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# WEAK CONVERGENCE FOR THE MAXIMA OF STATIONARY GAUSSIAN PROCESSES USING RANDOM NORMALIZATION

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Let $\{X_k, k > 1\}$ be a stationary Gaussian sequence with $EX_1 = 0$, $EX_1^2 = 1$, and $EX_1X_{n+1} = r_n$. Let $c_n = (2 \ln n)^{\frac{1}{2}}$, $b_n = c_n - \ln(4\pi \ln n)/2c_n$ and set $M_n = \max_{1<k<n} X_k$, $\bar{X}_n = \frac{1}{n}\Sigma_{k=1}^n X_k$, and $s_n^2 = \frac{1}{n}\Sigma_{k=1}^n (X_k - \bar{X}_n)^2$. If $r_n$ is not identically one and $(\ln n)/n\Sigma_{k=1}^n |r_k - r_n| = o(1)$, it is shown that

$$(1) \qquad \lim_{n\to\infty} P\left\{ c_n\left(\frac{M_n - \bar{X}_n}{s_n} - b_n\right) < x\right\} = \exp\{-e^{-x}\}.$$

If we further assume $(r_n \ln n)^{-1} = o(1)$ then it is shown that

$$(2) \quad \lim_{n\to\infty} P\left\{ r_n^{-\frac{1}{2}}\left(\frac{M_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) < x\right\} = \left(\frac{1-\gamma}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{(1-\gamma)u^2}{2}} du$$

where $\gamma = F(\{o\})$ is the atom at zero of the spectral distribution associated with $r$. A version of these results for continuous time processes is also presented.

**1. Introduction.** Let $\{X_k, k \geq 1\}$ be a stationary sequence of standard normal random variables. Let $r_n = EX_1X_{n+1}$, $c_n = (2 \ln n)^{\frac{1}{2}}$, $b_n = c_n - \ln(4\pi \ln n)/2c_n$ and set $M_n = \max_{1\leq k\leq n} X_k$. Under the condition $r_n \ln n = o(1)$, Berman [2] has shown

$$(1.1) \quad \lim_{n\to\infty} P\{c_n(M_n - b_n) \leq x\} = \wedge(x) = \exp\{-e^{-x}\}, \ -\infty < x < \infty.$$

On the other hand when it is no longer true that $r_n \ln n = o(1)$, a variety of possible limit laws for $M_n$ arise. Mittal and Ylvisaker [4] have shown for suitably smooth correlation functions that if $r_n \ln n = O(1)$, $M_n$ is attracted to a mixture of the double exponential and normal laws. Furthermore for convex correlation functions with $r_n = o(1)$ and $(r_n \ln n)^{-1} = o(1)$ and monotone for large $n$, they showed

$$(1.2) \qquad \lim_{n\to\infty} P\left\{r_n^{-\frac{1}{2}}\left(M_n - (1-r_n)^{\frac{1}{2}} b_n\right) \leq x\right\} = \Phi(x),$$

$-\infty < x < \infty$ where $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-u^2/2}\, du$.

In Section 2 we will show that the double exponential weak limit can be retained for sequences having a suitably smooth correlation function provided we allow random normalization. It is noteworthy that this behavior is independent of mixing conditions in that we require only the correlation function to be not identically one.

Basically (1.2) holds, because under the conditions for that theorem the sequence is bound together by what we may call a binding variable. It is this binding

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variable which effects the normal weak limit and by eliminating it at the outset, we find that the resulting sequence is sufficiently asymptotically independent to allow a statement like (1.1) to hold. More precisely letting $\bar{X}_n = 1/n\Sigma_{k=1}^n X_k$ and $s_n^2 = 1/n\Sigma_{k=1}^n(X_k - \bar{X}_n)^2$, the main result of Section 2 states that for correlation functions not identically one

$$\frac{\ln n}{n}\Sigma_{k=1}^n|r_k - r_n| = o(1)$$

is sufficient for

$$c_n\left(\frac{M_n - \bar{X}_n}{s_n} - b_n\right) \to \Lambda.$$

The random normalization employed above, $(M_n - \bar{X}_n)/s_n$, is referred to as the studentized maxima and has been considered previously by several authors. Grubbs [3] determined the exact distribution of $(M_n - \bar{X}_n)/s_n$ when the $X_i$'s are identically distributed independent normal random variables and has applied this work to the problem of testing for outlying observations. Berman [1] considered the asymptotic theory of studentized maxima for i.i.d. random variables $X_i$ with $EX_i = 0$, $EX_i^2 = 1$ and showed that $(M_n - b_n)/a_n$ attracted to an extremal distribution implies $1/a_n((M_n - \bar{X}_n)/s_n - b_n)$ attracted to the same distribution provided the normalizing constants satisfy $(b_n/a_n(n^{\frac{1}{2}})) = o(1)$. Our Theorem 2.1 represents a generalization of this result for dependent Gaussian variables. In Section 3 we present versions of the main results of Section 2 for continuous time processes.

**2. Limit distribution for $(M_n - \bar{X}_n)/s_n$.** Let $\{X_k, k \geqslant 1\}$ be a stationary sequence of standard normal random variables with correlation function $r_n = EX_1 X_{n+1}$. Define $\bar{r}_n(k) = \max_{k \leqslant j \leqslant n} |r_j - r_n|$. We will be concerned with correlation functions satisfying the smoothness condition

$$\frac{\ln n}{n}\Sigma_{k=1}^n |r_k - r_n| = o(1)$$

and therefore it will be of interest to us to know that the above condition is equivalent to the ostensibly stronger condition

$$\frac{\ln n}{n} \Sigma_{k=1}^n \bar{r}_n(k) = o(1).$$

This equivalence is contained in the following lemma which relates a rate of convergence to zero of the average of differences in the terms of a bounded sequence to a growth rate for the differences.

LEMMA 2.1. *For any bounded sequence of real numbers $\{a_k, k \geqslant 1\}$ the following conditions are equivalent*

$$\frac{\ln n}{n} \Sigma_{k=1}^n |a_k - a_n| = o(1), \tag{2.1}$$

$$\frac{\ln n}{n} \Sigma_{k=1}^n \bar{a}_n(k) = o(1), \quad \textit{and} \tag{2.2}$$

*for every $\varepsilon > 0$ and all $n$ sufficiently large*

$$(2.3) \qquad |a_k - a_n| < \varepsilon \max\left(\frac{\ln n}{\ln k} - 1, \frac{1}{\ln n}\right), \qquad 1 \leqslant k \leqslant n.$$

PROOF. Assume (2.1) holds. Fix any $\varepsilon > 0$ and set

$$A_n = A_n(\varepsilon) = \left\{k\colon 2 \leqslant k \leqslant n, |a_k - a_n| \geqslant \varepsilon \max\left(\frac{\ln n}{\ln k} - 1, \frac{1}{\ln n}\right)\right\}.$$

Define

$$\alpha_n = \alpha_n(\varepsilon) = \max\{k : k \in A_n\}, \quad \text{if } A_n \neq \varnothing$$
$$= 1, \qquad \text{otherwise.}$$

Then $\alpha_n = o(n)$ for if not then there is some $\delta > o$ such that along some subsequence $\{n'\}$ we have $\delta \leqslant \frac{\alpha_{n'}}{n'} \leqslant 1$. But

$$\frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} |a_j - a_{\alpha_{n'}}| \geqslant \frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} |a_{\alpha_{n'}} - a_{n'}| - \frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{n'} |a_j - a_{n'}|$$
$$\geqslant \varepsilon \frac{\ln \alpha_{n'}}{\ln n'} - \frac{n'}{\alpha_{n'}}\left(\frac{\ln n'}{n'} \Sigma_{j=1}^{n'} |a_j - a_{n'}|\right)$$
$$\geqslant \varepsilon/2$$

for $n'$ sufficiently large. Since this contradicts (2.1) we have $\alpha_n = o(n)$. In particular when $\alpha_n \in A_n$ we have for all $n$ sufficiently large

$$|a_{\alpha_n} - a_n| \geqslant \varepsilon\left(\frac{\ln n}{\ln \alpha_n} - 1\right).$$

Moreover, we have $\alpha_n = O(1)$ for if along some subsequence $\{n'\}$ we have $\alpha_{n'} \to \infty$ as $n' \to \infty$, then for all $n'$ large enough so that $2\alpha_{n'} < n' e^{-1}$

$$\frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} |a_j - a_{2\alpha_{n'}}| \geqslant \frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} |a_{\alpha_{n'}} - a_{2\alpha_{n'}}| - \frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} |a_j - a_{\alpha_{n'}}|$$
$$\geqslant o(1) + \frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} \left(|a_{\alpha_{n'}} - a_{n'}| - |a_{n'} - a_{2\alpha_{n'}}|\right)$$
$$\geqslant o(1) + \varepsilon \frac{\ln \alpha_{n'}}{\alpha_{n'}} \Sigma_{j=1}^{\alpha_{n'}} \left(\frac{\ln n'}{\ln \alpha_{n'}} - \frac{\ln n'}{\ln 2\alpha_{n'}}\right)$$
$$= o(1) + \varepsilon \ln \alpha_{n'} \ln n' \left(\frac{\ln 2}{\ln \alpha_{n'} \ln 2\alpha_{n'}}\right)$$
$$> \varepsilon/2.$$

This contradicts (2.1) so that we actually have for all $n$ sufficiently large

$$|a_k - a_n| < \varepsilon \max\left(\frac{\ln n}{\ln k} - 1, \frac{1}{\ln n}\right), \; 1 \leqslant k \leqslant n$$

establishing (2.3).

Thus for $n$ sufficiently large we have in view of (2.3)

$$\begin{aligned}\frac{\ln n}{n}\sum_{k=2}^{n}\overline{a_n}(k) &\leq \varepsilon\frac{\ln n}{n}\sum_{k=2}^{n}\left(\frac{\ln n}{\ln k}-1\right)+\varepsilon\\ &\leq \varepsilon\frac{\ln n}{n}\sum_{k=2}^{n}\frac{1}{\ln k}\sum_{j=k}^{n-1}(\ln(j+1)-\ln j)+\varepsilon\\ &\leq \varepsilon\frac{\ln n}{n}\sum_{j=2}^{n-1}(\ln(j+1)-\ln j)\sum_{k=2}^{j}\frac{1}{\ln k}+\varepsilon\\ &\leq \varepsilon\frac{\ln n}{n}\sum_{j=2}^{n-1}\frac{c_1}{\ln j}+\varepsilon\\ &\leq c_2\,\varepsilon+\varepsilon\end{aligned}$$

where $c_1$ and $c_2$ are some constants. Since $\varepsilon$ was arbitrary, this establishes (2.2). Hence the lemma follows since (2.1) is now immediate.

THEOREM 2.1. *Let $\{X_k, k \geq 1\}$ be a stationary sequence of standard normal random variables having correlation function $r_n$ assumed not to be identically one satisfying*

$$\frac{\ln n}{n}\sum_{k=1}^{n}|r_k - r_n| = o(1). \tag{2.4}$$

*Then letting $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n}X_k$ we have*

$$\lim_{n\to\infty} P\left\{c_n\left[\frac{M_n-\overline{X}_n}{(1-r_n)^{\frac{1}{2}}}-b_n\right]\leq x\right\}=\exp\{-e^{-x}\}, \tag{2.5}$$

$$-\infty < x < \infty.$$

PROOF. By the spectral representation theorem we may write

$$X_k = \int_{-\pi}^{\pi} e^{i\lambda k}\, Z(d\lambda)$$

where $Z$ is the random spectral measure associated with $\mathbf{X}$ and further

$$F(d\lambda) = E(Z(d\lambda))^2$$

is the spectral distribution associated with $r$. Now the ergodic theorem for wide sense stationary processes [6] yields $\overline{X}_n \to_{L^2} Z(\{o\})$ so that $E(\overline{X}_n)^2 \to E(Z(\{o\}))^2 = F(\{o\})$. Putting $\gamma = F(\{o\})$ we have by (2.4)

$$r_n - \gamma = o(1) \tag{2.6}$$

for

$$\begin{aligned}|\gamma - r_n| &\leq \left|\gamma - \frac{1}{n^2}\sum_{i,j} r_{(i-j)}\right| + \left|\frac{1}{n^2}\sum_{i,j}(r_{(i-j)} - r_n)\right|\\ &\leq o(1) + \frac{1}{n}\sum_{k=1}^{n}|r_k - r_n|\\ &= o(1).\end{aligned}$$

Since $r$ is assumed to be not identically one, we must have $o \leqslant \gamma < 1$. Stationarity then implies

$$\sup_{n \geqslant 1} |r_n| = \delta < 1. \tag{2.7}$$

Now let $\sigma_n^2(k) = E(X_k - \bar{X}_n)^2$. Then

$$\max_{1 \leqslant k \leqslant n} |\sigma_n^2(k) - (1 - r_n)| = o\left(\frac{1}{\ln n}\right) \tag{2.8}$$

for

$$\begin{aligned}
|\sigma_n^2(k) - (1 - r_n)| &= \left| \frac{1}{n^2} \Sigma_{i=1}^n \Sigma_{j=1}^n r_{(i-j)} - \frac{2}{n} \Sigma_{j=1}^n r_{(j-k)} + r_n \right| \\
&\leqslant \frac{1}{n^2} \Sigma_{i,j} |r_{(i-j)} - r_n| + \frac{2}{n} \Sigma_{j=1}^n |r_{(j-k)} - r_n| \\
&\leqslant \frac{1}{n} \Sigma_{j=1}^n |r_j - r_n| + \frac{2}{n} \Sigma_{j=1}^{k-1} |r_{(k-j)} - r_n| \\
&\quad + \frac{2}{n} \Sigma_{j=k}^n |r_{(j-k)} - r_n| \\
&\leqslant o\left(\frac{1}{\ln n}\right) + \frac{2}{n} \Sigma_{j=1}^{k-1} |r_j - r_n| + \frac{2}{n} \Sigma_{j=o}^n |r_j - r_n| \\
&= o\left(\frac{1}{\ln n}\right).
\end{aligned}$$

Now define $Y_{k,n} = \dfrac{1}{\sigma_n(k)}(X_k - \bar{X}_n)$, $1 \leqslant k \leqslant n$ and set $\rho_n(i_o, i_1) = EY_{i_o,n} Y_{i_1,n}$. Then

$$\max_{1 \leqslant i_o < i_1 \leqslant n} \left| \rho_n(i_o, i_1) - \left( \frac{r_{(i_1 - i_o)} - r_n}{1 - r_n} \right) \right| = o\left(\frac{1}{\ln n}\right) \tag{2.9}$$

for by (2.7) and (2.8)

$$\begin{aligned}
&\left| \rho_n(i_0, i_1) - \left( \frac{r_{(i_1 - i_0)} - r_n}{1 - r_n} \right) \right| \\
&\qquad \leqslant \left| \rho_n(i_0, i_1) - \left( \frac{r_{(i_1 - i_0)} - r_n}{\sigma_n(i_0)\sigma_n(i_1)} \right) \right| + 2 \left| \frac{1}{\sigma_n(i_0)\sigma_n(i_1)} - \frac{1}{1 - r_n} \right| \\
&\qquad \leqslant o\left(\frac{1}{\ln n}\right) \\
&\qquad\quad + \frac{2}{1-\delta} \left| \frac{1}{n^2} \Sigma_{j=1}^n \Sigma_{k=1}^n r_{(j-k)} - \frac{1}{n} \Sigma_{k=1}^n (r_{(k-i_0)} + r_{(k-i_1)}) + r_n \right| \\
&\qquad \leqslant o\left(\frac{1}{\ln n}\right) + \frac{10}{n}(1 - \delta)^{-1} \Sigma_{k=1}^n |r_k - r_n| \\
&\qquad = o\left(\frac{1}{\ln n}\right).
\end{aligned}$$

Now let $\bar{\rho}_n(k) = \max\{|\rho_n(i_0, i_1)|: 1 \leqslant i_0 < i_1 \leqslant n;\ i_1 - i_0 \geqslant k\}$.
Then by (2.9) we have

$$\bar{\rho}_n(k) \leqslant (1 - r_n)^{-1}\bar{r}_n(k) + o\left(\frac{1}{\ln n}\right), \qquad 1 \leqslant k \leqslant n. \tag{2.10}$$

Furthermore by (2.6), (2.7), and (2.9) we have for any fixed $\delta < \eta < 1$ and all $n$ sufficiently large

$$\bar{\rho}_n(1) \leqslant \eta. \tag{2.11}$$

Now

$$P\left\{c_n\left[\frac{M_n - \bar{X}_n}{(1 - r_n)^{\frac{1}{2}}} - b_n\right] \leqslant x\right\} = P\{X_k - \bar{X}_n \leqslant (1 - r_n)^{\frac{1}{2}}(b_n + x/c_n),\ 1 \leqslant k \leqslant n\}$$

$$= P\{c_n(\hat{Y}_{k,n} - b_n) \leqslant x + \theta_n(k, x),\ 1 \leqslant k \leqslant n\}$$

where

$$\theta_n(k, x) = (b_n c_n + x)\left(\frac{(1 - r_n)^{\frac{1}{2}}}{\sigma_n(k)} - 1\right)$$

is such that $\max_{1 \leqslant k \leqslant n}|\theta_n(k, x)| = o(1)$ so that to establish (2.5), it suffices to show for $M_n^* = \max_{1 \leqslant k \leqslant n} Y_{k,n}$ that

$$\lim_{n\to\infty} P\{c_n(M_n^* - b_n) \leqslant x\} = \exp\{-e^{-x}\}, \quad -\infty < x < \infty. \tag{2.12}$$

Fix $o < \alpha < (1 - \eta)/(1 + \eta)$. Then for all $n$ sufficiently large we have by Berman's lemma [2]

$$\begin{aligned}
&|P\{c_n(M_n^* - b_n) \leqslant x\} - \Phi^n(b_n + x/c_n)| \\
&\quad\leqslant \Sigma_{i<j}\frac{1}{2\pi}(1 - \rho_n^2(i,j))^{-\frac{1}{2}}|\rho_n(i,j)|\exp\left\{-\frac{(b_n + x/c_n)^2}{1 + |\rho_n(i,j)|}\right\} \\
&\quad\leqslant cn\Sigma_{j=1}^n\bar{\rho}_n(j)\exp\left\{-\frac{b_n^2}{1 + \bar{\rho}_n(j)}\right\} \\
&\quad\leqslant cn^{1+\alpha}\exp\left\{-\frac{b_n^2}{1 + \bar{\rho}_n(1)}\right\} + cn\Sigma_{j=[n^\alpha]}^n\bar{\rho}_n(j)\exp\left\{-\frac{b_n^2}{1 + \bar{\rho}_n(j)}\right\} \\
&\quad\leqslant c\exp\left\{-\frac{1}{2}\left(\frac{1 - \eta}{1 + \eta} - \alpha\right)\ln n\right\} \\
&\qquad + cn\Sigma_{j=[n^\alpha]}^n\bar{r}_n(j)\exp\left\{-\frac{b_n^2}{1 + \bar{r}_n(j)(1 - r_n)^{-1}}\right\} \\
&\quad\leqslant o(1) + cn\Sigma_{j=[n^\alpha]}^n\bar{r}_n(j)\exp\left\{-\frac{2\ln n}{1 + \bar{r}_n(j)(1 - r_n)^{-1}} + \ln\ln n\right\}.
\end{aligned} \tag{2.13}$$

To show that the sum in (2.13) is $o(1)$ we introduce two sequences of integers. First let $K = K(n)$ be that integer for which

$$(2.14) \qquad \ln_K n > 1 \geqslant \ln_{K+1} n$$

where $\ln_j n$ is defined recursively by $\ln_j n = \ln \ln_{j-1} n$. Then for any fixed $0 < \varepsilon < \left(\frac{1-\gamma}{8}\right)^2$, $\gamma = F(\{0\})$, we define integers $q(j) = q(j, n, \varepsilon)$ by

$$(2.15) \qquad q(1) = -1$$

$$\varepsilon^{q(j)/2} > \frac{6\varepsilon^{-\frac{1}{2}} \ln_j n}{\ln n} \geqslant \varepsilon^{(q(j)+1)/2}, \qquad j = 2, \cdots, K$$

and

$$(2.16) \qquad t_0 = t_0(n) = [n^\alpha]$$

$$t_i = t_i(n, \varepsilon) = [\exp\{(1 - \varepsilon^{i/2})\ln n\}], \qquad i = 1, \cdots, q(K)$$

$$t_{q(K)+1} = n.$$

Next we obtain upper bounds on quantities of interest to us. Firstly for all $n$ sufficiently large we have by (2.3)

$$(2.17) \qquad \bar{r}_n(t_0)(1 - r_n)^{-1} \leqslant \varepsilon\left(\frac{\ln n}{\ln t_0} - 1\right)(1 - r_n)^{-1} \leqslant \frac{2\varepsilon}{1-\gamma}\left(\frac{1-\alpha}{\alpha}\right)$$

$$\bar{r}_n(t_i)(1 - r_n)^{-1} \leqslant \varepsilon\left(\frac{\ln n}{\ln t_i} - 1\right)(1 - r_n)^{-1} \leqslant \frac{2\varepsilon}{1-\gamma}\varepsilon^{i/2} \qquad i = 1, \cdots, q(K).$$

Next note

$$q(2) < \ln_2 n$$

and for $2 \leqslant j < K$ we have

$$\varepsilon^{[q(j+1) - q(j) - 1]/2} = \frac{\varepsilon^{q(j+1)/2}}{\varepsilon^{(q(j)+1)/2}} > \frac{\ln_{j+1} n}{\ln_j n}$$

so that

$$(2.18) \qquad q(j+1) - q(j) - 1 < \ln_{j+1} n.$$

Now the sum at (2.13) may be bounded above by

$$(2.19) \qquad n\Sigma_{j=1}^{K-1}\Sigma_{i=q(j)+1}^{q(j+1)} t_{i+1}\bar{r}_n(t_i) \exp\left\{-\frac{2\ln n}{1 + \bar{r}_n(t_i)(1 - r_n)^{-1}} + \ln_2 n\right\}.$$

Consider the sum indexed by $j = 1$.

$$n\Sigma_{i=0}^{q(2)} t_{i+1}\bar{r}_n(t_i)\exp\left\{-\frac{2\ln n}{1+\bar{r}_n(t_i)(1-r_n)^{-1}}+\ln_2 n\right\}$$

$$\leqslant c\bar{r}_n(t_0)\Sigma_{i=0}^{q(2)}\exp\{(2\bar{r}_n(t_i)(1-r_n)^{-1}-\varepsilon^{(i+1)/2})\ln n+\ln_2 n\}$$

$$\leqslant c\frac{2\varepsilon}{1-\gamma}\left(\frac{1-\alpha}{\alpha}\right)\Sigma_{i=0}^{q(2)}\exp\left\{\left(\frac{4\varepsilon}{1-\gamma}\varepsilon^{i/2}-\varepsilon^{(i+1)/2}\right)\ln n+\ln_2 n\right\}$$

$$\leqslant c\varepsilon\Sigma_{i=0}^{q(2)}\exp\left\{\left(\frac{4\varepsilon}{1-\gamma}^{\frac{1}{2}}-1\right)\varepsilon^{(i+1)/2}\ln n+\ln_2 n\right\}$$

$$\leqslant c\varepsilon\Sigma_{i=0}^{q(2)}\exp\left\{-\frac{1}{2}\varepsilon^{(i+1)/2}\ln n+\ln_2 n\right\}$$

$$\leqslant c\varepsilon\Sigma_{i=0}^{q(2)}\exp\{-3\ln_2 n+\ln_2 n\}$$

$$\leqslant c\varepsilon\ln_2 n\ \exp\{-2\ln_2 n\}$$

$$\leqslant c\exp\{-\ln_2 n\} \tag{2.20}$$

where $c$ is a generic constant and where we have used the bounds at (2.17) and (2.18) and the defining property at (2.15). Next consider a sum indexed by $1 < j < K-1$.

$$n\Sigma_{i=q(j)+1}^{q(j+1)} t_{i+1}\bar{r}_n(t_i)\exp\left\{-\frac{2\ln n}{1+\bar{r}_n(t_i)(1-r_n)^{-1}}+\ln_2 n\right\}$$

$$\leqslant \bar{r}_n\left(t_{q(j)+1}\right)\Sigma_{i=q(j)+1}^{q(j+1)}\exp\{(2\bar{r}_n(t_i)(1-r_n)^{-1}-\varepsilon^{(i+1)/2})\ln n+\ln_2 n\}$$

$$\leqslant c\varepsilon^{\frac{1}{2}}\ln_j n\Sigma_{i=q(j)+1}^{q(j+1)}\exp\{-3\ln_{j+1} n\}$$

$$\leqslant c\varepsilon^{\frac{1}{2}}\exp\{-\ln_{j+1} n\}. \tag{2.21}$$

Finally consider the sum with index $j = K-1$.

$$n^2\Sigma_{i=q(K-1)+1}^{q(K)}\bar{r}_n(t_i)\exp\left\{-\frac{2\ln n}{1+\bar{r}_n(t_i)(1-r_n)^{-1}}+\ln_2 n\right\}$$

$$\leqslant en^2\ln n\bar{r}_n(t_{q(K-1)+1})\exp\left\{-\frac{2\ln n}{1+\bar{r}_n(t_{q(K-1)+1})(1-r_n)^{-1}}\right\}$$

$$\leqslant c\varepsilon^{\frac{1}{2}}. \tag{2.22}$$

Therefore by our bounds at (2.20), (2.21), and (2.22) we see that (2.19) is at most

$$c\varepsilon^{\frac{1}{2}}\Sigma_{j=2}^{K-2}e^{-\ln_j n}. \tag{2.23}$$

But for $j \leq K-2$ we have

$$\ln_j n = e^{\ln_{j+1} n} > (\ln_{j+1} n)^2 > e \ln_{j+1} n$$

since $\ln_{j+1} n \geq \ln_{K-1} n > e$. Iterating the above relation yields

$$\ln_j n > e^{K-1-j} \ln_{K-1} n > e^{K-j}.$$

Therefore we have that (2.23) is at most

$$\begin{aligned} c\varepsilon^{\frac{1}{2}}\Sigma_{j=2}^{K-2} \exp\{-e^{K-j}\} &\leq c\varepsilon^{\frac{1}{2}}\Sigma_{j=0}^{\infty} \exp\{-e^{j}\} \\ &\leq c\varepsilon^{\frac{1}{2}}. \end{aligned}$$

Since $\varepsilon$ was arbitrary, Theorem 2.1 now follows since $\Phi^n(b_n + x/c_n) \to \exp\{-e^{-x}\}$ as $n \to \infty$.

COROLLARY 2.1. *Suppose the hypothesis of Theorem* 2.1 *holds and that* $(r_n \ln n)^{-1} = o(1)$. *Then*

$$\lim_{n\to\infty} P\left\{r_n^{-\frac{1}{2}}\left(\frac{M_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) \leq x\right\} = (1-\gamma/2\pi)^{\frac{1}{2}}\textstyle\int_{-\infty}^{x} e^{-((1-\gamma)/2)u^2}\,du \tag{2.24}$$

*where* $\gamma = F(\{o\})$ *is the atom at zero of the spectral distribution associated with* $r$.

PROOF.

$$r_n^{-\frac{1}{2}}\left(\frac{M_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) = \left(r_n^{\frac{1}{2}}c_n\right)^{-1}c_n\left[\frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right] + (1-r_n)^{-\frac{1}{2}}r_n^{-\frac{1}{2}}\overline{X}_n.$$

Now the first term in the sum above converges in probability to zero by Theorem 2.1 and the assumption $(r_n \ln n)^{-1} = o(1)$ and since the variance of the second term tends to $(1-\gamma)^{-1}$ by (2.6), the corollary follows.

Let us note that the above corollary contains the proposition at (1.2) as a special case with $\gamma = o$, since the conditions for that result imply $o \leq r_k - r_n \leq r_n((\ln n/\ln k) - 1)$ for all sufficiently large $k \leq n$ which in turn implies (2.3) and hence (2.4).

From the preceding results it is clear that the factor $1 - r_n$ appearing in the normalization represents the reduction in variance of the $X_k$ variables by the subtraction of the sample mean. Consequently, it is natural to suggest the sample variance of $X_k - \overline{X}_n$ variables as an estimate for $1 - r_n$. That this estimate is appropriately close to $1 - r_n$ is the content of the next lemma.

LEMMA 2.2. *Under the hypothesis of Theorem* 2.1 *if* $s_n^2 = \frac{1}{n}\Sigma_{k=1}^{n}(X_k - \overline{X}_n)^2$,

$$\ln n\left[s_n^2 - (1-r_n)\right] \to_{L^2} o. \tag{2.25}$$

**Proof.** Let $\varepsilon > 0$ be arbitrary. Then for $n$ sufficiently large we have

$$\begin{aligned}
&(\ln n)^2 E\{s_n^2 - (1 - r_n)\}^2 \\
&\quad = (\ln n)^2 E\left\{\frac{1}{n}\Sigma_{k=1}^n\left[\left(X_k - \bar{X}_n\right)^2 - \sigma_n^2(k)\right] + \frac{1}{n}\Sigma_{k=1}^n\left[\sigma_n^2(k) - (1 - r_n)\right]\right\}^2 \\
&\quad \leqslant \varepsilon + \frac{(\ln n)^2}{n^2}\Sigma_{j=1}^n\Sigma_{k=1}^n E\left[\left(X_j - \bar{X}_n\right)^2 - \sigma_n^2(j)\right]\left[\left(X_k - \bar{X}_n\right)^2 - \sigma_n^2(k)\right] \\
&\quad = \varepsilon + \frac{(\ln n)^2}{n^2}\Sigma_{j=1}^n\Sigma_{k=1}^n 2\sigma_n^2(j)\sigma_n^2(k)\rho_n^2(j, k) \\
&\quad \leqslant \varepsilon + c\frac{(\ln n)^2}{n}\Sigma_{k=1}^n\bar{\rho}_n^2(k) \\
&\quad \leqslant 2\varepsilon + c\frac{(\ln n)^2}{n}\Sigma_{k=1}^n\bar{r}_n^2(k) \\
&\quad \leqslant 3\varepsilon + c\varepsilon\frac{(\ln n)^2}{n}\Sigma_{k=2}^n\left(\frac{\ln n}{\ln k} - 1\right)^2 \\
&\quad \leqslant 3\varepsilon + c\varepsilon\frac{(\ln n)^2}{n}\Sigma_{k=2}^n\frac{1}{(\ln k)^2}\left[\Sigma_{j=k}^{n-1}\ln(j + 1) - \ln j\right]^2 \\
&\quad \leqslant 3\varepsilon + c\varepsilon\frac{(\ln n)^2}{n}\Sigma_{k=2}^n\frac{1}{(\ln k)^2}\left[\Sigma_{j=k}^n\frac{1}{j}\right]^2 \\
&\quad \leqslant 3\varepsilon + c\varepsilon\frac{(\ln n)^2}{n}\Sigma_{k=2}^n\frac{1}{(\ln k)^2}\Sigma_{j=k}^n\Sigma_{l=k}^j\frac{1}{lj} \\
&\quad \leqslant 3\varepsilon + c\varepsilon\frac{(\ln n)^2}{n}\Sigma_{j=2}^n\frac{1}{j}\Sigma_{l=2}^j\frac{1}{l}\Sigma_{k=2}^l\frac{1}{(\ln k)^2} \\
&\quad \leqslant c\varepsilon
\end{aligned}$$

where $c$ is a constant that may vary from line to line. Hence the lemma follows since $\varepsilon$ was arbitrary.

**Theorem 2.2.** *Under the hypothesis of Theorem 2.1 we have*

$$\lim_{n\to\infty} P\left\{c_n\left(\frac{M_n - \bar{X}_n}{s_n} - b_n\right) \leqslant x\right\} = \wedge(x) = \exp\{-e^{-x}\}, \tag{2.26}$$

$$-\infty < x < \infty.$$

**Proof.** By Lemma 2.2 and (2.7) we have $(1 - r_n)^{\frac{1}{2}}/s_n \to_P 1$ and with Theorem 2,1, this implies

$$c_n\left(\frac{M_n - \bar{X}_n}{s_n} - \frac{(1 - r_n)^{\frac{1}{2}}}{s_n}b_n\right) \to_{\mathfrak{D}} \wedge. \tag{2.27}$$

Thus since

$$c_n\left(\frac{M_n - \overline{X}_n}{s_n} - b_n\right) = c_n\left(\frac{M_n - \overline{X}_n}{s_n} - \frac{(1 - r_n)^{\frac{1}{2}}}{s_n} b_n\right) + \left(\frac{(1 - r_n)^{\frac{1}{2}}}{s_n} - 1\right) b_n c_n$$

and the second term in the sum above converges in probability to zero by Lemma 2.2, the theorem follows by (2.27).

**3. Limit distribution for $(M_T - \overline{X}_T)/s_T$.** Let $\{X_t; o \leqslant t < \infty\}$ be a stationary Gaussian process with $EX_t = o$, $EX_t^2 = 1$, and with correlation function $r(t)$ satisfying

(3.1) $$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$$

for $t$ in a neighborhood of zero where $C$ is a positive constant and $o < \alpha \leqslant 2$. We will now present statements of the preliminary results needed for the proof of Theorem 3.1. For a continuous function $f$, define $\bar{f}_T(t) = \max_{t \leqslant s \leqslant T} |f(s) - f(T)|$. Then with obvious modifications of the proof given for discrete time we have

LEMMA 3.1. *The following conditions are equivalent*

(3.2) $$\frac{\ln T}{T} \int_0^T |f(t) - f(T)|\, dt = o(1)$$

(3.3) $$\frac{\ln T}{T} \int_0^T \bar{f}_T(t)\, dt = o(1).$$

*For any* $\varepsilon > 0$ *and all* $T$ *sufficiently large*

(3.4) $$|f_t - f_T| < \varepsilon \max\left(\frac{\ln T}{\ln t} - 1, \frac{1}{\ln T}\right), \qquad 1 \leqslant t \leqslant T.$$

In view of (3.1) we may take a version of $\mathbf{X}$ having continuous sample paths and may define random variables

$$M_T = \max_{0 \leqslant t \leqslant T} X_t, \quad \overline{X}_T = \frac{1}{T} \int_0^T X_t\, dt, \text{ and } s_T^2 = \frac{1}{T} \int_0^T \left(X_t - \overline{X}_T\right)^2 dt.$$

Further the spectral representation theorem allows us to write $X_t = \int_{-\infty}^{\infty} e^{i\lambda t} Z(d\lambda)$ where $Z$ is the random spectral measure and the ergodic theorem for wide sense stationary processes yields $\overline{X}_T \to_{L^2} Z(\{o\})$ so that as in the discrete case we have

(3.5) $$r_t - \gamma = o(1)$$

where $\gamma = E[Z(\{o\})]^2 = F(\{o\})$, the atom at zero of the spectral distribution associated with $r$. By (3.1) $r$ is not identically one so that $o \leqslant \gamma < 1$ and stationarity then implies that for any $\varepsilon > o$

(3.6) $$\sup_{t \geqslant \varepsilon} |r_t| = \delta = \delta(\varepsilon) < 1.$$

THEOREM 3.1. *Let* $\{X_t; o \leqslant t < \infty\}$ *be a separable stationary Gaussian process with* $EX_t = o$, $EX_t^2 = 1$ *and* $EX_0 X_t = r(t)$ *satisfying* (3.1) *and* (3.2). *Then*

(3.7) $$\lim_{T \to \infty} P\left\{ c_T\left(\frac{M_T - \overline{X}_T}{s_T} - \beta_T\right) \leqslant x \right\} = \exp\{-e^{-x}\}, \qquad -\infty < x < \infty$$

*where* $c_T = (2 \ln T)^{\frac{1}{2}}$ *and*

$$\beta_T = c_T + \frac{1}{c_T}\left\{\left(\frac{1}{\alpha} - \frac{1}{2}\right)\ln \ln T + \ln\left((2\pi)^{-\frac{1}{2}}\left(\frac{C}{1-\gamma}\right)^{1/\alpha} H_\alpha 2^{((2-\alpha)/2)}\right)\right\}$$

*where* $H_\alpha$ *is a positive finite constant that is determined in* [5] *and where* $\gamma = F(\{o\})$, *the atom at zero of the spectral distribution associated with* $r$.

PROOF. Define a process $Y_T(t) = \frac{1}{\sigma_T(t)}(X_t - \bar{X}_T)$, $o \leqslant t \leqslant T$ where $\sigma_T^2(t) = E(X_t - \bar{X}_T)^2$. Let $\rho_T(s, t) = EY_T(s)Y_T(t)$ and $\bar{\rho}_T(u) = \max\{|\rho_T(s, t)|: o \leqslant s < t \leqslant T, t - s \geqslant u\}$. Then it can be shown that

$$\max_{o \leqslant s, t \leqslant T}\left|\rho_T(s, t) - \frac{r(s-t) - r(T)}{1 - r(T)}\right| = o\left(\frac{1}{\ln T}\right). \tag{3.8}$$

Furthermore from (3.5), (3.6) and (3.8) it follows that for any $\varepsilon > o$

$$\bar{\rho}_T(\varepsilon) = \eta = \eta(\varepsilon) < 1 \tag{3.9}$$

for all $T$ sufficiently large. Also it can be shown that for any $\varepsilon > o$ there exists $\tau = \tau(\varepsilon) > o$ such that for all $T$ sufficiently large

$$\frac{1-\varepsilon}{1-\gamma}C|s-t|^\alpha < 1 - \rho_T(s, t) < \frac{1+\varepsilon}{1-\gamma}C|s-t|^\alpha \quad \text{when} \quad |s-t| \leqslant \tau. \tag{3.10}$$

Setting $M_T^* = \max_{0 \leqslant t \leqslant T} Y_T(t)$ we have as in the discrete case, that to show (3.7) it suffices to show

$$\lim_{T\to\infty} P\{c_T(M_T^* - \beta_T) \leqslant x\} = \exp\{-e^{-x}\}, \quad -\infty < x < \infty. \tag{3.11}$$

To prove (3.11) we follow the method established by Pickands [5]. For $o < a < \tau$ a fixed constant define $I_k = [k\tau + a, (k+1)\tau]$ and $I = I(T) = \cup_k I_k \cap [o, T]$. Then setting $u_T = \beta_T + x/c_T$ we have

$$\begin{aligned} o &\leqslant P\{\max_{t\in I} Y_T(t) \leqslant u_T\} - P\{\max_{0\leqslant t\leqslant T} Y_T(t) \leqslant u_T\} \\ &\leqslant \sum_{k=0}^{\left[\frac{T-a}{\tau}\right]} P\{\max_{k\tau \leqslant t \leqslant k\tau + a} Y_T(t) > u_T\} + P\{\max_{[(T-a)/\tau]\tau + a \leqslant t \leqslant T} Y_T(t) > u_T\}. \end{aligned} \tag{3.12}$$

Now let $\xi_1(t)$ be a separable stationary Gaussian process with correlation function $\tilde{r}_1(t)$ such that for $|t| \leqslant \tau$, $\tilde{r}_1(t) = 1 - \frac{1+\varepsilon}{1-\gamma}C|t|^\alpha$. Then by (3.10) and Slepian's lemma [6] the sum appearing at (3.12) is at most

$$\left(\frac{T}{\tau} + 1\right)P\{\max_{0\leqslant t \leqslant a} \xi_1(t) > u_T\} = 0(a) \text{ as } T \to \infty \tag{3.13}$$

where we have used Lemma 2.9 of [5] and the fact that

$$u_T^{2/\alpha}\psi(u_T) \sim (\text{CONST.})\frac{1}{T} \qquad \text{as } T \to \infty$$

where $\psi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}x}e^{-x^2/2}$.

Next let $G = G(T, a) = \{s \in I: s = aku_T^{-2/\alpha}, k = o, 1, 2, \cdots\}$. Then

$$o \leqslant P\{\max_{s\in G} Y_T(s) \leqslant u_T\} - P\{\max_{s\in I} Y_T(s) \leqslant u_T\}$$
$$\leqslant \Sigma_{k=0}^{[T/au_T^{2/\alpha}]} P\left\{ Y_T\left(kau_T^{-2/\alpha}\right) \leqslant u_T, \max_{kau_T^{-2/\alpha} \leqslant t \leqslant (k+1)au_T^{-2/\alpha}} Y_T(t) > u_T\right\}$$
$$\leqslant \left(T/au_T^{2/\alpha} + 1\right)P\left\{\xi_1(o) \leqslant u_T, \max_{0 \leqslant t \leqslant au_T^{-2/\alpha}}\xi_1(t) > u_T\right\} \tag{3.14}$$
$$= 0\left(\frac{M(a)}{a}\right) \quad \text{as} \quad T \to \infty$$

where we have used Slepian's lemma and Lemma 2.8 of [5]; and where $M(a)$ is defined in [5] and is such that $\dfrac{M(a)}{a} = o(1)$ as $a \to o$.

Now define a separable Gaussian process $\zeta(t)$ which on each interval $[k\tau, (k + 1)\tau)$ has covariance function $\rho_T$ and such that the variables $\zeta(t)$, $t \in [k\tau, (k + 1)\tau)$ and $\zeta(s)$, $s \in [j\tau, (j + 1)\tau)$ are independent if $k \neq j$. Then by Berman's lemma we have

$$|P\{\max_{s\in G}\zeta(s) \leqslant u_T\} - P\{\max_{s\in G} Y_T(s) \leqslant u_T\}|$$
$$\leqslant c\Sigma_{s,\, t\in G,\, |s-t|\geqslant a}|\rho_T(s, t)| \exp\left\{-\frac{u_T^2}{1 + |\rho_T(s, t)|}\right\}$$
$$\leqslant \frac{cTu_T^{4/\alpha}}{a^2}\bar{\rho}_T(a)\exp\left\{-\frac{u_T^2}{1 + \bar{\rho}_T(a)}\right\} \tag{3.15}$$
$$+ c\frac{Tu_T^{4/\alpha}}{a^2}\Sigma_{k=1}^n\bar{\rho}_T(k)\exp\left\{-\frac{u_T^2}{1 + \bar{\rho}_T(k)}\right\}$$
$$= o(1).$$

For $T$ sufficiently large we have by (3.9) that $\bar{\rho}_T(a) < 1$ so that (3.4) and (3.8) we may follow the procedure given in discrete time to obtain that the expression at (3.15) is $o(1)$.

Consider now

$$P\{\max_{s\in G}\zeta(s) \leqslant u_T\} \tag{3.16}$$
$$= \Pi_{k=0}^{\left[\frac{T-a}{\tau}\right]-1}P\{\max_{s\in G\cap I_k} Y_T(s) \leqslant u_T\}P\{\max_{s\in G\cap[\tau[(T-a)/\tau]+a,\, T]} Y_T(s) \leqslant u_T\}.$$

Next letting $\xi_2(t)$ be a separable stationary Gaussian process with correlation function $\tilde{r}_2(t)$ such that for $|t| \leqslant \tau$, $\tilde{r}_2(t) = 1 - (1 - \varepsilon)/(1 - \gamma)c|t|^\alpha$ and letting $\xi_1(t)$ be as before, then we have by (3.10) and (3.16)

$$P^{(T/\tau)}\{\max_{s\in G\cap I_0}\xi_1(s) \leqslant u_T\} \leqslant P\{\max_{s\in G}\zeta(s) \leqslant u_T\} \tag{3.17}$$
$$\leqslant P^{(T/\tau-2)}\{\max_{s\in G\cap I_0}\xi_2(s) \leqslant u_T\}.$$

Now by Lemma 2.5 of [5] we have

$$P\{\max_{s\in G\cap I_0}\xi_1(s) > u_T\} \sim \frac{H_\alpha(a)}{aH_\alpha}(1 + \varepsilon)^{1/\alpha}\frac{(\tau - a)}{T}e^{-x}$$

and

$$P\{\max_{s\in G\cap I_0}\xi_2(s) > u_T\} \sim \frac{H_\alpha(a)}{aH_\alpha}(1-\varepsilon)^{1/\alpha}\frac{(\tau-a)}{T}e^{-x}$$

Therefore by (3.17) and the above we have

$$\exp\left\{-\frac{H_\alpha(a)}{aH_\alpha}(1+\varepsilon)^{1/\alpha}\frac{(\tau-a)}{\tau}e^{-x}\right\} \leqslant P\{\max_{s\in G}\zeta(s) \leqslant u_T\}$$

$$\leqslant \exp\left\{-\frac{H_\alpha(a)}{aH_\alpha}(1-\varepsilon)^{1/\alpha}\frac{(\tau-a)}{\tau}e^{-x}\right\}. \tag{3.18}$$

Since $\lim_{a\to o}\frac{H_\alpha(a)}{a} = H_\alpha$, upon letting $a \to o$ at (3.18) we have

$$\exp\{-(1+\varepsilon)^{1/\alpha}e^{-x}\} \leqslant P\{\max_{s\in G}\zeta(s) \leqslant u_T\} \leqslant \exp\{-(1-\varepsilon)^{1/\alpha}e^{-x}\}. \tag{3.19}$$

Thus by (3.12), (3.13), (3.14), (3.15) and (3.19) we have

$$\exp\{-(1+\varepsilon)^{1/\alpha}e^{-x}\} \leqslant P\{c_T(M_T^* - \beta_T) \leqslant x\} \leqslant \exp\{-(1-\varepsilon)^{1/\alpha}e^{-x}\}. \tag{3.20}$$

Since $\varepsilon$ is arbitrary, Theorem 3.1 follows.

Finally by the same argument given in Section 2 we have

COROLLARY 3.1. *Suppose the hypothesis of Theorem* 3.1 *holds and that* $(r_T \ln T)^{-1} = o(1)$. *Then*

$$\lim_{T\to\infty} P\left\{r_T^{-\frac{1}{2}}\left(\frac{M_T}{(1-r_T)^{\frac{1}{2}}} - \beta_T\right) \leqslant x\right\} = \left(\frac{1-\gamma}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{x} e^{-\frac{(1-\gamma)}{2}u^2}\,du,$$

$$-\infty < x < \infty$$

*where* $\gamma = F(\{o\})$ *is the atom at zero of the spectral distribution associated with* $r$.

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