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BILLIARDS IN POLYGONS

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Some questions concerning the orbits of a billiard ball in a polygon are studied. It is shown that almost all such orbits come arbitrarily close to a vertex of the polygon, implying that the entropy of the corresponding geodesic flow is zero. For polygons with rational angles, we show by using interval exchange transformations that almost all orbits are spatially dense. Two applications are given.

The present paper answers some questions concerning a particular classical dynamical system, namely, billiards with one ball in a plane polygon. In the case where the region considered contains a convex obstruction, a certain number of results have been obtained (see e.g., [5], [7]). These systems turn out to have strong ergodic properties (i.e., Markov partitions can be constructed), due to the exponential scattering which occurs at the obstruction. However, if the boundary and/or obstructions have zero curvature, very little is known.

We consider in the following a point mass moving in a given non-self-intersecting plane polygon, with the usual rules of reflection when the point mass hits a side. It is shown first that, for each initial point and almost all initial directions, the orbit comes arbitrarily close to at least one vertex of the polygon. This yields in particular a coding procedure for the orbits, and implies that the corresponding dynamical system has zero entropy. Next, we concentrate on the simpler case where all angles of the polygon are rational multiples of π. This case can be reformulated in the context of interval exchange transformations (see [2] and [3]). We show here that for almost all starting conditions, the corresponding orbits are dense in the polygon. Finally, we discuss two interesting physical interpretations of our results.

Many problems remain to be solved, even in the case of a triangle. In particular, we have not been able to prove in general that if the angles of the polygon are irrational, the corresponding dynamical system is ergodic, or even that there exists a dense orbit (either in the polygon or in the phase space).

Some of the results given here have been presented in a preliminary way by one of us in [2]. Zemljakov and Katok have obtained analogous results by quite different techniques (see [9]). Research was done during a visit by the second author to Camerino, sponsored by the Italian C.N.R.

1. Definitions and notations. Let P denote the interior of a non-self-intersecting polygon in the plane with vertices A_i, sides a_i, and angles α_i,

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$i = 1, \cdots, n$. We consider the geodesic flow on P with the usual reflection rule on the boundary ∂P.

A *line element* $\omega = (x, \theta)$ of this flow is given by a point $x \in \bar{P}$ and an angle $\theta \in R/2\pi Z$. We think of a line element as a small arrow issuing from the point x and pointing in the direction θ, measured from a fixed reference direction which we shall take to be $\overrightarrow{A_1A_2}$.

On the boundary, we make the following identifications concerning line elements:

1°. If $x \in \partial P$ is not a vertex, then x lies on a unique side a_i of P which makes an angle, say β_i, with our reference direction $\overrightarrow{A_1A_2}$. We identify in this case the line elements $(x, \beta_i + \theta)$ and $(x, \beta_i - \theta)$ for each $0 < \theta < \pi$.

2°. If $x \in \partial P$ is a vertex, then we identify all line elements (x, θ), $0 \leqq \theta < 2\pi$.

The *phase space* Ω of our geodesic flow is then given by the set of all line elements. Under the obvious topology, Ω is compact and metrizable.

Consider now the one-parameter semigroup $(S_t)_{t \in R^+}$ of transformations on Ω defined as follows: if $(x_0, \theta_0) \in \Omega$, then $S_t(x_0, \theta_0) = (x_t, \theta_t)$ is obtained by starting at x_0 and drawing a continuous path inside the polygon consisting of straight line segments and of total length t, and ending at the point x_t in the direction θ_t. The straight line segments should (except for the first one, which begins at x_0 in the direction θ_0) begin and end (except for the last one, which ends at x_t in the direction θ_t) on the boundary ∂P, and the direction change at the boundary in passing from one segment to the next one is made in accordance with the identification in 1°. If the path should hit a vertex S before attaining length t, then we define $x_t = S$. In particular, $S_t(x_0, \theta_0) = (x_0, \theta_0)$ if x_0 is a vertex, for all $t \geqq 0$. The reader will easily see that these requirements define S_t uniquely for each $t > 0$ and that $(S_t)_{t \in R^+}$ is a semigroup of measurable transformations on Ω.

For $t \leqq 0$, we define S_t by setting $S_t(x_0, \theta_0) = (x_t, \theta_t)$ iff $S_{-t}(x_0, \theta_0 + \pi) = (x_t, \theta_t + \pi)$. Then on the set Ω' of line elements ω for which $S_t\omega$ is not a vertex for all $t \in R$, each transformation S_t is continuous and $(S_t)_{t \in R}$ is a one-parameter transformation group acting on Ω'. Moreover, Ω' is a dense G_δ in Ω.

Denote by dx normalized Lebesgue measure on $\bar{P}$ and by $d\theta$ normalized Haar measure on $R/2\pi Z$. Since $dx(\partial P) = 0$, there is a well-defined probability measure m on Ω corresponding to the product measure $dx \times d\theta$ on $\bar{P} \times R/2\pi Z$. It is easy to see that for each t, $S_t m = m$, and that $m(\Omega') = 1$. The triple $(\Omega, (S_t)_{t \in R}, m)$ will be called *billiards* on P.

Note that Ω is just the phase space of a mechanical system consisting of a Newtonian particle moving inside the polygon P with constant speed. The time evolution S_t is the one corresponding to absence of forces inside P and elastic reflection conditions at the boundary. We have defined S_t in such a way that vertices act as sinks, for simplicity. There is a more natural definition which consists in doubling each vertex and defining reflection at the vertex as a limit either from one side or from the other, but this involves a rather complicated description of the phase space. (The transformation in Section 4 acts on vertices correctly if we use this more natural definition.)

We now set

$$\Omega_0 = \{\omega = (x, \theta) \in \Omega : x \in \partial P\}$$

and

$$\Omega_0' = \Omega_0 \cap \Omega' .$$

It will be useful to define a transformation $T: \Omega_0 \to \Omega_0$ *induced* by the semigroup $(S_t)_{t \in R^+}$. If $(x, \theta) \in \Omega_0$, then we set

$$T(x, \theta) = S_{t_0(x,\theta)}(x, \theta) ,$$

where

$$t_0(x, \theta) = \inf\{t > 0 : S_t(x, \theta) \in \Omega_0\} .$$

Let m_0 be the probability measure on Ω_0 corresponding to the product measure $dx_0 x(\sin(\beta_i - \theta)\, d\theta/C)$, $(x_0 \in a_i)$, where C is a normalization constant. Then $Tm_0 = m_0$, $m_0(\Omega_0') = 1$, and T is continuous and invertible on Ω_0'. Ergodic or asymptotic properties of $(S_t)_{t \in R}$ are reflected in ergodic or asymptotic properties of T.

Finally, for $\omega \in \Omega$ we define

$$\begin{aligned} \mathrm{Orb}^+(\omega) &= \{S_t\omega : t \in R^+\} \\ \mathrm{Orb}^-(\omega) &= \{S_t\omega : t \in R^-\} \\ \mathrm{Orb}(\omega) &= \{S_t\omega : t \in R\} , \end{aligned}$$

and if $\omega = (x_0, \theta_0)$ and $S_t(\omega) = (x_t, \theta_t)$ $(t \in R)$,

$$\begin{aligned} \mathrm{Orb}_s^+(\omega) &= \{x_t : t \geqq 0\} \\ \mathrm{Orb}_s^-(\omega) &= \{x_t : t \leqq 0\} \\ \mathrm{Orb}_s(\omega) &= \{x_t : t \in R\} . \end{aligned}$$

These are called respectively the forward orbit, backward orbit, orbit, forward spatial orbit, backward spatial orbit, and spatial orbit of ω. The orbits of $\omega \in \Omega_0$ under T are defined similarly and will be denoted by the same symbols.

2. Statement of the problem and remarks. Let us begin by stating a few natural questions which one is quickly led to ask concerning billiards in P.

PROBLEM 1. Does there exist a line element $\omega \in \Omega$ whose orbit is dense in Ω?

PROBLEM 2. Does there exist a line element $\omega \in \Omega$ whose spatial orbit is dense in P?

PROBLEM 3. Are almost all orbits dense in Ω?

PROBLEM 4. Are almost all spatial orbits dense in P?

PROBLEM 5. Is the "billiards" dynamical system $(\Omega, (S_t)_{t \in R}, m)$ ergodic?

PROBLEM 6. If an orbit is (spatially) dense in $(P)\Omega$, is it uniformly distributed?

These are only a few examples of what one is really interested in, namely, the description of asymptotic and ergodic properties of a ball bouncing around in a polygon. We should begin at once by stating that, even in the case of a triangle, except for a few obvious examples and the small contribution we shall make below, nothing is known concerning the answers to any of the problems stated.

We continue by making three obvious remarks concerning the problems.

REMARK 1. One might be led to conjecture that perhaps all orbits of $\omega \in \Omega'$ would be dense, or spatially dense. In certain cases, at least, this is not true.

In particular, if P is an acute triangle, imagine a miniscule ring around each side of the triangle, take a piece of string, and pull it through each of the three loops. Now pull both ends until the string is taut and connect the ends. The rings will slip on the sides until an equilibrium is reached, and this equilibrium yields a periodic orbit under S_t.

Equivalently, this orbit is the one obtained by joining the bases of the three heights of the triangle.

There are even in this case uncountably many periodic orbits, which can be obtained from the one above by a slight perturbation in one of the points of the orbit, maintaining the same direction. All of these orbits have twice the length of the original one.

This remark leads us to two more problems which we have not been able to resolve.

PROBLEM 7. Does any polygon (and in particular, an obtuse triangle) have periodic orbits?

PROBLEM 8. Let us call two orbits *equivalent* if they have the same length. In an acute triangle (or in any polygon), do there exist infinitely many pairwise nonequivalent orbits?

For the examples discussed in Remark 3 below, the answer to Problem 8 can be seen to be "yes."

REMARK 2. A particular case of interest, which we shall go into more deeply in a later paragraph, is the one in which all the angles $\alpha_1, \alpha_2, \cdots$ of P are rational multiples of π. Let us call such a polygon a *rational polygon*. In this case, the answer to Problems 1, 2 and 5 is certainly "no." To see this, consider a point which arrives at side a_i with an angle θ, leaves at an angle of $-\theta$, and after bouncing off side a_{i-1} returns to side a_i at an angle $\theta + 2\alpha_i$. This situation is general, i.e., if we start off with an angle θ, then the only angles which we can obtain at later or earlier times are those of the form $\pm\theta + \sum \pm 2\varphi_{(j)}$ where $\varphi_{(j)}$ denotes the angle the side which is met at the jth reflection makes with our reference direction. If P is rational, the angles which are attainable form a finite subgroup of $R/2\pi Z$ translated by θ, and hence no orbit can be dense in Ω. We shall see later that the answers to Problems 2 and 4 in this case are "yes," whereas the answer to Problem 6 seems to be "no" in light of [4]. In passing we note also that if a ball is shot in the direction of a corner and does not hit the corner, then after a finite number of bounces it will come out of the corner.

REMARK 3. In some special cases we can arrive at our goal of describing the orbits with a good deal of accuracy and answer the problems posed. Consider a point which is about to bounce off a side, and its orbit. Instead of stopping at the side and reflecting, it is the same if we continue the orbit in a straight line and reflect the polygon P around the side. If P is a polygon whose *reflections pack the plane* (e.g., an equilateral triangle, or a rectangle, or a 45° right triangle, and a few others), then the corners of the reflections of P form a regular grid in the plane and we leave it to the reader to see that the problems can be solved. Note that all polygons having the property described are rational.

3. A general result. In this paragraph, we describe a result valid for all polygons which, although it falls short of answering our problems, seems to be of interest.

THEOREM. *Let $x \in \bar{P}$. Then for almost all $\theta \in R/2\pi Z$, the set*

$$\overline{\mathrm{Orb}_s(\omega)}$$

(*where $\omega = (x, \theta)$*), *contains at least one vertex of P.*

PROOF. Set $\mathcal{O}(\theta) = \overline{\mathrm{Orb}_s(\omega)}$. It suffices to show that for any fixed $\delta > 0$, the set

$$N = \{\theta : \mathrm{dist}\,(\mathcal{O}(\theta), \{A_i \mid i = 1, \cdots, n\}) \geqq \delta\}$$

has measure zero. By a well-known theorem (see, e.g., [6]), almost all points of N are points of density of N. That is, for almost all $\theta_0 \in N$,

$$\lim_{\varepsilon\to 0} \frac{|N \cap |\theta_0, \theta_0 + \varepsilon||}{\varepsilon} = 1\,,$$

where $|\cdot|$ denotes Haar measure on $R/2\pi Z$. Thus if N contains no points of density, then $|N| = 0$.

Suppose that θ_0 belongs to N. Let $\varepsilon > 0$ be fixed. Consider the line elements (x, θ_0) and $(x, \theta_0 + \varepsilon)$. We may assume that both line elements belong to Ω'. Thus there exist two increasing sequences $s_1, s_2, \cdots$ and $t_1, t_2, \cdots$ of positive numbers such that the orbit of (x, θ_0) meets the boundary of P at the successive times $s_1, s_2, \cdots$ and the orbit of $(x, \theta_0 + \varepsilon)$ meets ∂P at the successive times $t_1, t_2, \cdots$. Choose $n \geqq 1$ minimal such that the side which contains $S_{s_n}(x, \theta_0)$ is different from the side which contains $S_{t_n}(x, \theta_0 + \varepsilon)$. That such an n does exist follows by using the device of Remark 3: drawing straight lines from x in the directions θ_0 and $\theta_0 + \varepsilon$, and assuming that $S_{s_1}(x, \theta_0)$ and $S_{t_1}(x, \theta_0 + \varepsilon)$ lie on the same side of P, we may reflect P around that side. As we continue this process, the lines grow farther and farther apart, until the first time where a reflection of this type places a vertex of P in the cone created by the two lines. The following n will then have the desired property.

Now consider the set $N \cap [\theta_0, \theta_0 + \varepsilon]$. If $\theta_0 \leqq \theta \leqq \theta_0 + \varepsilon$, then the forward orbit of (x, θ) under S_t can be thought of as a ray of the cone described alone. If $\theta \in N$, then this ray cannot come within a distance of δ from the vertex of P which fell in the cone at the nth reflection. On the other hand, the distance across the whole cone at the nth reflection is at most the diameter of P. Therefore

$$\frac{|N \cap [\theta_0, \theta_0 + \varepsilon]|}{\varepsilon} < 1 - \frac{\delta}{\operatorname{diam}(P)},$$

and since the right-hand side does not depend on ε, θ_0 cannot be a point of density for N. □

We now describe an application of the above theorem. If $\omega \in \Omega'$, let us denote the sequence of sides of P which are visited by $S_t\omega$, $t \in R$, by:

$$\varphi(\omega) = (\cdots, k_{-1}, k_0, k_1, \cdots) \in (a_i \mid i = 1, \cdots, n)^Z$$

Now if $\omega \notin \Omega_0$ and if $0 < t < t_0(\omega)$, then $\varphi(\omega) = \varphi(S_t\omega)$. Denote further by φ_0 the restriction of φ to the set Ω_0, and by $\varphi_0{}^+ : \Omega_0 \to \{a_i \mid i = 1, \cdots, n\}^N$ the mapping obtained by only retaining the sequence $(k_0, k_1, k_2, \cdots)$.

COROLLARY. *$\varphi_0{}^+$ is almost surely injective*, i.e., *there exists a subset $\bar{\Omega}_0$ of Ω_0 with $m_0(\bar{\Omega}_0) = 1$ such that $\omega, \eta \in \bar{\Omega}_0$ and $\varphi_0{}^+(\omega) = \varphi_0{}^+(\eta)$ imply $\omega = \eta$.*

PROOF. If $\omega = (x, \theta_1)$ and $\eta = (y, \theta_2)$ and $\theta_1 \neq \theta_2$, the reflection argument shows easily that $\varphi_0{}^+(\omega) \neq \varphi_0{}^+(\eta)$. For this case we do not need the theorem. Suppose now that $\omega = (x, \theta)$ and $\eta = (y, \theta)$. A closer inspection of the reflection argument of the theorem shows that if $\theta \in N$ (for x) and $\theta \in N$ (for y), then a vertex of P must fall into the strip between the straight lines (x, θ) and (y, θ), so that also in this case $\varphi_0{}^+(\omega) \neq \varphi_0{}^+(\eta)$. □

COROLLARY. *The entropy of polygonal billiards is zero.*

PROOF. This follows from the fact that $\omega \in \Omega_0$ is almost surely determined by its forward side sequence $\varphi_0{}^+(\omega)$, which implies $h(T) = 0$, and Abramov's formula [1]. □

We remark that the last result implies an essential difference between billiards with one ball (which are "deterministic") and billiards with two balls (which are "random"; see, e.g., Kubo [5], Sinai [7]).

4. Rational billiards and interval exchange transformations. We shall prove that the study of asymptotic properties of rational billiards reduces to that of certain interval exchange transformations, which have been introduced in [3] (see also [2] and [4]). First we recall the basic definitions.

Let $Y = [0, 1[$ and let n be an integer greater than one. Suppose that $p = (p_1, p_2, \cdots, p_n)$ is a probability vector with $p_i > 0$ for $1 \leqq i \leqq n$, and let τ

be a permutation of the symbols $\{1, 2, \cdots, n\}$. We set

$$p^\tau = (p_1^\tau, \cdots, p_n^\tau) = (p_{\tau^{-1}(1)}, \cdots, p_{\tau^{-1}(n)})$$
$$q_0 = 0\,, \qquad q_i = \sum_{j=1}^{i} p_j$$
$$q_0^\tau = 0\,, \qquad q_i^\tau = \sum_{j=1}^{i} p_j^\tau = \sum_{j=1}^{i} p_{\tau^{-1}(j)}$$

and

$$Y_i = [q_{i-1}, q_i[$$
$$Y_i^\tau = [q_{i-1}^\tau, q_i^\tau[\,.$$

Then the map $T\colon Y \to Y$ defined by

$$Ty = y - q_{i-1} + q_{\tau(i)-1}^\tau \qquad y \in Y_i,\ 1 \leqq i \leqq n$$

is an order-preserving piecewise isometry of Y (on the "pieces" $Y_1, \cdots, Y_n$). It is called the (p, τ)-interval exchange transformation.

Obviously, any interval exchange transformation is invertible and its inverse is an interval exchange transformation. The map T is continuous except at the points $q_1, \cdots, q_{n-1}$ (called *separation points*) where it is continuous from the right.

We say that the interval exchange transformation T satisfies the *minimality condition* if

(M1) T is aperiodic (i.e., for each $y \in Y$, the orbit $\mathrm{Orb}\,(y) = \{T^n y : n \in Z\}$ is infinite), and

(M2) If F is a finite union of right open intervals with endpoints belonging to the countable set

$$D_\infty = \bigcup_{i=0}^{n-1} \mathrm{Orb}\,(q_i) \cup \{1\}\,,$$

then $TF = F$ implies $F = Y$ or $F = \varnothing$.

The result we shall need is the following one:

THEOREM ([3]). *T satisfies the minimality condition if and only if*

$$\mathrm{Orb}\,(y) \ \textit{ is dense in } \ Y \ \textit{ for all } \ y \in Y\,.$$

To obtain an interval exchange transformation from rational billiards, we consider first the dynamical system (Ω_0, T, μ_0) defined in Section 1. Choose a side a_1 and an initial angle θ_0, and restrict T to the subset $\tilde{\Omega}_0$ of Ω_0 consisting of all pairs of sides and angles actually visited starting from a_1 with direction θ_0. We denote by T_0 the restriction of T to $\tilde{\Omega}_0$. Now $\tilde{\Omega}_0$ consists of a certain number of sides a_1 (we shall see below that all sides are represented) together with angles θ_i^j, $j = 1, \cdots, k_i$, belonging to the side a_i. If we draw side by side k_i copies of the side a_i for each i, and then contract the jth copy of a_i by a factor $\sin\theta_i^j$, then by elementary physics T_0 becomes a piecewise isometry of this collection of intervals, and it is not hard to see that if they are correctly arranged, then T_0 is order preserving. Normalizing to unit length, we obtain an interval exchange transformation (whose separation points q_i correspond to vertices of the original polygon) which we shall also denote by T_0.

5. The density theorem. We first show that for almost all initial directions the interval exchange transformation generated by the billiard flow is minimal (in the sense of Section 3) and then show that this implies density of the orbits on the whole polygon and not merely on its sides.

LEMMA. *For all but a countable number of values of θ_0 the interval exchange transformation generated by the billiard flow (according to Section 4) is minimal.*

PROOF. We exclude all directions connecting two or more vertices in the rectified flow (see Remark 3, Section 2, and the proof of the theorem of Section 3). There is at most a countable number of such directions. We shall call such directions exceptional. We shall show that if θ_0 is not exceptional, T_0 is minimal.

Indeed condition (M1) is satisfied, since no vertex or separation point can be periodic and no other point can then be, as shown in [3]. Suppose now that F is as in condition (M2). Take $x \in \partial F$ and suppose that $x = T_0{}^k q_j$ for some $k \geqq 0$ and some j (the case $k < 0$ is treated analogously). Then either $T_0{}^{-1}x$ is a boundary point of F or it is in $D = \{q_0, q_1, \cdots, q_{n-1}\}$. Since F has only a finite number of boundary points, there must be a positive integer s such that $T_0{}^{-s}x = y \in D$. If $y = q_j$, this would imply a periodic orbit for q_j, so that to avoid the orbits of two vertices overlap (we have excluded such directions) y must be a separation point, $s = 1$ and x must be a vertex. Thus T_0 would leave invariant a subset of $\tilde{\Omega}_0$ made up of a certain number of whole sides with corresponding angles. But then this subset necessarily coincides with the whole of $\tilde{\Omega}_0$ because of the way we defined the set from the start. □

We remark that θ_0 has been chosen, by avoiding a countable number of values, in such a manner that the infinite distinct orbit condition of [3] is satisfied. This yields an alternative proof of the lemma.

We are now able to prove the density theorem.

THEOREM. *If $\omega = (x, \theta_0)$ and θ_0 is not exceptional,*

$$\overline{\mathrm{Orb}_s^+(\omega)} = \overline{\mathrm{Orb}_s^-(\omega)} = \bar{P}.$$

PROOF. From the preceding lemma it follows that $\overline{\mathrm{Orb}_0^{\pm}(\omega)} = \tilde{\Omega}_0$. However one easily sees that if θ_0 is not exceptional, no side can stay all the time "in the shade," i.e., all sides of the polygon are represented in $\tilde{\Omega}_0$. Indeed, suppose that side a_i, lying between vertices A_i and A_{i+1}, is in the shade, while some copy of side a_{i-1} is in $\tilde{\Omega}_0$. Then points in a_{i-1} in a neighbourhood of A_i are visited by the flow coming from some other side a_j with some angle $\theta_j{}^m$. If all points in a_i are in the shade then $\theta_j{}^m$ is a direction connecting A_i with a vertex of a_j, i.e., θ_0 is an exceptional direction. So, if θ_0 is not exceptional some point of a_i is visited and therefore the spatial flow is dense on a_i. In a similar way one can see that no internal region of the polygon can stay in the shade. Indeed such a region would be, by definition, an open set of polygonal shape, whose sides would consist of segments of trajectories connecting two vertices and the conclusion follows as above. □

This theorem has a simple physical interpretation: in a polygonal two-dimensional room with mirror walls and rational angles, a light ray travelling in a nonexceptional direction does not leave any part of the room in the shade.

As an application of this theorem note that the configuration space of two pointlike particles of masses m_1 and m_2 moving freely on a segment and bouncing elastically from each other and from the endpoints, is a right triangle, the angles of which depend on the ratio of the square root of the masses and the flow is a billiard [8]. We can thus conclude that if arc tan $(m_1^{\frac{1}{2}}/m_2^{\frac{1}{2}})$ is rational with π, for almost all initial velocities, the phase flow is spatially dense.

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