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COMPACTIFICATIONS FOR DUAL PROCESSES[1]

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We develop a general theory of duality for Markov processes satisfying Meyer's hypothesis (L) and possessing an excessive reference measure. We make use of a compactification introduced by Walsh which allows a right process and its moderate dual to have strong Markov versions on an enlarged state space. The representation theory for potentials of additive functionals due to Revuz and Sharpe can be extended to this setting. Using this theory, we show that the conatural additive functionals introduced by Garcia-Alvarez are natural additive functionals in the new topology. A general version of Motoo's theorem is given, and the Getoor-Sharpe approach to capacities is extended to this situation. Finally, we show that if the original process satisfies Hunt's hypothesis (H), then a version of the bounded maximum principle holds on the compactified space.

1. Introduction. Meyer and Garcia-Alvarez introduced a compactification in [6] in order to develop a theory of duality for processes satisfying hypothesis (L) and possessing an excessive reference measure. Garcia-Alvarez studied this duality in a subsequent paper [5]. We discuss an alternate compactification in order to simplify and complete some of their work.

In particular, we make use of a compactification introduced by Walsh [18], which allows a right process and its moderate dual to have strong Markov versions on an enlarged state space. Beginning with a process having finite lifetime and a σ-finite excessive reference measure, we construct the compactification in Sections 2 through 6. We have tried to give a "canonical" procedure for this similar to that given in [7]. Essentially, we introduce a cofine topology, and, in Sections 6 through 12, the resulting duality theory is explored. Walsh and Weil have used a similar compactification to represent terminal times [20]. In a slightly different setting, Smythe and Walsh [17] have shown that a version of Hunt's switching identity holds. These results are valid here, and we make use of them.

One major advantage of this approach is that we may completely dispense with the difficult theorem of Mokobodzki used by Garcia-Alvarez and Meyer, which they characterized as "un des progrès majeurs de ces dernières années, que tout noyau de théorie du potentiel possédant une mesure de base admet les mêmes fonctions excessives qu'un noyau compact" [6]. We discuss in Section 5 an example which illustrates the difference between the Garcia-Alvarez-Meyer approach and the compactification presented here. In short, their approach does not yield a complete duality on one state space: E may not embed naturally into the compactification $\bar{E}$. Thus, Garcia-Alavarez was unable to characterize completely the conatural additive functionals (which we can identify with the natural additive functionals).

Section 7 is devoted to extending the work of Revuz and Sharpe to this setting. Most of their results extend without difficulty, and we give a general version of Motoo's theorem which complements a result of Azéma [1] for processes satisfying hypothesis (L). In Section 8, we complete the characterization of conatural additive functionals in the new topology of $\bar{E}$. In Section 9 we give a proof of a technical result in probabilistic potential theory. We use it in Section 10 to show that if the original process X satisfies Hunt's hypothesis (H), then a version

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of the bounded maximum principle holds. This extends a result of Smythe [16]. (In fact, the compactification procedure may suggest why the results in [16] are true.) In Sections 11 and 12 we make some comments on capacities and representation of excessive functions. As a general theme, we have used the Revuz measure whenever possible. This approach does not seem to give the most general result for representation of excessive functions, however (cf. [6]). The approach of Garcia-Alvarez based on h-path transforms seems to be more powerful for this purpose.

The notation will be that of [2] and [7]: $b\bar{\mathscr{E}}$ denotes the bounded Borel functions on $\bar{E}$, $(\bar{\mathscr{E}} \times \bar{\mathscr{E}})_+$ the positive Borel functions on $\bar{E} \times \bar{E}$, $C(E)$ the continuous functions on E, etc.

2. The moderate Markov dual. A topological space E is a Lusin space if it is homeomorphic to a Borel subset of a compact metric space. Let $(E, \mathscr{E})$ be a Lusin topological space (containing an isolated point Δ to act as a cemetery for the process) together with its Borel field, and let

$$X = (\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, (P^x)_{x\in E}, (P_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0})$$

be a Borel right process on E with resolvent $(U^\alpha)_{\alpha>0}$. Recall that this means that X satisfies the following hypotheses ([7], pages 50–53):

HD1. For each probability μ on $(E, \mathscr{E})$, $(X_t, \mathscr{F}_t, P^\mu)$ is a right continuous Markov process with initial measure μ and transition function P_t: $b\mathscr{E} \to b\mathscr{E}$; that is,

$$P^\mu\{X_0 \in A\} = \mu(A) \qquad \text{for each} \quad A \in \mathscr{E}. \tag{2.1}$$

$$E^\mu\{f(X_{t+s}) \mid \mathscr{F}_s\} = P_t f(X_s) \qquad \text{for all} \quad t, s \geq 0 \quad \text{and} \quad f \in b\mathscr{E}. \tag{2.2}$$

HD2. Let f be an α-excessive function. For each probability μ on $(E, \mathscr{E})$, $t \to f(X_t)$ is right continuous almost surely P^μ.

It follows from HD1 and HD2 that X is a strong Markov process. We assume that X satisfies Meyer's hypothesis (L) (i.e., X has a reference measure) and that the lifetime ζ is finite P^x a.s. for all x in E. Without loss of generality, we assume $U(x, E) \leq 1$ for all x in E [6].

If $(Y_t)_{t\geq 0}$ is a Markov process with Borel transition semigroup $(Q_t)_{t\geq 0}$ and filtration $(\mathscr{G}_t)_{t\geq 0}$, then Y is said to be a moderate Markov process if the following statement is true.

(2.3) If T is a previsible stopping time for the filtration $(\mathscr{G}_t)_{t\geq 0}$, then $E\{f(Y_{T+t}) \mid \mathscr{G}_{T-}\} = Q_t f(X_{T-})$ for each $f \in b\mathscr{E}$.

We now recall the Smythe-Walsh construction of the moderate Markov dual for X [17]. Fix λ a reference probability measure for X. Define X_t^* for $t > 0$ by

$$X_t^*(\omega) \begin{array}{ll} = X_{\zeta(\omega)-t}(\omega) & 0 < t \leq \zeta(\omega) \\ = \Delta & t > \zeta(\omega) \end{array}$$

and set $\mathscr{G}_t = \sigma(X_s^* : s \leq t)$. Then there is a transition semigroup $Q_t : b\mathscr{E} \to b\mathscr{E}$ so that $X^* = (\Omega, (\mathscr{G}_t)_{t>0}, P^\lambda, (X_t^*)_{t>0})$ is a left continuous moderate Markov process [10, 17]. We denote by $\hat{X}$ the canonical moderate Markov realization of $(Q_t)_{t>0}$ on the space of left continuous paths in E [10]. Since $U(x, E) \leq 1$, Theorem (1.1) in [17] states that the measure $\nu = \lambda U$ is a σ-finite excessive reference measure on $(E, \mathscr{E})$. Moreover, X and $\hat{X}$ are in weak duality with respect to ν: for all $f, g \in b\mathscr{E}$ and for all $t > 0$.

$$\int_E f(x) P_t g(x)\, \nu(dx) = \int_E Q_t f(x)\, g(x)\, \nu(dx)$$

or, by integrating,

$$\int_E f(x) U^\alpha g(x)\, \nu(dx) = \int_E V^\alpha f(x)\, g(x)\, \nu(dx),$$

where $(V^\alpha)_{\alpha>0}$ denotes the resolvent associated with the semigroup Q_t. More generally, Smythe and Walsh construct in [17] a moderate Markov dual for any process having a σ-finite excessive reference measure.

It will often be convenient to discuss X_t^* governed by the measure P^ν. For statements true about X_t^* under P^ν imply results about X_t under P^ν (after all, X_t^* is just the reverse of X_t), and one can often argue that the statement must then be true for X_t P^x a.s. for all x (see, e.g., Proposition (4.1)).

3. The double Ray cone. We now construct the double Ray cone of functions on E introduced by Walsh [18]. We use the cone to define a new metric on E. After completing E in the new metric, we extend the resolvents U^α and V^α to be Ray resolvents on the completion.

Our construction of the double Ray cone is slightly different from the one given by Walsh, in that we do not pass to a quotient topology to recover the processes. We avoid it by adding enough 1-supermedian functions to the cone to separate points in E. This ensures that V^α extends to be a Ray resolvent on the completion.

The result of the double Ray cone may be disconcerting to the expert at first. The compactification obtained is by no means a "minimal" one. Degenerate branch points are, in fact, frequently introduced. (Recall that a point x is a degenerate branch point for a semigroup $(R_t)_{t\geq 0}$ if $R_t(x, dz) = \epsilon_y(dz)$ where $x \neq y$.) We shall try to point these out with some examples after the construction.

We shall refer to the notes of Getoor [7] whenever arguments are similar.

If $\mathbf{H}$ is any convex cone contained in $b\mathscr{E}_+$, define $\mathscr{U}(\mathbf{H})$, $\mathscr{V}(\mathbf{H})$ and $\Lambda(\mathbf{H})$ by

$$\mathscr{U}(\mathbf{H}) = \{U^{\alpha_1}f_1 + \cdots + U^{\alpha_n}f_n;\quad \alpha_j > 0,\ f_j \in \mathbf{H},\ 1 \leq j \leq n,\ n \geq 1\}$$

$$\mathscr{V}(\mathbf{H}) = \{V^{\alpha_1}f_1 + \cdots + V^{\alpha_n}f_n;\quad \alpha_j > 0,\ f_j \in \mathbf{H},\ 1 \leq j \leq n,\ n \geq 1\}$$

$$\Lambda(\mathbf{H}) = \{f_1 \wedge \cdots \wedge f_n;\ f_j \in \mathbf{H},\ 1 \leq n\}.$$

The following facts are collected in [7].

(3.1). LEMMA.

(i) $\Lambda(\mathbf{H})$ *is closed under pointwise minima.*

(ii) $\mathscr{U}(\mathbf{H})$, $\mathscr{V}(\mathbf{H})$ *and* $\Lambda(\mathbf{H})$*are convex cones contained in* $b\mathscr{E}_+$.

(iii) *If* $\mathbf{H}$ *is separable for the uniform topology on E, then so are* $\mathscr{U}(\mathbf{H})$, $\mathscr{V}(\mathbf{H})$ *and* $\Lambda(\mathbf{H})$.

The state space E is a Borel subset of a compact metric space $(\hat{E}, d)$. Let $\mathbf{C}_u$ denote the restrictions to E of the continuous functions on $\hat{E}$ which vanish at Δ. Since $(Q_t)_{t>0}$ is a moderate Markov semigroup, we can find a sequence (h_n) of bounded Borel measurable 1-supermedian functions for the resolvent $(V^\alpha)_{\alpha>0}$ which separate points in E and which are such that $h_n(X_t^*)$ is left continuous P^ν a.s. for each n [19]. Since $e^{-t}h_n(X_t^*)$ is a P^ν supermartingale for each n, $h_n(X_t^*)$ has right limits P^ν a.s. We take $h_1 = 1_E$; it is excessive for X in addition to being supermedian for $\hat{X}$.

Let $\mathbf{H}$ be the smallest positive cone containing the functions (h_n) and define

$$\begin{aligned}
\mathbf{R}_{-1} &= \mathscr{U}(\mathbf{C}_u^+) + \mathscr{V}(\mathbf{C}_u^+)\\
\mathbf{R}_0 &= \mathbf{R}_{-1} + \mathbf{H}\\
\mathbf{R}_1 &= \Lambda(\mathscr{U}(\mathbf{R}_0) + \mathscr{V}(\mathbf{R}_0) + \mathbf{R}_0)\\
&\ \vdots\\
\mathbf{R}_{n+1} &= \Lambda(\mathscr{U}(\mathbf{R}_n) + \mathscr{V}(\mathbf{R}_n) + \mathbf{R}_n)\\
&\ \vdots\\
\mathbf{R} &= \cup_{n\geq 0}\mathbf{R}_n.
\end{aligned}$$

The following proposition is easily derived from Lemma (3.1) and the fact that (U^α) and (V^α) are Borel resolvents ([7], pages 57–59).

(3.2). PROPOSITION. $\mathbf{R}$ *is a convex cone of bounded positive functions on E such that*

(i) $\mathbf{R} \subset b\mathscr{E}_+$

(ii) $U^\alpha\mathbf{R} \subset \mathbf{R}$, $V^\alpha\mathbf{R} \subset \mathbf{R}$ $\quad\forall\alpha > 0$.

(iii) f, $g \in \mathbf{R}$ *implies* $f \wedge g \in \mathbf{R}$.
(iv) $\mathbf{R}$ *is separable in the uniform topology.*
(v) $\mathbf{R}$ *separates points in* E.

We briefly sketch the construction of the Ray-Knight compactification of E with respect to $\mathbf{R}$. See [7], pages 59–62.

Fix a countable dense subset $(g_j)_{j\geq 1}$ in $\mathbf{R}$. These separate points in E. Let K be the compact metrizable space $\Pi[0, \|g_j\|]$ ($\|g_j\| = \sup\{|g_j(x)| : x \in E\}$) together with the metric $\bar{d}$ (compatible with the product topology) defined by

$$\bar{d}(x, y) = \sum_{j\geq 1} 2^{-j} \frac{|x_j - y_j|}{1 + |x_j - y_j|}$$

for $x = (x_j) \in K$, $y = (y_j) \in K$. Since the (g_j) separate points, we may define an injective map $\Phi : E \to K$ by $\Phi(x) = (g_j(x))_{j\geq 1}$. For $x, y \in E$, $\rho(x, y) = \bar{d}[\Phi(x), \Phi(y)]$ is a metric on E, and Φ is an isometry of (E, ρ) onto $(\Phi(E), \bar{d})$. Φ then extends to an isometry from the completion $\bar{E}$ of E onto F, the closure of $\Phi(E)$ in K. The space E is therefore densely embedded in the compact metric space $(\bar{E}, \rho)$. The topology induced by ρ on $\bar{E}$ will be called the D-Ray topology.

Each g_j on E has a uniformly continuous extension $\bar{g}_j$ to $\bar{E}$. In general, the extension of a ρ-uniformly continuous function g to $\bar{E}$ is denoted by $\bar{g}$. The following are simple consequences of the construction.

(3.3). Proposition.
(i) *If* $\bar{x}, \bar{y} \in \bar{E}$, $\rho(\bar{x}, \bar{y}) = \sum_{j\geq 1} 2^{-j}(|\bar{g}_j(\bar{x}) - \bar{g}_j(\bar{y})|/(1 + |\bar{g}_j(\bar{x}) - \bar{g}_j(\bar{y})|))$.
(ii) *Let* $\bar{f} \in \mathbf{C}(\bar{E})$. *Then* $\bar{f} \in \bar{\mathbf{R}}$ *if and only if* $\bar{f}|_E \in \mathbf{R}$.
(iii) $\bar{\mathbf{R}} - \bar{\mathbf{R}}$ *is a vector space of continuous functions on* $\bar{E}$ *dense in* $\mathbf{C}(\bar{E})$.

We now extend the resolvents $(U^\alpha)_{\alpha>0}$ and $(V^\alpha)_{\alpha>0}$ to $\bar{E}$. If $f \in \mathbf{R}$ and $\alpha > 0$, then $U^\alpha f$ and $V^\alpha f$ are in $\mathbf{R}$, so $\overline{U^\alpha f}$ and $\overline{V^\alpha f}$ are in $\bar{\mathbf{R}}$. Define $\bar{U}^\alpha$ and $\bar{V}^\alpha$ on $\bar{\mathbf{R}}$ by

$$\bar{U}^\alpha \bar{f} = \overline{U^\alpha f}; \quad f = \bar{f}|_E$$
$$\bar{V}^\alpha \bar{f} = \overline{V^\alpha f}; \quad f = \bar{f}|_E.$$

$(\bar{U}^\alpha)$ and $(\bar{V}^\alpha)$ extend to well-defined positive linear maps from $\mathbf{C}(\bar{E})$ to $\mathbf{C}(\bar{E})$. By the Riesz representation theorem, for each $\alpha > 0$, there are kernels $\bar{U}^\alpha(x, \cdot)$ and $\bar{V}^\alpha(x, \cdot)$ on $(\bar{E}, \bar{\mathscr{E}})$ such that for all $\bar{f} \in \mathbf{C}(\bar{E})$,

$$\bar{U}^\alpha \bar{f}(x) = \int \bar{U}^\alpha(x, dy)\bar{f}(y)$$
$$\bar{V}^\alpha \bar{f}(x) = \int \bar{V}^\alpha(x, dy)\bar{f}(y).$$

Also, $\alpha\bar{U}^\alpha$ and $\alpha\bar{V}^\alpha$ are sub-Markov kernels for each $\alpha > 0$.

If $(R^\alpha)_{\alpha>0}$ is a resolvent on a compact metric space F, let $\mathbf{S}^\alpha(F)$ be the cone of continuous α-supermedian functions on F. Recall that $(R^\alpha)_{\alpha>0}$ is said to be a Ray resolvent if $R^\alpha\mathbf{C}(F) \subset \mathbf{C}(F)$ for each $\alpha > 0$ and if $\mathbf{S} = \cup_\alpha \mathbf{S}^\alpha$ separates points in F. Such a resolvent has a right continuous strong Markov process (called a Ray process) associated with it ([7], Chapters 3–5).

(3.4). Proposition. *The families* $(\bar{U}^\alpha)_{\alpha>0}$ *and* $(\bar{V}^\alpha)_{\alpha>0}$ *are Ray resolvents on the compact metric space* $\bar{E}$.

Remark. The proof for $(\bar{U}^\alpha)_{\alpha>0}$ is as in [7]. We sketch it here.

Proof. $(\bar{U}^\alpha)_{\alpha>0}$ and $(\bar{V}^\alpha)_{\alpha>0}$ map $\mathbf{C}(\bar{E})$ into $\mathbf{C}(\bar{E})$ by construction. First, we check that

$(\bar{U}^\alpha)_{\alpha>0}$ and $(\bar{V}^\alpha)_{\alpha>0}$ satisfy the resolvent equation. If $\bar{f} \in \bar{\mathbf{R}}, f = \bar{f}|_E$,

$$\bar{U}^\alpha \bar{f} - \bar{U}^\beta \bar{f} = \overline{U^\alpha f - U^\beta f} = (\beta - \alpha)\overline{U^\alpha U^\beta f}.$$

But $U^\beta f \in \mathbf{R}$ and so by definition,

$$\overline{U^\alpha U^\beta f} = \bar{U}^\alpha(\overline{U^\beta f}) = \bar{U}^\alpha(\bar{U}^\beta \bar{f}).$$

The same holds for (V^α). This extends to $\bar{\mathbf{R}} - \bar{\mathbf{R}}$ by linearity and then to $\mathbf{C}(\bar{E})$ by continuity. Thus $(\bar{U}^\alpha)_{\alpha>0}$ and $(\bar{V}^\alpha)_{\alpha>0}$ are sub-Markov resolvents on $(\bar{E}, \bar{\mathscr{E}})$.

Since $\alpha U^\alpha f(x) \to f(x)$ for each f in $\mathbf{C}_u^+$, $\mathscr{U}(\mathbf{C}_u^+)$ separates points in E. Furthermore, each element g of $\mathscr{U}(\mathbf{C}_u^+)$ is β-U excessive for some β. By continuity, $\alpha\bar{U}^{\beta+\alpha}\bar{g} \le \bar{g}$. Therefore, $(\bar{U}^\alpha)_{\alpha>0}$ is a Ray resolvent on $\bar{E}$.

The $(h_n) \subset \mathbf{R}$ are V^1-supermedian. By continuity again, $\alpha V^{1+\alpha}h_n \le h_n$ implies that $\alpha\bar{V}^{1+\alpha}\bar{h}_n \le \bar{h}_n$. The $\bar{h}_n$ are continuous and separate points in $\bar{E}$. Therefore, $(\bar{V}^\alpha)_{\alpha>0}$ is also a Ray resolvent on $\bar{E}$. □

REMARK. The collection (h_n) is necessary in the construction of $\mathbf{R}$ only when $\mathscr{V}(\mathbf{C}_u^+)$ does not separate points in E. If $\mathscr{V}(\mathbf{C}_u^+)$ does separate points, $\mathbf{R}$ may be constructed as follows, and all results remain valid.

$$\begin{aligned} \mathbf{R}_0 &= \mathscr{U}(\mathbf{C}_u^+) + \mathscr{V}(\mathbf{C}_u^+) \\ &\vdots \\ \mathbf{R}_{n+1} &= \Lambda(\mathscr{U}(\mathbf{R}_n) + \mathscr{V}(\mathbf{R}_n) + \mathbf{R}_n) \\ \mathbf{R} &= \cup_{n\ge 1} \mathbf{R}_n. \end{aligned}$$

4. Comparison of processes. Let $\bar{P}_t$ and $\bar{Q}_t$ denote the strong Markov semigroups with resolvents $(\bar{U}^\alpha)_{\alpha>0}$ and $(\bar{V}^\alpha)_{\alpha>0}$ respectively ([7], Section 3). Let W be the set of right continuous maps $w: \mathbb{R}^+ \to \bar{E}$ with left limits in $\bar{E}$. Set $\bar{X}_t(w) = \bar{Y}_t(w) = w(t)$. Let $\mathscr{G}^0 = \sigma(\bar{X}_t : t \ge 0)$, $\mathscr{G}_t^0 = \sigma(\bar{X}_s : s \le t)$, and let $\mathscr{G}^\mu$ and $\mathscr{G}_t^\mu$ (resp. $\tilde{\mathscr{G}}^\mu$ and $\tilde{\mathscr{G}}_t^\mu$) denote their P^μ-completions (resp. Q^μ completions) in $\mathscr{G}^0$. If μ is a probability measure on $(\bar{E}, \bar{\mathscr{E}})$, P^μ (resp. Q^μ) denotes the probability on $(W, \mathscr{G}^\mu)$ (resp. $(W, \tilde{\mathscr{G}}^\mu)$) under which $\bar{X}$ (resp. $\bar{Y}$) is a strong Markov process with initial measure $\mu\bar{P}_0$ and transition function $\bar{P}_t$ (resp. with initial measure $\mu\bar{Q}_0$ and transition function $\bar{Q}_t$).

To compare X and $\bar{X}$, we need to examine the sample paths of $h_n(X_t)$ and $V^\alpha f(X_t)$. But $h_n(X_t^*)$ and $V^\alpha f(X_t^*)$ are left continuous with right limits P^ν a.s. [19, 10]. Therefore, by reversing time, we find that $h_n(X_t)$ and $V^\alpha f(X_t)$ are right continuous with left limits P^ν a.s. (see, e.g., [17], Theorem (3.1)).

(4.1). PROPOSITION. *Let $h \in b\mathscr{E}$. If $h(X_t)$ is right continuous with left limits on* $[0, \infty)$ P^ν a.s., *then $h(X_t)$ is right continuous with left limits on* $[0, \infty)$ P^x a.s. *for all x except in a polar set.*

PROOF. A standard argument shows that $P^x\{h(X_t)$ is right continuous with left limits on $(0, \infty)\} = 1$ for all x. To get right continuity of $h(X_t)$ at 0, let

$$\Omega_0 = \{\omega : \limsup_{s\downarrow\downarrow 0; s\in Q} h(X_s) = \liminf_{s\downarrow\downarrow 0; s\in Q} h(X_s) = h(X_0)\},$$

where $s \downarrow\downarrow 0$ means s decreases to zero and is strictly positive. Then $\Omega_0 \in \mathscr{F}_{0+}^0$ and hence has P^x-measure 0 or 1 by the Blumenthal 0-1 law. Furthermore, $x \to P^x(\Omega_0)$ is Borel. Let $A = \{x: P^x(\Omega_0) = 0\} \in \mathscr{E}$, and let $\Gamma = \{(t, \omega) : X_t(\omega) \in A\} \cap ((0, \infty] \times \Omega)$. Then Γ is an optional set. Fix $x \in E$. Then, given $\epsilon > 0$, there is a stopping time T such that $[[T]] \subset \Gamma$ and $P^x\{\Pi\Gamma\} \le P^x\{T < \infty\} + \epsilon$ where Π denotes projection on Ω. But $P^x\{T < \infty\} = E^x\{1_{\Omega_0} \circ \theta_T\} = E^x\{E^{X(T)}(1_{\Omega_0})\}$. But $X_T \in A$ P^x a.s. on $[T < \infty]$. Therefore, $T = \infty$ a.s. P^x. Since ϵ is arbitrary and we may do this for each $x \in E$, we conclude A is polar. □

It follows easily by induction that if $h \in \mathbf{R}$, then $h(X_t)$ is right continuous P^x a.s. for all x except in a polar set. This holds, therefore, for all of the (g_j) simultaneously except for one polar set $N = \cup_j N_j$ where each N_j is the exceptional polar set for g_j. If μ is a measure not charging N, we have from the definition of ρ and $\bar{E}$ that, almost surely P^μ, $t \to X_t$ is right D-Ray continuous with left ρ-limits in $\bar{E}$. We may state the following proposition analogous to [7], (11.2).

(4.2). PROPOSITION. *Let Ω_0 denote the set of those ω in Ω such that $t \to \omega(t)$ is right* D-*Ray continuous with left ρ-limits in $\bar{E}$. Then $\Omega_0 \in \mathscr{F}$ and $P^\mu(\Omega_0) = 1$ for each μ on $(E, \bar{\mathscr{E}})$ not charging N.*

Only minor changes are needed in the proof of Proposition (11.3) in [7] to state the following result.

(4.3). PROPOSITION. *Because P_t and Q_t are Borel, $E \in \bar{\mathscr{E}}$. $\mathscr{E}$ is the trace of $\bar{\mathscr{E}}$ on E and $\Phi: E \to F \subset K$ is $\mathscr{E}/\mathscr{B}(F)$ measurable, where $\mathscr{B}(F)$ is the σ-algebra of Borel subsets of F.*

Then a careful examination of the arguments in [7], pages 70–72, reveals that we may state the following. Let $\Omega^* = \{$maps $\omega: \mathbb{R}^+ \to E$ such that ω is right continuous in the original and D-Ray topologies with left limits in $\bar{E}$ in the ρ-topology$\}$. Set $W_t(\omega) = \omega(t)$, $\mathscr{F}^{0*} = \sigma(W_s: s \geq 0)$, $\mathscr{F}_t^{0*} = \sigma(W_s: s \leq t)$ where W_s is regarded as a map from Ω^* to $(\bar{E}, \bar{\mathscr{E}}^*)$.

(4.4). THEOREM. *For each probability μ on $(E, \mathscr{E})$ not charging N, let P^μ be the measure on $(\Omega^*, \mathscr{F}^{0*})$ constructed from μ and (P_t), and let $\bar{P}^\mu$ be the measure on $(\Omega^*, \mathscr{F}^{0*})$ constructed from μ and $(\bar{P}_t)$. Then $P^\mu = \bar{P}^\mu$ and $(W_t, \mathscr{F}_t^{0*}, P^\mu)$ is a Markov process with initial measure μ and having (P_t)* (resp. $(\bar{P}_t)$) *as transition semigroup if one takes $(E, \mathscr{E}^*)$* (resp. $(\bar{E}, \bar{\mathscr{E}}^*)$) *as state space.*

An extremely important corollary is the following.

(4.5). COROLLARY. *If $x \in E - N$, $P_t(x, \cdot) = \bar{P}_t(x, \cdot)$ $\forall\, t \geq 0$ and $U^\alpha(x, \cdot) = \bar{U}^\alpha(x, \cdot)$ $\forall\, \alpha > 0$. In particular, $\bar{P}_t(x, \cdot)$ and $\bar{U}^\alpha(x, \cdot)$ are carried by E if $x \in E - N$.*

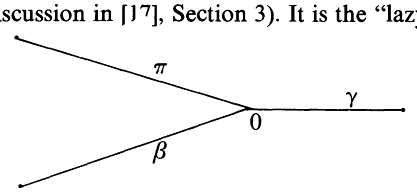
We shall now discuss the analogous situation for the reverse. To avoid measure theoretic difficulties with left continuous path spaces, we prefer to work with X_t^* under P^ν. Recall from Section 2 that it has moderate semigroup Q_t. Again following the discussion on pages 70–71 of [7] (and replacing "right continuous" with "left continuous"), it is easy to see that the following are true.

(i) Almost surely P^ν, X_t^* is left D-Ray continuous with right D-Ray limits on $(0, \infty)$.

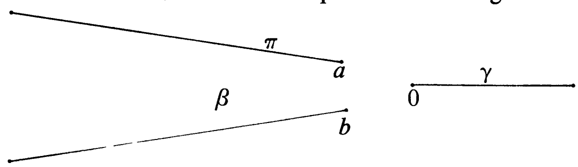
(ii) X_t^* is P^ν Markov with transition semigroup $\bar{Q}_{t-}$ on $(0, \infty)$ ($\bar{Q}_{t-}$ denotes the moderate version of the semigroup $\bar{Q}_t$).

Note that we restrict the "initial measure" to ν; the statements are not true for arbitrary initial measures. Define a map $Z_t^*: \Omega^* \to \bar{E}$ by $Z_t^*(\omega) = \omega(\zeta - t)$. Let Q^* denote the probability on Ω^* making Z_t^* Markovian with moderate semigroup $\bar{Q}_{t-}$ and entrance law the same as that of X^* under P^ν. Then the same arguments show that $Q^* = P^\nu$.

5. Examples. We shall give three examples here to illustrate what happens to the topology. The first example is given by Garcia-Alvarez [5] (but is due originally to Blumenthal and Getoor: consult the discussion in [17], Section 3). It is the "lazy crotch:"

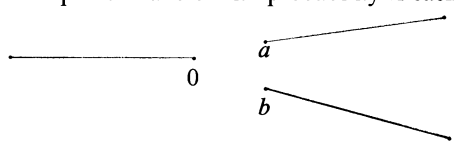


The process X is simply uniform motion to the right with speed one. It is a simple matter to compute the reverse semigroup. Since the resolvent for the reverse process does separate points in this instance, it is not necessary to add the sequence (h_n) of bounded 1-supermedian functions to the double Ray cone. The resolvent U^α of X maps bounded Borel functions to continuous functions. The resolvent of the reverse, V^α, maps bounded Borel functions to functions which are continuous except at the point 0. Moreover, $\lim_{x\to 0;\, x\in\pi} V^\alpha f(x)$, $\lim_{x\to 0;\, x\in\beta} V^\alpha f(x)$, and $\lim_{x\to 0;\, x\in\gamma} V^\alpha f(x) = V^\alpha f(0)$, exist. This is easy to understand since the reverse process "splits" at 0. Thus the double Ray cone changes the topology of the lazy crotch only in the neighborhood of zero, and after compactification we get

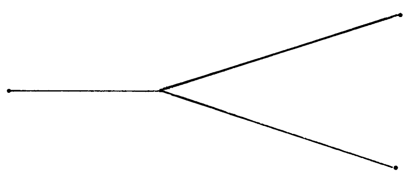


Note that we have introduced two degenerate branch points for the forward process, a and b. The point 0 is attached to γ rather than to π or β since $\lim_{x\to 0;\, x\in\gamma} V^\alpha f(x) = V^\alpha f(0)$.

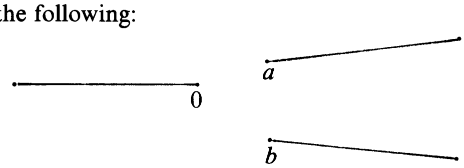
On the other hand, the original process might have been uniform motion to the right, which upon reaching 0 jumps to the points a and b with probability ½ each:



In the Garcia-Alvarez and Meyer formulation, the compactified space would be the lazy crotch again:

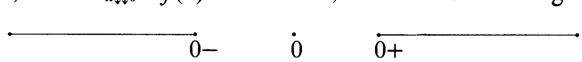


The original state space does not embed nicely in the lazy crotch (the points a and b are joined at 0), and the forward process is no longer strong Markov on $\bar{E}$. To construct a double Ray cone, we first choose functions to separate points. In this case, it is easy to choose *one* continuous function separating points which is 1-supermedian for V^α. It is a simple matter to check that both U^α and V^α map $bC(E)$ into bounded continuous functions on E having a limit at 0, and $\bar{E}$ is therefore the following:



Thus the Garcia-Alvarez and Meyer compactification is a "smaller" space (i.e., we can arrange it so that their cone of functions is actually contained in our cone). One disadvantage of this smaller space, as illustrated above, is that the forward process X may no longer be strong Markov on it.

Finally, we give an example of a less trivial nature. Let B_t be Brownian motion on the line, and let $T = \inf\{t > 0 : B_t = 0\}$. Set $Y_t = B_{t\wedge T}$, and let X_t be Y_t killed at an independent exponential time. Then X_t possesses a σ-finite excessive reference measure ([17], Theorem (1.1)), and its resolvent $U^\alpha : bC(E) \to bC(E)$. The resolvent of the reverse process, V^α, separates points in this case. If $f \in bC(E)$, then $V^\alpha f$ is continuous on the complement of $\{0\}$, $\lim_{x\uparrow\uparrow 0} V^\alpha f(x)$ exists, and $\lim_{x\downarrow\downarrow 0} V^\alpha f(x)$ exists. Thus, $\bar{E}$ takes the following form.



Notice that 0− and 0+ are degenerate branch points for the forward process: $P_t f(0+) = e^{-t} f(0) = P_t f(0-)$. Thus the compactifications obtained from the double Ray cone may at first seem strange because of the presence of degenerate branch points. These are, of course, present in a general Ray process (only when a right process is compactified is the set of degenerate branch points empty ([7] (10.9)). Thus all of the results of Sections 1 through 7 of [7] are valid for these processes.

6. Strong duality. The processes $\bar{X}$ and $\bar{Y}$ have the following property.

(6.1). PROPOSITION. *$\bar{X}$ and $\bar{Y}$ have the moderate Markov property*; i.e., *if T is a previsible stopping time relative to the filtration* ($\mathscr{G}_t^\mu$) (resp. $\bar{\mathscr{G}}_t^\mu$)), *then for $t > 0$, $f \in b\mathscr{E}$,*

$$\bar{P}^\mu\{f(\bar{X}_{T+t}) \mid \mathscr{G}_{T-}^\mu\} = \bar{P}_t f(\bar{X}_{T-})$$

$$\bar{Q}^\mu\{f(\bar{Y}_{T+t}) \mid \bar{\mathscr{G}}_{T-}^\mu\} = \bar{Q}_t f(\bar{Y}_{T-}).$$

PROOF. By a theorem due to Chung [4], we need only show that if (T_n) is a sequence of stopping times announcing T and $f \in C(\bar{E})$, then for $\alpha > 0$,

$$\lim_n \bar{U}^\alpha f(\bar{X}_{T_n}) = \bar{U}^\alpha f(\bar{X}_{T-}) \qquad \text{a.s.} \quad P^x$$

for each x on $\{T < \infty\}$. But $\bar{U}^\alpha f(x)$ is continuous, and so this follows immediately (similarly for $(\bar{Y}_t)$). □

Now define

$$\begin{aligned} \tilde{P}_t(x, \cdot) &= \bar{P}_t(x, \cdot) && \text{if} \quad x \in E - N \\ &= \epsilon_\Delta(\cdot) && \text{if} \quad x \notin E - N. \end{aligned}$$

Since $\bar{U}^\alpha(x, \cdot) = U^\alpha(x, \cdot) \ll \nu$ for $x \in E - N$ and N is polar, $(\tilde{P}_t)_{t \geq 0}$ is the semigroup of a right process on $E - N$. If $(\tilde{U}^\alpha)_{\alpha > 0}$ is the resolvent corresponding to $(\tilde{P}_t)$, $\tilde{U}^\alpha(x, \cdot) \ll \nu$ for each $x \in \bar{E}$.

REMARK. The above statement is, of course, not true when taken literally. We make the convention that all functions vanish at Δ. Thus the above statement of absolute continuity may be interpreted as a statement about measures on E. Since Δ is an isolated point in $\bar{E}$ (recall the function h_1), this causes no difficulties even when dealing with continuous functions on $\bar{E}$.

For f and g in $\mathbf{R}$ and for all $\alpha > 0$,

$$\int_E U^\alpha f(x) g(x) \nu(dx) = \int_E f(x) V^\alpha g(x) \nu(dx).$$

Because ν does not charge $\bar{E} - E$, we may rewrite this as

$$\int_{\bar{E}} \bar{U}^\alpha \bar{f}(x) \bar{g}(x) \nu(dx) = \int_{\bar{E}} \bar{f}(x) \bar{V}^\alpha \bar{g}(x) \nu(dx).$$

This extends by linearity and continuity to f and g in $C(\bar{E})$ and then via the monotone class theorem to f and g in $b\bar{\mathscr{E}}$. Therefore, $\bar{U}^\alpha$ and $\bar{V}^\alpha$ are in weak duality with respect to the excessive reference measure ν. Hence, $\tilde{U}^\alpha$ and $\bar{V}^\alpha$ are in weak duality.

DEFINITION. A set $B \in \bar{\mathscr{E}}$ is stable for a process Z if $P^x\{Z_t \in B \cup \Delta \ \forall\ t \geq 0\} = 1$ for all $x \in B$.

The following lemma, due to Smythe and Walsh ([17] Lemma (2.1)), in the case of a moderate Markov process in weak duality with a strong Markov process applies also in this situation of two strong Markov processes in weak duality with no change in proof.

(6.2). LEMMA. *The set B of x in $\bar{E}$ for which $\bar{V}^\alpha(x, \cdot) \ll \nu$ is a stable Borel set such that $\nu(\bar{E} -$*

$B) = 0$.

More precisely, the proof shows that B is stable for the process Y_{t-}. A simple modification of the proof shows that B is also stable for the process Y_t.

Then we define

$$\tilde{Q}_t(x, \cdot) = \bar{Q}_t(x, \cdot) \quad \text{if} \quad x \in B$$
$$= \epsilon_\Delta(\cdot) \quad \text{if} \quad x \notin B.$$

$(\tilde{Q}_t)_{t\geq 0}$ is the transition semigroup of a strong Markov process, and the corresponding resolvent $\tilde{V}^\alpha(x, \cdot) \ll \nu$. Thus $(\tilde{U}^\alpha)$ and $(\tilde{V}^\alpha)$ are in strong duality. Furthermore,

$$\tilde{U}^\alpha(x, \cdot) = \bar{U}^\alpha(x, \cdot) \quad \text{if} \quad x \in E - N,$$

and

$$\tilde{V}^\alpha(x, \cdot) = \bar{V}^\alpha(x, \cdot) \quad \text{if} \quad x \in B.$$

As an immediate corollary of Proposition (6.1) we have the following.

(6.3). COROLLARY. *If $\tilde{X}$ and $\tilde{Y}$ are the canonical realizations of $(\tilde{P}_t)$ and $(\tilde{Q}_t)$, then they have the moderate Markov property.*

REMARK. In the examples given in Section 5, there is no need to go through these modifications: the processes are in strong duality immediately after compactifying.

Thus, at the cost of introducing (possibly degenerate) branch points, we have produced two strong Markov processes in strong duality. Note that $\tilde{X}$ remains a P^x-right process for each $x \in E - N$, indistinguishable from the original process X on E. Thus, we may study X on $\bar{E}$.

7. Representation and absolute continuity of additive functionals. We revise the notation to be used henceforth. Let P_t (resp. $\hat{P}_t$) be the semigroup $\tilde{P}_t$ (resp. $\tilde{Q}_t$) of the last section. We denote the canonical strong Markov realization of P_t and $\hat{P}_t$ on the space of right continuous paths with left limits in $\bar{E}$ by X and $\hat{X}$ respectively. Then X and $\hat{X}$ are strong Markov processes in strong duality on $\bar{E}$ with respect to the σ-finite excessive reference measure ν. The processes may have branch points, but X is a right process when restricted to $E - N$.

The rest of this section is devoted to showing that a good part of the work of Revuz [12, 13] and Sharpe [14] extends to our setting with little or no change in proof. A major benefit of the compactification procedure is that the absolute continuity theorem for additive functionals (Theorem 7.8) is valid. This theorem is well known in the case of two standard processes in duality [14]. All functions which appear below are assumed to vanish at Δ.

DEFINITION. Let $\mathscr{A}$ be the collection of finite additive functionals of X.

If A is an additive functional (abbreviated as AF) of X and $F \in b(\bar{\mathscr{E}} \times \bar{\mathscr{E}}), f \in b\bar{\mathscr{E}}_+$, we define

$$\nu_A(F) = \sup_{t>0} t^{-1} E^\nu \int_{(0,\, t]} F(X_{s-}, X_s)\, dA_s$$

$$\nu_A^1(f) = \sup_{t>0} t^{-1} E^\nu \int_{(0,\, t]} f(X_{s-})\, dA_s$$

$$\nu_A^2(f) = \sup_{t>0} t^{-1} E^\nu \int_{(0,\, t]} f(X_s)\, dA_s.$$

Then ν_A, ν_A^1, and ν_A^2 are measures ([12], page 507). The AF A is said to be integrable if $\nu_A(1)$

$< \infty$; it is said to be σ-integrable if there is a decomposition of $\bar{E} \times \bar{E}$ into a countable union of Borel sets (F_i) with $\nu_A(F_i) < \infty$ for each i.

We also define potential operators for $\alpha \geq 0$.

$$M_A^\alpha F(x) = E^x \int_0^\infty e^{-\alpha t} F(X_{t-}, X_t)\, dA_t$$

$$U_A^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t)\, dA_t$$

$$W_A^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_{t-})\, dA_t.$$

DEFINITION. An AF A is said to be natural if, a.s., $t \to A_t(\omega)$ and $t \to X_t(\omega)$ have no discontinuities in common. Of course, the path $t \to X_t$ is understood to be in the topology of $\bar{E}$.

Let D be the diagonal in $\bar{E} \times \bar{E}$: $D = \{(x, x): x \in \bar{E}\}$. Then Sharpe's proof of the next result is valid here ([14], Proposition (3.3)).

(7.1). PROPOSITION. *Let A be σ-integrable. Then A is natural if and only if ν_A is carried by D in $\bar{E} \times \bar{E}$, and in this case $\nu_A^1 = \nu_A^2$. The same result holds if $A \in \mathscr{A}$.*

Fix densities $u^\alpha(x, y)$ for each $\alpha \geq 0$ such that
(i) $U^\alpha(x, dy) = u^\alpha(x, y)\nu(dy)$
(ii) $\hat{U}^\alpha(dx, y) = u^\alpha(x, y)\nu(dx)$
(iii) $x \to u^\alpha(x, y)$ is α-excessive for X
(iv) $y \to u^\alpha(x, y)$ is α-excessive for $\hat{X}$.
Given an additive functional A, define $u_A^\alpha = U_A^\alpha 1$.

(7.2). PROPOSITION. *Let A be an integrable* AF. *Then $u_A^\alpha \equiv U_A^\alpha 1 < \infty$* a.e. (ν) *if $\alpha > 0$ and $u_A^\alpha(x) = U^\alpha \nu_A^1(x) = \int u^\alpha(x, y)\nu_A^1(dy)$.*

PROOF. We indicate only the changes necessary in the proof of Revuz. If $\Phi \in b\bar{\mathscr{E}}_+ \cap L^1(d\nu)$, then $\Phi \hat{U}(X_{t+}^*)$ is right continuous P^ν a.s. (the right limit taken in the topology of $\bar{E}$). But then, by time reversal, $\Phi \hat{U}(X_{t-})$ is left continuous P^ν a.s. Therefore, one may follow the proof to find that $U^\alpha \nu_A^1 \leq u_A^\alpha$ a.e. (ν). But both functions are α-excessive and applying the resolvent to both sides gives the inequality everywhere. (Note that this is really a fact about the right process of $E - N$) The rest follows as in Revuz [12], page 157. □

The reader should consult [11] for an alternate proof of Proposition (7.2) under hypotheses which are slightly weaker than the classical duality hypotheses.

(7.3). THEOREM. *If $A \in \mathscr{A}$ and $F \in (\bar{\mathscr{E}} \times \bar{\mathscr{E}})_+$, then*

$$M_A^\alpha F(x) = \int\int_{\bar{E}\times\bar{E}} u^\alpha(x, y) F(y, z)\nu_A(dy, dz).$$

In particular, if $f \in \bar{\mathscr{E}}_+$,

$$W_A^\alpha f(x) = \int u^\alpha(x, y) f(y)\nu_A^1(dy)$$

$$U_A^\alpha f(x) = \int\int u^\alpha(x, y) f(z)\nu_A(dy, dz).$$

Another time reversal argument gives an extremely useful result. Let $\hat{B}$ be the set of cobranch points; i.e., $\hat{B} = \{x: \hat{P}_0(\cdot, x) \neq \epsilon_x\}$.

(7.4). PROPOSITION. *If A is integrable, then*

$$\nu_A(\hat{B} \times \bar{E}) = 0.$$

PROOF. If $\nu_A(\hat{B} \times \bar{E}) > 0$, then $P^\nu\{X_{t-} \in \hat{B}, \text{ some } t\} > 0$. But then by reversing we have $P^\nu\{X^*_{t+} \in \hat{B}, \text{ some } t\} > 0$. This cannot happen. □

The following three results hold here without change in proof ([14], page 83).

(7.5). PROPOSITION. *Let A, $B \in \mathscr{A}$, and suppose that for some fixed $\alpha \geq 0$, $u^\alpha_A = U^\alpha_A 1 < \infty$* a.e. *($\nu$) and that $U^\alpha_A f = U^\alpha_B f$ for all $f \in b\bar{\mathscr{E}}_+$. Then A and B are equivalent*; i.e., $A_t = B_t$ a.s. *on* $\{t < \zeta\}$.

From (7.5) and (7.3) follows the next result.

(7.6). PROPOSITION. *If $A \in \mathscr{A}$ amd $u^\alpha_A < \infty$* a.e. (ν), *then ν_A determines A uniquely.*

(7.7). PROPOSITION. *If $A \in \mathscr{A}$ is σ-integrable, then ν_A determines A uniquely.*

Let A and B be additive functionals of X.

DEFINITION. A is said to be absolutely continuous with respect to B (written $A \ll B$) if $M^\alpha_B F = 0$ implies $M^\alpha_A F = 0$ for all $F \in (\bar{\mathscr{E}} \times \bar{\mathscr{E}})_+$.

If A and B are σ-integrable and in $\mathscr{A}$, it follows that $A \ll B$ implies $\nu_A \ll \nu_B$. Thus $d\nu_A = F\, d\nu_B$ for some $F \in (\bar{\mathscr{E}} \times \bar{\mathscr{E}})_+$ which may be chosen to be everywhere finite. Set $C_t = \int_0^t F(X_{s-}, X_s)\, dB_s$. Then the Revuz measure for C is $F\, d\nu_B = d\nu_A$. Thus, by Proposition (7.7), $C_t = A_t$.

(7.8). THEOREM. *If $A \ll B$ are σ-integrable, then $A = F * B$, for some finite $F \in (\bar{E} \times \bar{E})_+$.*

Theorem (7.8) is not true in the Garcia-Alvarez topology. When A and B are continuous additive functionals, Theorem (7.8) is due to Motoo in the case of standard processes in duality. For comparison, Azema [1] had obtained the following absolute continuity theorem for processes satisfying hypothesis (L).

THEOREM. *Let A and B be natural additive functionals (in the original topology) such that $U_A \ll U_B$. Then there exists a previsible process (Z_s), $0 < s < \infty$, $Z \leq 1$, such that $A_t = \int_0^t Z(w, s)\, dB_s$. Moreover, Z is adapted to the σ-algebra generated by the measurable left continuous processes with right limits which are zero on (ζ, ∞) and which are homogeneous.*

Thus passing to the D-Ray topology allows the left and right limits of the process "to contain enough information" so that Z may be taken to be a function of the process, $Z_t = F(X_{t-}, X_t)$.

8. Conatural additive functionals. The aim of this section is to identify the conatural additive functionals with more familiar objects, namely, the natural additive functionals.

DEFINITION. Let $A \in \mathscr{A}$. Then A is said to be conatural if $U_A(x, \cdot) = W_A(x, \cdot)$ for all $x \in \bar{E}$.

This definition was first given by Garcia-Alvarez [5], page 968. These additive functionals form a convenient class to study since they ignore the discontinuities in the Ray topology. Note that the topology of the state space does determine which additive functionals are called conatural. In Garcia-Alvarez's case, the class of conatural additive functionals could not be identified with the natural additive functionals. This section shows that the topology we have introduced is a "natural" one in that an additive functional is conatural if and only if it is natural.

DEFINITION. Let $A \in \mathscr{A}$. Then A is said to be Ray discontinuous if A is purely discontinuous, and, a.s., the jumps of the sample paths $t \to A_t$ occur at jump times of X_t.

We shall require the following modification of [2], VI-1.15.

(8.1). PROPOSITION. *Let μ_1 and μ_2 be measures not charging $\hat{B}$. Assume $U^\alpha\mu_1 < \infty$ a.e. (ν) and $U^\alpha\mu_1 = U^\alpha\mu_2$ everywhere. Then $\mu_1 = \mu_2$.*

PROOF. Following [2], let h be a strictly positive bounded function on $\bar{E}$ with $\langle h, U^\alpha\mu_1\rangle_\nu < \infty$. Set $g = h\,\hat{U}^\alpha$. Let f be continuous on $\bar{E}$ with $0 \le f \le 1$. Then for $\gamma > 0$,

$$\mu_1[\gamma(fg)\hat{U}^{\gamma+\alpha}] = \langle \gamma fg, U^{\alpha+\gamma}\mu_1\rangle_\nu = \langle \gamma fg, U^{\alpha+\gamma}\mu_2\rangle_\nu = \mu_2[\gamma(fg)\hat{U}^{\gamma+\alpha}].$$

Since g is coexcessive, fg is cofinely continuous. Thus, letting $\gamma \to \infty$, $\gamma(fg)\hat{U}^{\gamma+\alpha} \to fg$ *off* $\hat{B}$. Moreover, $\gamma(fg)\hat{U}^{\alpha+\gamma} \le \gamma g\hat{U}^{\gamma+\alpha} \le g$. Thus, by the dominated convergence theorem (since $\mu_1(g) < \infty$), $\mu_1(fg) = \mu_2(fg)$ for all bounded continuous f since $\mu_1(\hat{B}) = \mu_2(\hat{B}) = 0$. Thus $g\,d\mu_1 = g\,d\mu_2$. But $g > 0$ *off* $\hat{B}$. Therefore $\mu_1 = \mu_2$. □

If $A \in \mathscr{A}$, it can be decomposed as $A^1 + A^2 + A^3$, where $A^1, A^2, A^3 \in \mathscr{A}$, A^1 is continuous, A^2 is natural and purely discontinuous, and A^3 is Ray discontinuous. Since natural additive functionals are clearly conatural, it suffices to examine functionals $A \in \mathscr{A}$ which are Ray discontinuous.

By the work of Walsh and Weil [20], on the representation of terminal times, a Ray discontinuous additive functional A may be written as $A_t = \sum_{s\le t} F(X_{s-}, X_s)$ for some $F \in (\bar{E} \times \bar{E})_+$, null on the diagnoal. If a is conatural and integrable, then for $f \in b\bar{\mathscr{E}}_+$,

$$E^x \sum f(X_s)F(X_{s-}, X_s) = E^x \sum f(X_{s-})F(X_{s-}, X_s).$$

But if these expressions are finite everywhere, then by the representation theorem (7.3),

$$\int\int u(x, y)f(y)\nu_A(dy, dz) = \int\int u(x, y)f(z)\nu_A(dy, dz).$$

By Proposition (7.4), ν_A does not charge $\hat{B} \times E$. Therefore, by Proposition (8.1),

$$\int\int_{\bar{E}\times\bar{E}} g(y)f(y)\nu_A(dy, dz) = \int_{\bar{E}\times\bar{E}} g(y)f(z)\nu_A(dy, dz).$$

Taking $f = 1_\Gamma$, $g = 1_\Lambda$ with $\Gamma \cap \Lambda = \varnothing$, we see that $\nu_A \equiv 0$. To summarize, we may state the following result.

(8.2). THEOREM. *Let $A \in \mathscr{A}$ be integrable. Then A is natural if and only if A is conatural.*

9. A result from potential theory. The real power of dual processes lies in Hunt's switching identity. In our setting, there is a switching identity for dual terminal times, given by Smythe and Walsh. It is not nearly as useful as in the case of standard processes in duality. This is due to the fact that one cannot say

$$P_A\,u(x, y) = u\hat{P}_A(x, y)$$

for general processes. It is this form, of course, which is most profitable for standard processes.

We present a new proof of a result useful in potential theory which was originally proved using the switching identity. We make additional rather restrictive hypotheses. These can be removed [9], but we shall not do that here.

(9.1). PROPOSITION. *Let $\alpha \geq 0$ and assume*

(i) $U^\alpha 1 > c > 0$ *for each* α *(c may depend on* α*).*

(ii) *The excessive reference measure* ν *is finite.*

Let μ *be a finite measure not charging* $\hat{B}$*. If* $U^\alpha\mu(x)$ *is bounded, then* μ *does not charge copolar sets.*

PROOF. Let $\alpha > 0$. Suppose $U^\alpha\mu(x)$ is bounded, and μ charges a copolar set $M \subset \bar{E} - \hat{B}$. Then, since $\hat{X}$ is a right process on $\bar{E} - \hat{B}$, we may find a coexcessive function f such that $\nu(f) < \infty$ and $M \subset \{f = \infty\}$. Take a sequence of bounded Borel functions (h_n) such that $h_n\hat{U}^\alpha \uparrow f$. Then $\mu(h_n\ \hat{U}^\alpha) \uparrow \infty$. But $\mu(h_n\ \hat{U}^\alpha) = \int\int u^\alpha(x, y)\mu(dy)h_n(x)\nu(dx)$. Therefore, $U^\alpha\mu$ bounded implies that $\int h_n(x)\nu(dx)$ increases to ∞. However,

$$\infty > \langle f, 1\rangle_\nu \geq \lim_n\langle h_n\ \hat{U}^\alpha, 1\rangle_\nu = \lim_n\langle h_n, U^\alpha 1\rangle_\nu.$$

But $\nu(h_n\ U^\alpha 1) > c\ \nu(h_n) \uparrow \infty$. Thus we have a contradiction, and, therefore, $U^\alpha\mu$ is unbounded.

In the case $\alpha = 0$, if $U\mu(x)$ is bounded, then $U^1\mu(x)$ is bounded, and μ, therefore, does not charge a copolar set. □

10. The bounded maximum principle. In this section, we assume the hypothesis of Proposition (9.1):

(i) $U^\alpha 1 > c > 0$ for each $\alpha > 0$.

(ii) ν is a finite measure.

Again, these assumptions may be removed (which we shall not do here).

Let μ be a finite measure on $\bar{E}$ with compact support K. We define the bounded maximum principle:

(M_α^*) If $U^\alpha\mu$ is bounded, then $\| U^\alpha\mu \| = \sup\{ U^\alpha\mu(x) : x \in K\}$, where $\|f\| = \sup\{f(x) : x \in \bar{E}\}$.

We also state Hunt's hypothesis (H).

(H) Every semipolar set is polar.

In the case of standard processes in duality, (H) implies (M_α^*) [3, 9]. We shall prove a version of this here. Since the proof in the standard case involves the switching identity, we may expect some slight difficulty.

Let A be a Borel set in $\bar{E}$, and let $S_A = \inf\{t > 0: (X_{t-}, X_t) \in A \times A\}$, $\hat{S}_A = \inf\{t > 0: (\hat{X}_{t-}, \hat{X}_t) \in A \times A\}$. We define the collection of strongly regular points for A, denoted A^R, as $\{x: E^x\{\exp(-S_A)\} = 1\}$.

(10.1). PROPOSITION. $A - A^R$ *is semipolar.*

PROOF. Set $\Psi_A(x) = E^x\{e^{-S_A}\}$. Then Ψ_A is 1-excessive. Set $A_n = \{\Psi_A(x) \leq 1 - (1/n)\} \cap A$. Then $A - A^R = \cup A_n$. Note that $\Psi_{A_N}(x) \leq 1 - (1/n)$ on $A_n \cup A_n^r$. Define a sequence of hitting times by setting $S_1 = S_{A_n}$, and $S_{k+1} = S_k + S_{A_n} \cdot \theta_{S_k}$ for $k \geq 1$. Then

$$E^x\{e^{-S_{k+1}}\} = E^x\{e^{-S_k}\ \Psi_{A_n}(X_{S_k})\}.$$

An argument analogous to the one given in [2] I-11.4 shows that $X_{S_k} \in A_n \cup A_n^r$. Therefore,

$$E^x\{e^{-S_k}\ \Psi_{A_n}(X_{S_k})\} \leq \left(1 - \frac{1}{n}\right) E^x\{e^{-S_k}\}.$$

It follows by induction that $E^x\{e^{-S_{k+1}}\} \leq (1 - (1/n))^k$, and the sequence (S_k) must increase to ∞. In particular, $\{t: (X_{t-}, X_t) \in A_n \times A_n\}$ is a countable set. Since $\{t: X_{t-} \neq X_t\}$ is countable, it follows that $\{t: X_t \in A_n\}$ is countable, and we conclude that A_n is semipolar by Dellacherie's theorem on semipolar sets. Therefore, $A = \cup A_n$ is semipolar.

We define the hitting operators P_A^α and $\hat{P}_A^\alpha$ by

$$P_A^\alpha f(x) = E^x\{e^{-\alpha S_A} f(X_{S_A})\}$$

$$f\hat{P}_A^\alpha(x) = \hat{E}^x\{e^{-\alpha \hat{S}_A} f(\hat{X}_{\hat{S}_A})\}.$$

These are *not* the usual hitting operators of [2].

Let μ be a measure on $\bar{E}$ not charging semipolar sets. Since semipolar is the same as cosemipolar ([16], Proposition (5.2)), μ does not charge B or $\hat{B}$. Thus, once we have Proposition (10.1), we may follow the proof of [3] (2.1). □

(10.2). PROPOSITION. *If μ does not charge semipolar sets, then* $\| U^\alpha\mu \| = \sup\{U^\alpha\mu(x) : x \in K\}$ *where K is the support of μ.*

PROOF. S_K and $\hat{S}_K$ are dual terminal times. Thus by the version of the switching identity proved by Smythe and Walsh, $P_K^\alpha U_\mu^\alpha = U^\alpha \hat{P}_K^\alpha \mu$. If we let ${}^RK$ denote the strongly coregular points of K, then $K - {}^RK$ is semipolar. Thus, if C is Borel,

$$\hat{P}_K^\alpha \mu(C) = \int \hat{P}_K^\alpha(C, x)\mu(dx) = \int_{K \cap {}^RK} \hat{P}_K^\alpha(C, x)\mu(dx)$$

since, by hypothesis, μ does not charge semipolars. As observed above, μ does not charge $\hat{B}$. Thus, if $x \in {}^RK \cap (\hat{B})^c$, $\hat{P}_K^\alpha(C, x) = 1_C(x)$. Therefore, $\hat{P}_K^\alpha\mu(B) = \mu(B \cap K \cap {}^RK) = \mu(B)$. Thus $U^\alpha\mu = P_K^\alpha U^\alpha \mu$. But $P_K^\alpha(x, \cdot)$ is carried by K, and the result follows. □

(10.3). THEOREM. *Hypothesis* (H) *implies* (M_α^*) *for measures μ not charging $\hat{B}$.*

PROOF. If $U^\alpha\mu$ is bounded, then μ does not charge semipolar sets by (H) and Proposition (9.1). Thus, $\| U^\alpha\mu \| = \sup\{U^\alpha\mu(x) : x \in K\}$. □

REMARK. Note that this theorem can be interpreted on the original space E.

11. Capacities. In this section we extend the Getoor-Sharpe approach to capacities [8]. Since they relied primarily on the Revuz measure, their results go through with little or no difficulty, and the results in this section, therefore, originate with them.

We now introduce an extremely useful technical device. Unfortunately, it requires another examination of the original compactification. Being originally given a right process $\tilde{X}$ on E, we constructed a moderate Markov dual $\hat{\tilde{X}}$ with respect to a σ-finite excessive reference measure $\nu = \lambda U$, where λ is a reference probability measure. Adjoin an isolated point Γ to E and set

$$\tilde{P}_t^* f(x) = \tilde{P}_t f(x) + f(\Gamma) E^x\{e^{-(t-\zeta(\omega))\vee 0}\} \quad \text{for} \quad x \in E$$

$$\tilde{P}_t^* f(\Gamma) = e^{-t} f(\Gamma).$$

That is, the process $\tilde{X}^*$ lives on E until it "dies" at $t = \zeta(\omega)$. Then it goes to Γ and is killed from there exponentially. We have simply "tacked on" some time at the death time ζ of $\tilde{X}$. Then $(\lambda + \epsilon_\Gamma)$ is a reference measure for $\tilde{X}^*$. As we remarked initially, there is no loss in generality in assuming $\tilde{U}^*$ a bounded kernel [6]. Thus, $\nu^* = (\lambda + \epsilon_\Gamma)\tilde{U}^*$ is a finite measure. Reversing, we compute the adjoint semigroup $\hat{\tilde{P}}_t^*$ in duality with respect to ν^* with $\tilde{P}_t^*$. Note that $\nu^* = \nu + K\epsilon_\Gamma$ for some constant K. Then

$$\hat{\tilde{P}}_t^* g(x) = \hat{\tilde{P}}_t g(x) \quad \text{for} \quad x \in E$$

$$\hat{\tilde{P}}_t^* g(\Gamma) = e^{-t} g(\Gamma) + \frac{1}{K} \int E^x\{e^{-(t-\zeta(\omega))\vee 0}\} g(x)\nu(dx).$$

We denote by X^* and $\hat{X}^*$ the strong Markov versions of $\tilde{X}^*$ and $\hat{\tilde{X}}^*$ on the new compactification $\bar{E}_\Gamma$. This trivial procedure has the pleasant property that certain excessive functions which are not natural potentials for X on $\bar{E}$ become natural potentials for X^* on $\bar{E}_\Gamma$.

For example, let $D \in \mathscr{E}$ and let ζ^* be the death time of X^*. The last exit time of X^* from D is defined to be

$$L_D = \sup\{t: X_t^* \in D\}.$$

Since $\Gamma \notin D$, $L_D < \zeta^*$ a.s. It is also easy to check that

$$M_D(x) = P^x\{T_D < \infty\}$$

is a natural potential; i.e. for each x, if (T_n) is an increasing sequence of stopping times with limit greater than or equal to ζ^*, $\lim_{n\to\infty} P_{T_n}^* M_D(x) = 0$. Notice that if ζ is previsible, then there is an increasing sequence of stopping times (S_n) announcing ζ, so that it may happen that $\lim_{n\to\infty} P_{S_n} M_D(x) > 0$ (e.g., take $D = E$). Thus Γ is a necessary artifice to prolong the life of the process enough to make $M_D(x)$ a natural potential. Since $M_D(x)$ is a natural potential for X^*, $M_D(x)$ is the potential of a previsible additive functional A_D [15]. Note that $M_D(x) = U_{A_D}1(x) = P^x\{T_D < \infty\} = P^X\{L > 0\}$. Then, in our situation, we may state the fundamental result of Getoor and Sharpe [8] as follows.

(11.1). PROPOSITION. *Let $D \in \mathscr{E}$, $L = L_D$, $A = A_D$. Then for each $x \in E$, $s > 0$, and $\Lambda \in \bar{\mathscr{E}}_\Gamma$,*
(i) $P^x\{X_{L-}^* \in \Lambda; L > 0\} = U_A^*(x, \Lambda)$;
(ii) $P^x\{X_{L-}^* \in \Lambda; L > s\} = \int P_s^*(x, dy) U_A^*(y, \Lambda)$;
(iii) $E^x\{e^{-\alpha L}; X_{L-}^* \in \Lambda; L > 0\} = U_A^{*\alpha}(x, \Lambda)$.

PROOF. Fix $x \in E$, and set $B_t = 1_{\{0<L\le t\}}$. Then

$$E^x\{B_\infty - B_t \mid \mathscr{F}_t\} = P^x\{L > t \mid \mathscr{F}_t\} = P^x\{L \circ \theta_t > 0 \mid \mathscr{F}_t\} = u_A(X_t^*),$$

since $L < \infty$ a.s. But

$$E^x\{A_\infty - A_t \mid \mathscr{F}_t\} = E^x\{A_\infty \circ \theta_t \mid \mathscr{F}_t\} = u_A(X_t^*).$$

Thus, for every nonnegative previsible process Y,

$$E^x \int_0^\infty Y_t\, dA_t = E^x \int_0^\infty Y_t\, dB_t = E^x\{Y_L; L > 0\}.$$

To get (i), (ii) and (iii), let $Y_t = 1_\Lambda(X_{t-}^*)$, $Y_t = 1_\Lambda(X_{t-}^*)\, 1_{(s,\infty)}(t)$, and $Y_t = e^{-\alpha t}\, 1_\Lambda(X_{t-}^*)$.

Let $D \in \mathscr{E}$, and let A be the previsible additive functional associated with M_D by $M_D = U_{A_D}1$. Then by our extension of the work of Revuz, there exists a capacitary measure π_D with

$$U_A f(x) = \int u(x, y) f(y) \pi_D(dy).$$

By (11.1(i)), we may write

$$P^x\{X_{L-} \in dy, L > 0\} = u(x, y)\pi_D(dy)$$

and the measure π_D therefore charges only $E \cup B$ (but not $\hat{B}$).

12. Representation of excessive functions. We assume the framework and notation of Section 11. Let f be a finite excessive function on $\bar{E}$ for X, and assume $f(X_t)$ is a uniformly integrable supermartingale P^x for each x. We extend f by setting $f(\Gamma) = 0$ and note that f is now a natural potential for X^*. Therefore, let A be the previsible additive functional with potential f. If u^* is the potential density for X^* on $\bar{E}_\Gamma$,

$$f(x) = \int u^*(x, y) \nu_A^1(dy).$$

This is less satisfying than the result of Garcia-Alvarez (he represents integrable excessive functions) but points out the close relationship between Revuz representation and Martin boundary theory. Of course, the h-path technique used by Garcia-Alvarez may be used here to get his result, so nothing is lost.

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