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INEQUALITIES FOR B-VALUED RANDOM VECTORS WITH APPLICATIONS TO THE STRONG LAW OF LARGE NUMBERS[1]

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Analogues of the Marcinkiewicz-Zygmund and Rosenthal inequalities for Banach space valued random vectors are proved. As an application some results on the strong law of large numbers are obtained. It is proved that the Marcinkiewicz SLLN holds for every p-integrable, mean zero B-valued rv if and only if B is of Rademacher type $p(1 \leq p < 2)$.

1. Introduction. Let B be a separable Banach space. In this paper we prove some inequalities for independent B-valued random vectors (rv's) which are weak analogues of the classical Marcinkiewicz-Zygmund inequality (see, e.g., [5], page 576) and the more recent inequality of Rosenthal ([11], page 150). Then we apply the inequalities to obtain some new results on the strong law of large numbers (SLLN) for B-valued random vectors.

The inequalities are proved in Section 2. Section 3 contains some SLLN for random vectors taking values in an arbitrary Banach space; in particular, we prove an analogue of the Marcinkiewicz SLLN. In an interesting recent paper, Kuelbs and Zinn [6] have shown that many classical SLLN hold for random vectors taking values in a general Banach space if one assumes $S_n/n \to 0$ in probability; this assumption often follows from appropriate geometric conditions imposed on the space. This is the point of view adopted in Section 3.

In Section 4 we show that the class of separable Banach spaces for which the Marcinkiewicz SLLN holds for every p-integrable, mean zero B-valued rv is precisely the class of spaces of Rademacher type $p(1 \leq p < 2)$. This result "interpolates" in a natural way between Mourier's SLLN [9] and the central limit theorem of Hoffmann-Jørgensen and Pisier [4] for spaces of type 2.

2. Some inequalities for B-valued random vectors. Theorem 2.1 presents inequalities for B-valued random vectors which are weak versions of (one side of) the Marcinkiewicz-Zygmund (see, e.g., [5], page 576) and Rosenthal [11] inequalities. Of course, the presence of the term $E\|S_n\|$ makes these inequalities less effective than the classical ones. Nevertheless, under certain conditions the inequalities can play the same role as the one-dimensional ones. This will be the case, for instance, if the aim is to prove $S_n/a_n \to 0$ a.s. $(0 < a_n \uparrow \infty)$ and one can show previously that $E\|S_n\|/a_n \to 0$; this idea will be illustrated in the next section.

Let us remark that while the proof of the case $p \neq 2$ in Theorem 2.1 depends on deep martingale inequalities, the case $p = 2$ can be proved in an elementary way.

THEOREM 2.1. *For every $p \geq 1$ there exists a positive constant C_p such that for any separable Banach space B and any finite sequence $\{X_j : 1 \leq j \leq n\}$ of independent B-valued* rv's *with $X_j \in L^p (j = 1, \cdots, n)$, the following inequality holds*:

(1) *For* $1 \leq p \leq 2$,

$$E\,|\,\|S_n\| - E\|S_n\|\,|^p \leq C_p \sum_{j=1}^n E\|X_j\|^p;$$

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if $p = 2$ then it is possible to take $C_2 = 4$.

(2) *For $p > 2$,*

$$E\,|\,\|S_n\| - E\|S_n\|\,|^p \leq C_p\{(\sum_{j=1}^n E\|X_j\|^2)^{p/2} + \sum_{j=1}^n E\|X_j\|^p\}.$$

A key step in the proof is provided by the following:

LEMMA 2.1. (*Yurinskii* [13]). *Let $\{X_j : 1 \leq j \leq n\}$ be independent B-valued* rv's *with $X_j \in L^1 (j = 1, \cdots, n)$. Let $\mathscr{F}_k$ be the σ-algebra generated by $\{X_1, \cdots, X_k\} (1 \leq k \leq n)$ and let $\mathscr{F}_0$ be the trivial σ-algebra.*

Then for $1 \leq k \leq n$,

$$|E\{\|S_n\| \,|\, \mathscr{F}_k\} - E\{\|S_n\| \,|\, \mathscr{F}_{k-1}\}| \leq \|X_k\| + E\|X_k\|.$$

The proof is based on elementary properties of conditional expectations.

PROOF OF THEOREM 2.1. One may assume $p > 1$ (the case $p = 1$ being trivial). Let $\eta_j = E\{\|S_n\| \,|\, \mathscr{F}_j\} - E\{\|S_n\| \,|\, \mathscr{F}_{j-1}\}$, $j = 1, \cdots, n$. Then $\{\eta_j : 1 \leq j \leq n\}$ is a martingale difference sequence and $\|S_n\| - E\|S_n\| = \sum_{j=1}^n \eta_j$.

By Burkholder's inequality (see [3] or [12], page 149),

$$E\,|\sum_{j=1}^n \eta_j|^p \leq B_p E\{(\sum_{j=1}^n \eta_j^2)^{p/2}\},$$

where B_p is a positive constant depending only on p.

By Lemma 2.1., $E|\eta_j|^p \leq 2^p E\|X_j\|^p$. Hence we have for $1 < p \leq 2$,

$$E\,|\sum_{j=1}^n \eta_j|^p \leq B_p E(\sum_{j=1}^n |\eta_j|^p) \leq 2^p B_p \sum_{j=1}^n E\|X_j\|^p.$$

If $p = 2$, then the rv's $\eta_j (j = 1, \cdots, n)$ are orthogonal; therefore

$$E(\|S_n\| - E\|S_n\|)^2 = \sum_{j=1}^n E\eta_j^2 \leq 4\sum_{j=1}^n E\|X_j\|^2.$$

By Burkholder's martingale-theoretic generalization of Rosenthal's inequality (see [3], page 40), for $p > 2$

$$E\,|\sum_{j=1}^n \eta_j|^p \leq B_p\{E(\sum_{j=1}^n E\{\eta_j^2 \,|\, \mathscr{F}_{j-1}\})^{p/2} + \sum_{j=1}^n E|\eta_j|^p\}.$$

By Lemma 2.1 and the independence of $\{X_1, \cdots, X_n\}$,

$$E\{\eta_j^2 \,|\, \mathscr{F}_{j-1}\} \leq 4E\|X_j\|^2.$$

Inequality (2) follows at once. □

REMARK. One may obtain the inequality

$$E\,|\,\|S_n\| - E\|S_n\|\,|^p \leq C_p n^{p/2-1} \sum_{j=1}^n E\|X_j\|^p \qquad \text{for} \quad p > 2$$

either by applying Hölder's inequality to inequality (2) or by proceeding as in the proof of (1) and applying Hölder's inequality and Lemma 2.1 to Burkholder's inequality.

3. Some laws of large numbers for *B*-valued random vectors. Theorem 3.1 below is a version of the Marcinkiewicz law of large numbers for random vectors taking values in an arbitrary separable Banach space. The proof is carried out by combining the most elementary case of Theorem 2.1 ($p = 2$) and an integrability lemma (Lemma 3.1) with well-known classical methods.

LEMMA 3.1. *Let $[X_j : j \geq 1\}$ be independent symmetric B-valued* rv's, $0 < a_j \uparrow \infty$. *Assume*

(a) $\|X_j\| \leq a_j$ a.s. $(j \geq 1)$,

(b) $S_n/a_n \to_P 0$.

Then for all $p > 0$, $E\|S_n/a_n\|^p \to 0$.

PROOF. As in Lemma 2.3. of [6]. Another proof may be constructed using the converse Kolmogorov inequality as in ([2], Theorem 2.3.). □

LEMMA 3.2. *Let $\{X_j : j \geq 1\}$ be independent symmetric B-valued* rv's. *Then for any* $r > 0$,

$$\frac{S_n}{n^r} \to 0 \quad \text{a.s. if and only if} \quad \frac{S_{2^{n+1}} - S_{2^n}}{2^{nr}} \to 0 \quad \text{a.s.}$$

This is proved just as in the real-valued case (see [12], page 158).

THEOREM 3.1. *Let B be a separable Banach space,* $1 \leq p < 2$. *Let* $\{X_j : j \geq 1\}$ *be independent identically distributed B-valued* rv's *with* $E \| X_1 \|^p < \infty$, $S_n = \sum_{j=1}^n X_j$. *Then*

$$S_n/n^{1/p} \to_P 0 \quad \text{if and only if} \quad S_n/n^{1/p} \to 0 \quad \text{a.s.}$$

PROOF. Assume $S_n/n^{1/p} \to_P 0$. By a standard argument, it is enough to prove the theorem for X_j symmetric ($j \geq 1$). We proceed under this additional assumption.

Let $Y_j = X_j I\{\| X_j \| \leq j^{1/p}\}$, $T_n = \sum_{j=1}^n Y_j$. Since $E \| X_1 \|^p < \infty$ it follows that $\sum_j P\{\| X_j \| > j^{1/p}\} < \infty$ and the Borel-Cantelli lemma implies that $P(\limsup_j \{X_j \neq Y_j\}) = 0$. Thus it is enough to prove $T_n/n^{1/p} \to 0$ a.s.

From $S_n/n^{1/p} \to_P 0$ it follows at once that $T_n/n^{1/p} \to_P 0$ and by Lemma 3.1 we have

$$E \| T_n \| / n^{1/p} \to 0. \tag{3.1}$$

A classical calculation (see, e.g., [12], page 128) shows that

$$\sum_{j=1}^\infty E \| Y_j \|^2 / j^{2/p} < \infty. \tag{3.2}$$

Let $V_n = \| T_{2^{n+1}} - T_{2^n} \| - E \| T_{2^{n+1}} - T_{2^n} \|$. By (3.1) and Lemma 3.2, the proof will be completed if we can show that

$$V_n/2^{n/p} \to 0 \quad \text{a.s.}$$

Now for any $\epsilon > 0$,

$$\begin{aligned} P\{| V_n | / 2^{n/p} > \epsilon\} &\leq \frac{1}{\epsilon^2 2^{2n/p}} E V_n^2 \\ &\leq \frac{1}{\epsilon^2 2^{2n/p}} \cdot 4 \sum_{j=2^n}^{2^{n+1}} E \| Y_j \|^2 \\ &\leq \frac{4 \cdot 2^{2/p}}{\epsilon^2} \sum_{j=2^n}^{2^{n+1}} E \| Y_j \|^2 / j^{2/p}; \end{aligned}$$

in the second step we have applied Theorem 2.1 for $p = 2$. Therefore,

$$\sum_{n=1}^\infty P\{| V_n | / 2^{n/p} > \epsilon\} \leq C \sum_{j=1}^\infty E \| Y_j \|^2 / j^{2/p} < \infty$$

by (3.2.) and an application of the Borel-Cantelli lemma ends the proof. □

In a similar vein, one may combine Theorem 2.1 (for $1 \leq p < \infty$) and classical arguments to prove Banach space versions of results due to Kolmogorov, Brunk, Chung and Prohorov in the real-valued nonidentically distributed case. The proof of Theorem 3.2 is similar to that of Theorem 3.1 and will be omitted. Part (a) of Theorem 3.2 has been proved by Kuelbs and Zinn [6] by a different method (it is also possible to obtain part (b) from Theorem 1 in [6]).

THEOREM 3.2. *Let B be a separable Banach space. Let* $\{X_j : j \geq 1\}$ *be independent B-valued* rv's, $S_n = \sum_{j=1}^n X_j$. *Assume that* $S_n/n \to_P 0$.

(a) *If* $1 \leq p \leq 2$, *then* $\sum_{j=1}^\infty E \| X_j \|^p / j^p < \infty$ *implies* $S_n/n \to 0$ a.s.

(b) *If* $p \geq 2$, *then* $\sum_{j=1}^{\infty} E \| X_j \|^p / j^{1+p/2} < \infty$ *implies* $S_n/n \to 0$ a.s.

4. Spaces of Rademacher type p and the Marcinkiewicz SLLN. It is well known that if B is of Rademacher type p $(1 \leq p \leq 2)$ then there exists a positive constant C such that for any closed subspace F and any mean zero, p-integrable independent B-valued rv's $X_1, \cdots, X_n$,

$$Eq_F^p \left(\sum_{j=1}^n X_j\right) \leq C \sum_{j=1}^n Eq_F^p (X_j),$$

where q_F is the seminorm defined by $q_F(x) = \inf\{\| x - y \| : y \in F\}$.

Let $\mathscr{F}$ be the class of finite-dimensional subspaces of B, directed upward by inclusion.

THEOREM 4.1. *Let B be a separable Banach space, $1 \leq p < 2$. The following conditions are equivalent*:

(1) *B is of Rademacher type p.*

(2) *For every independent identically distributed sequence $\{X_j,\ j \geq 1\}$ of B-valued* rv'*s with $E \| X_1 \|^p < \infty$, $EX_1 = 0$, one has $S_n/n^{1/p} \to 0$* a.s.

PROOF. (1) $\Rightarrow$ (2). By Theorem 3.1., we only need to show that $S_n/n \to_P 0$. For any $\epsilon > 0$, $F \in \mathscr{F}$, using the above remark,

$$\begin{aligned} P\{q_F(S_n/n^{1/p}) > \epsilon\} &\leq \frac{1}{n\epsilon^p} Eq_F^p (S_n) \\ &\leq \frac{1}{n\epsilon^p} \cdot C \sum_{j=1}^n Eq_F^p(X_j) \\ &= \frac{C}{\epsilon^p} Eq_F^p(X_1). \end{aligned}$$

From $E \| X_1 \|^p < \infty$ it easily follows that $\lim_{F\in\mathscr{F}} Eq_F^p (X_1) = 0$. Hence $\{\mathscr{L}(S_n/n^{1/p})\}$ is flatly concentrated ([1], page 279).

By the one-dimensional Marcinkiewicz SLLN, $f(S_n/n^{1/p}) \to 0$ a.s. for every $f \in B'$. By ([1], Theorem 2.4) it follows that $\mathscr{L}(S_n/n^{1/p}) \to_w \delta_0$, which implies $S_n/n^{1/p} \to_P 0$.

(2) $\Rightarrow$ (1). The arguments for the proof of (2) $\Rightarrow$ (1) are variants of arguments contained in Pisier [10]. We will indicate the main steps. If $p = 1$ there is nothing to prove, so let us assume $p > 1$.

Put $X = X_1$ and define $N_p(X) = \sup_n E \| S_n/n^{1/p} \|$. Let $M_p = \{X \in L^1 (\Omega, \mathscr{A}, P; B) : N_p(X) < \infty\}$. Then (M_p, N_p) can be shown to be a Banach space by a standard argument.

Assume $E \| X \|^p < \infty$, $EX = 0$; then $S_n/n^{1/p} \to 0$ a.s. and arguing as in [10], Proposition 2.1, one can show that $N_p(X) < \infty$. Let $L_0^p = \{X \in L^p(\Omega, \mathscr{A}, P; B) : EX = 0\}$, $u : L_0^p \to M_p$ the inclusion map. Then u is a closed operator and the closed graph theorem implies that u is continuous. Therefore there exists a positive constant C such that

$$E \| S_n \| \leq Cn^{1/p}(E \| X_1 \|^p)^{1/p}$$

for all $X \in L_0^p$. Applying Proposition 5.1. of [10] we conclude that B is of Rademacher type p. $\square$

REMARKS (a) Theorem 4.1. contains Mourier's SLLN [9]. (b) Theorem 4.1. should be compared with the following result, obtained by Mandrekar and Zinn [7] and by Marcus and Woyczynski [8]: for $1 \leq p < 2$, B is of the stable type p if and only if for every symmetric X satisfying $n^p P\{\| X \| > n\} \to 0$ as $n \to \infty$, $S_n/n^{1/p} \to_P 0$.

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