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A LIMIT THEOREM FOR THE MAXIMUM OF AUTOREGRESSIVE PROCESSES WITH UNIFORM MARGINAL DISTRIBUTIONS[1]

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A class of first-order autoregressive processes is given for which the extreme value limit theorems of Loynes and Leadbetter do not apply. A limit theorem is derived for these processes that depends on the parameter r, an integer greater than or equal to 2.

1. Introduction. The class of processes considered in this paper are stationary, first-order autoregressive processes with uniform marginal distributions. These processes shall be referred to as uniform AR(1) processes.

DEFINITION 1.1. The uniform AR(1) processes are defined recursively as $X_n^{(r)} = \frac{1}{r} X_{n-1}^{(r)} + \epsilon_n$ where r is an integer such that $r \geqslant 2$ and the ϵ_n's are a sequence of independent, identically distributed (i.i.d.) random variables and ϵ_n is independent of $X_{n-1}^{(r)}$. The distribution of the ϵ_n's is given by $P(\epsilon_n = k/r) = 1/r$ for $k = 0, 1, 2, \ldots, r-1$, for each $n \geqslant 1$. Let $X_0^{(r)}$ be distributed uniformly on the interval $[0, 1]$ (denoted $X_0^{(r)} \sim U[0, 1]$). This defines a family of strictly stationary autoregressive processes since $1/r < 1$ for each $r \geqslant 2$ and $x_n^{(r)} \sim U[0, 1]$ for each n. A simple characteristic function argument shows that $x_n^{(r)} \sim U[0, 1]$.

2. Leadbetter's theorems. Let M_n be the maximum of $X_0, X_1, \ldots, X_n$. Loynes (1965) showed that if a strong mixing condition is satisfied, M_n appropriately normalized can only converge to one of three extreme value type distributions. Further, he showed that if a sufficient condition is satisfied, the norming constants and the limiting distribution are the same as if the sequence were i.i.d. Loynes refers to the mixing condition as uniform mixing, but strong mixing is more commonly used in the literature. Leadbetter (1974) strengthened this result by weakening the mixing condition. Under the condition he calls $D(u_n)$, the first theorem of Loynes holds and if, in addition, $D'(u_n)$ holds [$D'(u_n)$ is similar to Loynes' condition] the second theorem of Loynes also holds. A converse to Leadbetter's theorem has recently been given by Davis (1979). Since the conditions $D(u_n)$ and $D'(u_n)$ will be used in subsequent sections, Leadbetter's definitions follow.

DEFINITION 2.1. A strictly stationary sequence $\{X_n\}$ is said to satisfy the condition $D(u_n)$ if for any integers $1 \le i_1 < i_2 < \cdots < i_p < j_1 < \cdots < j_q \le n$ with $j_1 - i_p \ge l$, $|F_{i_1,i_2,\ldots,i_p,j_1,\ldots,j_q}(u_n) - F_{i_1,i_2,\ldots,i_p}(u_n)F_{j_1,j_2,\ldots,j_q}(u_n)| \le \alpha_{n,l}$ where $\lim_{l\to\infty} \lim_{n\to\infty} \alpha_{n,l} = 0$ and $\{u_n\}$ is some sequence of real numbers. $F_{i_1,i_2,\ldots,i_p}(u_n)$ is notation for $P[X_{i_1} \le u_n, X_{i_2} \le u_n, \cdots, X_{i_p} \le u_n]$.

DEFINITION 2.2. $D'(u_n)$ is said to hold if

$$\limsup n_{n\to\infty} \textstyle\sum_{j=2}^{n} P[X_1 > u_{nk}, X_j > u_{nk}] = o\left(\frac{1}{k}\right).$$

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When $D'(u_n)$ holds in addition to $D(u_n)$ Leadbetter gets the following theorem.

THEOREM 2.3. *Let* $M_n = \max(X_0, X_1, \cdots, X_n)$ *and* $M_n^* = \max(X_0^*, X_1^*, \cdots X_n^*)$ *where* $\{X_n\}$ *is a strictly stationary sequence with* X_i *having distribution function* F *for each* i *and* $\{X_n^*\}$ *is an* i.i.d. *sequence with* X_i^* *also having distribution function* F. *Let* $u_n = u_n(x) = a_n x + b_n$. *If for each* x, $D(u_n)$ *and* $D'(u_n)$ *hold and* $P[M_n^* \le a_n x + b_n] \to G(x)$ *where* G *is one of the three extreme value type distributions, then* $P[M_n \le a_n x + b_n] \to G(x)$ *also.*

It is worthwhile noting that another theorem of Loynes holds with $D(u_n)$ replacing strong mixing. This is an improved version of Theorem 2 in Loynes (1965), page 995.

THEOREM 2.4. *Let* $\{X_n\}$ *be a strictly stationary sequence satisfying* $D(u_n)$ *and suppose for* $u_n = u_n(x)$ $P[X_1 > u_n] \approx (\tau(x)/n)$ *where* $A_n \approx B_n$ *means* $A_n/B_n \to 1$ *as* $n \to \infty$. *Then* $P[M_n \le u_n] \to \exp(-k\tau(x))$ *for some* $0 < k \le 1$, *provided it converges.*

This theorem was not given by Leadbetter (1974), but it follows quite easily. Leadbetter shows that $D(u_n)$ implies

$$\lim_{m\to\infty} P(M_n \le u_n) - P^l(M_m \le u_n) = 0 \tag{2.1}$$

for $n = ml$. Loynes (1965) showed that (2.1) for $l = 2$ is sufficient to determine that the limit can only have the form $\exp(-k\tau(x))$ for some $0 < k \le 1$. Since (2.1) is all he really uses in the proof, the combination of these two results yields Theorem 2.4.

O'Brien (1974) gives examples which show that all k between zero and one are possible. In Section 4 it will be shown that $P[M_n^{(r)} \le u_n] \to \exp(-(r-1)x/r)$ where $M_n^{(r)} = \max\{X_0^{(r)}, X_1^{(r)}, \cdots, X_n^{(r)}\}$, and $\{X_i^{(r)}\}$ is a uniform, AR(1) process. Theorem 2.4 holds for the uniform processes with $k = (r-1)/r$.

3. Checking Leadbetter's conditions.

THEOREM 3.1. *The uniform* AR(1) *processes satisfy the condition* $D(u_n)$ *with* $\alpha_{n,l} = \rho^l/(1-\rho)$ *and* $u_n = 1 - (x/n)$ *where* $\rho = 1/r$.

PROOF. For convenience we suppress the superscript r. Since $1/r > 0$, Theorem 1 in Chernick (1977) applies and $\{X_n\}$ is an associative process. Hence

$$\begin{aligned} 0 &\le F_{i_1,\ldots,i_p,j_1,\ldots,j_q}(u_n) - F_{i_1,\ldots,i_p}(u_n)F_{j_1,\ldots,j_q}(u_n) \\ &= F_{i_1,\ldots,i_p}(u_n)[P(X_{j_1} \le u_n, \cdots, X_{j_q} \le u_n \mid X_{i_1} \le u_n, \cdots, X_{i_p} \le u_n) \\ &\quad - P(X_{j_1} \le u_n, \cdots, X_{j_q} \le u_n)]. \end{aligned}$$

Let $W_{j_t} = X_{j_t} - \rho^{j_t - i_p} X_{i_p}$ for $t = 1, 2, \cdots, q$.

$$\begin{aligned} P(X_{j_1} &\le u_n, \cdots, X_{j_q} \le u_n \mid X_{i_1} \le u_n, \cdots, X_{i_p} \le u_n) \\ &\le P(W_{j_1} \le u_n, \cdots, W_{j_q} \le u_n) \\ &= P(X_{j_1} \le u_n + \rho^{j_1 - i_p} X_{i_p}, \cdots, X_{j_q} \le u_n + \rho^{j_q - i_p} X_{i_p}) \\ &\le P(X_{j_1} \le u_n, \cdots, X_{j_q} \le u_n) + \textstyle\sum_{t=1}^q P[u_n \le X_{j_t} \le u_n + \rho^{j_t - i_p}] \\ &\le P(X_{j_1} \le u_n, \cdots, X_{j_q} \le u_n) + \rho^l/(1-\rho). \end{aligned}$$

Computations show $\liminf_{k\to\infty} \limsup_{n\to\infty} nk \sum_{j=2}^n P(X_1 > u_{nk}, X_j > u_{nk}) \ge x/(r-1) > 0$. So $D'(u_n)$ fails.

4. The Limit theorem for the uniform AR(1) processes. For certain values of $m < n$, we consider $P[M_m \le 1 - (x/n)]$. We have $P[M_m \le 1 - (x/n)] = P[M_{m-1} \le 1 - (x/n), X_m \le 1 - (x/n)] = P[M_{m-1} \le 1 - (x/n), (X_{m-1}/r) + \epsilon_m \le 1 - (x/n)]$. Conditioning on ϵ_m gives

$$(4.1)\quad P\left[M_m \le 1 - \frac{x}{n}\right] = \frac{(r-1)}{r} P\left[M_{m-1} \le 1 - \frac{x}{n}\right] + \frac{1}{r} P\left[M_{m-1} \le 1 - \frac{x}{n}, X_{m-1} \le 1 - r\frac{x}{n}\right]$$

if $(r-1)x/n < 1$. We also have for each $1 \le i \le j-2$

$$(4.2)\quad P\left[M_m \le 1 - \frac{x}{n}, X_m \le 1 - r^i\frac{x}{n}\right] = \frac{(r-1)}{r} P\left(M_{m-1} \le 1 - \frac{x}{n}\right) + \frac{1}{r} P\left(M_{m-1} \le 1 - \frac{x}{n}, X_{m-1} \le 1 - r^{i+1}\frac{x}{n}\right)$$

where j is the integer for which $1 - r^{j-1}(x/n) \ge 0 > 1 - r^j(x/n)$.

These recursive type formulae enable one to calculate $P[M_m \le 1 - (x/n)]$ for every $m \le j-1$. Repeated application of (4.1) and (4.2) yield

$$(4.3)\quad P\left[M_m \le 1 - \frac{x}{n}\right] = 1 - \frac{\{(m+1)r - m\}x}{rn} \quad \text{for each} \quad m \le j-1.$$

Let $k = [n/j]$. We see that $P^k[M_{j-1} \le 1 - (x/n)] \to \exp(-((r-1)x/r))$ as $j, k \to \infty$.

THEOREM 4.1. *For the uniform* AR(1) *processes*

$$\{X_n^{(r)}\}_{n=0}^{\infty}\ r \ge 2, \quad P\left[M_n^{(r)} \le 1 - \frac{x}{n}\right] \to \exp\left(-\frac{(r-1)x}{r}\right) \quad \textit{for all} \quad x \ge 0.$$

PROOF. We will show for $k = [n/j]$

$$(4.4)\quad \left|P\left[M_n \le 1 - \frac{x}{n}\right] - P^k\left[M_{j-1} \le 1 - \frac{x}{n}\right]\right| \to 0 \quad \text{as } n \text{ and } j \to \infty.$$

We define $I_i = \{(i-1)j, \cdots, ij - 1 - m\}$ and $I_i^* = \{ij - m, \cdots, ij - 1\}$ for $i = 1, 2, \cdots, k$. We choose m so that $m \to \infty$ and $m/j \to 0$ as $j, n \to \infty$. We denote by $M(I_i)$ the maximum of the X_j's for $j \in I_i$.

$$\begin{aligned}
&\left|P\left[M_n \le 1 - \frac{x}{n}\right] - P^k\left[M_{j-1} \le 1 - \frac{x}{n}\right]\right| \\
&\le \left|P\left[M_n \le 1 - \frac{x}{n}\right] - P\left[M_{jk} \le 1 - \frac{x}{n}\right]\right| \\
&\quad + \left|P\left[\cap_{i=1}^k \left(M(I_i) \le 1 - \frac{x}{n}\right)\right] - P\left[M_{jk} \le 1 - \frac{x}{n}\right]\right| \\
&\quad + \left|P\left[\cap_{i=1}^k \left(M(I_i) \le 1 - \frac{x}{n}\right)\right] - P^k\left(M(I_1) \le 1 - \frac{x}{n}\right)\right| \\
&\quad + \left|P^k\left(M(I_1) \le 1 - \frac{x}{n}\right) - P^k\left(M_{j-1} \le 1 - \frac{x}{n}\right)\right|.
\end{aligned}$$

We will show that all four terms tend to zero as j, m and $n \to \infty$. Using (4.3) and following Lemma 2.4 of Leadbetter (1974), it is easy to see that the first, second, and fourth terms tend to zero.

It remains only to be shown that the third term tends to zero. Let $A_i = \{M(I_i) \le 1 - (x/n)\}$ for $i = 1, 2, \cdots, k$. We have

$$|P(\cap_{i=1}^k A_i) - \textstyle\prod_{i=1}^k P(A_i)| \le P(A_k \mid \cap_{i=1}^{k-1} A_i) - P(A_k) + P(A_{k-1} \mid \cap_{i=1}^{k-2} A_i) - P(A_{k-1}) + \cdots + P(A_2 \mid A_1) - P(A_2).$$

We shall show that for any $k \ge s \ge 2$

$$0 \le P(A_s \mid \cap_{i=1}^{s-1} A_i) - P(A_s) < \frac{3x}{n}$$

and hence

$$|P(\cap_{i=1}^k A_i) - \textstyle\prod_{i=1}^k P(A_i)| < \frac{3kx}{n} \le \frac{3x}{j}.$$

Let $q = j - 1 - m$. We have $P(A_s) = 1 - (q + 1)x/n) + (qx/rn)$ and

$$P\left(A_s \,\middle|\, \cap_{i=1}^{s-1} A_i\right) = \int_0^{1-x/n} P\left[X_{(s-1)j} \le 1 - \frac{x}{n}, \cdots, X_{sj-1-m} \le 1 - \frac{x}{n} \,\middle|\, X_{(s-1)j-1-m} = v\right] dF_s(v) \tag{4.5}$$

where

$$F_s(v) = P[X_{(s-1)j-1-m} \le v \mid \cap_{i=1}^{s-1} A_i].$$

By stationarity for each s the integrand in (4.5) is equal to

$$P\left[X_{q+m+1} \le 1 - \frac{x}{n}, \cdots, X_{2q+m+1} \le 1 - \frac{x}{n} \,\middle|\, X_q = v\right]. \tag{4.6}$$

For each l, $X_{q+1} = \rho^l X_q + W_l$ where W_l is a discrete uniform random variable with $P[W_l = (h/r^l)] = (1/r^l)$ for $h = 0, 1, 2, \cdots, (r^l - 1)$. Now $P[X_{q+m+1} \le 1 - (x/n), \cdots, X_{2q+m} \le 1 - (x/n) \mid X_q = v] = P[W_{m+1} \le 1 - (x/n) - \rho^{m+1}v, \cdots, W_{m+q} \le 1 - (x/n) - \rho^{m+q}v]$ and we observe that for $m + 1 \le l \le m + q$

$$P\left[W_l \le 1 - \frac{x}{n} - \rho^l v\right] = 1 \qquad \text{if} \quad v \le 1 - \frac{r^l x}{n}$$

$$= \frac{r^l - 1}{r^l} \qquad \text{if} \quad v > 1 - \frac{r^l x}{n}.$$

This observation together with (4.5) and (4.6) show that

$$\begin{aligned} P(A_s \mid \cap_{i=1}^{s-1} A_i) \le \int_0^{1-r^{j-1}x/n} dF_s(v) + \int_{1-r^{j-1}x/n}^{1-r^{j-2}x/n} P\left[W_{m+q} \le 1 - \frac{x}{n} - \rho^{m+q}v\right] dF_s(v) \\ + \int_{1-r^{j-2}x/n}^{1-r^{j-3}x/n} P\left[W_{m+q-1} \le 1 - \frac{x}{n} - \rho^{m+q-1}v,\ W_{m+q} \le 1 - \frac{x}{n} - \rho^{m+q}v\right] dF_s(v) \\ + \cdots + \int_{1-r^{m+1}x/n}^{1-x/n} P\left(W_{m+1} \le 1 - \frac{x}{n} - \rho^{m+1}v, \cdots, W_{m+q} \le 1 - \frac{x}{n} - \rho^{m+q}v\right) dF_s(v). \end{aligned} \tag{4.7}$$

It can be shown that for each $0 \le v \le 1$

$$v = P[X_{(s-1)j-1-m} \le v] \le F_s(v) \le P[W_q \le v] \le v + \rho^q. \tag{4.8}$$

The integrands in (4.7), $P[W_l \le 1 - (x/n) - \rho^l v, \cdots, W_{m+q} \le 1 - (x/n) - \rho^{m+q} v] \le P[W_l \le 1 - (x/n) - \rho^l v] = 1 - \rho^l$ over the interval of integration for each l. So (4.7) simplifies to give

$$P(A_s \mid \cap_{i=1}^{s-1} A_i) \le F_s\left(1 - r^{j-1}\frac{x}{n}\right) + \sum_{l=1}^{q-1}\left(1 - \frac{1}{r^{j-l}}\right)\left[F_s\left(1 - r^{j-l-1}\frac{x}{n}\right) - F_s\left(1 - r^{j-l}\frac{x}{n}\right)\right] + \left\{1 - F_s\left(1 - r^{m+1}\frac{x}{n}\right)\right\}\left(1 - \frac{1}{r^{m+1}}\right). \tag{4.9}$$

Repeated use of (4.8) in (4.9) together with some algebraic simplification yields

$$P(A_s \mid \cap_{i=1}^{s-1} A_i) \le 1 - \frac{(q+1)x}{n} + \frac{qx}{rn} + \frac{\rho^q}{r^{m+1}} + \rho^q \sum_{l=1}^{q-1} \frac{1}{r^{j-l}}$$
$$\le 1 - \frac{(q+1)x}{n} + \frac{qx}{rn} + \frac{1}{r^j}\left(2 - \frac{1}{r^{j-m-2}}\right).$$

Hence

$$P(A_s \mid \cap_{i=1}^{s-1} A_i) - P(A_s) < \frac{x}{n} + \frac{2}{r^j} \le \frac{3x}{n}.$$

This shows that the third term tends to zero as $n, j \to \infty$ and completes the proof of the theorem.

This result illustrates the importance of the condition $D'(u_n)$ in Theorem 3.1 Leadbetter (1974). It also shows that simple autoregressive processes can fail to satisfy the condition and, hence, different limits can be expected.

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