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THE GROWTH OF RANDOM WALKS AND LÉVY PROCESSES

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Let $\{X_i\}$ be a sequence of independent, identically distributed non-degenerate random variables taking values in $\mathbb{R}^d$ and $S_n = \Sigma_{i=1}^n X_i$, $M_n = \max_{1\le i\le n} |S_i|$. Define for $x > 0$, $G(x) = P\{|X_1| > x\}$, $K(x) = x^{-2}E(|X_1|^2 1\{|X_1| \le x\})$, $M(x) = x^{-1}|E(X_1 1\{|X_1| \le x\})|$, and $h(x) = G(x) + K(x) + M(x)$. Then if $\beta = \sup\{\alpha : \limsup x^\alpha h(x) = 0\}$, $\delta = \sup\{\alpha : \liminf x^\alpha h(x) = 0\}$, it is proved that $n^{-1/\alpha}M_n \to 0$ for $\alpha < \beta$, $\to \infty$ for $\alpha > \delta$, while the lim inf is 0 and the lim sup is ∞ for $\beta < \alpha < \delta$. Some alternative characterizations of the indices β, δ are obtained as well as the analogous results for Lévy processes.

1. Introduction. Let $\{X_i\}$ be a sequence of independent, identically distributed non-degenerate random variables taking values in $\mathbb{R}^d$ and $S_n = \Sigma_{i=1}^n X_i$, $M_n = \max_{i\le n} |S_i|$. The problem is to obtain bounds on the rate of growth of M_n. Let F denote the distribution function of X_1, and X denote a random variable with this distribution. Define, for $x > 0$,

$$G(x) = P\{|X| > x\}, \qquad K(x) = x^{-2}\int_{|y|\le x} |y|^2\, dF(y) \tag{1.1}$$
$$M(x) = x^{-1}\left|\int_{|y|\le x} y\, dF(y)\right|, \qquad h(x) = G(x) + K(x) + M(x).$$

First we will obtain the relatively simple bounds

$$P\{M_n \ge a\} \le Cnh(a), \qquad P\{M_n \le a\} \le \frac{C}{nh(a)}. \tag{1.2}$$

From these bounds it follows readily that if we let

$$\beta = \sup\{\alpha : \limsup_{x\to\infty} x^\alpha h(x) = 0\}, \qquad \delta = \sup\{\alpha : \liminf_{x\to\infty} x^\alpha h(x) = 0\}, \tag{1.3}$$

then the rate of growth of M_n relative to powers is almost completely determined by

$$\limsup n^{-1/\alpha}M_n = \begin{cases} 0 & \text{if } \alpha < \beta \\ \infty & \text{if } \alpha > \beta \end{cases}, \qquad \liminf n^{-1/\alpha}M_n = \begin{cases} 0 & \text{if } \alpha < \delta \\ \infty & \text{if } \alpha > \delta \end{cases}, \tag{1.4}$$

with probability one. Other results may be easily obtained from the bounds in (1.2). For example, the expected first passage time out of the ball of radius a, centered at the origin, is comparable to $\{h(a)\}^{-1}$. Functions similar to h have recently been used for a variety of purposes; for example, see [2], [7]–[10].

It is clear that the first result in (1.4) is unchanged if M_n is replaced by $|S_n|$. Complete information about the behavior of lim sup $a_n^{-1}|S_n|$ is available in Feller's (1946) paper on upper envelopes. Sharper forms of the second result in (1.4) are also known in special cases, for example if F is in the domain of attraction of a stable law [5]. The interest here is that both results are universal, the proof is relatively easy and shows the duality between the two results, and it probably gives all the information one would ordinarily want about

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the rate of growth of M_n. The problem of obtaining information about lim inf $n^{-1/\alpha} | S_n |$ seems to be much more difficult, and results are only known in special cases.

This work was motivated by the fundamental paper of Blumenthal and Getoor (1961) which defines certain indices for Lévy processes and determines some properties of the sample paths of the processes in terms of these indices. In particular, they obtain the analogue of the first result in (1.4). However, the correct index for the second result in (1.4) has not been found and this was mentioned as an open problem in Taylor's (1973) survey paper. This problem has a solution analogous to (1.4). G and K are defined as in (1.1) with F replaced by the Lévy measure, but the definition of M has to be changed somewhat. Then δ is defined analogously to (1.3); see (3.1) and (3.3).

The results for sums of independent random variables are given in the next section. Alternative definitions of the indices β, δ are also mentioned. The results for Lévy processes are in the final section. Since the proofs are similar, only the differences are noted in this case. The new index δ is compared to other indices that have been defined earlier and an example is given.

2. Sums of Independent Random Variables. We start with the proof of the fundamental lemma (1.2). To see that it really suffices to consider the one dimensional case, let $S_n = (S_n^1, \cdots, S_n^d)$ and $M_n^j = \max_{i \le n} | S_i^j |$ and then

(2.1) $$\max_{1 \le j \le d} M_n^j \le M_n \le \Sigma_{j=1}^d M_n^j.$$

Similarly, if $h_j(x)$ denotes the function defined as in (1.1) for X^j where $X = (X^1, \cdots, X^d)$, then straightforward but slightly tedious estimates show that there are constants c, C, depending only on the dimension d, such that

(2.2) $$ch(a) \le \Sigma_{j=1}^d h_j(a) \le Ch(a).$$

Using (2.1), (2.2), and (2.3) with the one dimensional version of (1.2) then gives the d dimensional version. Alternatively, one may simply replace $h(a)$ by $\Sigma\, h_j(a)$ in (1.2) and then there is no need to prove (2.2).

LEMMA. *Define h as in* (1.1). *Then there is a constant C, depending only on the dimension d, such that for all n and $a > 0$*

$$P\{M_n \ge a\} \le Cnh(a), \qquad P\{M_n \le a\} \le \frac{C}{nh(a)}.$$

PROOF. We may assume that the X_i are real valued as indicated above. Let $T_n = \Sigma_{i=1}^n X_i 1\{| X_i | \le a\}$. Then

$$| ET_n | = n \left| \int_{|x| \le a} x \, dF(x) \right| = naM(a)$$

so that if $| ET_n | \ge a/2$ we have

$$P\{M_n \ge a\} \le 1 \le 2nM(a) \le 2nh(a).$$

On the other hand, if $| ET_n | < a/2$, then we have by Kolmogorov's inequality

$$\begin{aligned} P\{M_n \ge a\} &\le P\{\max_{i \le n} | X_i | > a\} + P\{\max_{i \le n} | T_i | \ge a\} \\ &\le nG(a) + P\{\max_{i \le n} | T_i - ET_i | \ge a/2\} \\ &\le nG(a) + \frac{na^2K(a)}{(a/2)^2} \le 4nh(a). \end{aligned}$$

We will prove the second inequality with $h(2a)$ in place of $h(a)$. The reader may show that

$h(a)$ and $h(2a)$ are comparable; in fact, for $C > 1$

(2.3) $$\frac{1}{2C^2} \le \frac{h(Ca)}{h(a)} \le 2.$$

We abuse our notation slightly by letting T_n denote the sum of the X_i truncated at $2a$ instead of a in this part of the proof. The reason for this is that $M_n \le a$ implies that $|X_i| \le 2a$ for $i \le n$ and so $M_n = \max_{i\le n} |T_i|$. There are three cases depending roughly on which of $G(2a)$, $K(2a)$, and $M(2a)$ is dominant. First suppose that

(2.4) $$K(2a) \ge 2G(2a) + 2M(2a).$$

Then by Esseen's version of the concentration function inequality [2, page 295], we have

$$P\{|S_n| \le 2a\} \le \frac{C4a}{\{n(4a)^2K^s(4a)\}^{1/2}} = \frac{C}{\{nK^s(4a)\}^{1/2}}$$

where K^s is defined as in (1.1) for the symmetrized variable $X_1 - X_2$. (This inequality is considerably simpler than the concentration function inequality in its full generality. A proof is given at the end of this paper.) Now

$$K^s(4a) = (4a)^{-2}\int_{|X_1-X_2|\le 4a}(X_1 - X_2)^2\,dP \ge (4a)^{-2}\int_{|X_1|\le 2a,|X_2|\le 2a}(X_1 - X_2)^2\,dP$$

$$= \frac{1}{2}K(2a)\{1 - G(2a)\} - \frac{1}{2}\{M(2a)\}^2.$$

Since $K(2a) \le 1$ we have by (2.4) that

$$G(2a) \le 1/2, \qquad \{M(2a)\}^2 \le 1/4\{K(2a)\}^2 \le 1/4K(2a).$$

Thus

$$K^s(4a) \ge 1/8K(2a)$$

and so

(2.5) $$P\{|S_n| \le 2a\} \le \frac{8^{1/2}C}{\{nK(2a)\}^{1/2}}.$$

Now this gives the desired bound by letting $m = [n/2]$ and noting that

(2.6) $$P\{M_n \le a\} \le P\{|S_m| \le 2a, |S_n - S_m| \le 2a\}$$

since we also have under (2.4) that

$$K(2a) \ge 1/2K(2a) + G(2a) + M(2a) \ge 1/2h(2a).$$

Now we must deal with the case when (2.4) fails. If we also have $G(2a) \ge M(2a)$ then

$$P\{M_n \le a\} \le P\{\max_{i\le n}|X_i| \le 2a\} = \{1 - G(2a)\}^n \le e^{-nG(2a)} \le \frac{1}{nG(2a)}$$

and then

(2.7) $$G(2a) \ge 1/2G(2a) + 1/2M(2a) \ge 1/4G(2a) + 1/4M(2a) + 1/8K(2a) \ge 1/8h(2a)$$

since (2.4) fails. Finally if $M(2a) \ge G(2a)$, there are two possibilities. If $nM(2a) \le 1$, we have

(2.8) $$P\{M_n \le a\} \le 1 \le \frac{1}{nM(2a)}$$

while if $nM(2a) \ge 1$ we have

$$|ET_n| = n2aM(2a) \ge 2a$$

and so

$$P\{M_n \le a\} \le P\{|T_n| \le a\} \le P\{|T_n - ET_n| \ge ½|ET_n|\}$$
$$\le \frac{n(2a)^2K(2a)}{\{naM(2a)\}^2} = \frac{4K(2a)}{n\{M(2a)\}^2}.$$

Since (2.4) fails and M dominates G, we have

$$K(2a) \le 2G(2a) + 2M(2a) \le 4M(2a)$$

so we are led again to the bound in (2.8) with a constant of 16. Finally the argument in (2.7) gives a lower bound of $M(2a) \ge h(2a)/8$ in this case.

REMARK. The second bound of the lemma is not at all sharp. In fact, for any fixed integer k, we may split the interval $[1, n]$ into k blocks of length approximately n/k and the maximum of the sums of the X_i in each block will be at most $2a$ when $M_n \le a$; this is similar to (2.6). Thus the given bound leads to an improved bound of $C_k\{nh(2a)\}^{-k}$ where C_k now depends on k as well as d. As above, $h(2a)$ may be replaced by $h(a)$. As an example of the usefulness of this observation, we obtain the estimate for expected first passage times.

THEOREM 1. *Let*

$$S(a) = \min\{j : |S_j| > a\}.$$

Then there are constants c, C depending only on the dimension d such that

$$\frac{c}{h(a)} \le ES(a) \le \frac{C}{h(a)}.$$

PROOF. Note that $\{S(a) > n\} = \{M_n \le a\}$ and so

(2.9) $$ES(a) = \Sigma_{n=1}^{\infty} P\{S(a) \ge n\} = 1 + \Sigma_{n=1}^{\infty} P\{M_n \le a\}$$
$$\le N + \Sigma_{n=N}^{\infty} \frac{C}{n^2\{h(a)\}^2} \le N + \frac{C}{(N-1)\{h(a)\}^2}.$$

Since $h(a) \le 2$, we may take $N = [6/h(a)]$ and have $N - 1 \ge 2/h(a)$. This gives the upper bound. For the lower bound we use the first inequality of the lemma. If $n \le 1/2Ch(a)$, then $P\{M_n \ge a\} \le ½$, so by (2.9) we have

$$ES(a) \ge \frac{1}{2}\left(1 + \left[\frac{1}{2Ch(a)}\right]\right) \ge \frac{1}{4Ch(a)}.$$

Now we will prove the main result about the rate of growth of M_n.

THEOREM 2. *Let β, δ be defined as in* (1.3). *Then if $\alpha < \beta$, $n^{-1/\alpha}M_n \to 0$* a.s., *if $\alpha > \delta$, $n^{-1/\alpha}M_n \to \infty$* a.s., *while if $\beta < \alpha < \delta$, we have*

$$\liminf n^{-1/\alpha}M_n = 0, \qquad \limsup n^{-1/\alpha}M_n = \infty, \quad \text{a.s.}$$

PROOF. First suppose that $\alpha < \beta$ and take α_1, α_2 so that $\alpha < \alpha_2 < \alpha_1 < \beta$. By the lemma and the definition of β

$$P\{M_n \ge n^{1/\alpha_2}\} \le Cnh(n^{1/\alpha_2}) \le Cn^{1-\alpha_1/\alpha_2}$$

for large n. Thus, taking $n_k = 2^k$, we have $M_{n_k} \le n_k^{1/\alpha_2}$ for large k by Borel Cantelli. Then for $n_k \le n < n_{k+1}$, and k large

$$M_n \le M_{n_{k+1}} \le n_{k+1}^{1/\alpha_2} \le (2n)^{1/\alpha_2} \le n^{1/\alpha}2^{1/\alpha_2}n^{(\alpha-\alpha_2)/\alpha\alpha_2}$$

which gives the first result. The second is similar; if $\delta < \alpha_1 < \alpha_2 < \alpha$, then

$$P\{M_n \le n^{1/\alpha_2}\} \le \frac{C}{nh(n^{1/\alpha_2})} \le Cn^{-1+\alpha_1/\alpha_2}$$

for large n and the rest of the argument proceeds very much as above. Now suppose that $\beta < \alpha_1 < \alpha$. By the definition of β, there is a sequence $\{x_k\}$ with $x_k \to \infty$ and $x_k^{\alpha_1} h(x_k) \to \infty$. Let $n_k = [x_k^{\alpha_1}]$. Then for large k

$$P\{M_{n_k} \le x_k\} \le \frac{C}{n_k h(x_k)} \le \frac{2C}{x_k^{\alpha_1} h(x_k)} \to 0$$

so that

$$P\{M_{n_k} \ge n_k^{1/\alpha_1}\} \ge P\{M_{n_k} \ge x_k\} \to 1$$

and so $P\{M_{n_k} \ge n_k^{1/\alpha_1} \text{ i.o.}\} = 1$. This means that lim sup $n^{-1/\alpha_1} M_n \ge 1$ and so lim sup $n^{-1/\alpha} M_n = \infty$. The proof that lim inf $n^{-1/\alpha} M_n = 0$ for $\alpha < \delta$ is completely analogous.

Some general remarks about the indices β and δ will conclude this section. We expect that the slowest rate of growth for M_n should be $n^{1/2}$ when $E|X|^2 < \infty$ and $EX = 0$. This is confirmed since $h(x) \ge K(x) \ge cx^{-2}$ in any case so that $\beta \le 2$; in fact, even $\delta \le 2$. In case $E|X|^2 < \infty$ and $EX = 0$, we have $h(x) \sim x^{-2} E|X|^2$ so that $\beta = \delta = 2$ as we know from the laws of the iterated logarithm of Hartman-Wintner (1941) and Chung [6], respectively. If $E|X| < \infty$ and $EX \ne 0$ then by the strong law $n^{-1} S_n \to EX$ and so $n^{-1} M_n \to |EX|$. In this case it is not hard to check that $h(x) \sim M(x) \sim x^{-1}|EX|$ and so $\beta = \delta = 1$. If we then assume that $EX = 0$ whenever $E|X| < \infty$, then the reader may verify that

$$\beta = \sup\left\{\alpha \in [0, 2]: \int |x|^\alpha \, dF(x) < \infty\right\}. \tag{2.10}$$

In this form we see that the first half of (1.4) follows immediately from Feller's (1946) general result since $\Sigma\, G(n^{1/\alpha}) < \infty$ iff $E|X|^\alpha < \infty$. Another characterization of δ is given by

THEOREM 3. *Let* $M'_n = \max\{M_n, 1\}$. *Then*

$$\delta = \inf\{\alpha: \Sigma_n E\{M'_n\}^{-\alpha} < \infty\}.$$

PROOF. Let H_n denote the distribution function of M_n and $H = \Sigma H_n$. Then integration by parts shows that for $\alpha > 0$

$$E\{M'_n\}^{-\alpha} = \alpha \int_1^\infty x^{-\alpha-1} H_n(x)\, dx$$

and so

$$\Sigma_n E\{M'_n\}^{-\alpha} = \alpha \int_1^\infty x^{-\alpha-1} H(x)\, dx \tag{2.11}$$

with the understanding that if one side is infinite so is the other. Now for $a \ge 1$

$$\alpha \int_1^\infty x^{-\alpha-1} H(x)\, dx \ge \alpha \int_a^\infty x^{-\alpha-1} H(x)\, dx \ge H(a) a^{-\alpha}, \tag{2.12}$$

while by (2.9) and Theorem 1 we have

$$H(a) = ES(a) - 1 \ge \frac{c}{h(a)} - 1 \ge \frac{c_1}{h(a)} \tag{2.13}$$

for large a. By (2.11)–(2.13), we see that if $\Sigma\, E\{M'_n\}^{-\alpha}$ converges, then $h(a)a^\alpha$ is bounded below so that $\alpha \geq \delta$. On the other hand, if $\alpha_1 > \delta$ then $x^{\alpha_1}h(x) \geq 1$ for large x so that again by (2.9) and Theorem 1

$$H(a) = ES(a) - 1 \leq \frac{C}{h(a)} \leq Ca^{\alpha_1}$$

for large a. Then (2.11) shows that $\Sigma\, E\{M'_n\}^{-\alpha}$ converges for $\alpha > \alpha_1$.

It is also possible to calculate β and δ directly from the characteristic function. For simplicity, assume that $d = 1$ and let f be the characteristic function of X. Then $h(x)$ behaves like $|1 - f(x^{-1})|$ in nice cases but this is not true in general. However, there are universal constants c, C such that

$$ch(x) \leq x \left| \int_0^{x^{-1}} (1 - f(u))\, du \right| \leq Ch(x).$$

This is proved by considering separately the real and imaginary parts of $1 - f$ and using the standard estimates for $\sin x$ and $1 - \cos x$.

Finally, for $d > 1$, we consider the relation between the indices and their one-dimensional counterparts. If we let β_j, δ_j be the analogous indices for the components X^j, then it follows readily from (2.2) that $\beta = \min_{1\leq j\leq d} \beta_j$, $\delta \leq \min_{1\leq j\leq d} \delta_j$; this is also clear from Theorem 2. If h_j is regularly varying for all j (regular variation of G_j implies this), then $\beta_j = \delta_j$ for all j and so $\delta \geq \beta = \min \beta_j = \min \delta_j$. However, in general the times when the small values of M_n^j occur may vary for the different components, and M_n may not get as small as its "maximal" component, i.e. it is possible that $\delta < \min \delta_j$. As an example, let $x_n = 2^{2^n}$ and let X have mass

$$x_{2n+1}^{-1} \quad \text{at} \quad (0, \pm x_{2n+1}), \qquad n = 0, 1, 2, \cdots$$

$$x_{2n}^{-1} \quad \text{at} \quad (\pm x_{2n}, 0), \qquad n = 1, 2, \cdots,$$

with the remaining mass at zero. Then it is easy to see that

$$h_1(x) \sim 2x_{2n+2}^{-1} + 2x_{2n}x^{-2}, \qquad x_{2n} \leq x < x_{2n+2}$$

$$h_2(x) \sim 2x_{2n+1}^{-1} + 2x_{2n-1}x^{-2}, \qquad x_{2n-1} \leq x < x_{2n+1}.$$

Then it follows easily that $\beta = \beta_1 = \beta_2 = 1$ but $\delta = 4/3$, while $\delta_1 = \delta_2 = 8/5$.

3. Lévy Processes. A Lévy process is one with stationary independent increments, taking values in $\mathbb{R}^d$, and characteristic function $E\exp\{i(u, X_t)\} = \exp\{t\psi(u)\}$ where

$$\psi(u) = i(b, u) + \int \left(e^{i(u,x)} - 1 - \frac{i(u, x)}{1 + |x|^2} \right) d\nu(x)$$

with $b \in \mathbb{R}^d$ and ν a Borel measure on $\mathbb{R}^d$ satisfying

$$\int \frac{|x|^2}{1 + |x|^2}\, d\nu(x) < \infty.$$

It is also customary to include a Gaussian part, but since its behavior is well known we will omit this component in order to simplify the formulas. We will assume that $X_0 = 0$ and that we are dealing with a version which has almost all sample functions right continuous and having left limits.

We define, for $x > 0$,

$$G(x) = \nu\{y : |y| > x\}, \qquad K(x) = x^{-2} \int_{|y|\leq x} |y|^2\, d\nu(y)$$

$$M(x) = x^{-1} \left| b + \int_{|y|\leq x} \frac{y|y|^2}{1 + |y|^2}\, d\nu(y) - \int_{|y|>x} \frac{y}{1 + |y|^2}\, d\nu(y) \right| \tag{3.1}$$

$$h(x) = G(x) + K(x) + M(x).$$

Then if we let $M_t = \sup_{0 \le s \le t} |X_s|$, we have

$$P\{M_t \ge a\} \le Cth(a), \qquad P\{M_t \le a\} \le \frac{C}{th(a)}. \tag{3.2}$$

Furthermore, if we define

$$\beta = \inf\{\alpha : \limsup_{x \to 0} x^\alpha h(x) = 0\}, \qquad \delta = \inf\{\alpha : \liminf_{x \to 0} x^\alpha h(x) = 0\}, \tag{3.3}$$

then with probability one

$$\limsup_{t \to 0} t^{-1/\alpha} M_t = \begin{cases} 0 & \text{if } \alpha > \beta \\ \infty & \text{if } \alpha < \beta \end{cases}, \qquad \liminf_{t \to 0} t^{-1/\alpha} M_t = \begin{cases} 0 & \text{if } \alpha > \delta \\ \infty & \text{if } \alpha < \delta \end{cases}. \tag{3.4}$$

There are also results for $t \to \infty$ analogous to (1.4) if the indices are defined in terms of $x^\alpha h(x)$ for $x \to \infty$ as in (1.3). (The reason that some of the inequalities are reversed for small times is that x^α is increasing with α when x is large but decreasing when x is small.) If we define $S(a) = \inf\{t : |X_t| > a\}$ then we have $ES(a)$ comparable to $\{h(a)\}^{-1}$ as in Theorem 1. Also, as in (2.10),

$$\beta = \inf\left\{\alpha > 0 : \int_{|x| \le 1} |x|^\alpha \, d\nu(x) < \infty\right\}$$

where β is as in (3.3) provided that we remove a linear drift term when $\int_{|x| \le 1} |x| \, d\nu(x) < \infty$. This is the definition of β given by Blumenthal and Getoor (1961). Corresponding to Theorem 3, we have

$$\delta = \sup\left\{\alpha : \int_0^1 EM_t^{-\alpha} \, dt < \infty\right\}. \tag{3.5}$$

Combining the definition of δ with the fact that $ES(a)$ and $\{h(a)\}^{-1}$ are comparable shows that

$$\delta = \sup\{\alpha : \limsup_{a \to 0} a^{-\alpha} ES(a) < \infty\}. \tag{3.6}$$

This is also valid with $S(a)$ replaced by $\min\{S(a), 1\}$ and this makes it possible to compare δ with γ, the index introduced in [11] which gives the Hausdorff dimension of the range of X. The definition of γ was as in (3.6) with the first passage time $S(a)$ replaced by the sojourn time in the ball of radius a up to time one. Thus we have

$$\gamma \le \delta \le \beta.$$

If the Lévy process has increasing sample paths it is called a subordinator. In this case a lower subordinator index σ was defined by Blumenthal and Getoor (1961) and it was shown that $\gamma = \sigma$ in [11]. Since it was shown in Theorem 2 of [11] that γ may be obtained as δ is in (3.5) but with M_t replaced by $|X_t|$, we have $\gamma = \delta = \sigma$ for a subordinator. However, γ and δ may be different in general. If one symmetrizes the example given in Section 4 of [11], then $\gamma = 3/5$ as was (partially) shown in [11] while it is easy to check that $\delta = 2/3$; this calculation is done as in the example given for sums of independent random variables in Section 2.

The main thing that needs to be proved for Lévy processes is the pair of inequalities (3.2). The other proofs are then very much the same as in Section 2. As before, we consider only $d = 1$. The analogue of the truncation is to let $\psi(u) = \psi_1(u) + \psi_2(u)$ where

$$\psi_1(u) = ibu + \int_{|x| \le a} \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) d\nu(x) - \int_{|x| > a} \frac{iux}{1 + x^2} \, d\nu(x)$$

$$\psi_2(u) = \int_{|x| > a} (e^{iux} - 1) \, d\nu(x).$$

Then $X_t = X_t^1 + X_t^2$ with $X^1 \sim \psi_1$, $X^2 \sim \psi_2$, the two processes being independent with X_t^1 having finite variance and X_t^2 being compound Poisson. By differentiation of the characteristic function,

$$EX_t^1 = -it\psi_1'(0) = t\left\{b + \int_{|x|\le a} \frac{x^3}{1+x^2}\,d\nu(x) - \int_{|x|>a} \frac{x}{1+x^2}\,d\nu(x)\right\}$$

$$\operatorname{Var} X_t^1 = -t^2\{\psi_1'(0)\}^2 - t\psi_1''(0) - \{EX_t^1\}^2 = t\int_{|x|\le a} x^2\,d\nu(x)$$

so that $|EX_t^1| = taM(a)$ and $\operatorname{Var} X_t^1 = ta^2K(a)$. Now the proof of the first bound in (3.2) proceeds as in Section 2 since we know that for the compound Poisson process X_t^2

$$P\{X_s^2 \ne 0 \text{ for some } s \le t\} = 1 - e^{-tG(a)} \le tG(a).$$

The other inequality is also proved as in Section 2 with the truncation coming at $2a$ now instead of a. Since $M_t \le a$ implies that there are no jumps of magnitude larger than $2a$ before time t, we have

$$P\{M_t \le a\} = e^{-tG(2a)}P\{\sup_{0\le s\le t}|X_s^1| \le a\}.$$

The first factor may be used when G is dominant and the second when M is dominant as in Section 2. Thus we only need to consider the case when K is dominant. Rather than attempting to obtain the needed result from the concentration function inequality, it is easier to proceed directly in this case. This is essentially Esseen's (1968) argument but it is much easier here. For any random variable Y with characteristic function f we have

$$cu^2P\{|uY| \le 1\} \le EY^{-2}(1 - \cos uY) = E\int_0^u\int_0^v \cos wY\,dw\,dv$$

$$= \int_0^u\int_0^v \operatorname{Re} f(w)\,dw\,dv \le u\int_0^u |f(w)|\,dw$$

so that

$$P\{|Y| \le a\} \le Ca\int_0^{a^{-1}} |f(w)|\,dw.$$

If we take $Y = X_t$, then $f(w) = \exp\{t\psi(w)\}$ so that

$$|f(w)| = \exp\{t\operatorname{Re}\psi(w)\} \le \exp\left\{t\int_{|x|\le 2a}(\cos wx - 1)\,d\nu(x)\right\}$$

$$\le \exp\left\{-Ctw^2\int_{|x|\le 2a} x^2\,d\nu(x)\right\} \quad \text{for } |w| \le (2a)^{-1} \tag{3.7}$$

$$= \exp\{-Ctw^2 4a^2K(2a)\}.$$

Thus

$$P\{|X_t| \le 2a\} \le Ca\int_0^\infty \exp\{-Ct4a^2K(2a)w^2\}\,dw = \frac{C_1}{\{tK(2a)\}^{1/2}}.$$

This is the analogue of (2.5). The remainder of the proof is essentially the same as above.

REMARK. All that is needed to complete the proof of the analogous inequality for S_n is a slight modification of (3.7). If f, g denote the characteristic functions of S_n, X

respectively, then

$$|f(w)| = |g(w)|^n \le \exp\{-\tfrac{1}{2} n(1 - |g(w)|^2)\}$$
$$= \exp\left\{-\frac{1}{2} n \int (1 - \cos wx)\, dF^s(x)\right\}$$

where F^s is the distribution function of the symmetrized random variable $X_1 - X_2$ which has characteristic function $|g(w)|^2$. Now (3.7) and the rest of the proof are completed as above.

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