}$ such that y_{n_j} converges weakly to z and $z \in K$. Thus $\{y_{n_j}\}$ converges weakly to z in the weak topology on B induced by B^* as $i: H_\mu \to B$ is continuous by (i). Since B^* separates points of B we have $y = z$ so $y \in K$ and K is closed.

Since $SB^* \cap K$ is dense in K it now suffices to prove that if $\{y_n\} \subseteqq SB^* \cap K$ then $\{y_n\}$ has a convergent subsequence.

Let U denote the unit ball of B^* with the weak-star topology. Since B is separable we have that U is a compact metric space in the weak-star topology. For $x \in B$, $f \in B^*$ let $\theta x(f) = f(x)$. Then $\theta: B \to C(U)$ is an isometry from B into the Banach space $C(U)$ with the supremum norm. Thus to show $\{y_n\}$ has a B-convergent subsequence we need only show that $\{\theta y_n\}$ is an equicontinuous and uniformly bounded sequence in $C(U)$ (apply Ascoli's theorem).

Let $f, g \in U$. Then since $\{y_n\} \subseteqq K \cap SB^*$ we have $y_n = Sr_n$ for $r_n \in B^*$ and such that $\int_B r_n^2(x)\,d\mu(x) \leqq 1$. Hence

$$\begin{aligned}|\theta y_n(f) - \theta y_n(g)| &= |(f-g)(Sr_n)|\\ &= \left|\int_B (f-g)(x) r_n(x)\,d\mu(x)\right| \qquad (2.7)\\ &\leqq \left\{\int_B [(f-g)(x)]^2\,d\mu(x)\right\}^{\frac{1}{2}}.\end{aligned}$$

Now

$$\int_B [(f-g)(x)]^2\, d\mu(x) \leqq \|f-g\|_{B^*}^2 \int_B \|x\|^2\, d\mu(x)$$

so setting $g = 0$ we have from (2.7) that

$$\sup_{f\in U} |\theta y_n(f)| \leqq \left(\int_B \|x\|^2\, d\mu(x)\right)^{\frac{1}{2}} .$$

Thus $\{\theta y_n : n \geqq 1\}$ is uniformly bounded on U and it remains to prove $\{\theta y_n : n \geqq 1\}$ is equicontinuous on U.

Recall that the weak star topology on U is equivalent to that given by the metric

$$d(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|f(x_j) - g(x_j)|}{1 + |f(x_j) - g(x_j)|}$$

where $\{x_1, x_2, \cdots\}$ is dense in B.

Fix ε such that $0 < \varepsilon \leqq 1$. In view of (2.7) to establish equicontinuity of $\{\theta y_n : n \geqq 1\}$, we need only show that there exists a $\delta > 0$ such that $d(f, g) < \delta$ implies

$$\int_B [(f-g)(x)]^2\, d\mu(x) < \varepsilon .$$

Our first step is to choose a compact set C in B such that

$$\int_{B-C} \|x\|^2\, d\mu(x) < \varepsilon/2 .$$

Then we observe that since weak-star convergence of elements in U is equivalent to uniform convergence on compact subsets of B we have a $\delta > 0$ such that $d(f, g) < \delta$ implies

$$\int_C [(f-g)(x)]^2\, d\mu(x) < \varepsilon/2 .$$

Combining these two inequalities we have for $f, g \in U$ and $d(f, g) < \delta$ that

$$\int_B [(f-g)(x)]^2\, d\mu(x) < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Thus K is compact as asserted.

If μ and ν have the same covariance function, then for every $f \in B^*$ we have

$$\int_B x f(x)\, d\mu(x) = \int_B x f(x)\, d\nu(x) .$$

This follows since applying $g \in B^*$ to both sides we get $T(f, g)$, the common covariance function of μ and ν. Since such elements are dense in $H_\mu(H_\nu)$ and the norms induced by μ and ν are identical on these elements $H_\mu = H_\nu$ as asserted.

The verification of (vi) follows from well-known results on Gaussian measures. See, for example, Theorem 3 of [15] and Lemma 2 of [14] for details as well as further references.

We conclude our discussion of the limit set K with some examples. In each case the details are left to the reader.

EXAMPLES. (1) If $B = \mathbb{R}^n$ with the usual Euclidean norm $\|\cdot\|$ and μ is as in Lemma 2.1, then

$$\begin{aligned} K &= \{\Sigma^{\frac{1}{2}}(x) : x \in \mathbb{R}^n, \|x\| \leqq 1\} \\ &= \{\{\textstyle\sum_{j=1}^n c_j R(i, j) : 1 \leqq i \leqq n\} : \sum_{i,j=1}^n c_i c_j R(i, j) \leqq 1\} \end{aligned}$$

where Σ is the linear transformation associated with the covariance matrix determined by μ, and $R(i, j) = \int_B x_i x_j \, d\mu(x)$.

(2) If $B = l_p$ $(1 \leqq p < \infty)$, and μ is a measure on l_p satisfying the conditions of Lemma 2.1 and such that the coordinate mappings are uncorrelated, then

$$K = \{\{x_k\} \in l_p : \textstyle\sum_{k \geqq 1} (x_k/\sigma_k)^2 \leqq 1\},$$

where $\sigma_k^2 = \int_{l_p} x_k^2 \, d\mu(x)$.

(3) If $B = C[0, 1]$ and μ denotes Wiener measure (the distribution induced by Brownian motion) then

$$K = \{f \in C[0, 1] : f(t) = \textstyle\int_0^t g(s)\,ds \text{ and } \int_0^1 g^2(s)\,ds \leqq 1\}.$$

(4) If $B = C[0, 1]$ and μ denotes the distribution of the Brownian bridge, then

$$K = \{f \in C[0, 1] : f(t) = \textstyle\int_0^t g(s)\,ds,\ \int_0^1 g^2(s)\,ds \leqq 1, \text{ and } \int_0^1 g(s)\,ds = 0\}.$$

In example (3) ((4)) the set K is precisely the set K_R where $R(s, t) = \min(s, t)$ $(R(s, t) = \min(s, t) - st)$. In general, if μ is any probability measure on $C[0, 1]$ satisfying Lemma 2.1, then

$$R(s, t) = \int_{C[0,1]} x(s)x(t)\,d\mu(x)$$

is continuous on $[0, 1] \times [0, 1]$ and the set K constructed in Lemma 2.1 is equal to the set K_R.

3. A basic convergence result and some corollaries. We first give a general result which will have corollaries dealing with sums of independent identically distributed B-valued random variables as well as with other stochastic processes. In the applications of Theorem 3.1 which we have in mind the Y_n's should be viewed as approximately Gaussian with approximately a fixed covariance structure, and the ϕ_n's are positive constants taken to provide the necessary convergence.

THEOREM 3.1. *Let K denote the unit ball of the Hilbert space $H_\mu \subseteqq B$ where μ is a mean-zero measure on B such that $\int_B \|x\|^2\,du(x) < \infty$. Let $\{Y_n : n \geqq 1\}$ be a sequence of B-valued random variables such that for some sequence of positive constants $\{\phi_n\}$ we have*

$$P\left\{\omega : \limsup_n f\left(\frac{Y_n(\omega)}{\phi_n}\right) \leqq \sup_{x \in K} f(x)\right\} = 1 \qquad f \in B^*. \tag{3.1}$$

Then:

I. *We have*

$$P\left\{\omega : C\left(\left\{\frac{Y_n(\omega)}{\phi_n}\right\}\right) \nsubseteq K\right\} = 0, \tag{3.2}$$

and hence $P\{\omega : \{Y_n(\omega)/\phi_n : n \geqq 1\}$ *is conditionally compact in* $B\} = 1$ iff

$$P\left\{\omega : \lim_n d\left(\frac{Y_n(\omega)}{\phi_n}, K\right) = 0\right\} = 1. \tag{3.3}$$

Here $d(x, K) = \inf_{y \in K} \|x - y\|$.

II. *If* $P\{\omega : \limsup_n f(Y_n(\omega)/\phi_n) = \sup_{x\in K} f(x)\} = 1$ *for* f *in* B^* *and if*

$$P\left\{\omega : \left\{\frac{Y_n(\omega)}{\phi_n} : n \geqq 1\right\} \text{ is conditionally compact in } B\right\} = 1 , \tag{3.4}$$

then H_μ *infinite dimensional implies*

$$P\left\{\omega : C\left(\left\{\frac{Y_n(\omega)}{\phi_n} : n \geqq 1\right\}\right) = K\right\} = 1 . \tag{3.5}$$

PROOF. Let $K(\omega) = C(\{Y_n(\omega)/\phi_n : n \geqq 1\})$ for $\omega \in \Omega$. If $K(\omega) = \phi$, then, of course, $\phi = K(\omega) \subseteqq K$. Now $B - K$ is open and B is separable so

$$B - K = \textstyle\bigcup_{r=1}^\infty N_r ,$$

where each N_r is a closed sphere in B. Then

$$\{\omega : K(\omega) \nsubseteqq K\} = \textstyle\bigcup_{r=1}^\infty \{\omega : K(\omega) \cap N_r \neq \phi\} ,$$

and hence if P^* denotes the outer measure induced by P

$$P^*(\omega : K(\omega) \nsubseteqq K) \leqq \textstyle\sum_{r=1}^\infty P^*(\omega : K(\omega) \cap N_r \neq \phi) .$$

If $P^*(\omega : K(\omega) \nsubseteqq K) > 0$, then $P^*(\omega : K(\omega) \cap N_r \neq \phi) > 0$ for some r and this will produce a contradiction.

To verify this last assertion choose $g \in B^*$ such that

$$\sup_{x\in K} g(x) = \gamma_1 < \gamma_2 = \inf_{x\in N_r} g(x) . \tag{3.6}$$

Then

$$\{\omega : K(\omega) \cap N_r \neq \phi\} \subseteqq \left\{\omega : \limsup_n g\left(\frac{Y_n(\omega)}{\phi_n}\right) \geqq \gamma_2\right\} ,$$

so $P^*(\omega : K(\omega) \cap N_r \neq \phi) > 0$ implies

$$P\left(\omega : \limsup_n g\left(\frac{Y_n(\omega)}{\phi_n}\right) \geqq \gamma_2\right) > 0 .$$

This contradicts (3.1) since (3.6) holds for g. Thus we have

$$P^*(\omega : K(\omega) \nsubseteqq K) = 0 ,$$

and since we assume our probability space to be complete this gives (3.2).

If (3.4) holds, then (3.2) implies (3.3), and the proof of (I) is complete.

Now we establish (II). To do so we need the linear operators Π_N and Q_N defined in (2.4) with $\{S\alpha_k : k \geqq 1\}$ a C.O.N.S. in H_μ such that each $\alpha_k \in B^*$.

Fix $\varepsilon > 0$. First we shown there exists N_0 such that $N \geqq N_0$ implies

$$Q_N K \subseteqq \{x \in B : ||x|| < \varepsilon\} . \tag{3.7}$$

If (3.7) does not hold, then we have a sequence $\{x_j\}$ such that

$$x_j \in Q_j K \qquad \text{and} \qquad ||x_j|| \geqq \varepsilon \qquad\qquad j = 1, 2, \cdots .$$

Now $Q_j K \subseteqq K$ for all $j \geqq 1$ and K compact implies there exists a subsequence j' such that

$$\lim_{j' \to \infty} x_{j'} = z$$

in B. Thus $||z|| \geqq \varepsilon$ and since $\{x_j : j \geqq N\} \subseteqq Q_N K$ for $N = 1, 2, \cdots$ $(Q_1 K \supseteqq Q_2 K \supseteqq \cdots)$ with each $Q_N K$ compact we have $z \in \bigcap_{N \geqq 1} Q_N K$. This is impossible since $\bigcap_{N \geqq 1} Q_N K = \{0\}$ and $||z|| \geqq \varepsilon > 0$. Hence (3.7) holds as indicated.

Therefore for $N \geqq N_0$ we have

$$\text{(3.8)} \qquad \left\{\omega : \limsup_k d\left(Q_N\left(\frac{Y_k(\omega)}{\phi_k}\right), Q_N K\right) \leqq \varepsilon\right\} \subseteqq \left\{\omega : \limsup_k \left\|Q_N\left(\frac{Y_k(\omega)}{\phi_k}\right)\right\| \leqq 2\varepsilon\right\}.$$

Since (3.4) holds we have, as mentioned previously, that (3.3) holds. Since Q_N maps B into B continuously we have

$$\text{(3.9)} \qquad P\left\{\omega : \limsup_k d\left(Q_N\left(\frac{Y_k(\omega)}{\phi_k}\right), Q_N K\right) = 0\right\} = 1,$$

and hence for $N \geqq N_0$ (3.8) implies

$$\text{(3.10)} \qquad P\left\{\omega : \limsup_k \left\|Q_N\left(\frac{Y_k(\omega)}{\phi_k}\right)\right\| \leqq 2\varepsilon\right\} = 1.$$

Choose $h \in K$ and take $N \geqq N_0$ such that $||Q_N h|| \leqq \varepsilon$. Then for an ω-set of probability one we have

$$\text{(3.11)} \qquad \left\|\frac{Y_k(\omega)}{\phi_k} - h\right\| \leqq \left\|\Pi_N\left(\frac{Y_k(\omega)}{\phi_k} - h\right)\right\| + \left\|Q_N\left(\frac{Y_k(\omega)}{\phi_k}\right)\right\| + ||Q_N h|| \leqq \left\|\Pi_N\left(\frac{Y_k(\omega)}{\phi_k} - h\right)\right\| + 3\varepsilon$$

for all k sufficiently large (the largeness of k depends, of course, on ω).

Since K is separable (3.5) follows from (3.11) if

$$\text{(3.12)} \qquad P\left\{\omega : \left\|\Pi_N\left(\frac{Y_k(\omega)}{\phi_k} - h\right)\right\| < \varepsilon \text{ for infinitely many } k\right\} = 1$$

for any $\varepsilon > 0$.

Now $\Pi_N B = \Pi_N H_\mu$ and all norms on a finite dimensional space are equivalent so (3.12) holds if

$$\text{(3.13)} \qquad P\left\{\omega : \left\|\Pi_N\left(\frac{Y_k(\omega)}{\phi_k} - h\right)\right\|_\mu \leqq \varepsilon \text{ i.o. in } k\right\} = 1$$

for each $\varepsilon > 0$.

To show (3.13) we first prove that for every $g \in \Pi_{N+1} K$ such that $||g||_\mu = 1$ we have

$$\text{(3.14)} \qquad P\left(\omega : \left\|\Pi_{N+1}\left(\frac{Y_k(\omega)}{\phi_k} - g\right)\right\|_\mu \leqq \varepsilon \text{ i.o. in } k\right) = 1$$

for each $\varepsilon > 0$. Then (3.13) follows from (3.14) by taking $g = \Pi_N h + c\alpha_{N+1}$ where c is such that $||g||_\mu = 1$. That is,

$$\left\|\Pi_{N+1}\left(\frac{Y_k(\omega)}{\phi_k}\right) - g\right\|_\mu^2 = \left\|\Pi_N\left(\frac{Y_k(\omega)}{\phi_k}\right) - \Pi_N h\right\|_\mu^2 + \left|\alpha_{N+1}\left(\frac{Y_k(\omega)}{\phi_k}\right) - c\right|^2$$

so the event in (3.13) contains the event in (3.14).

Therefore (3.14) is to be established to complete the proof. Take $g \in \Pi_{N+1}K$ such that $||g||_\mu = 1$. Then $g = \sum_{k=1}^{N+1} \alpha_k(g)S\alpha_k$ where $\sum_{k=1}^{N+1} \alpha_k^2(g) = 1$. Futhermore, $g = Sf_0$ where $f_0 = \sum_{k=1}^{N+1} \alpha_k(g)\alpha_k$ is in B^*. Thus if (3.14) does *not* hold there exists a $\delta > 0$ such that

$$P\left\{\omega : \limsup_k f_0\left(\frac{Y_k(\omega)}{\phi_k}\right) \leqq 1 - \delta\right\} > 0 . \tag{3.15}$$

That is,

$$f_0\left(\frac{Y_k(\omega)}{\phi_k}\right) = \sum_{j=1}^{N+1} \alpha_j(g)\alpha_j\left(\frac{Y_k(\omega)}{\phi_k}\right) = \left(\Pi_{N+1}\left(\frac{Y_k(\omega)}{\phi_k}\right), g\right)_\mu , \tag{3.16}$$

and hence $f_0(Y_k(\omega)/\phi_k)$ denotes the length of $\Pi_{N+1}(Y_k(\omega)/\phi_k)$ in the direction g (computed in H_μ). Letting

$$A\left\{\omega : \lim_k d\left(\frac{Y_k(\omega)}{\phi_k}, K\right) = 0 \text{ and } \liminf_{k\to\infty}\left\|\Pi_N \frac{Y_k(\omega)}{\phi_k} - g\right\| \geqq \varepsilon\right\}$$

we have that $P(A) > 0$ if (3.14) fails. Therefore for each $\omega \in A$ there exists a $\delta > 0$ (depending only on N and ε) such that $\limsup_k f_0(Y_k(\omega)/\phi_k) \leqq (g, g)_\mu - \delta = 1 - \delta$. Thus $P(A) > 0$ implies (3.15). Now (3.15) contradicts the condition

$$P\left\{\omega : \limsup_k f_0\left(\frac{Y_k(\omega)}{\phi_k}\right) = \sup_{x\in K} f_0(x)\right\} = 1$$

since $\sup_{x\in K} f_0(x) = \sup_{x\in K}(x, g)_\mu = 1$. Thus (3.14) holds and the proof is complete.

Proof of Corollary 3.1. If H_μ is infinite dimentional, then Corollary 3.1 of Section 1 is an immediate corollary of Theorem 3.1.

To see this recall that K is compact so (1.4) implies (1.6). For the remainder let $Y_n = S_n/n^{\frac{1}{2}}$ and $\phi_n = (2\,\mathrm{LL}\,n)^{\frac{1}{2}}$ in Theorem 3.1. Then by the Hartman–Wintner result applied to the i.i.d. real valued random variables

$$f(X_1), f(X_2), \cdots$$

we have

$$P\left\{\omega : \limsup_n f\left(\frac{Y_n(\omega)}{\phi_n}\right) = (\textstyle\int_B [f(y)]^2\, d\mu(y))^{\frac{1}{2}}\right\} = 1$$

for each $f \in B^*$ where $\mu = \mathscr{L}(X_1)$. By Lemma 2.1 (iv) we have

$$\{\textstyle\int_B [f(y)]^2\, d\mu(y)\}^{\frac{1}{2}} = \sup_{x\in K} f(x)$$

and hence the conditions of Theorem 3.1 hold proving the corollary.

If $\dim H_\mu < \infty$ then Corollary 3.1 follows from a result of H. Finkelstein [7]. That is, if $\dim H_\mu < \infty$, then $P(S_n \in H_\mu) = 1$ for every n and we can work with the H_μ norm instead of the B-norm on H_μ because all locally convex Hausdorff topologies compatible with the vector space structure are equivalent on finite dimensional vector spaces.

For an example where the normalizing constants ϕ_n appearing in Theorem 3.1 are something other than $(2\,\mathrm{LL}\,n)^{\frac{1}{2}}$ we turn to a generalization of some of the recent work of T. L. Lai [20].

COROLLARY 3.2 *Let $\{Y_n : n \geqq 1\}$ be a sequence of B-valued random variables and assume μ is a mean-zero Gaussian measure on B generated by H_μ. Let Q_N ($N \geqq 1$) be the maps defined in* (2.4) *and suppose K is the unit ball of H_μ. Furthermore, assume there exists a sequence of positive constants $\{a_k\}$ such that $\sum_k a_k < \infty$, and for all λ large, and given integer N, there exists a constant $C(N)$ satisfying*

$$P\{\omega : \|Q_N Y_k(\omega)\| \geqq \lambda\} \leqq C(N)\mu(x : \|Q_N x\| > \lambda/2) + a_k \tag{3.17}$$

uniformly in $k = 1, 2, \cdots$. If for every $\varepsilon > 0$ and $f \in B^*$ there exists a positive constant $c(f, \varepsilon)$ such that

$$\begin{aligned} P\{\omega : |f(Y_k(\omega))| &\geqq (1+\varepsilon)\lambda \sup\nolimits_{x\in K} f(x)\} \\ &\leqq c(f,\varepsilon)\{a_k + \mu\{y : |f(y)| > (1+\varepsilon/2)\lambda \sup\nolimits_{x \in K} f(x)\}\} \end{aligned} \tag{3.18}$$

for all λ large and uniformly in $k = 1, 2, \cdots$, then

$$P\left\{\omega ; \lim\nolimits_n \left\| \frac{Y_n(\omega)}{(2\,\mathrm{L}\,n)^{\frac{1}{2}}} - K \right\| = 0\right\} = 1 . \tag{3.19}$$

REMARK. If, in addition to the assumptions in Corollary 3.2, we have $P\{\omega : \limsup_n f(Y_n(\omega)/\phi_n) = \sup_{x\in K} f(x)\} = 1$ for all $f \in B^*$, H_μ is infinite dimensional, and $\phi_n = (2\,\mathrm{L}\,n)^{\frac{1}{2}}$, then

$$P\left(\omega : C\left(\frac{Y_n(\omega)}{(2\,\mathrm{L}\,n)^{\frac{1}{2}}} : n \geqq 1\right) = K\right) = 1 . \tag{3.20}$$

For applications of Corollary 3.2 and conditions sufficient for (3.20) we refer the reader to Section 4.

PROOF OF COROLLARY 3.2. Using Theorem 3.1 we need only verify that (3.1) and (3.4) hold with $\phi_n = (2\,\mathrm{L}\,n)^{\frac{1}{2}}$.

To establish (3.4) we first observe that given $\varepsilon > 0$ and Π_N defined as in (2.4) we have

$$\begin{aligned} P\left(\frac{Y_k}{(2\,\mathrm{L}\,k)^{\frac{1}{2}}} \notin K^\varepsilon \text{ i.o. in } k\right) &\leqq P\left(\frac{\Pi_N Y_k}{(2\,\mathrm{L}\,k)^{\frac{1}{2}}} \notin K^{\varepsilon/2} \text{ i.o. in } k\right) \\ &\quad + P\left(\left\|\frac{Q_N Y_k}{(2\,\mathrm{L}\,k)^{\frac{1}{2}}}\right\| \geqq \frac{\varepsilon}{2} \text{ i.o. in } k\right). \end{aligned} \tag{3.21}$$

Here $K^\varepsilon = \{y \in B : \|y - K\| < \varepsilon\}$, and since K is compact and $\varepsilon > 0$ is arbitrary to prove (3.4) it suffices to show that

$$P\left\{\frac{Y_k}{(2\,\mathrm{L}\,k)^{\frac{1}{2}}} \notin K^{\varepsilon} \text{ i.o. in } k\right\} = 0\,. \tag{3.22}$$

Now (3.22) follows from (3.21) and the Borel–Cantelli lemma if we show there exists an N such that

$$P\left(\frac{\Pi_N Y_k}{(2\,\mathrm{L}\,k)^{\frac{1}{2}}} \notin K^{\varepsilon/2} \text{ i.o. in } k\right) = 0 \tag{3.23}$$

and

$$\textstyle\sum_k P(||Q_N Y_k|| > (\varepsilon/2)(2\,\mathrm{L}\,k)^{\frac{1}{2}}) < \infty\,. \tag{3.24}$$

Choose any integer N and $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\begin{aligned} P\left(\frac{\Pi_N Y_k}{(2\,\mathrm{L}\,k)^{\frac{1}{2}}} \notin K^{\varepsilon/2} \text{ i.o. in } k\right) \\ \leqq P(||\Pi_N Y_k||_\mu \geqq (1+\delta)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \text{ i.o. in } k)\,. \end{aligned} \tag{3.25}$$

The existence of such a $\delta > 0$ follows since K is the unit ball of H_μ, $\Pi_N : B \to \Pi_N H_\mu$, and norms on finite dimensional spaces are equivalent. Now if the right-hand side of (3.25) is positive there exists a continuous linear functional f on $\Pi_N B$ such that the norm of f on $\Pi_N H_\mu$ is 1 and

$$P(|f(\Pi_N Y_k)| \geqq (1+\delta/2)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \text{ i.o. in } k) > 0\,. \tag{3.26}$$

Now extend f to be linear on all of B by the defining equation $f(x) = f(\Pi_N x)$. Then f extended satisfies $||f||_\mu = 1$ so $\sup_{x \in K} f(x) = 1$. Hence by (3.18)

$$\begin{aligned} P(|f(Y_k)| \geqq (1+\delta/2)(2\,\mathrm{L}\,k)^{\frac{1}{2}}) \\ \leqq C(f, \delta/2)\{a_k + \mu(y : |f(y)| > (1+\delta/4)(2\,\mathrm{L}\,k)^{\frac{1}{2}})\}\,. \end{aligned} \tag{3.27}$$

Since f has a mean-zero Gaussian distribution with variance one on B with respect to measure μ we easily have

$$\textstyle\sum_k \mu(y : |f(y)| > (1+\delta/4)(2\,\mathrm{L}\,k)^{\frac{1}{2}}) < \infty\,. \tag{3.28}$$

Thus (3.27), (3.28), $\sum_k a_k < \infty$, and $f(\Pi_N x) = f(x)$ on B imply that

$$\textstyle\sum_k P(\omega : |f(\Pi_N Y_k(\omega)) \geqq (1+\delta/2)(2\,\mathrm{L}\,k)^{\frac{1}{2}}) < \infty\,. \tag{3.29}$$

Using the Borel–Cantelli lemma (3.29) implies (3.26) is a contradiction, and hence the right-hand side of (3.25) is zero. This implies (3.23) and it remains to establish (3.24).

Using (3.17) and $\sum_k a_k < \infty$ it suffices to prove that there exists an integer N such that

$$\textstyle\sum_k \mu(x : ||Q_N x|| > (\varepsilon/4)(2\,\mathrm{L}\,k)^{\frac{1}{2}}) < \infty\,. \tag{3.30}$$

Fix $\beta > 1$ and choose λ so that $2\lambda(\varepsilon/4)^2 \geqq \beta$. Since $||Q_N x|| \to 0$ with μ-probabiiity one [12] we have by [21] that there exists an N_0 such that $N \geqq N_0$ implies

$$\textstyle\int_B \exp\{\lambda||Q_N x||^2\}\mu(dx) < \infty\,. \tag{3.31}$$

For fixed $N \geqq N_0$ and λ we then obtain

$$(3.32) \qquad \mu(x\colon ||Q_N x|| \geqq (\varepsilon/4)(2\,\mathrm{L}\,k)^{\frac{1}{2}}) \leqq C\exp\{-2\lambda(\varepsilon/4)^2\,\mathrm{L}\,k\} \leqq \frac{C}{k^B}$$

where C is a positive constant. Hence (3.30) holds and the corollary is proved.

4. Applications of the general results. Some situations where Theorem 3.1 and Corollaries 3.1 and 3.2 easily apply are examined here. We first consider sums of i.i.d. random variables and applications of Corollary 3.1.

For the next theorem we need the notation and terminology of [16] and [17].

The norm $||\cdot||$ on B is *twice directionally differentiable* on $B - \{0\}$ if for $x, y \in B$, $x + ty \neq 0$, we have

$$\frac{d}{dt}||x + ty|| = D(x + ty)(y)\,,$$

where $D\colon B - \{0\} \to B^*$ is measurable from the Borel subsets of B generated by the norm topology to the Borel subsets of B^* generated by the weak-star topology, and

$$(4.0) \qquad \frac{d^2}{dt^2}||x + ty|| = D^2_{x+ty}(y, y)\,,$$

where D_x^2 is a bounded bilinear form on $B \times B$. We call D_x^2 the *second directional derivative* of the norm, and without loss of generality we can assume D_x^2 is a symmetric bilinear form. That is, if T_x is a bilinear form which satisfies (4.0) then $\Lambda_x(y, z) = [T_x(y, z) + T_x(z, y)]/2$ also satisfies (4.0) and Λ_x is symmetric. Hence in all that follows we assume D_x^2 is a symmetric bilinear form. Of course, if the norm is actually twice Frechet differentiable on B with second derivative at x given by Λ_x, then it is well known that Λ_x is a symmetric bilinear form on $B \times B$ and in this case D_x^2 would be equal to Λ_x since symmetric bilinear forms are uniquely determined on the diagonal of $B \times B$.

If $D_x^2(y, y)$ is continuous in x ($x \neq 0$) and for all $r > 0$ and $x, h \in B$ such that $||x|| \geqq r$ and $||h|| \leqq r/2$ we have

$$|D^2_{x+h}(h, h) - D_x^2(h, h)| \leqq C_r||h||^{2+\alpha}$$

for some fixed $\alpha > 0$ and some constant C_r we say *the second directional derivative is* Lip (α) *away from zero.*

We now can state Theorem 4.1.

THEOREM 4.1. *Let B denote a real separable Banach space with norm $||\cdot||$. Let $||\cdot||$ be twice directionally differentiable on $B - \{0\}$ with the second directional derivative D_x^2 being* Lip (1) *away from zero and such that*

$$(4.1) \qquad \sup_{||x||=1} ||D_x^2|| < \infty\,.$$

Let $X_1, X_2, \cdots$ be i.i.d. *B-valued random variables such that*

$$(4.2) \qquad E(X_1) = 0 \quad \textit{and} \quad E||X_1||^2 < \infty\,,$$

and set $S_n = X_1 + \cdots + X_n$ $(n \geqq 1)$. *Let*

$$T(f, g) = E(f(X_1)g(X_1)) \qquad f, g \in B^* \tag{4.3}$$

denote the common covariance function and assume $\mathscr{L}(S_n/n^{\frac{1}{2}})$ *converges weakly to a measure* μ. *For each integer n let*

$$\begin{aligned} Y_n &= X_1 && \text{iff} \quad \|X_1\| \leqq n^{\frac{1}{2}} \\ &= 0 && \textit{otherwise} \end{aligned}$$

and define the covariance functions

$$T_n(f, g) = E(f(Y_n - E(Y_n))g(Y_n - E(Y_n))) \qquad f, g \in B^* . \tag{4.4}$$

Then, μ *is a mean zero Gaussian measure on B with covariance function T. Further, if each* T_n *is the covariance function of a mean-zero Gaussian measure* μ_n *on B such that* $\{\mu_n\}$ *converges weakly to* μ, *then* (1.3) *and* (1.4) *hold with K the unit ball of* H_μ.

Remark. It was shown in [17] that real separable Hilbert spaces as well as the L^p spaces $(3 \leqq p < \infty)$ satisfy the smoothness conditions required of the norm in Theorem 4.1. In addition, it is known that in these spaces the classical central limit theorem holds for i.i.d. random variables under the conditions given in (4.2). Hence for such spaces the assumptions in Theorem 4.1 reduce to the X_k's being i.i.d. random variables satisfying (4.2), and that the Gaussian measures $\{\mu_n\}$ converge weakly to the Gaussian measure μ. In case the space B is a real separable Hilbert space or a sequence space of the type l^p $(3 \leqq p < \infty)$ then by [13], Corollary 5.2) we always have $\{\mu_n\}$ converging weakly to μ so our assumptions are precisely the classical ones in these spaces. In [17] a result similar to Theorem 4.1 was proved, but a slightly stronger moment assumption was used. Finally we note that the trunaction argument employed in the proof of Theorem 4.1 leans heavily on some of the ideas in [11].

Proof of Theorem 4.1. First of all observe that μ must be a mean-zero Gaussian measure with covariance function T since its finite dimensional distributions are of this form.

As a result of Corollary 3.1 and Lemma 2.1(v) we need only verify (1.9). Since K is compact in B by Lemma 2.1 we need only prove that for each $\varepsilon > 0$

$$P\left\{\frac{S_n}{(2n \operatorname{LL} n)^{\frac{1}{2}}} \notin K^\varepsilon \text{ i.o. in } n\right\} = 0 . \tag{4.5}$$

Arguing as in [16] we have (4.5) holding if

$$\textstyle\sum_r P(B_r) < \infty , \tag{4.6}$$

where

$$B_r = \left\{\frac{S_n}{(2n_r \operatorname{LL} n_r)^{\frac{1}{2}}} \notin K^\varepsilon \text{ for some } n : n_r \leqq n \leqq n_{r+1}\right\} ; \tag{4.7}$$

$n_r = [\beta^r]$ denotes the greatest integer $\leqq \beta^r$, and $\beta > 1$. Further, we have as in

[16] that for any fixed $\varepsilon > 0$ there exists a $\beta > 1$ sufficiently close to one and a $\delta > 0$ such that for all r sufficiently large

$$(4.8) \qquad P(B_r) \leqq 2P\left(\frac{S_{n_{r+1}}}{(n_{r+1})^{\frac{1}{2}}} \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}} K^{\delta}\right).$$

Let Π_N and Q_N be defined as in (2.4). Then for each integer N we have

$$(4.9) \qquad P\left(\frac{S_{n_{r+1}}}{(n_{r+1})^{\frac{1}{2}}} \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}} K^{\delta}\right) \leqq P\left(\Pi_N\left(\frac{S_{n_{r+1}}}{(n_{r+1})^{\frac{1}{2}}}\right) \notin (2\,\mathrm{LL}\,n_r)^{\frac{1}{2}} \Pi_N(K^{\gamma})\right) + P\left(\left\|Q_N\left(\frac{S_{n_{r+1}}}{(n_{r+1})^{\frac{1}{2}}}\right)\right\| \geqq \frac{\delta}{2}(2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\right),$$

where $\gamma > 0$ is such that if $K^{\gamma} = \{x \in H_\mu : \|x - K\|_\mu < \gamma\}$ then $\Pi_N(K^{\gamma}) \subseteqq \Pi_N(K^{\delta/2})$. The existence of such a $\gamma > 0$ is obvious since $\Pi_N B$ is finite-dimensional and hence all norms are equivalent on $\Pi_N B$ (γ may depend on N but this will be no problem). Further,

$$(4.10) \qquad \Pi_N K^{\gamma} = \{x \in \Pi_N H_\mu : \|x\|_\mu < 1 + \gamma\}.$$

Now (4.6) follows from (4.8), (4.9) and (4.10) provided there exists an N such that

$$(4.11) \qquad \sum_r P\left(\left\|\frac{Q_N S_{n_{r+1}}}{(n_{r+1})^{\frac{1}{2}}}\right\| > \frac{\delta}{2}(2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\right) < \infty$$

and

$$(4.12) \qquad \sum_r P\left(\left\|\Pi_N\left(\frac{S_{n_{r+1}}}{(n_{r+1})^{\frac{1}{2}}}\right)\right\|_\mu \geqq (1 + \gamma)(2\,\mathrm{LL}\,n_r)^{\frac{1}{2}}\right).$$

We will establish (4.11) as the argument for (4.12) is entirely similar.

Recall $\delta > 0$ is fixed and next we will choose N. Define for each $j, n \geqq 1$

$$(4.13) \qquad \begin{aligned} X_{j,n} &= X_j \qquad \text{if } \|X_j\| \leqq n^{\frac{1}{2}} \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then the truncated random variables $\{X_{j,n} : j \geqq 1\}$ are independent and identically distributed, and have common covariance function T_n as given in (4.4), Let

$$(4.14) \qquad \alpha_n = E(X_{j,n}) = \int_{\{\|x\| \leqq n^{\frac{1}{2}}\}} x\nu(dx) \qquad n = 1, 2, \cdots$$

where ν is the common distribution of X_j's. Let $\{f_j : j \geqq 1\}$ be a weak-star dense subset of the unit ball of B^* such that $\|x\| = \sup(f_j(x))$ and fix s such that $0 < s < ((\delta^2 \log 3)/192\beta)^{\frac{1}{2}}$. Since $\lim_{N\to\infty} \|Q_N x\| = 0$ with μ-probability one [12] we next fix N sufficiently large so that

$$(4.15) \qquad \mu(x \in B : \|Q_N x\| \leqq s) \geqq \tfrac{7}{8}.$$

Since $\{\mu_n : n \geqq 1\}$ converges weakly to the measure μ and $\mu(x \in B : \|Q_N x\| \leqq s)$ is a continuous function of s ([8], Corollary 5.1), we have for all sufficiently large n that

$$\mu_n(x \in B\colon \|Q_N x\| \leqq s) \geqq \tfrac{3}{4}. \tag{4.16}$$

Now $\|x\| = \sup_j |f_j(x)|$ for every $x \in B$ and hence by equation (4.8) of [21] we have

$$\mu_n(x \in B\colon \|Q_N x\| \geqq t) \leqq \exp\left\{\frac{-t^2}{24s^2}\log 3\right\} \leqq \exp\left\{\frac{-8\beta t^2}{\delta^2}\right\} \tag{4.17}$$

for $n = 1, 2, \cdots$ and $t \geqq 0$.

Assuming N is fixed so that (4.15) and hence (4.17) holds we let

$$S_n' = X_{1,n} + \cdots + X_{n,n} \qquad n \geqq 1. \tag{4.18}$$

Then we have

$$\begin{aligned} P\left(\left\|\frac{Q_N S_n}{n^{\frac{1}{2}}}\right\| \geqq t\right) &\leqq P\left(\left\|Q_N \frac{S_n'}{n^{\frac{1}{2}}}\right\| \geqq t\right) + nP(\|X_1\| > n^{\frac{1}{2}}) \\ &\leqq P\left(\left\|Q_N\left(\frac{S_n' - n\alpha_n}{n^{\frac{1}{2}}}\right)\right\| \geqq t - n^{\frac{1}{2}}\|Q_N\alpha_n\|\right) \\ &\quad + nP(\|X_1\| > n^{\frac{1}{2}}) \end{aligned} \tag{4.19}$$

where α_n is as in (4.14). Further, since $\int_B xv(dx) = E(X_1) = 0$ we have $\alpha_n = -\int_{\{\|x\|>n^{\frac{1}{2}}\}} xv(dx)$ and hence

$$\begin{aligned} \limsup_n n^{\frac{1}{2}}\|\alpha_n\| &\leqq \limsup_n n^{\frac{1}{2}} \int_{\{\|x\|>n^{\frac{1}{2}}\}} \|x\| v(dx) \\ &\leqq \limsup_n \int_{\{\|x\|>n^{\frac{1}{2}}\}} \|x\|^2 v(dx) = 0 \end{aligned} \tag{4.20}$$

as $\int_B \|x\|^2 v(dx) < \infty$. Since Q_N is continuous and linear from B into B thus we have $\lim_n n^{\frac{1}{2}} Q_N \alpha_n = 0$ and for all but finitely many n we have

$$P\left(\left\|\frac{Q_n S_n}{n^{\frac{1}{2}}}\right\| \geqq t\right) \leqq P\left(\left\|Q_N\left(\frac{S_n' - n\alpha_n}{n^{\frac{1}{2}}}\right)\right\| \geqq \frac{3t}{4}\right) + nP(\|X_1\| > n^{\frac{1}{2}}) \tag{4.21}$$

uniformly in $t \geqq 1$. Applying ([16], Theorem 2.1) we obtain

$$P\left(\left\|\frac{Q_N S_n}{n^{\frac{1}{2}}}\right\| \geqq t\right) \leqq \mu_n\left(x \in B\colon \|Q_N x\| \geqq \frac{t}{2}\right) + \frac{C_n}{n^{\frac{1}{2}}} + nP(\|X_1\| > n^{\frac{1}{2}}), \tag{4.22}$$

where

$$C_n \leqq CE\|Q_N(X_{1,n} - \alpha_n)\|^3 \tag{4.23}$$

for an absolute constant C which is independent of $t \geqq 1$. Now

$$\begin{aligned} E\|Q_N(X_{1,n} - \alpha_n)\|^3 &\leqq 4\{E\|Q_N X_{1,n}\|^3 + \|Q_N\alpha_n\|^3\} \\ &\leqq 4C'\{E\|X_{1,n}\|^3 + \|\alpha_n\|^3\}, \end{aligned} \tag{4.24}$$

where C' is independent of n. Combining (4.22), (4.23), (4.24) and (4.17) we obtain for all $t \geqq 1$ and all but finitely many n that

$$\begin{aligned} P\left(\left\|\frac{Q_N S_n}{n^{\frac{1}{2}}}\right\| \geqq t\right) &\leqq 2\exp\left\{\frac{-2\beta t^2}{\delta^2}\right\} + \frac{4C'}{n^{\frac{1}{2}}}(E\|X_{1,n}\|^3 + \|\alpha_n\|^3) \\ &\quad + nP(\|X_1\| > n^{\frac{1}{2}}). \end{aligned} \tag{4.25}$$

Using (4.25) and setting $t = (\delta/2)(2\,\mathrm{LL}\, n_r)^{\frac{1}{2}}$ we have (4.11) if

$$\sum_r n_r P(||X_1|| > n_r^{\frac{1}{2}}) > \infty \tag{4.26}$$

and

$$\sum_r \frac{E||X_{1,n_r}||^3}{n_r} < \infty \tag{4.27}$$

since all other terms form a convergent series (recall $n_r = [\beta^r]$ where $\beta > 1$).

To verify (4.26) and (4.27) we first note that $P\{||X_1|| > k^{\frac{1}{2}}\}$ is decreasing in k and that

$$\sum_k P(||X_1|| > k^{\frac{1}{2}}) = \sum_k P(||X_1||^2 > k) \leqq E||X_1||^2 < \infty .$$

Thus for some positive constant ρ

$$\begin{aligned} \infty > \sum_{k\geqq 1} P(||X_1|| > k^{\frac{1}{2}}) &= \sum_{r=1}^{\infty} \sum_{j=n_{r-1}+1}^{n_r} P(||X_1|| > j^{\frac{1}{2}}) \\ &\geqq \sum_{r=1}^{\infty} (n_r - n_{r-1}) P(||X_1|| > n_r^{\frac{1}{2}}) \\ &\geqq \rho \cdot \sum_r n_r P(||X_1|| > n_r^{\frac{1}{2}}) \end{aligned}$$

since $n_r - n_{r-1} = [\beta^r] - [\beta^{r-1}] \sim [\beta^r](1 - 1/\beta)$. Hence (4.26) holds.

Letting $a_n = E||X_{1,n}||^3$ we see

$$a_n \leqq \sum_{k=1}^{n} k^{\frac{3}{2}} P(k-1 < ||X_1||^2 \leqq k) ,$$

and hence

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-\frac{3}{2}} a_n &\leqq \sum_{n=1}^{\infty} n^{-\frac{3}{2}} \sum_{k=1}^{n} k^{\frac{3}{2}} P(k-1 < ||X_1||^2 \leqq k) \\ &\leqq \sum_{k=1}^{\infty} k^{\frac{3}{2}} P(k-1 < ||X_1||^2 \leqq k) \cdot \sum_{n=k}^{\infty} n^{-\frac{3}{2}} \\ &= O(\sum_{k\geqq 1} kP(k-1 \leqq ||X_1||^2 \leqq k)) \\ &= E||X_1||^2 < \infty . \end{aligned}$$

Since a_n is increasing in n we have for some $\rho > 0$ that

$$\begin{aligned} \infty > \sum_{n\geqq 1} \frac{a_n}{n^{\frac{3}{2}}} &\geqq \sum_{r=1}^{\infty} \sum_{j=n_r+1}^{n_{r+1}} \frac{a_j}{j^{\frac{3}{2}}} \\ &\geqq \sum_{r=1}^{\infty} (n_{r+1} - n_r) \frac{a_{n_r}}{n_{r+1}^{\frac{3}{2}}} \\ &= \rho \cdot \sum_{r=1}^{\infty} \frac{a_{n_r}}{n_r^{\frac{1}{2}}} \end{aligned}$$

since $(n_{r+1} - n_r)/n_{r+1}^{\frac{3}{2}} = (1/n_{r+1}^{\frac{1}{2}})(1 - n_r/n_{r+1}) \sim (1/n_{r+1}^{\frac{1}{2}})(1 - 1/\beta)$. Hence (4.27) holds and (4.11) is established.

Since N is fixed we can prove (4.12) using classical Berry–Esseen estimates and the truncation argument provided above. Hence the theorem is proved.

PROOF OF THEOREM 4.2. Since K_R is a compact subset of $C[0, 1]$, then (1.14) clearly implies (1.15).

Hence assume (1.15 i) or (1.15 ii) hold. In view of Lemma 1.1 we have (1.14) if (1.12 i) holds with $K = K_R$ and any finite subset T of $[0, 1]$.

Using (1.13) example (1) of Section 2, and the law of the iterated logarithm for random variables taking values in a finite dimensional space we have (1.12 i) if, given

$$K_R = \{\sum_n c_n \phi_n : \sum_n c_n{}^2/\lambda_n \leqq 1\},$$

then the set $K_R(T) \equiv \{(x(t))_{t \in T} : x \in K_R\}$ is just the set $K_\Sigma \equiv \{\Sigma^{\frac{1}{2}}(x) : x \in \mathbb{R}^n, ||x|| \leqq 1\}$ for $T = (t_1, \cdots, t_n)$ and $\Sigma = (\sigma_{ij})$, $\sigma_{ij} = E(X(t_i)X(t_j)) = R(t_i, t_j)$.

First we show $K_R(T) \subseteqq K_\Sigma$. To do so we first make precise some notation. Let $||\cdot||_R$ denote the norm for H_R and let R_T denote the restriction of the kernel R to $T \times T$. Let H_{R_T} be the RKHS of the kernel R_T, i.e.,

$$H_{R_T} = \{\{\sum_{j=1}^n c_j R(t_i, t_j) : 1 \leqq i \leqq n\} : (c_1, \cdots, c_n) \in R^n\}$$

with norm

$$||\{\sum_{j=1}^n c_j R(t_i, t_j) : 1 \leqq i \leqq n\}||_{H_{R_T}} = (\sum_{i,j=1}^n c_i c_j R(t_i, t_j))^{\frac{1}{2}}$$

If $x \in H_R$, then by [1], page 351, we have the vector $(x(t_1), \cdots, x(t_n)) \equiv x_T$ such that

$$||x_T||_{H_{R_T}} = \inf\{||f||_{H_R} : f \in H_R, f_T = x_T\}.$$

Hence $x \in K_R$ implies $||x_T||_{H_{R_T}} \leqq 1$ so

$$K_R(T) \subseteqq K_{R_T} = K_\Sigma \quad \text{(by example (1))}.$$

On the other hand, if $x \in K_{R_T}$ then $x = (x_1, \cdots, x_n)$ where

$$x_i = \sum_{j=1}^n c_j R(t_i, t_j) \qquad i = 1, \cdots, n,$$

and $||x||^2_{H_{R_T}} = \sum_{i,j=1}^n c_i c_j R(t_i, t_j) \leqq 1$. If

$$g(s) = \sum_{j=1}^n c_j R(s, t_j) \qquad 0 \leqq s \leqq 1,$$

then $g_T = x$ and $g \in K_R$ since $||g||^2_{HR} = \sum_{i,j=1}^n c_i c_j R(t_i, t_j) \leqq 1$. Hence the theorem is proved.

PROOF OF COROLLARIES 4.1 AND 4.2. Let $T_r = \{i/2^r : 0 \leqq i \leqq 2^r\}$ and define $\Lambda_r = \Lambda_{T_r}$ for $r \geqq 1$. The corollaries hold if we establish (1.15 ii) or, equivalently, if for every $\varepsilon > 0$ we prove there exists $r(\varepsilon)$ such that $r \geqq r(\varepsilon)$ and $\Lambda = \Lambda_r$ implies

$$P\{||H_n - \Lambda(H_n)||_\infty \geqq \varepsilon \text{ i.o. in } n\} = 0. \tag{4.28}$$

To prove (4.28) we first prove that for any $\varepsilon > 0$

$$\lim_n P\{||S_n - \Lambda(S_n)||_\infty \geqq \varepsilon(2n \operatorname{LL} n)^{\frac{1}{2}}\} = 0. \tag{4.29}$$

Now (4.29) follows since for $\Lambda = \Lambda_r$ and $D_j = [(j-1)/2^r, j/2^r]$ we have

$$\begin{aligned} &P\{||S_n - \Lambda(S_n)||_\infty > \varepsilon(2n \operatorname{LL} n)^{\frac{1}{2}}\} \\ &\quad \leqq \sum_{j=1}^{2^r} P\left\{\sup_{t \in D_j} \left|S_n(t) - S_n\left(\frac{j-1}{2^r}\right)\right| > \frac{\varepsilon}{2}(2n \operatorname{LL} n)^{\frac{1}{2}}\right\} \\ &\quad \leqq \sum_{j=1}^{2^r} 2E\left\{\left|\frac{S_n(j/2^r) - S_n((j-1)/2^r)}{(\varepsilon/2)(2n \operatorname{LL} n)^{\frac{1}{2}}}\right|^2\right\}^{\frac{1}{2}} = O\left(\frac{1}{(\operatorname{LL} n)^{\frac{1}{2}}}\right), \end{aligned}$$

where the last inequality is a result of the maximal inequality ([4], page 353) applied to the submartingale $|S_n(t) - S_n((j-1)/2^r)|^2((j-1)/2^r \leqq t \leqq j/2^r)$.

Now we prove (4.28) provided $\Lambda = \Lambda_r$ where r is sufficiently large so that $r \geqq r(\varepsilon)$ implies

$$\sigma_{j,r}^2 = E\left(\left(X_1\left(\frac{j}{2^r}\right) - X_1\left(\frac{j-1}{2^r}\right)\right)^2\right) \leqq \left(\frac{\varepsilon}{96}\right)^2 \frac{1}{(1+\varepsilon)^2} \tag{4.30}$$

for $j = 1, \cdots, 2^r$. That $\lim_{r\to\infty} \sup_j \sigma_{j,r}^2 = 0$ follows immediately from the continuity of $R(s, t)$ on $[0, 1] \times [0, 1]$. Henceforth we set $\Lambda = \Lambda_r$ where $r \geqq r(\varepsilon)$ and (4.30) is satisfied.

Let $A_n = \{\|S_n/(2n \operatorname{LL} n)^{\frac{1}{2}} - \Lambda(S_n)/(2n \operatorname{LL} n)^{\frac{1}{2}}\|_\infty \geqq \varepsilon\}$ for $n = 1, 2, \cdots$. Then $\limsup_n A_n \subseteqq \limsup_k B_k$ where

$$B_k = \left\{\left\|\frac{S_n}{(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}} - \frac{\Lambda(S_n)}{(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}}\right\|_\infty \geqq \varepsilon \text{ for some } n(n_k \leqq n \leqq n_{k+1})\right\}$$

and $n_k = [\beta^k]$ with $1 < \beta \leqq 2$. Now by the same proof as given in [2], page 45, for real valued random variables

$$P(B_k) \leqq \frac{1}{1-d} P\left(\|S_{n_{k+1}} - \Lambda(S_{n_{k+1}})\|_\infty \geqq \frac{\varepsilon}{2}(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\right), \tag{4.31}$$

where $d = \sup_{n_k \leqq n \leqq n_{k+1}} P\{\|(I-\Lambda)S_{n_{k+1}} - (I-\Lambda)S_n\|_\infty > (\varepsilon/2)(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\}$. By (4.29) $d \leqq \frac{1}{2}$ for k sufficiently large and hence for large k

$$\begin{aligned} P(B_k) &\leqq 2P\|S_{n_{k+1}} - \Lambda(S_{n_{k+1}})\|_\infty > \frac{\varepsilon}{2}(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\Big) \\ &\leqq 2 \textstyle\sum_{j=1}^{2^r} P\left\{\sup_{t \in D_j}\left|S_{n_{k+1}}(t) - S_{n_{k+1}}\left(\frac{j-1}{2^r}\right)\right| > \frac{\varepsilon}{4}(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\right\} \end{aligned} \tag{4.32}$$

where $D_j = [(j-1)/2^r, j/2^r]$ $(j = 1, \cdots, 2^r)$.

In case the $\{X_j(t)\}$ are independent increment processes we have for large k (by the same argument used in (4.31)) that

$$\begin{aligned} &P\left(\sup_{t \in D_j}\left|S_{n_{k+1}}(t) - S_{n_{k+1}}\left(\frac{j-1}{2^r}\right)\right| > \frac{\varepsilon}{4}(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\right) \\ &\quad\leqq 2P\left(\left|S_{n_{k+1}}\left(\frac{j}{2^r}\right) - S_{n_{k+1}}\left(\frac{j-1}{2^r}\right)\right| > \frac{\varepsilon}{8}(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\right) \\ &\quad\leqq 2P\left\{\left|\frac{S_{n_{k+1}}(j/2^r) - S_{n_{k+1}}((j-1)/2^r)}{(n_{k+1} \cdot \sigma_{j,r}^2)^{\frac{1}{2}}}\right| > \frac{\varepsilon}{8}\left(\frac{2n_k \operatorname{LL} n_k}{n_{k+1}\sigma_{j,k}^2}\right)^{\frac{1}{2}}\right\} \\ &\quad\leqq 2P\left\{\left|\frac{S_{n_{k+1}}(j/2^r) - S_{n_{k+1}}((j-1)/2^r)}{(n_{k+1}\sigma_{j,r}^2)^{\frac{1}{2}}}\right| > (1+\varepsilon)(2 \operatorname{LL} n_k)^{\frac{1}{2}}\right\} \end{aligned} \tag{4.33}$$

provided $(\varepsilon/8)(n_k/n_{k+1}\sigma_{j,r}^2)^{\frac{1}{2}} > 1 + \varepsilon$. Now this last inequality results from (4.30) since $n_k/n_{k+1} \geqq \frac{1}{2}$.

Now (4.28) follows if $\sum_k P(B_k) < \infty$ and these probabilities sum by applying (4.33) and the truncation method applied in the proof of Theorem 4.1. Here things are much easier as $S_{n_{k+1}}(j/2^r) - S_{n_{k+1}}((j-1)/2^r) = \sum_{s=1}^{n_{k+1}} [X_s(j/2^r) - X_s((j-1)/2^r)]$ and the random variables summed are real valued, mean zero, and have variance $\sigma^2_{j,r}$. Hence the classical Berry–Esseen estimates [6] apply when we execute the truncation technique in this setup. Thus (4.28) holds if the processes are independent increment processes and therefore Corollary 4.2 is proved.

If we have $(2+\delta)$ moments and the $\{X_k(t)\colon k \geqq 1\}$ are martingales we have by the maximal inequality for submartingales ([4], page 353) that

$$\begin{aligned} &P\left\{\sup_{t\in D_j}\left|S_{n_{k+1}}(t) - S_{n_{k+1}}\left(\frac{j-1}{2^r}\right)\right| > \frac{\varepsilon}{4}(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}\right\} \\ &\quad\leqq \frac{4(n_{k+1})^{\frac{1}{2}}}{\varepsilon(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}} E\left(\frac{(S_{n_{k+1}}(j/2^r) - S_{n_{k+1}}((j-1)/2^r))^2}{n_{k+1}}\right)^{\frac{1}{2}} \cdot P(A_{k,j,\varepsilon})^{\frac{1}{2}} \qquad (4.34) \\ &\quad\leqq \frac{4(n_{k+1})^{\frac{1}{2}}}{\varepsilon(2n_k \operatorname{LL} n_k)^{\frac{1}{2}}} \cdot \sigma_{j,r} \cdot P(A_{k,j,\varepsilon})^{\frac{1}{2}} = O(P(A_{k,j,\varepsilon})^{\frac{1}{2}}, \end{aligned}$$

where

$$A_{k,j,\varepsilon} = \left\{\left|S_{n_{k+1}}\left(\frac{j}{2^r}\right) - S_{n_{k+1}}\left(\frac{j-1}{2^r}\right)\right| > \frac{\varepsilon}{4}(2n_k \operatorname{LL} n_k^{\frac{1}{2}})\right\}.$$

Let $Y_\alpha = X_\alpha(j/2^r) - X_\alpha((j-1)/2^r)$ $(\alpha = 1, 2, \cdots)$. Then the Y_α's are i.i.d. with mean zero and $E(Y_\alpha^2) = \sigma^2_{j,r} \leqq (\varepsilon/96)^2(1/(1+\varepsilon)^2)$. Hence for large k

$$\begin{aligned} P(A_{k,j,\varepsilon}) &= P\left(\left|\frac{Y_1 + \cdots + Y_{n_{k+1}}}{(n_{k+1}\sigma^2_{j,r})^{\frac{1}{2}}}\right| > \frac{\varepsilon}{4}\left(\frac{2n_k \operatorname{LL} n_k}{n_{k+1}\sigma^2_{j,r}}\right)^{\frac{1}{2}}\right) \qquad (4.35) \\ &= P\left(\left|\frac{Y_1 + \cdots + Y_{n_{k+1}}}{(n_{k+1}\sigma^2_{j,r})^{\frac{1}{2}}}\right| \geqq 4(1+\varepsilon)(2 \operatorname{LL} n_k)^{\frac{1}{2}}\right) \end{aligned}$$

since (4.30) and $n_k/n_{k+1} \geqq \frac{1}{2}$ imply $(\varepsilon/4)(n_k/n_{k+1}\sigma^2_{j,r})^{\frac{1}{2}} \geqq 4(1+\varepsilon)$. Since the Y_α's have $(2+\delta)$ moments, (4.35) and the Berry–Esseen estimates [6] imply that

$$P(A_{k,j,\varepsilon}) \leqq P\left(|U| > 4(1+\varepsilon)(2 \operatorname{LL} n_k)^{\frac{1}{2}}\right) + O\left(\frac{1}{n_k^{\delta/2}}\right), \qquad (4.36)$$

where U is a normal random variable with mean zero and variance 1. Thus

$$\begin{aligned} P(A_{k,j,\varepsilon}) &\leqq \frac{1}{(1+\varepsilon)(2 \operatorname{LL} n_k)^{\frac{1}{2}}} e^{-8(1+\varepsilon)^2(2 \operatorname{LL} n_k)} + O\left(\frac{1}{n_k^{\delta/2}}\right) \\ &\leqq \frac{1}{(\operatorname{L} n_k)^{16(1+\varepsilon)^2}} + O\left(\frac{1}{n_k^{\delta/2}}\right) \qquad (4.37) \\ &\leqq O\left(\frac{1}{k^{16}\log\beta}\right). \end{aligned}$$

Thus by (4.32), (4.34) and (4.37) we have

$$\sum_k P(B_k) < \infty.$$

Hence (4.28) holds in the martingale case, and Corollary 4.1 is proved.

PROOF OF COROLLARY 4.3. Let $X_k(t, \omega) = 1_{[0,t]}(U_k(\omega)) - t$ for $k \geqq 1$. Then

$n(F_n(t, \omega) - t) = \sum_{k=1}^n X_k(t, \omega)$ and the covariance structure of each X_k is given by (1.19). Furthermore, it is shown in (J. Kiefer, ZfW **24** 1972) that $X_k(t)/(1-t)$ is a martingale in t, $0 \leqq t < 1$, so an easy application of Corollary 4.1 gives

$$P\left(\omega : \left\{\frac{n(F_n(t, \omega) - t)}{(2n \operatorname{LL} n)^{\frac{1}{2}}} : n \geqq 1\right\} \text{ asymptotically equicontinuous on } [0, 1-\delta]\right) = 1 \tag{4.38}$$

for any $\delta > 0$. Symmetry considerations immediately give that (4.38) also holds on the set $[\delta, 1]$, and hence Theorem 4.2 yields (1.14) to complete the proof.

We now turn to applications of Corollary 3.2.

Let μ_1 and μ_2 be probability measures on the Borel subsets of the metric space (M, d). Let $\mathscr{C}$ denote the closed sets of (M, d), and define for each $\varepsilon > 0$ and subset A of M the set $A^\varepsilon = \{y \in M : d(y, A) < \varepsilon\}$ where, of course, $d(y, A) = \inf_{x \in A} d(y, x)$. Let $\varepsilon_{12} = \inf\{\varepsilon > 0 : \mu_1(F) \leqq \mu_2(F^\varepsilon) + \varepsilon \ \forall F \in \mathscr{C}\}$ and $\varepsilon_{21} = \inf\{\varepsilon > 0 : \mu_2(F) \leqq \mu_1(F^\varepsilon) + \varepsilon \ \forall F \in \mathscr{C}\}$. Define $L(\mu_1, \mu_2) = \max(\varepsilon_{12}, \varepsilon_{21})$. Then L is the Prokhorov metric on the class of all Borel probability measures on M and weak convergence for these measures is equivalent to L-convergence [24] provided (M, d) is complete and separable.

If X is a B-valued random variable, then the probability distribution X induces on B is denoted by $\mathscr{L}(X)$.

THEOREM 4.3. *Let $\{Y_k : k \geqq 1\}$ be a sequence of B-valued random variables and assume μ is a mean-zero Gaussian measure on B generated by H_μ. Let K denote the unit ball of H_μ. If*

$$L(\mathscr{L}(Y_k), \mu) = b_k \tag{4.39}$$

where $\sum_k b_k < \infty$ and L is the Prokhorov metric for measures on $(B, \|\cdot\|)$, then (3.19) *holds. If the Y_k's are independent random variables as well, then* (3.20) *holds.*

COROLLARY 4.4. *Let $\{Y_k : k \geqq 1\}$, μ, H_μ and K be as in Theorem* 4.3. *If $\mathscr{L}(Y_k) = \mu$ for all $k \geqq 1$ then* (3.19) *holds. If we also have*

$$\lim_{m\to\infty; k-m\to\infty} E(\{E(f(Y_k) \mid \mathscr{F}_m)\}^2) = 0 \tag{4.40}$$

for every $f \in B^*$ where $\mathscr{F}_m = \mathscr{F}(Y_k : k \leqq m)$, then (3.20) *holds.*

REMARK. Corollary 4.4 is due to T. L. Lai in [20].

PROOF OF THEOREM 4.3. It follows immediately from the definition of the Prokhorov metric L and (4.39) that for all large λ and given integer N we have (3.17) with $C(N) = 1$ and $a_k = b_k$. Recall here that $Q_N : B \to B$ continuously so $\{x : \|Q_N x\| \geqq \lambda\}$ is a closed subset of B. Similary, if $f \in B^*$ then $\{y : |f(y)| \geqq (1+\varepsilon)\lambda \sup_{x \in K} f(x)\}$ is a closed set in B so (3.18) also follows from (4.39) with $C(f, \varepsilon) = 1$ and $a_k = b_k$. Thus (3.19) holds as asserted. To prove (3.20) when the Y_k's are independent we first show that

$$P(\omega : \limsup_n f(Y_n/(2 \operatorname{L} n)^{\frac{1}{2}}) = \sup_{x \in K} f(x)) = 1 \qquad f \in B^* . \tag{4.41}$$

Let $f \in B^*$ and fix $\varepsilon > 0$. Since $L(\mathscr{L}(Y_k), \mu) = b_k$ we have for all large k that

$$(4.42) \qquad P(f(Y_k) \geqq (1-\varepsilon)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \sup_{x \in K} f(x)) + b_k$$
$$\geqq \mu\left(x : f(x) \geqq \left(1 - \frac{\varepsilon}{2}\right)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \sup_{x \in K} f(x)\right).$$

Now $f(x)$ has a Gaussian distribution with mean zero and variance $\int_B f^2(y)\mu(dy) = \sup_{y \in K} f(y)$ so we have

$$(4.43) \qquad \sum_k \mu\left(x : f(x) \geqq \left(1 - \frac{\varepsilon}{2}\right)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \sup_{x \in K} f(x)\right)$$
$$= \sum_k \frac{2}{(2\pi)^{\frac{1}{2}}} \int_{(1-\varepsilon/2)(2\,\mathrm{L}\,k)^{\frac{1}{2}}}^{\infty} e^{-s^2/2}\, ds = \infty .$$

Combining (4.42) and (4.43) and using the fact that $\sum_k b_k < \infty$ we have

$$(4.44) \qquad \sum_k P(f(Y_k) \geqq (1-\varepsilon)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \sup_{x \in K} f(x)) = \infty .$$

Since the events involved in (4.44) are independent we have by the Borel–Cantelli lemma that

$$P(f(Y_k) \geqq (1-\varepsilon)(2\,\mathrm{L}\,k)^{\frac{1}{2}} \sup_{x \in K} f(x) \text{ i.o. in } k) = 1 .$$

Hence (4.41) is verified and (3.20) holds provided H_μ is infinite dimensional. This completes the proof of the theorem since (4.39), with $\sum_k b_k < \infty$, and fairly standard computations give (3.20) when $\dim H_\mu < \infty$.

PROOF OF COROLLARY 4.4. That (3.19) holds in this situation is obvious since we have $L(\mathscr{L}(Y_k), \mu) = 0$ for $k = 1, 2, \cdots$. We next use (4.40) to prove (4.41) for all $f \in B^*$.

Fix $f \in B^*$. If $\sup_{x \in K} f(x) = 0$, then since K is symmetric we have $f(x) = 0$ for all $x \in K$, and hence

$$E(f^2(Y_n)) = \int_B f^2(y)\mu(dy) = \sup_{x \in K} f^2(x) = 0 .$$

Thus (4.41) follows immediately if $\sup_{x \in K} f(x) = 0$. If $\sup_{x \in K} f(x) > 0$ define

$$X_k = \frac{f(Y_k)}{\sup_{x \in K} f(x)} \qquad k \geqq 1 .$$

Fix $\varepsilon > 0$, $\varepsilon < \frac{1}{2}$. Choose an integer $\lambda \geqq 1$ and by (4.40) an integer n_0 such that $n \geqq n_0$ implies

$$E(E(X_{\lambda n} \mid \mathscr{F}_{\lambda(n-1)})^2) \leqq \frac{\varepsilon}{2} .$$

Let $U_n = E(X_{\lambda n} \mid \mathscr{F}_{\lambda(n-1)})$ and $V_n = X_{\lambda n} - U_n$ for $n \geqq n_0$. Then $E(U_n) = E(V_n) = E(X_n) = 0$ and since $n \geqq n_0$, $E(U_n^2) \leqq \varepsilon/2$ and $E(V_n^2) \geqq 1 - \varepsilon/2$ as U_n and V_n are independent Gaussian random variables with $E(X_{\lambda n}^2) = 1$. Let

$$C_n = \left\{\frac{X_{\lambda n}}{(2\,\mathrm{L}\,\lambda n)^{\frac{1}{2}}} > 1 - 4\varepsilon\right\}$$

$$E_n = \left\{ \frac{|U_n|}{(2 \operatorname{L} \lambda n)^{\frac{1}{2}}} > \varepsilon \right\}$$

$$F_n = \left\{ \frac{V_n}{(2 \operatorname{L} \lambda n)^{\frac{1}{2}}} > 1 - 2\varepsilon \right\} .$$

Then $C_n \supseteqq F_n \cap E_n^c$ and $P(C_n \text{ i.o.}) \geqq P(F_n \cap E_n^c \text{ i.o.})$. Now $P(E_n^c$ for all large $n) = 1$ since $P(E_n \text{ i.o.}) = 0$. That is, $\sum_n P(E_n) < \infty$ as U_n is Gaussian with mean zero and variance less than or equal to $\varepsilon/2$, and hence by the Borel–Cantelli lemma $P(E_n \text{ i.o.}) = 0$. Now $P(F_n \text{ i.o.}) = 1$ iff $\sum_n P(F_n) = \infty$ as the random variables $\{V_n : n \geqq n_0\}$ are independent and hence the events F_n are independent. That is, the V_n's are Gaussian with mean zero so they are easily seen to be independent by checking they are orthogonal. Further,

$$P(F_n) = P\left(\frac{V_n}{(2 \operatorname{L} \lambda n)^{\frac{1}{2}}} > 1 - 2\varepsilon \right) = \frac{1}{(2\pi E(V_n^2))^{\frac{1}{2}}} \int_{(1-2\varepsilon)(2 \operatorname{L} \lambda n)^{\frac{1}{2}}}^{\infty} e^{-u^2/2E(V_n^2)}\, du$$

$$\geqq \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{(1-\gamma)(2 \operatorname{L} \lambda n)^{\frac{1}{2}}}^{\infty} e^{-s^2/2}\, ds ,$$

where $1 - \gamma = (1 - 2\varepsilon)/(1 - \varepsilon/2)^{\frac{1}{2}}$ and we are using the fact that

$$E(V_n^2) \geqq 1 - \frac{\varepsilon}{2}$$

in our change of variables. Since $\gamma > 0$ we have

$$\textstyle\sum_n P(F_n) = +\infty$$

so $P(F_n \text{ i.o.}) = 1$. Thus $P(C_n \text{ i.o.}) = 1$ and hence (4.41) holds as $\varepsilon > 0$ was arbitrarily small. This completes the proof of the corollary since (4.41) now implies (3.20) if $\dim H_\mu = \infty$, and if $\dim H_\mu < \infty$, then fairly standard modifications of the previous argument yield (3.20) as well.

Another application of Corollary 3.1 is given in our next result which establishes the law of the iterated logarithm for $C(S)$ valued random variables under conditions exactly the same as those used to establish the central limit theorem in this setting. Its proof is in the same spirit as that of Theorem 2.1 of [19] so it will be omitted. Further references and examples are also contained in [19].

Let S denote a compact metric space with metric d. Let $C(S)$ denote the space of real-valued continuous functions on S, and for $f \in C(S)$ define $\|f\|_\infty = \sup_{t \in S} |f(t)|$. If S is a pseudo-metric space with pseudo-metric ρ, then $N(\rho, S, \varepsilon)$ denotes the minimal number of balls of ρ-radius less than ε which cover S. The ε-entropy of (S, ρ) is

$$H(\rho, S, \varepsilon) = \log N(\rho, S, \varepsilon)$$

where $\log x$ denotes the natural logarithm of x.

If S is a metric space under d and ρ is a pseudo metric on S we say ρ *is continuous with respect to* d if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(s, t) < \delta$ implies $\rho(s, t) < \varepsilon$. If S is compact under d (with topology τ_d) then it is easy to see that ρ is continuous with respect to d iff τ_d is stronger than τ_ρ.

THEOREM 4.4. *Let X be a $C(S)$ valued random variable such that*

$$E(X(s)) = 0 \quad \text{and} \quad E(X^2(s)) < \infty \quad s \in S .$$

Suppose there exists a nonnegative random variable M such that for given $s, t \in S$ and sample point ω we have

$$|X(s, \omega) - X(t, \omega)| \leqq M(\omega)\rho(s, t)$$

with $E(M^2) < \infty$ and ρ a pseudo-metric on S such that ρ is continuous with respect to d. If

(a) $\int_0 H^{\frac{1}{2}}(S, \rho, u)\, du < \infty$,

(b) $X_1, X_2, \cdots$ *are independent identically distributed such that* $\mathscr{L}(X_k) = \mathscr{L}(X)$, *and if*

(c) K *is the unit ball of* $H_{\mathscr{L}(X)}$, *then*

$$P\left\{\lim_n d\left(\frac{S_n}{(2n \operatorname{LL} n)^{\frac{1}{2}}}, K\right) = 0\right\} = 1$$

and

$$P\left\{C\left(\left\{\frac{S_n}{(2n \operatorname{LL} n)^{\frac{1}{2}}} : n \geqq 1\right\}\right) = K\right\} = 1 .$$

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