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LIMIT THEOREMS FOR BRANCHING PROCESSES IN A RANDOM ENVIRONMENT

BY DAVID TANNY[1]

University of Rochester

In this paper, growth of branching processes in random environment is considered. In particular it is shown that this process either "explodes" at an exponential rate or else becomes extinct w.p. 1. A classification theorem outlining the cases of "explosion or extinction" is given. To prove these theorems, the associated branching process (the process conditioned on each particle having infinite descent) and the reduced branching process (the particles of the process having infinite descent) are introduced. The method of proof used, in general, is direct probabilistic computation, in contrast with the classical functional iteration method.

0. Introduction. Consider a population of particles with initial population size Z_0. Each of the particles is identical and splits independently according to the offspring distribution ξ_0. The offspring of Z_0 are referred to as the 1st generation of the branching processes in a random environment. In general, Z_n, the size of the nth generation, is the number of progeny of the Z_{n-1} identical particles of the $(n-1)$st generation, each of which split independently according to the offspring distribution ξ_{n-1}. $\bar{\xi} = (\xi_0, \xi_1, \cdots)$, the "environmental sequence" is assumed to be a stationary and ergodic stochastic process. ξ_i is referred to as the environment of the ith generation. For given $\bar{\xi}$, the process $\{Z_n\}_{n=0}^{\infty}$ generated in this manner is called the branching process conditioned on the environment $\bar{\xi}$. $\{Z_n\}_{n=0}^{\infty}$ is called the branching process in a random environment. The latter model is sometimes referred to as the Athreya–Karlin model of the process ([1], [3]) in contrast to the Smith–Wilkinson model ([10]) which assumed that the environments $\{\xi_i\}_{i=0}^{\infty}$ were independent and identically distributed. In this paper we are concerned with the following problems for the Athreya–Karlin model of the process:

(1) Must the process $\{Z_n\}_{n=0}^{\infty}$ either "explode" or become extinct?

(2) If (1) is true, what are necessary and sufficient conditions for noncertain extinction of the process?

(3) What is the rate of growth of the process $\{Z_n\}_{n=0}^{\infty}$ on $\{\omega : Z_n(\omega) \to \infty$ as $n \to \infty\}$?

Theorem 5.3 answers (1) affirmatively (except for the degenerate case $P(p_0(\xi_0) = 1) = 1$ for which $P(Z_n \equiv 1\ \forall\, n \mid \bar{\xi}) = 1$ w.p. 1.) This "explosion or extinction"

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theorem was first proved by Athreya and Karlin ([1]) under the hypotheses $E(\log^+ m(\xi_0)) < \infty$, where $m(\xi_0)$ is the expected number of offspring of a particle conditioned on the environment ξ_0. However, Theorem 5.3 asserts the result without any hypotheses on the process.

(3) and part of (2) are answered by the classification theorem (Theorem 5.5).

THEOREM 5.5. *(Classification theorem). Let* $\{Z_n\}_{n=0}^\infty$ *be a branching process for a random environment for which* $E(\log m(\xi_0))$ *exists. Then*

(i) $E(\log m(\xi_0)) < 0$ *implies* $P(q(\bar{\xi}) = 1) = 1$, *where* $q(\bar{\xi}) = P(Z_n \to 0$ *as* $n \to \infty \mid \bar{\xi})$,

(ii) $E(\log m(\xi_0) = 0$ *implies that either* $P(q(\bar{\xi}) = 1) = 1$ *or* $P(p_1(\xi_0) = 1) = 1$, *in which case* $P(Z_n \equiv \forall\, n \mid Z_0 = 1, \bar{\xi}) = 1$ w.p. 1, *and*

(iii) $E(\log m(\xi_0)) > 0$ *implies that* $\lim_{n\to\infty} 1/n \log Z_n = E(\log m(\xi_0))$ a.e. *on* $\{\omega : Z_n(\omega) \to +\infty$ *as* $n \to \infty\}$.

In the special case of Galton–Watson processes, part (iii) asserts that $\lim_{n\to\infty} 1/n \log Z_n$ exists and equals $\log m$ a.e. on $\{\omega : Z_n(\omega) \to \infty$ as $n \to \infty\}$. This result can also be proved by using the normalization theorem of Heyde-Senata ([3], Chapter 6).

In Section 6, we make further partial progress on problem (2). A sufficient condition for noncertain extinction is stated (Proposition 6.2) from which the sufficient condition of Athreya and Karlin ([1]) follows as an easy corollary.

1. Mathematical description of branching processes in a random environment. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a suitably large probability space. Let Π denote the space of probability distributions $\bar{p} = \{p_i\}_{i=0}^\infty$ where $\sum_{i=1}^\infty ip_i < \infty$. Let $(\ell^\infty, \mathscr{B}_\infty)$ be the space of bounded real sequences with $\mathscr{B}_\infty$ the Borel σ-algebra generated by the product topology. Then $\Pi \subset \ell^\infty$ (in fact $\Pi \in \mathscr{B}_\infty$) and so we can endow Π with the trace σ-algebra $\Pi \cap \mathscr{B}_\infty$. (The trace σ-algebra on Π is the Borel σ-algebra generated by the induced topology.)

Let $\{\xi_i\}_{i=0}^\infty$ be a sequence of random variables, $\xi_i : \Omega \to \Pi$, and denote by $\bar{\xi}$ the process $(\xi_0, \xi_1, \cdots)$. $\bar{\xi}$ is assumed to be stationary and ergodic. $\varphi_{\xi_0}(s) = \sum_{i=0}^\infty p_i(\xi_0)s^i$ is called the probability generating function associated with the environment ξ_0. $m(\xi_0) = \sum_{i=0}^\infty ip_i(\xi_0)$ is called the mean corresponding to the environment ξ_0 and is equal to the expected number of offspring of a particle given the environment ξ_0.

Consider a sequence of random variables $\{Z_n\}_{n=0}^\infty$ on Ω having the nonnegative integers as their state space, and suppose that this process satisfies

$$E(s^{Z_{n+1}} \mid \mathscr{F}_n(\bar{\xi})) = [\varphi_{\xi_n}(s)]^{Z_n} \tag{1}$$

for $n \geq 0$, $|s| \leq 1$, where $\mathscr{F}_n(\bar{\xi})$ is the σ-algebra generated by $Z_0, Z_1, \cdots, Z_n$ and $\bar{\xi}$; i.e., conditioned on the past and on the environmental sequence $\bar{\xi}$, Z_{n+1} may be viewed as the sum of Z_n independent and identically distributed random variables, each having $\varphi_{\xi_n}(s)$ as its probability generating function. Then the process $\{Z_n\}_{n=0}^\infty$ conditioned on the environment $\bar{\xi}$ is called a nonhomogeneous branching process, and the unconditioned process $\{Z_n\}_{n=0}^{\infty}$ is referred to as the branching process in a random environment. The proof of the existence of such a process is standard and we refer the interested reader to Harris (1963).

The following notation will be used frequently.

Z^+ the positive integers
$X \in \mathscr{F}$ the random variable X is measurable with respect to the σ-algebra $\mathscr{F}$
$\sigma(X_1, X_2, \cdots, X_n)$ the σ-algebra generated by the random variables $X_1, X_2, \cdots, X_n$
$E(\ |X_1, X_2, \cdots, X_n)$ the expected value conditioned on $\sigma(X_1, X_2, \cdots, X_n)$
$E(\ |A; X_1, X_2, \cdots, X_n)$ $E(\ |X_1, X_2, \cdots X_n, \chi_A)$ on A, where χ_A is the indicator function of the set A, and 0 on A^c
$\mathscr{F}_n(\xi)$ $\sigma(Z_0, Z_1, \cdots, Z_n; \xi)$
T the shift transformation $T(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$
$[x]$ the greatest integer contained in x
$f^+(x)$ equals $f(x)$ if $f(x) \geqq 0$; otherwise equals 0
$f^-(x)$ equals $-f(x)$ if $f(x) \leqq 0$; otherwise equals 0
$f'(x)$ the first derivative of the function f
$f^{(n)}(x)$ the nth order derivative of the function f.

By an iteration of (1), it can be shown that, for any set of integers $i \leqq n_1 < n_2 < \cdots < n_k$, and positive integer m

$$E(s_1^{Z_{n_1}} s_2^{Z_{n_2}} \cdot \cdots \cdot s_k^{Z_{n_k}} \mid Z_0 = m; \xi) = [E(s_1^{Z_1} s_2^{Z_2} \cdot \cdots \cdot s_k^{Z_{n_k}} \mid Z_0 = 1; \xi)]^m \tag{2}$$

where $|s_i| \leqq 1$ for $1 \leqq i \leqq k$. Using (1) and (2), Athreya and Karlin (1971) noted the following proposition.

PROPOSITION 1.1.

$$E(s^{Z_{n+1}} \mid Z_0 = m; \xi) = [\varphi_{\xi_0}(\varphi_{\xi_1}(\cdots(\varphi_{\xi_n})s)\cdots)]^m .$$

Though the process $\{Z_n\}_{n=0}^{\infty}$ is not, in general a Markov process, the branching process conditioned on the environment ξ is Markovian by (1) and has independent lines of descent by (2) (see [1]). The unconditioned process $\{Z_n\}_{n=0}^{\infty}$ is easily seen to be Markov, however, in the special case when $\{\xi_i\}_{i=0}^{\infty}$ are independent and identically distributed (i.i.d.).

2. Extinction probability of the branching process in a random environment. From (1) it is easily seen that $E(s^{Z_{n+1}} \mid Z_n = 0) = 1$ for $|s| \leqq 1$. But for $0 < s_0 < 1$, $0 < s_0^{Z_{n+1}} \leqq 1$, with equality holding iff $Z_{n+1} = 0$. Thus $Z_{n+1} = 0$ a.e. on $\{\omega : Z_n = 0\}$, from which we conclude that, excluding a set of measure zero, $Z_{n_0}(\omega) = 0$ for some n_0, implies that $Z_n(\omega) = 0$ for all $n \geqq n_0$. In the latter case, the process $\{Z_n\}_{n=0}^{\infty}$ is said to have died out or to have become extinct. Let $q(\xi)$ be the extinction probability given the environment; i.e.,

$$q(\xi) = P(Z_n = 0 \text{ for some } n > 0 \mid \xi) = \lim_{n\to\infty} \varphi_{\xi_0}(\varphi_{\xi_1}(\cdots \varphi_{\xi_n}(0)\cdots)) . \tag{3}$$

Let $A = \{\xi_0 : p_0(\xi_0) = 1\}$.

PROPOSITION 2.1. *$P(A) > 0$ implies that $P(q(\xi) = 1) = 1$.*

PROOF. Since the environmental process $\bar{\xi}$ is stationary and ergodic, the Birkhoff ergodic theorem implies that

$$\frac{1}{n} \sum_{i=0}^{n-1} \chi_A(\xi_i) \to P(A) \quad \text{w.p. 1 as } n \to \infty \tag{4}$$

where $\chi_A(\cdot)$ is the indicator function of the set A. Hence w.p. 1 there exists $i_0(\bar{\xi})$ such that $p_0(\xi_{i_0}) = 1$ and so $Z_{i_0+1} = 0$ w.p. 1.

REMARK 2.2. In view of Proposition 2.1, we will, from this point on, assume that $p_0(\xi_0) < 1$ w.p. 1.

Let T be the shift operator, $T(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$. T is measure preserving and ergodic. It is easy to see ([1]) that the sets $\{\bar{\xi} : q(\bar{\xi}) = 1\}$ and $\{T\bar{\xi} : q(T\bar{\xi}) = 1\}$ differ by a set of measure zero and so $P(q(\bar{\xi}) < 1)$ equals 0 or 1. Furthermore, $q(\bar{\xi})$ satisfies the functional equation $\beta(\bar{\xi}) = \varphi_{\xi_0}(\beta(T\bar{\xi}))$ and is known to be ([1]) the unique solution satisfying $P(\beta(\bar{\xi}) < 1) = 1$ if such a solution exists.

Smith and Wilkinson (1968) showed that for $\{\xi_i\}_{i=0}^{\infty}$ i.i.d. and $E(\log^+ m(\xi_0)) < \infty$, a necessary and sufficient condition for noncertain extinction of the process is that $E|\log(1 - p_0(\xi_0))| < \infty$ and $E(\log m(\xi_0)) > 0$. Athreya and Karlin (1971) demonstrated the sufficiency of the condition for $\bar{\xi}$ stationary and ergodic. However, the condition is no longer necessary for noncertain extinction (see Example 7.2). The following is a necessary, but not sufficient (see Example 7.4) condition for noncertain extinction of the process.

THEOREM 2.3. *If* $E(\log^+ m(\xi_0)) < \infty$ *and* $P(q(\bar{\xi}) < 1) = 1$, *then* $\lim_{n\to\infty} 1/n \log(1 - p_0(\xi_n)) = 0$ w.p. 1.

The proof of Theorem 2.3 is postponed until Section 5.

3. The associated branching process in a random environment. Consider a branching process in a random environment $\{Z_n\}_{n=0}^{\infty}$ for which $P(q(\bar{\xi}) < 1) = 1$ and let $\{\varphi_{\xi_n}(s)\}_{n=0}^{\infty}$ denote the associated probability generating functions. Define, for $n \geqq 0$

$$\hat{\varphi}_{T^n\bar{\xi}}(s) = \frac{\varphi_{\xi_n}((1 - q(T^{n+1}\bar{\xi}))s + q(T^{n+1}\bar{\xi})) - q(T^n\bar{\xi})}{1 - q(T^n\bar{\xi})}. \tag{5}$$

It is easily checked that $\hat{\varphi}_{T^n\bar{\xi}}$ is a probability generating function. In the case of nonrandom environment (i.e., $\varphi_{\xi_0}(s) \equiv \varphi(s)$ w.p. 1 for some fixed p.g.f. φ) the transformation of φ to $\hat{\varphi}$ is the transformation introduced by Harris (1948), namely, the conditioning of the generating function on infinite descent.

Define a sequence of random variables $\{\hat{\xi}_n\}_{n=0}^{\infty}$, $\hat{\xi}_n : \Omega \to \Pi$ by

$$\hat{\xi}_n(\omega) = \left\{\frac{\hat{\varphi}^{(n)}_{T^n\bar{\xi}}(0)}{n!}\right\}_{n=0}^{\infty}.$$

Since $\bar{\xi} = (\xi_0, \xi_1, \cdots)$ is a stationary and ergodic process, then $\bar{\hat{\xi}} = (\hat{\xi}_0, \hat{\xi}_1, \cdots)$ is also stationary and ergodic ([4]). Now consider a branching process in a random environment $\{\hat{Z}_n\}_{n=0}^{\infty}$ with $\hat{Z}_0 \equiv 1$ on (Ω, F, P) having $\bar{\hat{\xi}}$ as its environmental sequence. The process is unique and is called the associated branching process in a random environment. We note that $\bar{\hat{\xi}}$ is completely determined by $\bar{\xi}$, and so $\bar{\xi}$ will be used frequently to denote the environmental sequences of both processes. Also, it is convenient to adopt the notation $\hat{\varphi}_{\xi_n}(s) = \varphi_{\hat{\xi}_n}(s)$, $\hat{p}_i(\xi_j) = p_i(\hat{\xi}_j)$, etc.

PROPOSITION 3.1. *$\{\hat{Z}_n\}_{n=0}^{\infty}$ is an increasing process. Furthermore either $P(\hat{Z}_n \to +\infty$ as $n \to \infty \mid \bar{\xi}) = 1$ w.p. 1 or $P(\hat{Z}_n \equiv 1\ \forall n \geqq 0 \mid \bar{\xi}) = 1$ w.p. 1 with the latter occurring* iff $P(\hat{p}_1(\xi_0) = 1) = 1$.

PROOF. We note that $\hat{\varphi}_{\xi_n}(0) = 0 = \hat{p}_0(\xi_n)$ where $\hat{p}_0(\xi_n)$ is the probability that a particle in the nth generation gives birth to 0 children, given the environment $\hat{\xi}_n$. Hence each $\hat{Z}$-particle gives birth to at least one child so that $\{\hat{Z}_n\}_{n=0}^{\infty}$ is increasing. Thus, for each $\omega \in \Omega$, either $\hat{Z}_n(\omega) \to +\infty$ as $n \to \infty$ or there exists a positive integer $N(\omega)$ such that $\hat{Z}_n(\omega) = \hat{Z}_{N(\omega)}(\omega)$ for all $n \geqq N(\omega)$.

Let $A_k = \{\omega : \hat{Z}_n(\omega) = \hat{Z}_k(\omega)\ \forall n \geqq k\}$ and $A = \bigcup_{k=0}^{\infty} A_k$. Since $\{A_k\}_{k=0}^{\infty}$ is an increasing sequence of sets,

$$
\begin{aligned}
0 \leqq P(A \mid \bar{\xi}) &= \lim_{k\to\infty} P(A_k \mid \bar{\xi}) \\
&= \lim_{k\to\infty} P(\text{each particle in the } n\text{th generation produces} \\
&\qquad \text{exactly one offspring } \forall\, n \geqq k \mid \bar{\xi}) \\
&\leqq \limsup_{k\to\infty} P(\text{a single particle of the } k\text{th generation} \\
&\qquad \text{and each of its descendants produce exactly} \\
&\qquad \text{one offspring } \mid \bar{\xi}) \\
&= \limsup_{k\to\infty} \textstyle\prod_{i=k}^{\infty} \hat{p}_1(\xi_i)\,.
\end{aligned} \tag{7}
$$

If $P(\hat{p}_1(\xi_0) = 1)$, then it follows that $P(\hat{Z}_n \equiv 1\ \forall\, n \geqq 0 \mid \bar{\xi}) = 1$ w.p. 1 since $\{\hat{\xi}_n\}_{n=0}^{\infty}$ is stationary. Hence it suffices to assume that there exists $0 < \delta < 1$ such that $P(\hat{p}_1(\xi_0) < \delta) > 0$. By the Birkhoff ergodic theorem

$$
\frac{1}{n} \sum_{i=0}^{n-1} \chi_{[\hat{p}_1(\xi_i) < \delta]}(\hat{\xi}_i) \to P(\hat{p}_1(\xi_0) < \delta) \qquad \text{w.p. 1} \quad \text{as} \quad n \to \infty \tag{8}
$$

and so $P(\hat{p}_1(\xi_0) < \delta) > 0$ implies that $P(\hat{p}_1(\xi_n) < \delta$ i.o.$) = 1$. Therefore, for any integers $k \geqq 0$, $M \geqq 0$, $\prod_{i=k}^{\infty} \hat{p}_1(\xi_i) < \delta^M$ w.p. 1, so by (7) $P(A \mid \bar{\xi}) = 0$ w.p. 1. This implies that $P(\hat{Z}_n \to +\infty$ as $n \to \infty \mid \bar{\xi}) = 1$ w.p. 1.

4. The reduced branching process.

DEFINITION 4.1. A tree T is a connected graph containing no cycles; i.e., it consists of a set $V = \{v_1, v_2, \cdots\}$ called vertices and a set E, called edges, such that

(i) $v, w \in V$ implies that there exists $v_1, v_2, \cdots, v_n \in V$ such that $vv_1, v_1v_2, \cdots, v_nw$ are distinct edges; i.e., there is a path from v to w.

(ii) for $v, w \in V$ there is only one path from v to w.

DEFINITION 4.2. A directed tree is a tree T with a partial order "$\leqq$" on V, the vertices of T, such that

(i) if $v_1 v_2$ is an edge, then either $v_1 \leqq v_2$ or $v_2 \leqq v_1$
(ii) there exists a minimal element v_0; i.e., $v_0 \leqq v$ for all $v \in V$
(iii) if $v_1 \leqq v_2$ and w is on the path joining v_1 and v_2, then $v_1 \leqq w$ and $w \leqq v_2$.

The length of a path is the number of edges it contains. A vertex v of a directed tree T is said to be at the nth level of T if the path from v_0 to v is of length n. The length of a directed tree T is sup $\{n:$ there exists a vertex v_n of T at the nth level of $T\}$.

Let $Z_0 = 1$ and let $\omega \in \Omega$. Consider the following random directed tree $T(\omega)$. The initial particle of the process is represented by v_0, the minimal element. Each particle of the nth generation of the process is represented by a vertex at the nth level of $T(\omega)$. Let v be a vertex at the nth level of $T(\omega)$. Then for w a vertex at the $(n+1)$st level, vw is an edge iff w corresponds to one of the offspring of the particle in the nth generation of the process which v represents. Furthermore, for v, w in the nth and $(n+1)$st level of T respectively, v is comparable to w iff vw is an edge, in which case, $v \leqq w$. $T(\omega)$ is thus described completely and is called the family tree corresponding to ω of the process. Similarly a particle in the nth generation of the process gives rise to a directed subtree of $T(\omega)$, called its tree of descendants ([5]).

DEFINITION 4.3. For $\omega \in \Omega$, a particle in the nth generation of the branching process in a random environment $\{Z_n\}_{n=0}^{\infty}$ is said to be "*a particle of infinite descent*" if its tree of descendants has infinite length. We call these particles W-particles.

Let $W_n(\omega) =$ the number of particles of infinite descent in the nth generation of the process $\{Z_n(\omega)\}_{n=0}^{\infty}$. Any W-particle in the nth generation has a parent in the $(n-1)$st generation of the Z-process. This parent must also have infinite descent (since at least one of its offspring does) and so is a W-particle; i.e., the W-particles in the nth generation are produced solely by W-particles in the $(n-1)$st generation. Let a and b denote two particles in the nth generation of the process $\{Z_n\}_{n=0}^{\infty}$ and let $\mathscr{G}_a(\bar{\xi})$ and $\mathscr{G}_b(\bar{\xi})$ be the σ-fields of their respective "futures" conditioned on the environment $\bar{\xi}$ (see [11] Definition 1.7.7). Then $\mathscr{G}_a(\bar{\xi})$ and $\mathscr{G}_b(\bar{\xi})$ are independent σ-fields. Letting A and B denote respectively the event that a is a W-particle and the event that b is a W-particle, we note that $A \in \mathscr{G}_a(\bar{\xi})$ and $B \in \mathscr{G}_b(\bar{\xi})$ so that A and B are independent. Furthermore, conditioned on being W-particles and on the environment $\bar{\xi}$, the "futures" of a and b as W-particles (i.e., their lines of descent in the $\{W_n\}_{n=0}^{\infty}$ process) are sub σ-fields of $\mathscr{G}_a(\bar{\xi})$ and $\mathscr{G}_b(\bar{\xi})$ respectively and hence are independent. Using these facts and the property that a and b have the branching property within the $\{Z_n\}_{n=0}^{\infty}$ process, it is readily seen that the process $\{W_n\}_{n=0}^{\infty}$ conditioned on $\bar{\xi}$ has the branching property. Hence we have shown, heuristically at least, that the process $\{W_n\}_{n=0}^{\infty}$ conditioned on $\bar{\xi}$ is a nonhomogeneous branching process. (A formal proof of this fact is given in [11], Section (1.7).) We call the process $\{W_n\}_{n=0}^\infty$, conditioned in the environment $\bar{\xi}$, the reduced branching process.

PROPOSITION 4.4. *(i) W_n is increasing as n increases and $W_n \leqq Z_n$ and (ii) $\{\omega : W_0(\omega) \neq 0\} = \{\omega : Z_n \to +\infty$ as $n \to \infty\}$.*

PROOF. Each particle of the $(n-1)$st generation of the $\{W_n\}_{n=0}^\infty$ process has at least one offspring in the nth generation, since it has infinite descent. Hence $W_{n-1} \leqq W_n$. Since each W-particle is a Z-particle, $W_n \leqq Z_n$. Part (ii) is clear from the definition of $\{W_n\}_{n=0}^\infty$.

The next theorem relates the reduced branching process to the associated branching process in a random environment.

THEOREM 4.5. *The process $\{W_n\}_{n=0}^\infty$ conditioned on $\{W_0 \equiv 1\}$ and on the environment $\bar{\xi}$ is equivalent to the process $\{\hat{Z}_n\}_{n=0}^\infty$ conditioned on $\bar{\xi}$ in the sense of finite dimensional distributions.*

PROOF. Let $A = \{\omega : W_0(\omega) = 1\}$. The processes $\{W_n\}_{n=0}^\infty$ conditioned on A and $\bar{\xi}$ and $\{\hat{Z}_n\}_{n=0}^\infty$ conditioned on $\bar{\xi}$ are Markov processes with initial distribution $P(\hat{Z}_0 \equiv 1 \mid \bar{\xi}) = 1 = P(W_0 = 1 \mid A, \bar{\xi})$. Thus it suffices to prove that these two Markov processes have the same transition probabilities; i.e., for any nonnegative integers j, k, n with $n \geqq 1$,

$$P(W_n = k \mid W_{n-1} = j, A, \bar{\xi}) = P(\hat{Z}_n = k \mid \hat{Z}_{n-1} = j, \bar{\xi}) . \tag{9}$$

Since $\{W_n\}_{n=0}^\infty$ conditioned in $\bar{\xi}$ is Markov, the left hand side of (9) equals $P(W_n = k \mid W_{n-1} = j, \bar{\xi})$. Hence (9) is equivalent to

$$\textstyle\sum_{k=0}^\infty P(W_n = k \mid W_{n-1} = j, \bar{\xi}) s^k = \sum_{k=0}^\infty P(\hat{Z}_n = k \mid \hat{Z}_{n-1} = j, \bar{\xi}) s^k \tag{10}$$

for all real s, $|s| \leqq 1$. Letting $\psi_{n-1}(s) = \sum_{k=1}^\infty P(W_n = k \mid W_{n-1} = 1, \bar{\xi}) s^k$ and using the branching property of the processes $\{W_n\}_{n=0}^\infty$ and $\{\hat{Z}_n\}_{n=0}^\infty$ conditioned on $\bar{\xi}$, (10) becomes

$$[\psi_{n-1}(s)]^j = [\hat{\varphi}_{\xi_{n-1}}(s)]^j \tag{11}$$

so it suffices to prove that $\psi_{n-1}(s) = \hat{\varphi}_{\xi_{n-1}}(s)$ for all $n \geqq 1$ and $|s| \leqq 1$. Clearly

$$\begin{aligned}\psi_{n-1}(s) = \textstyle\sum_{k=1}^\infty s^k \sum_{r=1}^\infty P(W_n = k \mid Z_{n-1} = r, W_{n-1} = 1, \bar{\xi}) \\ \times P(Z_{n-1} = r \mid W_{n-1} = 1, \bar{\xi}) .\end{aligned} \tag{12}$$

Conditioned on $\bar{\xi}$, and given $Z_{n-1} = r$, $\{Z_k\}_{k=n}^\infty$ may be viewed as the sum of r independent nonhomogeneous branching processes, each one generated by a particle of the $(n-1)$st generation. Denote the r particles of the $(n-1)$st generation by $i = 1, \cdots, r$. For $i = 1, \cdots, r$ let X_i^{n-1} be the number of offspring in the nth generation of the Z-process produced by particle i of the $(n-1)$st generation, and let N_i^{n-1} be the number of these particles having infinite descent. Then $\{X_i^{n-1}\}_{i=1}^r$ are independent and identically distributed random variables and are conditionally independent of $\{Z_i\}_{i=0}^{n-1}$ given $\bar{\xi}$. Also since conditioned on the environment $\bar{\xi}$, the Z-particles have independent lines of descent, $\{N_i^{n-1}\}_{i=0}^{r}$ are i.i.d. and are conditionally independent of $\{Z_i\}_{i=0}^{n-1}$ given $\bar{\xi}$. Hence for $k \geqq 1$

$$
\begin{aligned}
&P(W_n = k \mid W_{n-1} = 1, Z_{n-1} = r, \bar{\xi}) \\
&\qquad = P(W_n = k \mid W_{n-1} = 1, Z_{n-1} = 1, \bar{\xi}) \\
&\qquad = \frac{P(W_n = k, W_{n-1} = 1 \mid Z_{n-1} = 1, \bar{\xi})}{P(W_{n-1} = 1 \mid Z_{n-1} = 1, \bar{\xi})} \\
&\qquad = \frac{P(N_1^{n-1} = k \mid \bar{\xi})}{P(W_{n-1} = 1 \mid Z_{n-1} = 1, \bar{\xi})} \\
&\qquad = \frac{\sum_{t=k}^{\infty} P(X_1^{n-1} = t, N_1^{n-1} = k \mid \bar{\xi})}{1 - q(T^{n-1}\bar{\xi})} \\
&\qquad = \frac{\sum_{t=k}^{\infty} P(Z_n = t, W_n = k \mid Z_{n-1} = 1, \bar{\xi})}{1 - q(T^{n-1}\bar{\xi})} .
\end{aligned} \tag{13}
$$

Substituting (13) into (12) and performing the summation over r yields

$$
\psi_{n-1}(s) = \frac{1}{1 - q(T^{n-1}\bar{\xi})} \sum_{k=1}^{\infty} s^k \sum_{t=k}^{\infty} P(Z_n = t, W_n = k \mid Z_{n-1} = 1, \bar{\xi}) . \tag{14}
$$

But

$$
\begin{aligned}
&P(Z_n = t, W_n = k \mid Z_{n-1} = 1, \bar{\xi}) \\
&\qquad = \binom{t}{k}(1 - q(T^n\bar{\xi}))^k q(T^n\bar{\xi})^{t-k} P(Z_n = t \mid Z_{n-1} = 1, \bar{\xi}) .
\end{aligned} \tag{15}
$$

Thus

$$
\begin{aligned}
\psi_{n-1}(s) &= \frac{1}{1 - q(T^{n-1}\bar{\xi})} \sum_{t=1}^{\infty} P(Z_n = t \mid Z_{n-1} = 1, \bar{\xi}) \\
&\qquad \times \sum_{k=1}^{t} \binom{t}{k}[(1 - q(T^n\bar{\xi}))s]^k q(T^n\bar{\xi})^{t-k} \\
&= \frac{1}{1 - q(T^{n-1}\bar{\xi})} \sum_{t=1}^{\infty} P(Z_n = t \mid Z_{n-1} = 1, \bar{\xi}) \\
&\qquad \times \{((1 - q(T^n\bar{\xi}))s + q(T^n\bar{\xi}))^t - q(T^n\bar{\xi})^t\} \\
&= \frac{\varphi_{\xi_{n-1}}((1 - q(T^n\bar{\xi}))s + q(T^n\bar{\xi})) - \varphi_{\xi_{n-1}}(q(T^n\bar{\xi}))}{1 - q(T^{n-1}\bar{\xi})} \\
&= \hat{\varphi}_{\xi_{n-1}}(s) .
\end{aligned} \tag{16}
$$

The proposition now follows from the remarks made earlier.

5. Growth of the branching process in a random environment. We begin with an upper bound on the growth rate of $\{Z_n\}_{n=0}^{\infty}$.

THEOREM 5.1. *Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process in a random environment for which $E(\log m(\xi_0))$ exists. Then* $\limsup_{n\to\infty} 1/n \log Z_n \leqq E(\log m(\xi_0))$ w.p. 1.

PROOF. By Remark 2.2, $m(\xi_0) > 0$ w.p. 1. Thus the random variables $X_n = Z_n/\prod_{i=0}^{n-1} m(\xi_i)$, $n \geqq 0$, are well defined except on a set of measure zero on which we define them to be 0. It is easily seen that $\{X_n\}_{n=0}^{\infty}$ is a positive martingale with respect to the σ-fields $\{\mathscr{F}_n(\bar{\xi})\}_{n=0}^{\infty}$ and so $X_n \to X_\infty$ w.p. 1 as $n \to \infty$ where X_∞ is a random variable satisfying $E(X_\infty) < \infty$ ([4]]. Now

$$(17)\qquad \frac{1}{n}\log \prod_{i=0}^{n-1} m(\xi_i) = \frac{1}{n}\sum_{i=0}^{n-1}\log m(\xi_i) \to E(\log m(\xi_0)) \quad \text{w.p. 1} \quad \text{as} \quad n\to\infty$$

by the Birkhoff ergodic theorem. Therefore

$$(18)\qquad \limsup_{n\to\infty}\frac{1}{n}\log Z_n \leqq \limsup_{n\to\infty}\frac{1}{n}\log\prod_{i=0}^{n-1} m(\xi_i) + \limsup_{n\to\infty}\frac{1}{n}\log X_\infty = E(\log m(\xi_0))$$

and so the theorem is proved.

COROLLARY 5.2. *If* $E(\log m(\xi_0)) < 0$, *then* $P(q(\bar{\xi}) = 1) = 1$.

PROOF. On $A=\{\omega: Z_n \not\to 0 \text{ as } n\to\infty\}$, $\liminf_{n\to\infty} 1/n \log Z_n \geqq \lim_{n\to\infty} 1/n\log 1 \geqq 0$ w.p. 1. But, by Theorem 5.1, on A $\limsup_{n\to\infty} 1/n\log Z_n \leqq E(\log m(\xi_0)) < 0$ w.p. 1 so we must have $P(A) = 0$.

The next theorem generalizes to branching processes in a random environment the classical "extinction or explosion" result for Galton–Watson processes [5, Chapter 1, Theorem 6.2.]. The theorem was first proved by Athreya and Karlin [1] under the hypothesis $E(\log^+ m(\xi_0)) < \infty$. We now give a proof of the result without any hypotheses on the process.

THEOREM 5.3. *Let* $\{Z_n\}_{n=0}^\infty$ *be a branching process in a random environment. Then either* (i) $P(Z_n \to 0$ *or* $Z_n \to +\infty$ *as* $n \to \mid \bar{\xi}) = 1$ w.p. 1 *independent of* Z_0 *or* (ii) $P(Z_n \equiv 1\ \forall n \mid Z_0 = 1, \bar{\xi}) = 1$ w.p. 1. *Case* (ii) *occurs* iff $P(p_1(\xi_0) = 1) = 1$.

LEMMA 5.4. *If* $P(q(\bar{\xi}) < 1) = 1$ *and* $P(p_1(\xi_0) < 1) > 0$, *then* $P(\hat{p}_1(\xi_0) < 1) > 0$.

PROOF. If $P(p_0(\xi_0) + p_1(\xi_0) = 1) = 1$, then $E(\log m(\xi_0))$ exists, and since $P(p_1(\xi_0) < 1) > 0$, it follows that $E(\log m(\xi_0)) < 0$. But, by Corollary 5.2, this implies that $P(q(\bar{\xi}) = 1) = 1$. Hence $P(p_0(\xi_0) + p_1(\xi_0) < 1) > 0$. This last statement is equivalent to $P(\varphi_{\xi_0}^{(2)}(s) > 0 \text{ for } 0 < s < 1) > 0$.

Now

$$(19)\qquad \hat{p}_2(\xi_0) = \frac{\hat{\varphi}_{\xi_0}^{(2)}(0)}{2!} = \frac{(1-q(T\bar{\xi}))^2}{2!(1-q(\bar{\xi}))}\varphi_{\xi_0}^{(2)}(q(T\bar{\xi}))$$

so that if $P(p_0(\xi_0) = 0) < 1$, then $P(\hat{p}_2(\xi_0) > 0) > 0$. If $P(p_0(\xi_0) = 0) = 1$, then $\hat{p}_1(\xi_0) = p_1(\xi_0)$ w.p. 1 and thus the lemma is proved.

PROOF OF THEOREM 5.3. Clearly $P(p_1(\xi_0) = 1) = 1$ implies that $P(Z_n \equiv 1\ \forall n \mid Z_0 = 1, \bar{\xi}) = 1$ w.p. 1. Also if $P(Z_n \equiv 1\ \forall n \mid Z_0 = 1, \bar{\xi}) = 1$, w.p. 1, then $P(Z_1 = 1 \mid Z_0 = 1, \bar{\xi}) = 1$ w.p. 1 so that $P(p_1(\xi_0) = 1) = 1$. Hence it suffices to assume that $P(p_1(\xi_0) < 1) > 0$. Furthermore, if $P(q(\bar{\xi}) = 1) = 1$ then $P(Z_n \to 0$ as $n \to \infty \mid \bar{\xi}) = 1$ w.p. 1 independent of Z_0 so consider the case $P(q(\bar{\xi}) < 1) = 1$.

By Lemma 5.4 and Proposition 3.1, $P(\hat{Z}_n \to +\infty \text{ as } n \to \infty \mid \bar{\xi}) = 1$ w.p. 1 and applying Theorem 4.5 yields $P(W_n \to +\infty \text{ as } n \to \infty \mid W_0 \neq 0, \bar{\xi}) = 1$ w.p. 1. Since $Z_n \geqq W_n$ for all n, it follows that $P(Z_n \to +\infty \text{ as } n \to \infty \mid W_0 \neq 0, \bar{\xi}) = 1$ w.p. 1. On $\{W_0 = 0\}$ there exists $N(\omega)$ such that $Z_{N(\omega)}(\omega) = 0$; i.e., $P(Z_n \to 0$ as $n \to \infty \mid W_0 = 0, \bar{\xi}) = 1$ w.p. 1. Hence $P(Z_n \to +\infty$ or $Z_n \to 0$ as $n \to \infty \mid \bar{\xi}) =$ w.p. 1 independent of Z_0. □

Having established the "extinction or explosion" theorem, our next objective is to classify the outcomes and obtain a rate of growth on $\{\omega : Z_n \to +\infty$ as $n \to \infty\}$.

THEOREM 5.5. (*Classification theorem.*) *Let* $\{Z_n\}_{n=0}^{\infty}$ *be a branching process in a random environment for which* $E(\log m(\xi_0))$ *exists. Then*

(i) $E(\log m(\xi_0)) < 0$ *implies* $P(q(\bar{\xi}) = 1) = 1$,

(ii) $E(\log m(\xi_0)) = 0$ *implies that either* $P(q(\bar{\xi}) = 1) = 1$ *or* $P(p_1(\xi_0) = 1) = 1$, *in which case* $P(Z_n \equiv 1 \ \forall\, n \mid Z_0 = 1, \bar{\xi}) = 1$ w.p. 1,

(iii) $E(\log m(\xi_0)) > 0$ *implies that* $\lim_{n\to\infty} 1/n \log Z_n = E(\log m(\xi_0))$ a.e. *on* $\{\omega : Z_n \to +\infty$ as $n \to \infty\}$.

REMARK. Case (iii) does not attempt to measure the set of nonextinction since a large class of examples may be constructed with $E(\log m(\xi_0)) > 0$ and $P(q(\bar{\xi}) = 1) = 1$ (see Example 7.3).

Before proceeding with the proof of Theorem 5.5, it is necessary to prove two preliminary results. The first is a version of Theorem 5.5 under stronger hypotheses. The second relates the growth parameter of the associated process to the growth parameter of the branching process in a random environment and, as well, contains a restatement and proof of Theorem 2.3.

THEOREM 5.6 *Let* $\{Z_n\}_{n=0}^{\infty}$ *be a branching process in a random environment with* $Z_0 \equiv 1$ *such that* $P(p_0(\xi_0) = 0) = 1$. *Then* $E(\log m(\xi_0))$ *exists*, $E(\log m(\xi_0)) \geqq 0$ *and*

(i) $E(\log m(\xi_0)) = 0$ *implies that* $P(p_1(\xi_0) = 1) = 1$, *in which case* $P(Z_n \equiv 1 \ \forall\, n \mid \bar{\xi}) = 1$ w.p. 1,

(ii) $E(\log m(\xi_0)) > 0$ *implies that* $\lim_{n\to\infty} 1/n \log Z_n = E(\log m(\xi_0))$ w.p. 1.

PROOF. Since $P(p_0(\xi_0) = 0) = 1$, $m(\xi_0) \geqq 1$, w.p. 1. Hence $\log^- m(\xi_0) = 0$ w.p. 1, and so $E(\log m(\xi_0))$ exists and $E(\log m(\xi_0)) = E(\log^+ m(\xi_0)) \geqq 0$. Now if $E(\log m(\xi_0)) = 0$, then $m(\xi_0) = 1$ w.p. 1, and this implies that $P(p_1(\xi_0) = 1) = 1$ (which further implies that $P(Z_n \equiv 1 \ \forall\, n \mid \bar{\xi}) = 1$ w.p. 1). Suppose $E(\log m(\xi_0)) > 0$. Consider the branching process in a random environment $\{Z_n\}_{n=0}^{\infty}$ truncated at a positive integer A, and denote this process by $\{Z_n{}^A\}_{n=0}^{\infty}$; i.e., for $n \geqq 1$, $k \geqq 0$

$$
\begin{aligned}
P(Z_n{}^A = k \mid Z_{n-1}^A = 1, \bar{\xi}) &= P(Z_n = k \mid Z_{n-1} = 1, \bar{\xi}) && \text{if } k < A \\
&= \textstyle\sum_{j=A}^{\infty} P(Z_n = j \mid Z_{n-1} = 1, \bar{\xi}) && \text{if } k = A \qquad (20)\\
&= 0 && \text{if } k > A\,.
\end{aligned}
$$

(This truncation is equivalent to allowing the Z-process to evolve under the restriction that if any particle gives birth to more than A offspring, the particle is considered to have given birth to exactly A offspring.) Clearly $E(\log m^A(\xi_0))$ exists, where $m^A(\xi_0) = E(Z_1^A \mid Z_0^A = 1, \bar{\xi})$ is the truncated mean. $\{Z_n^A\}_{n=0}^\infty$ is a branching process in a random environment satisfying the conditions of Theorem 1 of [2], so for almost all $\bar{\xi}$, $\lim_{n\to\infty} Z_n^A/\prod_{i=0}^{n-1} m^A(\xi_i) = W^A$ w.p. 1, where W^A is a nonnegative random variable such that $P(W^A = 0 \mid \bar{\xi})$ where $q^A(\bar{\xi})$ is the extinction probability of the Z^A-process conditioned on $\bar{\xi}$. But $P(p_0(\xi_0) = 0) = 1$ so that $q^A(\bar{\xi}) = P(Z_n^A \to 0 \text{ as } n \to \infty \mid Z_0^A = 1, \bar{\xi}) = 0$ w.p. 1 and hence $P(W^A > 0 \mid \bar{\xi}) = 1$ w.p. 1. Therefore $\log W^A$ is well defined, and, for almost all $\bar{\xi}$,

$$\frac{1}{n}\log Z_n^A - \frac{1}{n}\log \prod_{i=0}^{n-1} m(\xi_i) - \frac{1}{n}\log W^A \to 0 \qquad \text{as } n \to \infty \text{ w.p. } 1\,. \tag{21}$$

But $\{\log m^A(\xi_i)\}_{i=0}^\infty$ is a stationary and ergodic sequence since $\bar{\xi}$ is stationary and ergodic ([4], Theorems 6.6 and 6.31) so by the Birkhoff ergodic theorem

$$\begin{aligned}\frac{1}{n}\log \prod_{i=0}^{n-1} m^A(\xi_i) \\ = \frac{1}{n}\sum_{i=0}^{n-1} \log m^A(\xi_i) \to E(\log m^A(\xi_0)) \qquad \text{as} \quad n \to \infty \quad \text{w.p. } 1\,.\end{aligned} \tag{22}$$

Now $1/n \log W^A \to 0$ as $n \to \infty$ w.p. 1, so by (21) and (22), $\lim_{n\to\infty} 1/n \log Z_n^A = E(\log m^A(\xi_0))$ w.p. 1. Since $\log m^A(\xi_0)$ is a nonnegative monotone increasing sequence in A, it follows that $E(\log m^A(\xi_0))$ increases to $E(\log m(\xi_0))$ as A increases to infinity. Also for $n \geqq 0$, $Z_n \geqq Z_n^A$. Hence

$$\liminf_{n\to\infty} \frac{1}{n}\log Z_n \geqq \liminf_{n\to\infty} \frac{1}{n}\log Z_n^A \geqq E(\log m(\xi_0)) \quad \text{w.p. } 1\,. \tag{23}$$

But by Theorem 5.1, $\limsup_{n\to\infty} 1/n \log Z_n \leqq E(\log m(\xi_0))$ and so $\lim_{n\to\infty} 1/n \log Z_n = E(\log m(\xi_0))$.

PROPOSITION 5.7. *If $P(q(\bar{\xi}) < 1) = 1$ and $E(\log m(\xi_0))$ exists, then $E(\log m(\xi_0)) = E(\log \hat{m}(\xi_0))$. If in addition $E(\log^+ m(\xi_0)) < \infty$, then $E(|\log m(\xi_0)|) < \infty$ and $\lim_{n\to\infty} 1/n \log(1 - p_0(\xi_n)) = 0$ w.p. 1.*

PROOF. Since $P(q(\bar{\xi}) < 1) = 1$, the associated branching process in a random environment $\{\hat{Z}_n\}_{n=0}^\infty$ is well defined. Using (5), we conclude that for any $\bar{\xi}$ for which $q(T^n\bar{\xi}) < 1$ for all n (almost all $\bar{\xi}$),

$$\hat{m}(\xi_{n-1}) = \frac{1 - q(T^n\bar{\xi})}{1 - q(T^{n-1}\bar{\xi})}\, m(\xi_{n-1})\,. \tag{24}$$

and so

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} \log \hat{m}(\xi_i) = \frac{1}{n}\log(1 - q(T^n\bar{\xi})) - \frac{1}{n}\log(1 - q(\bar{\xi})) \\ + \frac{1}{n}\sum_{i=0}^{n-1} \log m(\xi_i)\,.\end{aligned} \tag{25}$$

Now $\{\log \hat{m}(\xi_i)\}_{i=0}^\infty$ and $\{\log m(\xi_i)\}_{i=0}^\infty$ are stationary ergodic sequences because $\bar{\xi}$

is stationary and ergodic ([4]). Also $P(\hat{p}_0(\xi_0) = 0) = 1$ implies that $\hat{m}(\xi_0) \geqq 1$ and so $E(\log \hat{m}(\xi_0))$ exists. By hypothesis, $E(\log m(\xi_0))$ exists. Thus, by the Birkhoff ergodic theorem, $\lim_{n\to\infty} 1/n \sum_{i=0}^{n-1} \log \hat{m}(\xi_i) = E(\log \hat{m}(\xi_0))$ w.p. 1 and $\lim_{n\to\infty} 1/n \sum_{i=0}^{n-1} \log m(\xi_i) = E(\log m(\xi_0))$ w.p. 1. Also $1/n \log (1 - q(T^n\bar{\xi}))$ equals $1/n \log (1 - q(\bar{\xi}))$ in distribution and so $1/n \log (1 - q(T^n\bar{\xi}))$ converges to zero in distribution, and hence, in probability, as $n \to \infty$. Thus the right-hand side of (26) converges to $E(\log m(\xi_0))$ in probability as $n \to \infty$. Since the left-hand side of (26) converges w.p. 1 and hence in probability, to $E(\log \hat{m}(\xi_0))$ as $n \to \infty$, it follows that $E(\log m(\xi_0)) = E(\log \hat{m}(\xi_0))$.

Suppose $E(\log^+ m(\xi_0)) < \infty$. Since $P(q(\bar{\xi}) < 1) = 1$, it follows that $E(\log m(\xi_0)) > 0$ by Corollary 5.2. Hence $E(\log^- m(\xi_0)) < \infty$ and so $E|\log m(\xi_0)| < \infty$. But this implies that $E \log \hat{m}(\xi_0) < \infty$. Also for $n \geqq 0$,

$$q(T^n\bar{\xi}) \geqq P(Z_1 = 0 \mid Z_0 = 1, T^n\bar{\xi}) = p_0(\xi_n) \tag{26}$$

and so

$$\begin{aligned} 0 &\leqq \limsup_{n\to\infty} \left| \frac{1}{n} \log (1 - p_0(\xi_n)) \right| \leqq \limsup_{n\to\infty} \left| \frac{1}{n} \log (1 - q(T^n\bar{\xi})) \right| \\ &= \limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=0}^{n-1} \log \hat{m}(\xi_i) - \frac{1}{n} \sum_{i=0}^{n-1} \log m(\xi_i) \right. \\ &\qquad \left. - \frac{1}{n} \log (1 - q(\bar{\xi})) \right| \qquad (27) \\ &= |E(\log \hat{m}(\xi_0)) - E(\log m(\xi_0))| \quad \text{w.p. } 1 \\ &= 0 . \end{aligned}$$

Therefore $\lim_{n\to\infty} 1/n \log (1 - p_0(\xi_n)) = 0$ w.p. 1.

PROOF OF THEOREM 5.5. (i) If $E(\log m(\xi_0))$ exists and is negative then $P(q(\bar{\xi})) = 1) = 1$ by Corollary 5.2. (ii) Suppose $P(q(\bar{\xi}) < 1) = 1$. Then the associated branching process in a random environment $\{\hat{Z}_n\}_{n=0}^{\infty}$ is well defined and, by Proposition 5.7, $E(\log \hat{m}(\xi_0)) = E(\log m(\xi_0)) = 0$. But $\hat{m}(\xi_0) \geqq 1$ w.p. 1, so we must have $\hat{m}(\xi_0) \equiv 1$ w.p. 1. Therefore $P(\hat{p}_1(\xi_0) = 1) = 1$. Using the contrapositive form of Lemma 5.4, we conclude that $P(p_1(\xi_0) = 1) = 1$ and thus (ii) is proved. (iii) If $P(q(\bar{\xi}) = 1) = 1$, then there is nothing to prove since it implies that $P(Z_n \to 0 \text{ as } n \to \infty \mid \bar{\xi}) = 0$ w.p. 1 independent of Z_0. So it suffices to assume that $P(q(\bar{\xi}) < 1) = 1$. Then the associated process $\{\hat{Z}_n\}_{n=0}^{\infty}$ is well defined and $E(\log \hat{m}(\xi_0)) = E(\log m(\xi_0)) > 0$. By Theorem 5.6, $\lim_{n\to\infty} 1/n \log \hat{Z}_n = E(\log \hat{m}(\xi_0))$ w.p. 1. Applying Theorem 4.4 yields $\lim_{n\to\infty} 1/n \log W_n = E(\log m(\xi_0))$ a.e. on $\{\omega : W_0 \neq 0\}$. Since $E(\log m(\xi_0)) > 0$, $W_n \to +\infty$ as $n \to \infty$ a.e. on $\{\omega : W_0 \neq 0\}$ and so $Z_n \to +\infty$ as $n \to +\infty$ a.e. on $\{\omega : W_0 \neq 0\}$ since $Z_n \geqq W_n$ for all n. Clearly $\{\omega : Z_n \to +\infty \text{ as } n \to \infty\} \subseteqq \{\omega : W_0 \neq 0\}$. Thus the sets $\{\omega : W_0 \neq 0\}$ and $\{\omega : Z_n \to +\infty \text{ as } n \to \infty\}$ differ by a set of measure zero. Hence

$$\liminf_{n\to\infty} \frac{1}{n} \log Z_n \geqq \liminf_{n\to\infty} \frac{1}{n} \log W_n = E(\log m(\xi_0)) \quad \text{a.e.} \tag{28}$$

on $\{\omega : Z_n \to +\infty \text{ as } n \to \infty\}$, and applying Theorem 5.1 concludes the proof of (iii).

REMARK. Part (iii) of the classification theorem states that whenever "explosion" of the process occurs, the rate of explosion is exponential. Under the hypothesis $E((1/m(\xi_0)) \sum_{j=2}^{\infty} p_j(\xi_0) j \log j) < \infty$, Athreya and Karlin ([2]), obtained the following sharper growth rate result: for almost all $\bar{\xi}$, $\lim_{n\to\infty} Z_n / \prod_{i=0}^{n-1} m(\xi_i) = W$ w.p. 1, where $P(W = 0 \mid \bar{\xi}) = q(\bar{\xi})$ and $E(W \mid \bar{\xi}) = 1$ w.p. 1. The following theorem, proved by the author in [13], shows that appropriate normalizing constants for the process always exist as long as the environmental process is reversible.

DEFINITION 5.8. $\bar{\xi} = (\xi_0, \xi_1, \cdots)$ is a reversible stochastic process if, for any nonnegative integer K, the joint distributions of $\{\xi_0, \xi_1, \cdots, \xi_K\}$ and $\{\xi_K, \xi_{K-1}, \cdots, \xi_0\}$ are equal.

THEOREM 5.9. *Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process in a random environment with a reversible stationary and ergodic environmental sequence $\bar{\xi}$. Then there exist random normalizing constants $C_n(\bar{\xi})$ such that, for almost all $\bar{\xi}$,* $\lim_{n\to\infty} Z_n / C_n(\bar{\xi}) = W$ w.p. 1 *where* (i) *W is a random variable satisfying $P(W = 0 \mid \bar{\xi}) = q(\bar{\xi})$ and* (ii) $\lim_{n\to\infty} 1/n \log C_n(\bar{\xi}) = E(\log m(\xi_0))$ w.p. 1.

6. Noncertain extinction of the branching process in a random environment. Disregarding the trivial case $P(p_1(\xi_0) = 1) = 1$, Theorem 5.5 implies that a necessary condition for noncertain extinction is $E(\log m(\xi_0)) > 0$ provided this expectation exists. By the monotone convergence theorem, $E(\log (m(\xi_0) \wedge A)) \uparrow E(\log (m(\xi_0)))$ as $A \uparrow \infty$, where $a \wedge b$ denotes $\min (a, b)$. Thus for all sufficiently large A, $E_A = E(\log (m(\xi_0) \wedge A)) > 0$.

DEFINITION 6.1. For real $t \geqq 0$, let $\gamma_A{}^t(\xi_0) = \inf_{c \in Z^+} \{c \geqq 1 : \int_0^c x \, dF_{\xi_0}(x) + c(1 - F_{\xi_0}(c)) \geqq (m(\xi_0) \wedge A)/e^{tE_A}\}$ where F_{ξ_0} is the probability distribution having p.g.f. φ_{ξ_0}. It is easily seen that $\gamma_A{}^t(\xi_0)$ decreases to some random variable denoted by $D_A(\xi_0)$ as t increases to 1. (In fact, since $\gamma_A{}^t(\xi_0) \in \mathbb{Z}^+$, there exists $t_0(\xi_0)$ such that $\gamma_A{}^t(\xi_0) = D_A(\xi_0)$ for all $t_0(\xi_0) \leqq t < 1$.) By truncation arguments and application of the extended Kaplan–Karlin technique ([8], [12]), the following proposition may be proved.

PROPOSITION 6.2. *Suppose there exists some real $A > 0$ satisfying*

(i) $E(\log (m(\xi_0) \wedge A)) > 0$ *and*
(ii) $\limsup_{n\to\infty} (|\log D_A(\xi_n)|/n) < \infty$.

Then $P(q(\bar{\xi}) < 1) = 1$.

The sufficient condition of Athreya and Karlin ([1]) follows as an easy corollary.

COROLLARY 6.3. *If* $E(|\log (1 - p_0(\xi_0))|) < \infty$ *and* $E(\log m(\xi_0)) > 0$ *then* $p(q(\bar{\xi}) < 1) = 1$.

PROOF. Let $\{Z_n{}^A\}_{n=0}^{\infty}$ denote the process $\{Z_n\}_{n=0}^{\infty}$ truncated at $A > 0$, and let

$m^A(\xi_0) = E(Z_1{}^A \mid Z_0{}^A = 1, \xi)$ be the truncated mean. Since $m^A(\xi_0) \geqq (1 - p_0(\xi_0))$, $E(\log^- m^A(\xi_0)) \leqq E(\log^- (1 - p_0(\xi_0))) < \infty$ and hence $E(\log m^A(\xi_0))$ increases to $E(\log m(\xi_0))$ as $A \uparrow \infty$. Choose $A > 0$ sufficiently large such that $E(\log m^A(\xi_0)) > E(\log m(\xi_0))/2 > 0$. Let $q^A(\bar{\xi})$ denote the extinction probability of the process $\{Z_n{}^A\}_{n=0}^\infty$ conditioned on $\bar{\xi}$. Now $E(\log (m^A(\xi_0) \wedge A)) = E(\log m^A(\xi_0)) > 0$, and clearly $1 \leqq \gamma_A{}^t(\xi_0) \leqq A$ for the process $\{Z_n{}^A\}_{n=0}^\infty$. Thus the conditions of Proposition 6.2 are satisfied and so $P(q^A(\bar{\xi}) < 1) = 1$. But $Z_n \geqq Z_n{}^A$ w.p. 1 for $n \geqq 0$, and so $q(\bar{\xi}) \leqq q^A(\bar{\xi})$ w.p. 1. Hence $P(q(\bar{\xi}) < 1) = 1$.

7. Examples. The examples in this section rely heavily on the properties of the following Markov chain.

EXAMPLE 1. Consider a Markov chain $\{X_n\}_{n=1}^\infty$ with state space the positive integers and with the following stationary transition probabilities:

$$P(0, k) = p_k \qquad k = 1, 2, \cdots$$
$$P(k, k-1) = 1 \qquad k = 1, 2, \cdots$$

where $\{p_k\}_{k=1}^\infty$ is a sequence of positive reals such that

$$\text{(29)} \quad \begin{array}{ll} \text{(i)} & \sum_{k=1}^\infty p_k = 1\,, \\ \text{(ii)} & \sum_{k=1}^\infty k p_k < \infty \quad \text{and} \\ \text{(iii)} & \sum_{k=1}^\infty k^2 p_k = +\infty\,. \end{array}$$

Let $0 \leqq T_1 < T_2 < \cdots$ be the successive times that the Markov chain occupies state 0. In order that there exists an invariant probability distribution for the Markov chain, it is necessary and sufficient that each state be positive recurrent. In this case, the latter is equivalent to $E_0 T_1 < \infty$. But $E_0 T_1 = \sum_{k=1}^\infty (k+1) p_k$, which, by condition (ii) is finite. Thus there exists a finite stationary initial distribution which will be denoted by μ. Since $\{X_n\}_{n=1}^\infty$ is clearly an indecomposable Markov chain, the Markov process $\{X_n\}_{n=0}^\infty$ having the initial distribution μ is stationary and ergodic ([4, Propositions 7.11, 7.16]). Furthermore, $E''(X_0) = +\infty$ and $\lim_{n\to\infty} X_n/n = 0$ w.p. 1 ([12]).

EXAMPLE 2. A branching process in a random environment $\{Z_n\}_{n=0}^\infty$ with stationary Markovian environments such that

(i) $0 < E(\log m(\xi_0)) < \infty$,
(ii) $E|\log (1 - p_0(\xi_0))| = +\infty$, and
(iii) $P(q(\bar{\xi} < 1) = 1$.

This example shows that the condition of Smith–Wilkinson is not a necessary condition for noncertain extinction of the process (see Theorem 2.3).

Let $\{X_n\}_{n=0}^\infty$ be the Markov process of Example 1 and let $\{\xi_n\}_{n=0}^\infty$ be a sequence of random environments with ξ_n defined by its associated probability generating function

$$\varphi_{\xi_n}(s) = (1 - e^{-X_n}) + e^{-X_n} s^{\lambda_n} \qquad (30)$$

where $\lambda_n = ([4e^{X_n}] + 1)$. (This corresponds to placing probability mass e^{-X_n} at the point λ_n and the rest of the probability mass at the origin.)

Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process in a random environment with environmental sequence $\{\xi_n\}_{n=0}^{\infty}$; i.e., conditioned on the environment ξ_n, a particle in the nth generation gives birth to λ_n particles with probability e^{-X_n} or to 0 particles with probability $1 - e^{-X_n}$. By repeating the argument of Theorem 1 of [12], it follows readily that $P(q(\bar{\xi}) < 1)$ equals 1. Furthermore, $E|\log(1 - p_0(\xi_0))| = EX_0 = +\infty$, and

$$0 \leqq 4e^{X_n} \cdot e^{-X_n} \leqq m(\xi_n) \leqq (4e^{X_n} + 1)e^{-X_n} \tag{31}$$

so that $0 < E(\log m(\xi_0)) < \infty$.

EXAMPLE 3. A branching process in a random environment $\{Z_n\}_{n=0}^{\infty}$ such that $\infty > E(\log m(\xi_0)) > 0$ and $P(q(\bar{\xi}) = 1) = 1$.

If $\{Z_n\}_{n=0}^{\infty}$ is any branching process in a random environment having i.i.d. environments $\{\xi_n\}_{n=0}^{\infty}$, and such that $0 < E(\log m(\xi_0)) < \infty$, then $P(q(\bar{\xi}) = 1) = 1$ iff $E|\log(1 - p_0(\xi_0))| = +\infty$. In particular, let $\{X_n\}_{n=0}^{\infty}$ be a sequence of positive i.i.d. random variables with $EX_0 = +\infty$, and with this choice of $\{X_n\}_{n=0}^{\infty}$, let $\{\xi_n\}_{n=0}^{\infty}$ be defined by (30). Then the process $\{Z_n\}_{n=0}^{\infty}$ satisfies $\infty > E(\log m(\xi_0)) > 0$ and $E|\log(1 - p_0(\xi_0))| = +\infty$, so by the previous remarks, $\{Z_n\}_{n=0}^{\infty}$ satisfies $P(q(\bar{\xi}) = 1) = 1$.

EXAMPLE 4. A branching process in a random environment $\{Z_n\}_{n=0}^{\infty}$ with stationary Markov environments such that

(i) $0 < E(\log m(\xi_0)) < +\infty$
(ii) $1/n \log(1 - p_0(\xi_n)) \to 0$ as $n \to \infty$ w.p. 1, and
(iii) $P(q(\bar{\xi}) = 1) = 1$.

Let $\{X_n\}_{n=0}^{\infty}$ be the Markov process described in Example 1. Let $\{Z_n\}_{n=0}^{\infty}$ be a branching process in a random environment with environmental sequence $\bar{\xi}$, where ξ_n has as its associated probability generating function

$$\varphi_{\xi_n}(s) = (1 - e^{-X_n}) + (e^{-X_n} - e^{-X_n^4})s + e^{-X_n^4}s^{\lambda_n} \tag{32}$$

where $\lambda_n = ([2e^{X_n^4}] + 1)$.

$\{Z_n\}_{n=0}^{\infty}$ satisfies (i) since

$$0 < 2e^{X_n^4} \cdot e^{-X_n^4} \leqq m(\xi_n) \leqq 2 + 2e^{X_n^4}e^{-X_n^4} \tag{33}$$

so that $0 < E(\log m(\xi_0)) < +\infty$.

Since $\lim_{n\to\infty} X_n/n = 0$ w.p. 1, and since $\log(1 - p_0(\xi_n)) = -X_n$, it follows that $\lim_{n\to\infty} 1/n \log(1 - p_0(\xi_n)) = 0$ w.p. 1. It remains to prove part (iii).

We first note that the sequence $\{X_{T_k+1}\}_{k=1}^{\infty}$ is i.i.d. and that

$$EX_{T_1+1}^2 = \textstyle\sum_{k=1}^{\infty} k^2 p_k = +\infty\ , \tag{34}$$

so that

$$\limsup_{k\to\infty} \frac{X_{T_k+1}^2}{k} = +\infty \quad \text{w.p. } 1\ . \tag{35}$$

Since $\lim_{k\to\infty} (T_k + 1)/k = E_0 T_1$ w.p. 1, where $0 < E_0 T_1 < +\infty$, it follows that

$$\limsup_{k\to\infty} \frac{X^2_{T_k+1}}{T_k + 1} = +\infty \quad \text{w.p. } 1 . \tag{36}$$

Thus, w.p. 1, there exists a strictly increasing sequence of positive integers $\{n_k\}_{k=0}^{\infty} = n_k(\{X_i\}_{i=0}^{\infty})\}_{k=0}^{\infty}$ such that $X^4_{n_k} \geqq n_k{}^2 e_k{}^2 \geqq n_k{}^2 e_k{}^2$ for all $k \geqq 0$, where $\{e_k\}_{k=0}^{\infty}$ is a sequence of positive constants such that $e_k \uparrow +\infty$ as $k \to \infty$.

For $k \geqq 0$, let

$$A_k = \{\omega \in \Omega : \text{ for } n_k \leqq j \leqq n_k + [(n_k)^{\frac{1}{2}}/2], \text{ no particle in the } j\text{th generation produces more than one child and } Z_{n_k+[(n_k)^{\frac{1}{2}}/2]} > 0\} .$$

$$P(A_k \mid Z_{n_k}, \bar{\xi}) \leqq Z_{n_k} P(Z_j = 1 \text{ for } n_k \leqq j \leqq n_k + [(n_k)^{\frac{1}{2}}/2] \mid Z_{n_k} = 1, \bar{\xi}) . \tag{37}$$

Now, by Theorem 5.1, $\limsup_{n\to\infty} 1/n \log Z_n \leqq E$ so for all sufficiently large $k = k(\omega)$,

$$Z_{n_k} \leqq e^{2n_k E} . \tag{38}$$

Also

$$\begin{aligned} P(Z_j = 1 \text{ for } n_k \leqq j \leqq n_k + [(n_k)^{\frac{1}{2}}/2] \mid Z_{n_k} = 1, \bar{\xi}) &= \textstyle\prod_{j=n_k+1}^{n_k+[(n_k)^{\frac{1}{2}}/2]} (e^{-X_j} - e^{-X_j^4}) \\ &\leqq \textstyle\prod_{j=1}^{[(n_k)^{\frac{1}{2}}/2]} e^{-(n_k e_k)^{\frac{1}{2}}+j} \\ &\leqq \textstyle\prod_{j=[(n_k e_k)^{\frac{1}{2}}-\frac{1}{2}(n_k)^{\frac{1}{2}}]+1}^{[(n_k e_k)^{\frac{1}{2}}]-1} e^{-j} \\ &\leqq e^{-(n_k(e_k)^{\frac{1}{2}})/8} \end{aligned} \tag{39}$$

for sufficiently large k. Thus

$$P(A_k \mid Z_{n_k}, \bar{\xi}) = O(e^{-n_k(e_k)^{\frac{1}{2}}/8+2n_k E}) \tag{40}$$

as $k \to \infty$ and since $e_k \uparrow +\infty$ as $k \to \infty$, it follows that $\sum_{k=1}^{\infty} P(A_k \mid Z_{n_k} \bar{\xi}) < \infty$. By the extended Borel–Cantelli lemma, $P(A_k \text{ i.o.} \mid \bar{\xi}) = 0$.

Now on $\{\omega : Z_n \not\to 0 \text{ as } n \to \infty\}$, $Z_{n_k+[(n_k)^{\frac{1}{2}}/2]} > 0$ for all $k \geqq 0$. Furthermore, since $\limsup_{n\to\infty} 1/n \log Z_n \leqq E$ w.p. 1, there exists w.p. 1 an integer $n_0(\omega)$ such that $Z_n \leqq 2^{nE}$ for all $n \geqq n_0(\omega)$. Since

$$X_n{}^4 \geqq \frac{n_k{}^2 e_k{}^2}{16} > 4nE \tag{41}$$

for $n_k < n < n_k + [(n_k)^{\frac{1}{2}}/2]$ and all sufficiently large k, it follows from (32) and a simple Borel–Cantelli argument that any particle in the nth generation, $n_k < n < n_k + [(n_k)^{\frac{1}{2}}/2]$ and k sufficiently large must give birth to at most one particle. Thus

$$P(Z_n \not\to 0 \text{ as } n \to \infty \mid \bar{\xi}) \leqq P(A_n \text{ i.o.} \mid \bar{\xi}) = 0 \quad \text{w.p. } 1 . \tag{42}$$

Therefore $\{Z_n\}_{n=0}^{\infty}$ satisfies $P(q(\bar{\xi}) = 1) = 1$.

8. Remarks. While the growth theorems presented here concerned the growth of $\log Z_n$, it is of interest to consider limit theorems for $(F(Z_n) - g(n))$ for an appropriate class of functions F and g. This problem was considered by Waugh ([14]) for continuous time (nonrandom environment) birth and death processes with $g(t) \equiv t$; however Waugh's motivation stemmed from convergence theorems for continuous time (nonrandom environment) branching processes.

In closing, it should be noted that a number of theorems presented here in terms of branching Processes in a random environment can be generalized to "branching processes in a varying environment" (see [6]).

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Department of Mathematics
The University of Rochester
Rochester, New York 14627