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THE TAIL σ-FIELD OF TIME-HOMOGENEOUS ONE-DIMENSIONAL DIFFUSION PROCESSES[1]

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We give necessary and sufficient conditions for tail σ-fields of time-homogeneous one-dimensional diffusion processes and birth and death processes to be trivial and study related questions.

1. Introduction. Let $X(t) = X(t, \omega)$, $t \in T \subset \mathbb{R}^+$, $\omega \in \Omega$, be a Markov process on a probability space $(\Omega, \mathfrak{F}, P)$ having time-homogeneous transition probabilities. By P_π, P_x we denote the probabilities uniquely determined by the transition kernels P^t and the initial measure π, respectively the point measure in x, on the state space $(S, \mathfrak{B})$. E_π, Var_π are the expectation, variance relative to P_π.

Let I be an interval on the real line with boundaries b, c, possibly infinite, endowed with the Borel σ-field. A diffusion is a Markov process $X(t)$, $t \in \mathbb{R}^+$, on $I = S$ with continuous paths and Feller transition probabilities. These processes, sometimes called Feller processes (Breiman (1968)), possess the strong Markov property.

We denote the smallest σ-field generated by $X(u)$, $s \leqslant u \leqslant t \in \mathbb{R}^+(\infty)$, by $\mathfrak{F}_s^t$. The intersection of all $\mathfrak{F}_s^\infty$, $s \in \mathbb{R}^+$, is the tail σ-field $\mathfrak{F}_\infty$. The invariant σ-field $\mathfrak{F}_\theta$ consists of all shift invariant sets. The shift θ_s, $s \in \mathbb{R}^+$, maps Ω to Ω by the transformation

$$X(t, \omega) = X(t - s, \Theta_s(\omega)) \qquad \omega \in \Omega, s \leqslant t \in \mathbb{R}^+. \tag{1.1}$$

In general we have $\mathfrak{F}_\Theta \subset \mathfrak{F}_\infty$. For an example $\mathfrak{F}_\Theta \neq \mathfrak{F}_\infty$ see Orey (1971), page 21, Example 4.1.

A σ-field is trivial relative to μ, if the σ-field consists of two elements μ almost surely. The tail (or invariant) σ-field $\mathfrak{F}_\infty(\mathfrak{F}_\Theta)$ is said to be trivial or to obey the zero-one law, if $\mathfrak{F}_\infty(\mathfrak{F}_\Theta)$ is trivial for P_π, π any initial measure.

Before we handle continuous Markov processes we refer to some work done for discrete-time Markov processes. Let T be the natural numbers and $(S, \mathfrak{B})$ a measurable space, A $\mathfrak{B}$-measurable function f is harmonic, if for all $n \in \mathbb{N}$

$$h(x) = \int_S h(y) P^n(x, dy) \tag{1.2}$$

holds, where $P^n(\cdots)$ is the usual n-step transition kernel. The function $P_x(B)$, $B \in \mathfrak{F}_\Theta$, is an example of a harmonic function. All bounded harmonic functions are constant a.s. if and only if the invariant σ-field is trivial (Orey (1971)).

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A similar result is true for the tail σ-field. All bounded space-time harmonic functions are constant a.s. if and only if the tail σ-field is trivial (Orey (1971)). A space-time harmonic function is a harmonic function for the (space-time) process $X^*(t) = X(t + \tau)$, where τ is a random variable taking values in the natural numbers.

Now consider a diffusion process. The invariant σ-field is trivial, iff all bounded harmonic functions are constant, i.e., all $\mathfrak{B}$ measurable functions m satisfying

$$\text{(1.3)} \qquad \begin{aligned} &\text{(a) } 0 \leqslant p \leqslant 1 \\ &\text{(b) } p \quad \text{in the domain of} \quad \mathcal{G} \\ &\text{(c) } \mathcal{G}(p) = 0 \end{aligned}$$

where $\mathcal{G}$ is the generator of the process, are identically constant a.s. The theorem is stated in Itô-McKean ((1965), page 303–305. The first condition in 8.7 is fulfilled because we have only two boundary points). But the given proof is only valid for the invariant σ-field. We shall use this theorem in Section 2.

Before we start to consider the tail σ-field of diffusions let us have a look at random walks and birth and death processes on the real line.

A random walk is a Markov process $X(t)$, $t \in \mathbb{N}$, with state space $S = \mathbb{Z}$. Only the one-step transitions from x to $x + 1$ and $x - 1$ are allowed with probabilities p_x and $q_x = 1 - p_x$. For simplicity assume $0 < p_x < 1$ for all $x \in \mathbb{Z}$. The process is called right drifting, if

$$\text{(1.4)} \qquad P_0(\exists n > 0 \qquad X(n) = 1) = 1$$

holds (left drifting analogous).

Using harmonic functions it is easy to show that $\mathfrak{F}_\Theta$ is trivial if and only if the process is right or left drifting. The related result (Rösler (1977)) for the tail σ-field is: $\mathfrak{F}_\infty$ is trivial, iff the process is right and left drifting (recurrent) or only right (left) drifting and the condition $\Sigma_{n \in \mathbb{N}} q_n = \infty (\Sigma_{n \in \mathbb{N}} p_{-n} = \infty)$ is satisfied. An example in which $\mathfrak{F}_\Theta$ is trivial and $\mathfrak{F}_\infty$ is not is obvious.

In Section 2 we formulate our necessary and sufficient condition for the triviality of $\mathfrak{F}_\infty$ and start with the proof in Section 3. Section 4 gives the idea for the remaining part of the proof which then is finished in Sections 5 and 6 and is based on the fact that $P_0(t_y^* < 0)$ is in general unimodal (Rösler (1978)) and even strong unimodal if zero is absorbing (Keilson (1971)). The method used works for birth and death processes too. Theorem 2.2 remains true for those processes.

2. Main result. We will use frequently the stopping times t_y^* and $t_{(x,y)}^*$, x, $y \in S$, relative to $\mathfrak{F}_0^t$, $t \in \mathbb{R}^+$, the first passage of y and the first exit from (x, y).

For simplicity we work with regular processes, which means

$$\text{(2.1)} \qquad P_x(t_y^* < \infty) > 0 \qquad x \in \operatorname{int} S, y \in S.$$

A diffusion possesses a natural scale (Breiman (1968)), if

$$P_y(t^*_{(x,z)} = t^*_x) = \frac{z-y}{z-x} \qquad x < y < z \in S. \tag{2.2}$$

We shall call a diffusion right drifting (left) if there exist $x, y \in S$, $x < y (y < x)$ with

$$P_x(t^*_y < \infty) = 1. \tag{2.3}$$

It is easy to prove that a right drifting (left) regular diffusion will satisfy (2.3) for all $x < y \in S (y < x)$. There is a close connection between right drifting and the behavior of the left boundary b.

PROPOSITION 2.1. *A regular diffusion $X(t)$ on the natural scale is right drifting* iff *the left boundary b of S is minus infinity or an element of S and then not absorbing.*

In order to see this we distinguish four cases: (i) $b = -\infty$; (ii) $b \notin S, b > -\infty$; (iii) $b \in S$ absorbing; (iv) $b \in S$ not absorbing. In the first two cases we use the fact that $t^*_{(x,y)}$ is almost surely less than infinity (Breiman (1968)). Further, $t^*_{(x,y)}$ tends to t^*_y if x converges to b, because of the continuity of the paths and $b \notin S$. Thus for $y > 0$, assuming $0 \in \operatorname{int} S$,

$$\begin{aligned} P_0(t^*_y < \infty) &= \lim_{x\to b} P_0(t^*_{(x,y)} = t^*_y < \infty) \\ &= \lim_{x\to b} x(y-x)^{-1} \begin{array}{ll} = 1 & b = -\infty, \\ < 1 & b > -\infty. \end{array} \end{aligned} \tag{2.4}$$

If b is absorbing one reaches b with positive probability, but never leaves b. If b is not absorbing one leaves b almost surely and thus by the Markov property $P_a(t^*_y < \infty) = 1$ for $a < y \in S$.

THEOREM 2.2. *Let $X(t)$ be a regular diffusion process. Assume that* 0 *is an interior point. Let $\bar{X}(t)$ be the process defined on $S \cap [0, \infty)$ by the scale- and speed-measure of the $X(t)$ process on* $(0, \infty)$, *with* 0 *a reflecting boundary.*

The tail σ-field is trivial, if and only if one of the conditions (a), (b), (c) *hold.*

(a) *The process is right and left drifting (recurrent, persistent (Itô-McKean* (1965))).

(b) *The right (left) boundary c (b) is an element of S and the process is right (left) drifting.*

(c) *The right (left) boundary c (b) is not an element of S, the process is right (left) drifting and*

$$\lim_{y\to c\,(b)} \operatorname{Var}_0\left(\bar{t}^*_y\right) = \infty \tag{2.5}$$

for the $\bar{X}(t)$ process.

NOTE. "Zero an interior point" is no restriction. This theorem can be formulated for nonregular diffusion processes too. But then one gets only a trivial tail σ-field relative to initial measures π with restrictions to the support of π. This note is also true for the following theorem.

THEOREM 2.3. *Let $X(t)$ be a regular diffusion. The invariant σ-field is trivial iff the diffusion is right or left drifting.*

PROOF OF THEOREM 2.3. Because of $\mathscr{F}_\Theta \subset \mathscr{F}_\infty$ and of symmetry it suffices to consider (i) right drifting processes where c is not an element of S and (ii) neither right nor left drifting processes.

(i) If the process is also left drifting then the diffusion is recurrent and the invariant σ-field is trivial (Itô-McKean). If the process is not left drifting then $X(t)$ converges to the right boundary. Therefore a bounded harmonic function h is constant ($h(X(t))$ is a converging martingale). This implies a trivial invariant σ-field.

(ii) Define

$$\begin{aligned} A_y^+ &:= \{\omega\ \exists t_0 \forall t > t_0\ X(t) > y\} \\ A_y^- &:= \{\omega\ \exists t_0 \forall t > t_0\ X(t) < y\}. \end{aligned} \tag{2.6}$$

A_y^+ and A_y^- are disjoint invariant (even tail) sets. By the strong Markov property we obtain for $x < y < z \in S$

$$P_y(A_y^+) \geqslant P_y(t^*_{(x,z)} = t_z^* < \infty) P_z(t_y^* = \infty) > 0$$
$$P_y(A_y^-) \geqslant P_y(t^*_{(x,z)} = t_x^* < \infty) P_x(t_y^* = \infty) > 0$$

and thus $\mathscr{F}_\infty$, $\mathscr{F}_\Theta$ are not trivial.

3. **Proof of Theorem 2.2.** The neither right nor left drifting case was treated already in the proof of Theorem 2.3, see (2.6). By symmetry and using Proposition 2.1 it suffices to treat.

(3.1) right and left drifting processes,

(3.2) right and not left drifting processes, where c is not an element of S.

We come back to (3.1) in Section 6, to (3.2) in Section 5. As a preparatory step we reformulate the problem using stopping times. Afterwards the proofs of (3.1), (3.2) will differ.

PROPOSITION 3.1. *For a regular diffusion process the following conditions are equivalent*:

(α) *$\mathscr{F}_\infty$ is trivial*;

(β) $\lim_{s\to\infty} \sup_{A\in\mathscr{F}_s^\infty} |P_\pi(A\cap B) - P_\pi(A)P_\pi(B)| = 0$ *for all $B \in \mathscr{F}$ and for all probability measures π*;

(γ) $g(x, y, t_1, t_2) = \lim_{s\to\infty} \sup_D |P^{s+t_1}(x, D) - P^{s+t_2}(y, D)| \equiv 0 \quad x, y \in S, t_1, t_2 \in \mathbb{R}^+$;

(δ) *let* $h(x, t^*, t_1, t_2) := \|P_x(t^* + t_1 \in \cdot\,) - P_x(t^* + t_2 \in \cdot\,)\|$ *where t^* is a stopping time and $\|\cdot\|$ denotes the total variation norm. Let t^i, $i \in I \subset \mathbb{R}$, be stopping times satisfying*

(i) $P_y(t^i < \infty) = 1$ *for i large enough*;

(ii) *t^i is increasing for $i > i_0 = i_0(y)$ large enough* a.s. P_y;

(iii) *t^i increases to infinity*;

(iv) $\tau^{i,i+j} = t^{i+j} - t^i$ *is a stopping time for* $X'(t) = X(t + t^i)$;

(v) $\exists k(i) \in S \quad P_y(X(t^i) = k(i)) = P_y(t^i < \infty) \quad y \in S$. *Then*

$$\lim_{i\to\infty} h(x, t^i, t_1, t_2) \equiv 0 \quad x \in S,\ t_1, t_2 \in \mathbb{R}^+$$

holds.

NOTE. We define $P_x(X(t) \in \cdot\,) \equiv 0$ for negative t.

REMARK. If the process is right but not left drifting, and further $c \notin S$, then the stopping times $t_y^* = t^{y'}, y' \in S, y' > y$ fulfill conditions (i–v) of δ. However, for a recurrent process (iii) is not true in general. Take for example $c \in S$. The random variables t_y^* tend to t_c^* if y tends to c. But t_c^* is almost surely finite.

PROOF. The proof uses well-known standard methods and is given here only for reasons of completeness.

$\alpha \Leftrightarrow \beta$. The statement (β) is a well-known necessary and sufficient criterion for the triviality of $\mathcal{F}_\infty$, obtained by martingale arguments (Orey (1971)).

Because of the Markov property it suffices to require only that (β) holds for $B \in \cup_{t>0}\mathcal{F}_t^t$. In order to see this, we observe

$$\mathcal{D} = \{B \in \mathcal{F} : \beta \quad \text{is valid for} \quad B\}$$

contains a generating class closed under finite intersections. Further, $\mathcal{D}$ contains the union of disjoint sets and B-A is contained for $A \subset B, A \in \mathcal{D}, B \in \mathcal{D}$. Again by the Markov property it suffices to require (β) for $A \in \mathcal{F}_s^s$. (Use the Markov property and the Hahn decomposition.)

$\beta \Leftrightarrow \gamma$. For a signed measure ν with total mass zero we have as a consequence of the Hahn decomposition

$$\|\nu(\cdot)\| = 2 \sup_A |\nu(A)|. \tag{3.3}$$

For a convolution of ν with a probability measure μ we have

$$\|\nu * \mu\| \leqslant \|\nu\|. \tag{3.4}$$

By this argument it follows that

$$2 \sup_D |P^{s+t}(x, D) - P^s(y, D)| = \|(P^t(x, \cdot) - P^0(y, \cdot))^* P^s(\cdot, \cdot)\| \tag{3.5}$$

is monotonically decreasing for $s \geqslant 0$. Therefore $g(\cdot)$ is well defined, measurable and in fact the limit of sup $|\cdot|$.

By definition we have the property

$$0 \leqslant g(x, y, t_1, t_2) = g(y, x, t_2, t_1) = g(x, y, t_1 - t_2, 0) \leqslant 1,\ t_1 > t_2. \tag{3.6}$$

Part "$\Rightarrow$": choose $\Pi(z) = \Pi(y) = \frac{1}{2}$ and $B = \{X(0) = z\}$. By (β) one obtains $g(z, y, 0, 0) = 0$. Let $t > 0$.

$$\begin{aligned} g(x, y, t, 0) &= \lim_s \sup_D |\textstyle\int P^s(z, D) P^t(x, dz) - P^s(y, D) \int P^t(x, dz)| \\ &\leqslant \textstyle\int g(z, y, 0, 0) P^t(x, dz) = 0. \end{aligned} \tag{3.7}$$

Part "$\Leftarrow$": let $B \in \mathcal{F}_t^t, A \in \mathcal{F}_s^s, t < s$. Use again the Markov property, the triangle inequality and the lemma of Fatou in the same manner as in 3.7.

$\gamma \Leftrightarrow \delta$. The function $h(\cdot, t^i, \cdot, \cdot)$ is increasing in i for i large enough. In order to see this, we remark that t^{i+j} is the sum of t^i and $\tau^{i,i+j}$.

$$\begin{aligned} P_x(t^{i+j} \in \cdot) &= \int P_x(t^{i+j} - t^i \in \cdot - t^i | X(s) \cdot 0 \leqslant s \leqslant t^i) dP_x \\ &= \int P_{X(t^i)}(\tau^{i,i+j} \in \cdot - t^i) dP_x \\ &= \int P_{k(i)}(\tau^{i,i+j} \in \cdot - t^i) dP_x \\ &= P_{k(i)}(\tau^{i,i+j} \in \cdot) * P_x(t^i \in \cdot). \end{aligned} \tag{3.8}$$

$k(i)$ is the constant for which $P_x(X(t^i) = k(i)) = 1$ holds.

The key equations for $\gamma \Leftrightarrow \delta$ are

$$\begin{aligned} P^{s+t}(x, D) &= P_x(t^i > s + t, X(s + t) \in D) \\ &\quad + \int P^{s+t-t^i}(k(i), D) P_x(t^i \in \cdot), \end{aligned} \tag{3.9}$$

$$\begin{aligned} P_x(t^i + t \in D) &= P_x(t^i < s + t, t^i + t \in D) \\ &\quad + P_{X(s+t)}(t^i - s \in D) P^{s+t}(x, \cdot). \end{aligned} \tag{3.10}$$

The proof now is straightforward and so we shall omit it.

4. Unimodal functions. The most important property of t_y^* is the fact that $P_x(t_y^* < \cdot)$ is a continuous unimodal function (Rösler (1977)). A distribution function F is called unimodal, if F is convex for t less than a t_0 and concave for t greater than t_0 (Ibragimov (1956)). At t_0 itself there may be a jump. A unimodal function F possesses a density f relative to Lebesgue measure. This density is a.s. increasing on $t < t_0$, then monotone decreasing and

$$F(t) = \int_{-\infty}^{t} f(x) dx. \tag{4.1}$$

Let X be a random variable with a continuous unimodal distribution F. We shall estimate the total variation between the distributions of X and $X + t$.

$$\begin{aligned} T^t(F) &:= \|P(X \in \cdot) - P(X + t \in \cdot)\| \\ &= \int_R |f(x) - f(x - t)| dx \\ &= \int_{-\infty}^{t_0} (f(x) - f(x - t)) dx + \int_{t_0}^{t_0+t} |f(x) - f(x - t)| dx \\ &\quad + \int_{t_0+t}^{\infty} (f(x - t) - f(x)) dx \\ &\leqslant 2(F(t_0 + t) - F(t_0 - t)) \\ &\leqslant 4t \sup_{x \in \mathbf{R}} f(x). \end{aligned} \tag{4.2}$$

For our purposes we will choose $F(t) = P_x(t_y^* \leqslant t)$, $x < y$. If x is a reflective point, even a little bit more is known (Keilson (1971)): namely, $P_0(t_y^* < \cdot)$ is strong unimodal. A function F is called strong unimodal, if every convolution of it with a unimodal function is again unimodal. A convolution with a point measure shows that strong unimodality implies unimodality. Ibragimov (1956) gave a useful unimodality characterisation. A distribution function F is strong unimodal if and only if F is continuous and $\Psi(\cdot) = \log F'(\cdot)$ is a concave function on the interval $E = \{F' \neq 0\}$. Further the property that Ψ is concave (Schönberg (1951), Lemma 1) implies that F' is twice positive. If twice positive functions are also density functions then we call them Pólya frequency functions of order 2, in short (Karlin, Proschan, Barlow (1961)) PF_2 functions. They gave a useful inequality between moments and the extreme values of F':

THEOREM 4.1. *Let $f(t)$ be a PF_2 function, continuous on $[0, \infty)$, and vanishing on $(-\infty, 0)$. If*

$$\mu_i = \int_0^\infty t^i f(t)\,dt$$

exists for $i = 1, 2$, then

$$\mu_2 \leqslant 2\mu_1^2$$

and $f(0) \leqslant \mu_1^{-1}$.

NOTE. Every PF_2 function f is continuous on $\operatorname{int}\{f \neq 0\}$.

We need this result in the following version.

PROPOSITION 4.2. (a) *Let F be a continuous strong unimodal function, $F = f$ a PF_2 function. Let a be the point, such that f is increasing on $(-\infty, a)$ and decreasing on (a, ∞). The moments*

$$\nu_i^+ = \int_a^\infty (t-a)^i f(t)\,dt$$

$$\nu_i^- = \int_{-\infty}^a (a-t)^i f(t)\,dt$$

shall exist for $i = 1, 2$. Then it is true that

$$\operatorname{ess\,sup}_{x \in R} f(x) \leqslant \frac{(\nu_0^+)^2}{\nu_1^+} \wedge \frac{(\nu_0^-)^2}{\nu_1^-}. \tag{4.3}$$

(b) *Let f_y, $y \in \mathbb{N}$, be a sequence of functions as described above, then*

$$\int_{-\infty}^\infty t^2 f_y(t)\,dt - \left(\int_{-\infty}^\infty t f_y(t)\,dt\right)^2 \to \infty \qquad \text{as } y \to \infty$$

implies $\operatorname{ess\,sup}_{x \in \mathbb{R}} f_y(x) \to 0 \quad$ *as* $\; y \to \infty$.

PROOF. The functions $f^+ = f \cdot 1_{(a,\infty)}$, $f^- = f \cdot 1_{(-\infty, a)}$ are both twice positive and, after a normalisation, PF_2 functions. An application of the above theorem taking the note into account shows the truth of (4.3) and further

$$\frac{\nu_2^+}{\nu_0^+} + \frac{\nu_2^-}{\nu_0^-} \leqslant 2\left(\frac{\nu_1^+}{\nu_0^+}\right)^2 + 2\left(\frac{\nu_1^-}{\nu_0^-}\right)^2.$$

Further, we have

$$\nu_2^+ + \nu_2^- = \int_{-\infty}^\infty t^2 f(t)\,dt - \left(\int_{-\infty}^\infty t f(t)\,dt\right)^2 + \left(a - \int_{-\infty}^\infty t f(t)\,dt\right)^2.$$

Combining the last two inequalities and (4.3), the proposition follows.

5. Main part of the proof of Theorem 2.2. Throughout this section $X(t)$ denotes a right but not left drifting process with no absorbing points. Zero is an interior point. We will show that, for triviality of F_∞ only the behaviour at the right boundary c is important. Therefore we may change zero into a reflecting point and take into consideration the related process $\bar{X}(t)$. The $\bar{X}$ related functions are barred. Using the unimodality it suffices to look at the supremum of the function $(\partial/\partial t)P_0(\bar{t}_y^* < \cdot\,)$ for y near c. The unimodality of $P_x(\bar{t}_y^* < \cdot\,)$ is needed only for this step. Now $P_0(\bar{t}_y^* < \cdot\,)$ is even a strong unimodal function; thus by Section 4 it suffices to look at the behaviour of $\mathrm{Var}_0(\bar{t}_y^*)$ for y tending to c.

PROPOSITION 5.1. *The following conditions are equivalent*:

(i) $\mathcal{F}_\infty$ *is trivial*;

(ii) $\lim_{y\to c} h(\cdot, t_y^*, \ldots) \equiv 0$;

(iii) $\lim_{y\to c} \bar{h}(\cdot, \bar{t}_y^*, \ldots) \equiv 0$;

(iv) $\lim_{y\to c} \max_t P_x(|\bar{t}_y^* - t| < a) \equiv 0, \qquad a \in \mathbb{R}^+, x \in S$;

(v) $\lim_{y\to c} \max_t P_0(|\bar{t}_y^* - t| < a) \equiv 0, \qquad a \in \mathbb{R}^+$;

(vi) $\lim_{y\to c} \mathrm{Var}_0\,(\bar{t}_y^*) = \infty$.

PROOF. (i) $\Leftrightarrow$ (ii). This is Proposition 3.1.

(ii) $\Leftrightarrow$ (iii). We will establish the following equality:

$$(5.2)\quad \begin{aligned}\lim_{x\to c}\lim_{y\to c} h(x, t_y^*, .d, .,) &= \lim_x \lim_y h(x, t^*{}_{(0,y)}, .,.)\\ &= \lim_x \lim_y \bar{h}\big(x, \bar{t}^*{}_{(0,y)}, .,.\big) = \lim_{x\to c}\lim_{y\to c} \bar{h}\big(x, \bar{t}_y^*, .,.\big).\end{aligned}$$

This proves (ii) $\Leftrightarrow$ (iii) because $\lim_y h(x, t_y^*, .,.)$ is an increasing function in x. In order to see this, note that

$$P_x(t_y^* + t < \cdot\,) = P_x(t_z^* < \cdot\,)^* P_z(t_y^* + t < \cdot\,) \qquad x < z < y \in S$$

and then use $\|p^*q\| < \|p\|$, where p is a signed and q a probability measure.

Without loss of generality assume the diffusion is on the natural scale. Thus

$$\sup_{x<y} P_x\big(t_y^* = t^*{}_{(0,y)}\big) = \frac{c-x}{c-0} \to 0 \qquad \text{as } x \to c.$$

This gives us the first equality and, with an analogous argument, the third. The second follows from the fact that the distribution of $t^*_{(0,y)}$ and $\bar{t}^*_{(0,y)}$ is the same.

(iii) $\Leftrightarrow$ (iv). This is a consequence of the unimodality of $P_x(\bar{t}_y^* < \cdot\,)$ (Rösler (1978)). Let f be the density of this function and t_0 be the point such that $f(x)$ is increasing left of t_0 and decreasing on the right side.

$$\begin{aligned}&\bar{h}\big(x, \bar{t}_y^*, 0, a\big)\\ &= \textstyle\int |f(x) - f(x-a)|dx \leqslant \int_{-\infty}^{t_0}(f(x) - f(x-a))dx + \int_{t_0}^{t_0+a}(|f(x)| + |f(x-a)|)dx\\ &\quad + \textstyle\int_{t_0+a}^{\infty}(f(x-a) - f(x))dx = 2P\big(\bar{t}_y^* - t_0 \leqslant 2a\big) \leqslant 2\sup_t P\big(|\bar{t}_y^* - t| \leqslant 2a\big)\\ &\leqslant 2P\big(|\bar{t}_y^* - t_0| \leqslant 4a\big) = 2\textstyle\int_{t_0-2a}^{t_0+2a} f(x)dx = 2\int_{-\infty}^{t_0}(f(x) - f(x-2a))dx\\ &\quad + 2\textstyle\int_{t_0+2a}^{\infty}(f(x-2a) - f(x))dx \leqslant 2\int |f(x) - f(x-2a)|dx = 2\bar{h}\big(x, \bar{t}_y^*, 0, 2a\big).\end{aligned}$$

This is true for all $x \in S$, $a \in R$.

(iv) $\Rightarrow$ (v). Obvious.

(v) $\Rightarrow$ (iv). Let t_1, $a \in R^+$, $0 < x < y \in S$.

$$\begin{aligned}\max_t P_x\big(|\bar{t}_y^* - t| < a\big) &= \max_t P_0\big(|\bar{t}_y^* - \bar{t}_x^* - t| < a\big)\\ &\leqslant \max_t \Sigma_{n=-\infty}^{\infty} P_0\big(|\bar{t}_y^* - nt_1 - t| < a + t_1\big)\\ &\quad \cdot P_0\big(\bar{t}_x^* \in [nt_1, (n+1)t_1)\big)\\ &\leqslant \max_t P_0\big(|\bar{t}_y^* - t| < a + t_1\big)\\ &\quad \cdot \Sigma P_0\big(\bar{t}_x^* \in [nt_1, (n+1)t_1)\big).\end{aligned}$$

This is true for arbitrary $t_1 \in R^+$.

(vi) $\Rightarrow$ (v). $P_0(\bar{t}_y^* < \cdot\,)$ is a strong unimodal continuous function (Keilson (1971)). Its density is a Pólya frequency function of order 2. Take any sequence y_i converging to c. Proposition 4.2b gives

$$\max_t P_0\big(|\bar{t}_{y_i}^* - t| < a\big) \leqslant 2a \sup_t \frac{\partial}{\partial t} P_0\big(\bar{t}_{y_i}^* < \cdot\big) \to 0 \qquad \text{as } y_i \to c. \tag{5.3}$$

(v) $\Rightarrow$ (vi). We suppose $\lim_{y\to c} \operatorname{Var}_0(\bar{t}_y^*) < \infty$. The random variables $Z_i = \bar{t}_{y_i}^* - \bar{t}_{y_{i-1}}^*$ are independent and the sum of their variances is finite. Therefore $S_n = \Sigma_{i=1}^n (Z_i - E_0(Z_i)) = \bar{t}_{y_n}^* - E_0(\bar{t}_{y_n}^*)$ converges to a random variable S_0 with a continuous distribution, say G, which is strong unimodal too (Ibragimov (1956)).

Now

$$\max_t P_0\big(|\bar{t}_{y_n}^* - t| < a\big) \to_n \max_t (G(t+a) - G(t-a)) > 0. \tag{5.4}$$

Actually we obtain a little bit more: $\lim_n(\bar{t}_{y_n}^* - E_0(\bar{t}_{y_n}^*))$ exists almost surely and is $\mathcal{F}_\infty$ measurable. Therefore

$$A_d = \big\{\lim_n\big(\bar{t}_{y_n}^* - E_0\big(\bar{t}_{y_n}^*\big)\big) < d\big\}$$

is a tail event, satisfying $P(A_d) = G(d)$. G is continuous, so $\mathcal{F}_\infty$ possesses no atoms.

COROLLARY 5.2. *Let $X(t)$ be a right and not left drifting regular diffusion process. Assume that zero is an interior point and that the right boundary is not absorbing. If* $\lim_{y\to c}\operatorname{Var}_0(\bar{t}_y^*) < \infty$, *then for all $d \in [0, 1]$ there exists a terminal set A_d^x obeying $P_x(A_d^x) = d$ and the tail σ-field possesses no atoms.*

REMARK. Recently Fristedt-Orey (1978) showed that the tail σ-field is generated by $\lim_n(\bar{t}_{y_n}^* - E_0(\bar{t}_{y_n}^*))$ in the case $\lim_y \operatorname{Var}_0(\bar{t}_y^*) < \infty$.

COROLLARY 5.3. *Let $X(t)$ be a right not left drifting regular diffusion process. Zero is an interior point, the boundaries b, c are not absorbing. Assume further*

$$\lim_{y\to c,\, b}\operatorname{Var}_0\big(\bar{t}_y^*\big) = \infty$$

for the $\bar{X}^+$ and $\bar{X}^-$ process on the positive and negative part of R, 0 reflecting. Then *the tail σ-field is generated almost surely by the two atoms*

$$\{\lim_t X(t) = c\}, \quad \{\lim_t X(t) = b\}.$$

PROOF. The processes $\bar{X}^+, \bar{X}^-$ obey the 0–1 law. From the strong Markov property and from

$$P_x(X = \bar{X}^+) \to 1 \quad \text{as } x \to c, \qquad P_x(X = \bar{X}^-) \to 1 \quad \text{as } x \to b,$$

one obtains the corollary.

For an example of a right but not left drifting process satisfying $\lim_{y \to c} \mathrm{Var}_0(t_y^*) < \infty$ see Rösler (1977) or Fristedt and Orey (1978, Theorem 2).

6. Recurrent diffusions.

PROPOSITION 6.1. *Every right and left drifting regular diffusion process possesses a trivial tail σ-field.*

PROOF. For fixed $x < y \in S$ define stopping times

$$\begin{aligned} t^{2n+1} &= \inf\{t > t^{2n} : X(t) = y\} \qquad n \in \mathbb{N} \\ t^{2n} &= \inf\{t > t^{2n-1} : X(t) = x\} \qquad n \in \mathbb{N} \\ t^0 &= 0. \end{aligned} \tag{6.1}$$

These random variables fulfill the conditions 3.1.δ(i–v).

The random variables $Z_i = t^{i+1} - t^i$ are independent and each possesses a unimodal distribution function. However, it is not known whether the nth partial sum $S_n = t^{n+1}$ possesses a unimodal distribution or not. However, S_n is the sum of independent random variables with a unimodal distribution. This fact is used in Rösler (1977) to prove a trivial tail σ-algebra. But we will use an idea similar to that for discrete Markov chains. Let $Y_i = t^{2i+2} - t^{2i}$. Now $\mathcal{F}_\infty$ is trivial is equivalent to $\lim_i h(x, t^{2i},.,.) = 0$ and this is equivalent to a trivial tail σ-algebra of the process $S_n = \Sigma^n Y_i$ (similar to Proposition 3.1). The Hewitt-Savage zero-one law (Breiman (1968), Corollary 3.50) gives us the triviality of the tail σ-field. This uses heavily the fact that the distribution of the random variables Y_i are identical.

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