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A RELATION BETWEEN BROWNIAN BRIDGE AND BROWNIAN EXCURSION

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It is shown that Brownian excursion is equal in distribution to Brownian bridge with the origin placed at its absolute minimum. This explains why the maximum of Brownian excursion and the range of Brownian bridge have the same distribution, a fact which was discovered by Chung and Kennedy. The result is proved by establishing similar relations for "Bernoulli excursions" and "Bernoulli bridges" constructed from symmetric Bernoulli walks, and exploiting known weak convergence results. Some technical complications arise from the fact that Bernoulli bridges assume their minimum value with positive probability more than once.

1. Introduction and corollaries. Let W be standard Brownian motion, W_0 Brownian bridge and $W_0^{\oplus}$ Brownian excursion, i.e., the $C[0, 1]$ – valued random variables defined by

$$(W_0(t))_{0 \leqslant t \leqslant 1} =_d (W(t) - tW(1))_{0 \leqslant t \leqslant 1},$$

$$(W_0^{\oplus}(t))_{0 \leqslant t \leqslant 1} =_d \left((\tau_+ - \tau_-)^{-\frac{1}{2}} |W((1-t)\tau_- + t\tau_+)|\right)_{0 \leqslant t \leqslant 1}.$$

Here τ_- is the last zero of W before 1 and τ_+ the first after 1, and $=_d$ denotes equality in distribution. Chung (1976) and Kennedy (1976) noticed that the maximum of Brownian excursion has the same distribution as the range of Brownian bridge:

$$\max_{0 \leqslant t \leqslant 1} W_0^{\oplus}(t) =_d \max_{0 \leqslant t \leqslant 1} W_0(t) - \min_{0 \leqslant t \leqslant 1} W_0(t); \tag{1}$$

but so far no probabilistic explanation was found for this identity. The main result of the present paper is

THEOREM 1. *Let τ_m be the location of the absolute minimum of W_0. Then τ_m is unique with probability* 1 (w.p.1) *and*

$$W_0^{\oplus} =_d W_0(\tau_m + \cdot \bmod 1) - W_0(\tau_m). \tag{2}$$

Formula (1) follows immediately from (2) by application of the continuous function $f \mapsto \max_{0 \leqslant t \leqslant 1} f(t)$ on $C[0, 1]$ to both sides of (2). Another corollary is obtained by means of the continuous function $f \mapsto \int_0^1 f(t)dt$ on $C[0, 1]$, namely

$$\int_0^1 W_0^{\oplus}(t)dt =_d \int_0^1 W_0(t)dt - \min_{0 \leqslant t \leqslant 1} W_0(t). \tag{3}$$

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Hence, as $W_0 =_d - W_0$,

$$E\int_0^1 W_0^{\oplus}(t)dt = E \max_{0 \leqslant t \leqslant 1} W_0(t)$$
$$= \tfrac{1}{2} E \max_{0 \leqslant t \leqslant 1} W_0^{\oplus}(t) = \tfrac{1}{2}\left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}}.$$

The last formula follows also from Durrett and Iglehart (1977, Proposition (3.4)) or from Chung (1976, formula (6.2)). The distribution of $\int_0^1 W_0^{\oplus}(t)dt$ seems to be unknown (see, however, Chung (1976, Section 6) for results about moments). In order to obtain results from (3) one would need the joint distribution of the two terms on the right-hand side. This does not seem a simpler problem than the original one.

2. The main lines of the proof. Let $(S_n)_{n=0}^{\infty}$ be a symmetric Bernoulli walk starting at the origin ($S_0 = 0$ w.p. 1). By linear interpolation between the integers we extend $(S_n)_{n=0}^{\infty}$ to a random continuous function $S(\cdot)$ on $[0, \infty)$. As a particular case of results of Liggett (1968) we have

THEOREM 2. *In* $C[0, 1]$

$$\left[(2n)^{-\frac{1}{2}} S(2n\cdot) | S_{2n} = 0\right] \to_d W_0.$$

Here and in the sequel $\to_d$ denotes convergence in distribution as $n \to \infty$. The left-hand side is a shorthand notation for the conditional distribution of $(2n)^{-\frac{1}{2}} S(2n\cdot)$ over $C[0, 1]$, given $S_{2n} = 0$. As a consequence of a result of Kaigh (1976) we have

THEOREM 3. *In* $C[0, 1]$

$$\left[(2n)^{-\frac{1}{2}} S(2n\cdot) | S_{2n} = 0, S > 0 \text{ on } (0, 2n)\right] \to_d W_0^{\oplus}.$$

PROOF. Kaigh (1976, Theorem 2.6) obtains the theorem with $S \neq 0$ instead of $S > 0$ for more general lattice random walks. The change into $S > 0$ for a symmetric Bernoulli walk is obvious.

When we compare Theorems 2 and 3 to the sides of (2) in Theorem 1, the following approach suggests itself.

Let T be the unit interval [0, 1] with the endpoints identified, and let for $f \in C(T)$ $\tau_m(f)$ be the first time at which the minimum of f is attained. Set for $f \in C(T)$

$$\pi(f) := f(\tau_m(f) + \cdot \bmod 1) - f(\tau_m(f)).$$

Then π maps $C(T)$ into $C(T)$. Clearly π is continuous at those f that assume their minimum value only once. In the next section we will show

LEMMA 0. *W_0 assumes its minimum value only once* w.p.1.

It follows that π is continuous at W_0 w.p. 1. Now (2) reads

$$W_0^{\oplus} =_d \pi(W_0).$$

Hence Theorem 1 would follow from the continuous mapping theorem (Billingsley (1968, Theorem 5.1)), if π applied to the left-hand side of Theorem 2 would be equal in distribution to the left-hand side of Theorem 3. However, complications arise from the fact that the left-hand side of Theorem 2 assumes its minimum value more than once with positive probability. Therefore we have to modify Theorem 2.

Let A be the set of all $f \in C(T)$ that have a unique absolute minimum. Then A is Borel in $C(T)$ (see (12) and (13)). In the next section we will prove the following two theorems.

THEOREM 4. *In* $C(T)$

$$\left[(2n)^{-\frac{1}{2}}S(2n\cdot)|S_{2n} = 0, S(2n\cdot) \in A\right] \to_d W_0.$$

THEOREM 5. *In* $C(T)$

$$\pi\left[(2n)^{-\frac{1}{2}}S(2n\cdot)|S_{2n} = 0, S(2n\cdot) \in A\right]$$
$$=_d\left[(2n)^{-\frac{1}{2}}S(2n\cdot)|S_{2n} = 0, S > 0 \text{ on } (0, 2n)\right].$$

Combining Theorems 3, 4, 5 and the continuous mapping theorem now yields (2) in $C(T)$, and hence in $C[0, 1]$, since $C(T)$ can be identified with the Borel set $\{f : f(0) = f(1)\}$ in $C[0, 1]$.

3. Proof of Theorems 4 and 5 and Lemma 0. Consider the symmetric Bernoulli walk S. For positive integers n we define

$$\begin{aligned}
B_{2n} &= \{k : 0 \leqslant k \leqslant 2n, S_k = \min\nolimits_{0 \leqslant j \leqslant 2n} S_j\},\\
\tau_{l,2n} &= \inf B_{2n},\\
\tau_{r,2n} &= \sup B_{2n} \quad \text{if} \quad B_{2n} \neq \{0, 2n\},\\
&= 0 \quad \text{if} \quad B_{2n} = \{0, 2n\},\\
\zeta_{2n} &= \tau_{r,2n} - \tau_{l,2n},\\
L_{2n}(t) &= S(\tau_{l,2n} + t) - S(\tau_{l,2n}) \text{ for } 0 \leqslant t \leqslant \zeta_{2n},\\
K_{2n}(t) &= S(t) \quad \text{for} \quad 0 \leqslant t \leqslant \tau_{l,2n},\\
&= S(t + \zeta_{2n}) \quad \text{for} \quad \tau_{l,2n} \leqslant t \leqslant 2n - \zeta_{2n}.
\end{aligned}$$

$\tau_{l,2n}$ and $\tau_{r,2n}$ are the locations of the most left-hand and most right-hand absolute minima in $[0, 2n]$, modified such that on $[S_{2n} = 0]$

$$[S(2n\cdot) \in A] = [\tau_{l,2n} = \tau_{r,2n}] = [\zeta_{2n} = 0].$$

The graph of L_{2n} is the piece of the path between $\tau_{l,2n}$ and $\tau_{r,2n}$. When this piece is left out and the remaining pieces are put together, we obtain the graph of K_{2n}.

LEMMA 1. (a) *The random functions* K_{2n} *and* L_{2n} *are conditionally independent, given* $S_{2n} = 0$ *and* ζ_{2n}.

(b) $$[K_{2n}|S_{2n} = 0, \zeta_{2n} = 2k] =_d [(S(t))_{0 \leqslant t \leqslant 2n-2k}|S_{2n-2k} = 0, S((2n-2k)\cdot) \in A]$$

for $0 \leqslant k \leqslant n$.

(c) $$[L_{2n}|S_{2n} = 0, \zeta_{2n} = 2k] =_d [(S(t))_{0 \leqslant t \leqslant 2k}|S_{2k} = 0, S \geqslant 0 \text{ on } [0, 2k]]$$

for $0 \leqslant k \leqslant n$.

PROOF. Let R_S be the set of possible values of $[(S(t))_{0 \leqslant t \leqslant 2n} : S_{2n} = 0, \zeta_{2n} = 2k]$, i.e., the set of paths of length $2n$ with $S_{2n} = 0$ and $\zeta_{2n} = 2k$. Similarly, let R_K and R_L be the sets of possible values of the left-hand sides of (b) and (c), which obviously are also the sets of possible values of the right-hand sides. The function determined by $(S(t))_{0 \leqslant t \leqslant 2n} \mapsto (K_{2n}, L_{2n})$ restricted to $[S_{2n} = 0, \zeta_{2n} = 2k]$ maps R_S one-to-one onto $R_K \times R_L$. Given $S_{2n} = 0, \zeta_{2n} = 2k$, $(S(t))_{0 \leqslant t \leqslant 2n}$ assumes its values in R_S with equal probability, so under the same condition (K_{2n}, L_{2n}) assumes its values in $R_K \times R_L$ with equal probability. Now all assertions of the lemma follow.

LEMMA 2. *Let*

$$\pi_{2n}(S) = S(\tau_{l,2n} + \cdot \bmod 2n) - S(\tau_{l,2n}).$$

If we let the domain of π_{2n} *be the paths of length* $2n$ *with* $S_{2n} = 0$ *and* $\tau_{l,2n} = \tau_{r,2n}$, *then the range of* π_{2n} *consists of all paths of length* $2n$ *with* $S_{2n} = 0$ *and* $S > 0$ *on* $(0, 2n)$ *and the mapping* π_{2n} *is* $2n$ *to* 1.

PROOF. Clearly π_{2n} maps into the claimed range; π_{2n} restricted to the claimed range is the identity map, thus onto. All pre-images of f under π_{2n} are given by $f(k + \cdot \bmod 2n) - f(k)$ for $k = 0, 1, \cdots, 2n-1$. They are all different, since they have their unique minimum at different places.

PROOF OF THEOREM 5. All possible paths on the right-hand side of Theorem 5, and on the left-hand side behind π, are equally probable. Application of π on the left-hand side produces the possible paths on the right-hand side by Lemma 2, and moreover with equal probability, since, for fixed n, π sends a constant number of paths into one image.

LEMMA 3.

$$[\zeta_{2n}|S_{2n} = 0] \to_d T - 2, \tag{4}$$

where T *is the time of the first return to* 0 *in the symmetric Bernoulli walk, so*

$$P[T = 2n] =: f_{2n} = \frac{1}{2n-1}\binom{2n}{n}2^{-2n} \tag{5}$$

$$= \frac{2}{n}\binom{2n-2}{n-1}2^{-2n} \qquad \text{for } n = 1, 2, \ldots$$

and

(6) $$Ez^T = 1 - (1 - z^2)^{\frac{1}{2}} \quad \textit{for} \quad |z| \leqslant 1.$$

PROOF. The statements about T are well known (cf. Feller (1968)). From Feller (1968, (III.9.1)) we know that

(7) $$P[S_{2n} = 0, S \geqslant 0 \text{ on } [0, 2n]] = \frac{1}{n+1}\binom{2n}{n}2^{-2n} \quad \text{for} \quad n = 0, 1, \cdots.$$

From (5) we have

$$\#\{S \text{ on } [0, 2n] : S_{2n} = 0, S > 0 \text{ on } (0, 2n)\} = \tfrac{1}{2}f_{2n}2^{2n} = \frac{1}{n}\binom{2n-2}{n-1} \quad \text{for} \quad n = 1, 2, \cdots.$$

By Lemma 2 the mapping π_{2n} is $2n$ to 1, so

(8) $$\#\{S \text{ on } [0, 2n] : S_{2n} = 0, \tau_{l, 2n} = \tau_{r, 2n}\} = 2\binom{2n-2}{n-1} \quad \text{for} \quad n = 1, 2, \cdots.$$

By applying Lemma 1, (7) and (8) we obtain

$$P[\zeta_{2n} = 2k | S_{2n} = 0] = \frac{P[S_{2n-2k} = 0, \tau_{l, 2n-2k} = \tau_{r, 2n-2k}]P[S_{2k} = 0, S \geqslant 0 \text{ on } [0, 2k]]}{P[S_{2n} = 0]}$$

$$= 2\binom{2n - 2k - 2}{n - \quad k - 1} \cdot \frac{1}{k+1}\binom{2k}{k} \Big/ \binom{2n}{n}$$

$$= \frac{2}{k+1}\binom{2k}{k}\frac{(n(n-1)\cdots(n-k))^2}{2n(2n-1)\cdots(2n-2k-1)}.$$

Keeping k fixed and letting $n \to \infty$ we obtain

$$P[\zeta_{2n} = 2k | S_{2n} = 0] \to \frac{1}{2(k+1)}\binom{2k}{k}2^{-2k} = f_{2k+2}.$$

This proves (4).

LEMMA 4. *In* $C(T)$

$$[(2n - \zeta_{2n})^{-\frac{1}{2}}K_{2n}((2n - \zeta_{2n})\cdot) | S_{2n} = 0] \to_d W_0.$$

PROOF. Denoting the sup norm on $C(T)$ by $\|\cdot\|$ we have on $[S_{2n} = 0]$

(9) $$\|(2n - \zeta_{2n})^{-\frac{1}{2}}K_{2n}((2n - \zeta_{2n})\cdot) - (2n)^{-\frac{1}{2}}S(2n\cdot)\| \leqslant \|(2n)^{-\frac{1}{2}}S(2n\cdot)\| \, |(1 - \zeta_{2n}/2n)^{-\frac{1}{2}} - 1| + \zeta_{2n}/(2n)^{\frac{1}{2}}.$$

By (4) we have $[\zeta_{2n}/(2n)^a | S_{2n} = 0] \to_d 0$ for $a > 0$. Hence by Billingsley (1968, Theorem 4.4) and Theorem 2

$$\left[\left((2n)^{-\frac{1}{2}}S(2n\cdot), \zeta_{2n}/2n\right)|S_{2n} = 0\right] \to_d (W_0, 0)$$

in $C(T) \times \mathbb{R}$. It follows that both sides of (9) converge in distribution to 0. This combined with Theorem 2 and Billingsley (1968, Theorem 4.1) proves the lemma.

LEMMA 5. *Let ϕ be a bounded continuous function on $C(T)$ and let*

$$X_{2n} :=_d \left[(2n)^{-\frac{1}{2}}S(2n\cdot)|S_{2n} = 0, S(2n\cdot) \in A\right].$$

Then, as $n \to \infty$,

$$\Sigma_{k=0}^{n} E\phi(X_{2n-2k})f_{2k+2} \to E\phi(W_0). \tag{10}$$

PROOF. By Lemmas 4 and 1 we have

$$\begin{aligned} E\phi(W_0) &\leftarrow E\left[\phi\left((2n - \zeta_{2n})^{-\frac{1}{2}}K_{2n}((2n - \zeta_{2n})\cdot)\right)|S_{2n} = 0\right] \\ &= \Sigma_{k=0}^{n} E\left[\phi\left((2n - 2k)^{-\frac{1}{2}}K_{2n}((2n - 2k)\cdot)\right)|S_{2n} = 0, \zeta_{2n} = 2k\right] \\ &\quad \cdot P[\zeta_{2n} = 2k|S_{2n} = 0] \\ &= \Sigma_{k=0}^{n} E\phi(X_{2n-2k})P[\zeta_{2n} = 2k|S_{2n} = 0]. \end{aligned} \tag{11}$$

As (4) concerns weak convergence of probability distributions on a countable set, the convergence is also in total variation. Since $E\phi(X_{2n-2k})$ is uniformly bounded, it follows that the difference between the left-hand side of (10) and the most right-hand side of (11) vanishes as $n \to \infty$. This proves (10).

LEMMA 6. *Let $(a_k)_{k=0}^{\infty}$ be a sequence of nonnegative reals such that $a_0 > 0$ and $\Sigma_{k=0}^{\infty} a_k = 1$. In order that for all real sequences $(x_n)_{n=0}^{\infty}$*

$$\lim_{n\to\infty}\Sigma_{k=0}^{n} a_k x_{n-k} = \lim_{n\to\infty} x_n$$

whenever at least one side exists and is finite, it is necessary and sufficient that the power series expansion in z of $1/\Sigma_{k=0}^{\infty} a_k z^k$ converges absolutely for $z = 1$.

PROOF. This is a special case of Hardy (1949, Theorem 21, page 67). Take $p_k = a_k$, $q_k = 1$ if $k = 0$, 0 if $k > 0$.

PROOF OF THEOREM 4. Apply Lemma 6 to (10) with $a_k = f_{2k+2}$, so $\Sigma_{k=0}^{\infty} a_k z^k = (1 - (1 - z)^{\frac{1}{2}})/z$ (cf.(6)). Then $1/\Sigma_{k=0}^{\infty} a_k z^k = 1 + (1 - z)^{\frac{1}{2}} = 2 - \Sigma_{k=0}^{\infty} f_{2k} z^k$, which converges absolutely for $z = 1$. Hence $E\phi(X_{2n}) \to E\phi(W_0)$ for all bounded continuous functions ϕ on $C(T)$. This proves Theorem 4.

PROOF OF LEMMA 0. We will first derive similar results for Brownian motion W. This forces us to consider $C[0, 1]$ rather than $C(T)$, which can and will be identified with $\{f \in C[0, 1] : f(0) = f(1)\}$. For $0 < t \leqslant 1$ let A_t be the set of those

$f \in C[0, 1]$ whose restrictions to $[0, t]$ have a unique absolute minimum. Recall $A \subset C(T)$ from the lines above Theorem 4. With the above identification we have

$$A = (A_1 \cup \{f : f(t) > f(0) \text{ for } 0 < t < 1\}) \cap C(T).$$

Consequently,

$$A^c \cap C(T) \subset A_1^c \cap C(T) \subset \bigcup_{\text{rational } r \in (0,1)} A_r^c. \tag{12}$$

From

$$A_t^c = \bigcup_{\text{rational } r \in (0,t)} \{f : \min_{0 \leqslant u \leqslant r} f(u) = \min_{r \leqslant u \leqslant t} f(u)\} \tag{13}$$

and arguments like in the proof of Freedman (1971, (1.52)) it follows that

$$P[W \in A_t^c] = 0 \qquad \text{for} \quad t > 0. \tag{14}$$

Now let $0 < t < 1$. It is well known that for real x

$$[(W(u))_{0 \leqslant u \leqslant t} | W(t) = x] =_d [(W_0(u))_{0 \leqslant u \leqslant t} | W_0(t) = x] \tag{15}$$

(compute the correlation functions of the Gaussian processes on both sides). The distributions of $W(t)$ and $W_0(t)$ are normal, whence equivalent. Because of (15) the distributions of $(W(u))_{0 \leqslant u \leqslant t}$ and $(W_0(u))_{0 \leqslant u \leqslant t}$ over $C[0, t]$ are equivalent, too, so (14) implies $P[W_0 \in A_t^c] = 0$ for $0 < t < 1$. Now (12) and $P[W_0 \in C(T)] = 1$ imply $P[W_0 \in A] = 1$, and the lemma is proved.

REFERENCES

[1] Billingsley, P.(1968). *Convergence of Probability Measures*. Wiley, New York.

[2] Chung, K. L. (1976). Excursions in Brownian motion. *Ark. Mat.* **14** 155–177.

[3] Durrett, R. T. and Iglehart, D. L. (1977). Functionals of Brownian meander and excursion. *Ann. Probability* **5** 130–135.

[4] Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, **1**, 3rd ed. Wiley, New York.

[5] Freedman, D. (1971). *Brownian Motion and Diffusion*. Holden-Day, San Francisco.

[6] Hardy, G. H. (1949). *Divergent Series*. Oxford Univ. Press.

[7] Kaigh, W. D. (1976). An invariance principle for random walk conditioned by a late return to zero. *Ann. Probability* **4** 115–121.

[8] Kennedy, D. P. (1976). The distribution of the maximum Brownian excursion. *J. Appl. Probability* **13** 371–376.

[9] Liggett, T. M. (1968). An invariance principle for conditioned sums of random variables. *J. Math. Mech.* **18** 559–570.

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