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NOTE ON THE RAY-KNIGHT COMPACTIFICATION

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We give an example to show that, given a nonhomogeneous Markov process X on E, one cannot, in general, produce a right continuous strong Markov version of X on a compactification of E. In particular, the space-time regularization fails to produce one compact Hausdorff state space for the nonhomogeneous process.

Ray introduced in [5] a procedure for the regularization of time homogeneous Markov processes. Knight, building on Ray's work, produced the procedure now known as the Ray-Knight compactification [4]. Let X_t be a time homogeneous Markov process taking values in a Borel subset E of a compact metric space $(\hat{E}, \hat{d})$ and having a measurable transition function (these hypotheses may be weakened a bit). The Ray-Knight procedure produces a new compact metric space $(\bar{E}, \bar{d})$ in which E is densely embedded. The process $\bar{X}_t = \lim_{s\downarrow\downarrow t;\, s\in\mathbb{Q}} X_s$, where the limit is taken in the topology of $\bar{E}$, is a right continuous strong Markov process with resolvent carrying continuous functions on $\bar{E}$ to continuous functions on $\bar{E}$. Moreover, $\bar{X}_t$ and X_t agree with probability 1 for all t except for a countable number of times [4]. If the original process is right continuous and strong Markov on E, then $\bar{X}_t$ and X_t are indistinguishable as processes on $\bar{E}$.

The importance of this procedure has begun to be realized fully in recent years, notably in its connection with Shih's theorem and the study of right processes [3, 7]. A procedure to modify a nonhomogeneous process on a metric completion of the state space producing a right continuous strong Markov version of the process would also be of great value. Unfortunately, the extension of the Ray-Knight technique to the regularization of nonhomogeneous Markov processes outlined in [4], page 551, is not, in general, correct, as the example below shows. Furthermore, the example shows that one cannot expect a right continuous strong Markov version on one state space, in general.

We construct a nonhomogeneous strong Markov process on a two point discrete state space $E = \{x, y\}$. Let A be a Borel set on $\mathbb{R}^+$ containing the point 0 with the property that for all $0 < t < s$, $\lambda(A \cap (t, s)) > 0$ and $\lambda(A^c \cap (t, s)) > 0$, where

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λ denotes Lebesgue measure. Define

$$
\begin{aligned}
P_{t',t'+t}(x,\cdot) &= \varepsilon_x(\cdot) && \text{if } t' \in A \text{ and } t'+t \in A, \text{ or} \\
& && \text{if } t' \in A^c \text{ and } t'+t \in A^c. \\
&= \varepsilon_y(\cdot) && \text{if } t' \in A \text{ and } t'+t \in A^c, \text{ or} \\
& && \text{if } t' \in A^c \text{ and } t'+t \in A. \\
P_{t',t'+t}(y,\cdot) &= \varepsilon_y(\cdot) && \text{if } t' \in A \text{ and } t'+t \in A, \text{ or} \\
& && \text{if } t' \in A^c \text{ and } t'+t \in A^c. \\
&= \varepsilon_x(\cdot) && \text{if } t' \in A \text{ and } t'+t \in A^c, \text{ or} \\
& && \text{if } t' \in A^c \text{ and } t'+t \in A.
\end{aligned}
$$

If the process starts at x (resp. y), it will be at x (resp. y) for all times $t \in A$ and will be at y (resp. x) for all times $t \in A^c$.

Define the space-time transition semigroup on $Ex(0, \infty)$ as in [4] by setting

$$p_t((x, t'), B \times \{t' + t\}) = P_{t',t'+t}(x, B).$$

If f is a bounded measurable function on $Ex(0, \infty)$, the resolvent for p_t is the following ($\alpha > 0$):

$$
\begin{aligned}
U^\alpha f(x,t') &= \int_{(0,\infty)\cap(A-t')} e^{-\alpha t} f(x, t'+t)\,dt \\
&\quad + \int_{(0,\infty)\cap(A^c-t')} e^{-\alpha t} f(y, t'+t)\,dt \qquad \text{if } t' \in A \\
&= \int_{(0,\infty)\cap(A-t')} e^{-\alpha t} f(y, t'+t)\,dt \\
&\quad + \int_{(0,\infty)\cap(A^c-t')} e^{-\alpha t} f(x, t'+t)\,dt \qquad \text{if } t' \in A^c. \\
U^\alpha f(y,t') &= \int_{(0,\infty)\cap(A-t')} e^{-\alpha t} f(y, t'+t)\,dt \\
&\quad + \int_{(0,\infty)\cap(A^c-t')} e^{-\alpha t} f(x, t'+t)\,dt \qquad \text{if } t' \in A \\
&= \int_{(0,\infty)\cap(A-t')} e^{-\alpha t} f(x, t'+t)\,dt \\
&\quad + \int_{(0,\infty)\cap(A^c-t')} e^{-\alpha t} f(y, t'+t)\,dt \qquad \text{if } t' \in A^c.
\end{aligned}
$$

Let $\mathbf{R}$ be the smallest positive cone closed under $(U^\alpha)_{\alpha>0}$ and pointwise minima, containing $U^\alpha f$ for f bounded and continuous on $Ex[0, \infty]$. This is known as the Ray cone, and by Knight's fundamental lemma, it is separable in the uniform topology on $Ex(0, \infty)$. It is easy to see that $\mathbf{R}$ induces a topology on $Ex(0, \infty)$ with the following property. Every open neighborhood of a point (x, t) contains a point (y, s), and every open neighborhood of a point (y, s) contains a point (x, t). Thus, the quotient space (by the projection map T) is $F = \{x, y\}$. But $T^{-1}(\{x\})$ and $T^{-1}(\{y\})$ are not open sets. Therefore F, together with the quotient topology, is not a Hausdorff space. If we then form the quotient of F to get a Hausdorff space, the points x and y combine, and the process sits at one point. This process bears no interesting relation to the original process. Thus, the Ray-Knight procedure applied to the space-time version of the process on $Ex(0, \infty)$ does not yield a compact Hausdorff space $\bar{E}$ in general.

This process intertwines space and time so that, after retopologizing $Ex(0, \infty)$, the space and time variable can no longer be separated. Notice that if E and F are identified in the obvious manner, then the original process on E and the "regularized" process are indistinguishable. Only the topology of E has been changed to produce F. Such difficulties can occasionally be overcome by adding a countable collection of functions which are supermedian for the space-time process to the Ray cone [6]. In this example, however, addition of such functions to the Ray cone will not separate points x and y in F, due to the choice of the Borel set A.

To separate points using the Ray cone, one would need to add positive supermedian functions k_n, uniformly bounded by 1, so that $T^{-1}(\{x\})$ or $T^{-1}(\{y\})$ is open in $Ex(0, \infty)$ together with the Ray metric. This is equivalent to finding a decreasing function K on $(0, \infty)$ such that if $\kappa(u, v) = |K(u) - K(v)|$ for positive u and v, then A or A^c is an open subset of $(0, \infty)$ together with the κ-metric. But given $t \in A$, for any $\varepsilon > 0$, there is an $s \in A^c$ with $|t - s| < \varepsilon$. If K is continuous at t, $\{s : \kappa(s, t) < \varepsilon\}$ contains a point of A^c. Therefore, K would have to be discontinuous at every $t \in A$, an uncountable number of points. This cannot be since K is decreasing. The argument for A^c is similar.

It is a simple matter to show that the nonhomogeneous process X_t admits *no* compactification on which X_t has a right continuous modification $\overline{X}_t$. For let $\overline{E}$ be any compactification such that $P^x\{X_t = \overline{X}_t\} = 1$ for all but countably many times. Choose $t_0 \in A$ with X_{t_0} equal to $\overline{X}_{t_0}$ a.s. P^x. Then $X_{t_0} = x$ a.s. P^x. There exists a sequence of times t_n contained in A^c such that the t_n decrease to t_0 and $X_{t_n} = \overline{X}_{t_n}$ a.s. P^x for each n. Therefore $\overline{X}_{t_n} = y$ a.s. P^x, and, by right continuity of $\overline{X}_t$, $\overline{X}_{t_0} = y$ a.s. P^x.

Note that the original nonhomogeneous process has only constant homogeneous 1-supermedian functions which demonstrates that the theorem of Walsh [6] asserting the existence of 1-supermedian functions separating points in E for time homogeneous strong Markov processes will not extend to the nonhomogeneous case.

REMARK. It is perhaps unreasonable to expect to produce a topology on E in which a nonhomogeneous Markov process has a right continuous strong Markov version. Indeed, the strong Markov process above cannot be made right continuous on E without radically altering the nature of the process. The space-time regularization may suffice in many instances, as suggested by Dynkin [1, 2]. We may interpret the regularization in the following manner. Fix a countable sequence (g_n) dense in $\mathbf{R}$. Each g_n is a nonhomogeneous α-supermedian function. Let $\|g_n\| = \sup\{g_n(t, x) : t \in (0, \infty), x \in E\}$. Define the map

$$\Gamma_t : E \to \Pi[0, \|g_n\|]$$

by setting $\Gamma_t(x) = (g_n(t, x))_{n \geqslant 1}$. If X_t is the nonhomogeneous Markov process on E, $\Gamma_t(X_t)$ has a right continuous version $\overline{X}_t$, defined by setting

$$\overline{X}_t = \lim_{s \downarrow\downarrow t;\, s \in \mathbf{Q}} \Gamma_s(X_s).$$

This is, of course, the space-time regularization, and $\bar{X}_t$ is now a homogeneous process. We may interpret the addition of the time coordinate in the space-time process as a mechanism for moving the state space E in $\Pi\, [0, \|g_n\|]$ to produce a homogeneous process. The process $\bar{X}_t$ has state space $\bar{E}_t$ at time t, where

$$\bar{E}_t = \bigcap_{n \geqslant 1} \overline{\bigcup_{0<u<1/n;\, u \in \mathbf{Q}} \Gamma_{t+u}(E)}\,.$$

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