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A RENEWAL THEOREM FOR AN URN MODEL[1]

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For an urn model (arising typically in the sequential estimation of the size of a finite population), along with an invariance principle for a partial sequence of nonnegative random variables, a renewal theorem relating to some stopping times is established. A representation of these random variables in terms of linear combinations of some martingale-differences provides the key to a simple solution.

1. Introduction. The following *urn model* arises typically in the *sequential estimation* of the *total size of a finite population*. An urn contains an unknown number N of white balls and no others. We repeatedly draw a ball at random, observe its color and replace it by a black ball, so that before each draw, there are N balls in the urn. Let W_n be the number of white balls observed in the first n draws, $n \geq 1$ ($W_0 = 0$, $W_1 = 1$ and $W_k \leq k$, $k \geq 2$). For large N (and n, satisfying $n/N \to \alpha: 0 < \alpha < \infty$), the limiting normality of the standardized form of W_n has been studied by a host of workers; we may refer to Rényi (1962) where other references are also cited. For every $c > 0$, consider the *stopping variable*

$$t_c = \inf\{n : n \geq (c+1) W_n\}, \tag{1.1}$$

so that t_c can take on only the values $[(c+1)k]$, $k = 1, 2, \cdots$, and $W_{t_c} = m$ whenever $t_c = [m(c+1)]$.

In the context of sequential estimation of N, Samuel (1968) has considered the stopping variable t_c and [see her (5.10)] made a conjecture that the standardized form of t_c is asymptotically (as $N \to \infty$) normally distributed, for every (fixed) $c > 0$. Since $W_n \geq W_{n-1}$, for every $n \geq 1$, a *renewal theorem* on the W_n would naturally provide an affirmative answer to her conjecture.

With this motivation, we consider an *invariance principle* for the partial sequence $\{W_k, k \leq n\}$ and incorporate the same in the formulation of the renewal theorem on the stopping times $\{t_c;\ c > 0\}$. In this context, we consider a *representation* of W_n in terms of a *linear combination* of some *martingale-differences* and this provides us with a simple tool for the derivation of the main results.

Along with the preliminary notions, this representation and the main theorems are presented in Section 2 and their proofs are then considered in Section 3. The last section deals with some general remarks.

2. The main results. We may write

$$W_n = W_{n-1} + w_n, \quad n \geq 1; \qquad W_0 = w_0 = 0, \quad W_1 = 1, \tag{2.1}$$

where

$$w_n = \begin{cases} 1, & \text{if a white ball appears at the } n\text{th draw,} \\ 0, & \text{otherwise; for } n \geq 1. \end{cases} \tag{2.2}$$

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Also, note that conditional on the $W_j, j \le n-1$, w_n can assume the two values 1 and 0 with respective conditional probabilities $(N - W_{n-1})/N$ and $N^{-1}W_{n-1}$. Thus, if we let

$$Y_n = w_n - N^{-1}(N - W_{n-1}), \qquad n \ge 1, \tag{2.3}$$

then the Y_n, $n \ge 1$ form a martingale-difference sequence. From (2.1) through (2.3), we obtain by induction that

$$\begin{aligned} W_n &= 1 + Y_n + (1 - N^{-1})W_{n-1} = \cdots \\ &= \textstyle\sum_{j=1}^n (1-N^{-1})^{j-1}Y_{n-j+1} + \sum_{j=1}^n (1-N^{-1})^{j-1} \\ &= N\{1 - (1-N^{-1})^n\} + \textstyle\sum_{j=1}^n (1-N^{-1})^{j-1}Y_{n-j+1}, \qquad n \ge 1. \end{aligned} \tag{2.4}$$

This representation is a key to our subsequent analysis.

Now, for every $K (0 < K < \infty)$ and $N (\ge 1)$, we consider a stochastic process $Z_n = \{Z_N(t), t \in [0, K]\}$ by letting

$$Z_N(t) = N^{-1/2}\{W_{[Nt]} - N(1 - (1 - N^{-1})^{[Nt]})\}, \qquad t \in [0, K], \tag{2.5}$$

where $[s]$ stands for the largest integer contained in s. Then, Z_N belongs to the $D[0, K]$ space, endowed with the Skorokhod J_1-topology. Let $Z = \{Z(t), t \in [0, K]\}$ be a Gaussian function with zero drift and covariance function

$$EZ(s)Z(t) = e^{-t}\{1 - (1+s)e^{-s}\}, \quad \text{for} \quad 0 \le s \le t \le K. \tag{2.6}$$

Then, we have the following.

THEOREM 2.1. *For the given urn model, for every $K: 0 < K < \infty$, as $N \to \infty$,*

$$Z_N \to_{\mathcal{D}} Z, \text{ in the } J_1\text{-topology on } D[0, K]. \tag{2.7}$$

Note that $a(x) = e^x - (1 + x)$, $x \ge 0$ is a continuous and monotone function of x. Thus, if $Z^* = \{Z^*(t), 0 \le t \le K^*\}$ be defined by letting

$$Z^*(t) = e^{a^{-1}(t)}Z(a^{-1}(t)), \qquad 0 \le t \le K^* = a(K), \tag{2.8}$$

then, by using the transformation in Section 5 of Doob (1949), we obtain that $EZ^*(t) = 0$ for every t and

$$EZ^*(s)Z^*(t) = s \wedge t, \quad \text{for every} \quad s, t \text{ in } [0, K^*]. \tag{2.9}$$

Thus, Z^* is a standard Wiener process on $[0, K^*]$. Hence, on letting

$$Z_N^*(t) = e^{a^{-1}(t)}Z_N(a^{-1}(t)), \qquad 0 \le t \le K^* \tag{2.10}$$

we obtain from (2.7), (2.8) and (2.9) that

$$Z_N^* \to_{\mathcal{D}} Z^*, \quad \text{in the } J_1\text{-topology on } D[0, K^*]. \tag{2.11}$$

Let us next consider a renewal theorem for the W_n. Note that [by (2.4)]

$$EW_n = N\{1 - (1 - N^{-1})^n\} \simeq N\{1 - e^{-n/N}\} \tag{2.12}$$

is a nonlinear function of n, and hence, the usual renewal theorem [a dual to Theorem 2.1] (see Billingsley, 1968, page 148, or Vervaat, 1972) may not be applicable to the sequence of stopping times in (1.1). For this reason, we consider the following.

Let $I^* = [a, b]$ be a subinterval of $[0, 1]$, where $0 < a < b < 1$. For every $m \in I^*$, define t_m by the solution of the equation

$$m = \{1 - e^{-t_m}\}/t_m, \qquad m \in I^*. \tag{2.13}$$

Note that $t_1 = 0$, while, as m moves from 1 to 0, t_m monotonically goes to ∞. Further, $mt_m = 1 - e^{-t_m} \le 1$, for every $m \in [0, 1]$. Then, for every $N (\ge 1)$, we consider a stochastic process $X_N = \{X_N(m),\, m \in I^*\}$ by letting

$$\tau_{Nm} = \inf\{n(\geq 1) : mn \geq W_n\}, \qquad X_N(m) = N^{-1/2}\{\tau_{Nm} - Nt_m\}, \qquad m \in I^*. \tag{2.14}$$

Note that the W_n depend on N and m plays the role of $(1 + c)^{-1}$ in (1.1). Finally, let $X = \{X(m),\, m \in I^*\}$ be a Gaussian function with 0 drift and covariance function

$$EX(m)X(m') = e^{-t_m}\{1 - (1 + t_{m'})e^{-t_{m'}}\}/\{(m - e^{-t_m})(m' - e^{-t_{m'}})\}, \quad \text{for} \quad m \leq m'. \tag{2.15}$$

Then, we have the following

THEOREM 2.2. *For every $0 < a < b < 1$, as $N \to \infty$,*

$$X_N \to_{\mathscr{D}} X, \textit{ in the } J_1\textit{-topology on } D[a, b]. \tag{2.16}$$

The proofs of the theorems are considered in the next section.

3. Proofs of the theorems. We need to prove that for Z_N (and X_N), the finite-dimensional distributions (f.d.d.) converge to those of Z (and X) and that Z_N (and X_N) are *tight*. For the case of Z_N, for arbitrary $r(\geq 1)$, $0 \leq t_1 < \cdots < t_r \leq K$ and $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_r)' \neq \mathbf{0}$, let $n_j = [Nt_j]$, $1 \leq j \leq r$ and consider the linear compound $Z_N^0(\mathbf{t}, \boldsymbol{\lambda}) = \sum_{j=1}^r \lambda_j Z_N(t_j)$, where, by (2.4) and (2.5), we have

$$Z_N^0(\mathbf{t}, \boldsymbol{\lambda}) = N^{-1/2} \sum_{j=1}^r \lambda_j \sum_{i=1}^{n_j} (1 - N^{-1})^{i-1} Y_{n_j - i+1} = \sum_{i=1}^n c_{Ni} Y_i, \quad \text{say}, \tag{3.1}$$

where $n = n_r$ and the c_{Ni} depend on N, $n_1, \cdots, n_r$ and $\boldsymbol{\lambda}$. Since the Y_i are bounded r.v.'s and are martingale differences, we may use the dependent central limit theorem of Dvoretzky (1972) and for this, we need to show only that

$$\gamma_N^2 = \sum_{i=1}^n c_{Ni}^2 E(Y_i^2 \mid \mathscr{B}_{i-1}) \to \gamma^2, \quad \text{as} \quad N \to \infty, \tag{3.2}$$

where $\mathscr{B}_{i-1} = \mathscr{B}(Y_j, j \leq i - 1)$, $i \geq 1$,

$$\gamma^2 = \boldsymbol{\lambda}'\boldsymbol{\nu}\boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{\nu} = ((EZ(t_j)Z(t_{j'})))_{j,j'=1,\cdots,r}. \tag{3.3}$$

Note that by (3.1), $\max_{i \leq n} c_{Ni}^2 \to 0$ as $N \to \infty$, and hence, the boundedness of the Y_i eliminates the need for the verification of the conditional form of the Lindeberg condition in the Dvoretzky theorem.

Note that by (2.3),

$$E(Y_i^2 \mid \mathscr{B}_{i-1}) = W_{i-1}(N - W_{i-1})/N^2, \quad \text{for every} \quad i \geq 1, \tag{3.4}$$

so that on using the representation for the W_n in (2.4), we may express γ_N^2 as the sum of a principal (non-stochastic) term and two other stochastic terms which are respectively linear and quadratic functions of the Y_i. This non-stochastic term converges to γ^2, while each of the two stochastic terms converges in mean square to 0. Hence, γ_N^2 converges in mean square to γ^2 and this ensures (3.2). This completes the proof of the convergence of the f.d.d.'s of $\{Z_N\}$ to those of Z.

Note that by definition $Z_N(0) = 0$, with probability 1, for every $N \geq 1$. Hence, to establish the tightness of Z_N, it suffices to show that for every $(1 \leq) k < q = k + r(\leq NK)$,

$$E[Z_N((k + r)/N) - Z_N(k/N)]^4 \leq C^* r^2/N^2, \tag{3.5}$$

where C^* is a finite positive constant, independent of N; see Theorems 12.3, and 15.6 and (14.9) of Billingsley (1968) in this context. By (2.4) and (2.5), we have for every $r \geq 1$, $k \geq 1$,

$$\begin{aligned} &Z_N((k + r)/N) - Z_N(k/N) \\ &\qquad = N^{-1/2} \sum_{i=1}^r (1 - N^{-1})^{i-1} Y_{k+r-i+1} - \{1 - (1 - N^{-1})^r\} Z_N(k/N), \end{aligned} \tag{3.6}$$

where

$$1 - (1 - N^{-1})^r \leq r/N, \quad \text{for every} \quad r \geq 1. \tag{3.7}$$

Note that the Y_i are martingale differences (and bounded r.v.), so that by (2.3) and a theorem of Dharmadhikari, Fabian and Jogdeo (1968), we have

$$\begin{aligned} E[N^{-1/2} \textstyle\sum_{i=1}^{r} (1-N^{-1})^{i-1} Y_{k+r-i+1}]^4 &\le (Cr \textstyle\sum_{i=1}^{r} (1-N^{-1})^{4(i-1)} EY_{k+r-i+1}^4)/N^2 \\ &\le Cr[1-(1-N^{-1})^{4r-3}]/N^2[1-(1-N^{-1})^4] \\ &\le Cr[1-(1-N^{-1})^{4r}]/N^2[1-(1-N^{-1})^4] \\ &\le 4Cr^2/N^2, \quad \text{by (3.7).} \qquad (C<\infty) \end{aligned} \tag{3.8}$$

Similarly, by using (2.4), (2.5) and the same theorem of Dharmadhikari, Fabian and Jogdeo (1968), we obtain that

$$EZ_N^4(k/N) \le N^{-2}Ck \textstyle\sum_{i=1}^{k} (1-N^{-1})^{4(i-1)} EY_{k-i+1}^4 \le 4Ck^2/N^2. \tag{3.9}$$

Therefore, by (3.6) through (3.9), we obtain that (3.5) holds with $C^* = 8C$. This completes the proof of Theorem 2.1. In fact, we have actually shown that in (2.7), one may also take the uniform topology instead of the J_1-topology.

To prove Theorem 2.2, we consider the function $g(t, m) = mt + e^{-t} - 1$, $m \in (0, 1)$, $t \in (0, \infty)$. Note that $g(0, m) = 0$ for every m and $g_{10}(t, m) = (\partial/\partial t)g(t, m) = m - e^{-t}$ is $\uparrow$ with $g_{10}(0, m) = m - 1 < 0$, for every $m < 1$. Thus, by (2.13), we obtain that $0 = g(t_m, m) = g(t_m, m) - g(0, m) = \int_0^{t_m} g_{10}(t, m)\, dt$, so that $g_{10}(t_m, m) > 0$ for every $m \in (0, 1)$. In fact, for every $0 < a < b < 1$, we have

$$\inf\{g_{10}(t_m, m) : a \le m \le b\} = g_{10}^* > 0, \tag{3.10}$$

where g_{10}^* may depend on a and b. Also, by Theorem 2.1, for every $K(<\infty)$,

$$\sup\{|Z_N(t)| : 0 \le t \le K\} = O_p(1), \tag{3.11}$$

and, by (3.6) and (3.11),

$$\begin{aligned} \max\{|Z_N((k+1)/N) - Z_N(k/N)| : 0 \le k \le NK\} &\le N^{-1/2}\{\max_{1\le k\le NK} |Y_k|\} \\ &+ N^{-1}\{\max_{1\le k\le NK} |Z_N(k/N)|\} \le N^{-1/2} + N^{-1}[O_p(1)] = O_p(N^{-1/2}). \end{aligned} \tag{3.12}$$

Further, we may rewrite τ_{Nm} in (2.14) as

$$\tau_{Nm} = \inf\{n : Z_N(n/N) \le N^{1/2}(mN^{-1}n - (1-(1-N^{-1})^n))\}. \tag{3.13}$$

Note that for every $m \in [a, b]$, $\tau_{Nb} \le \tau_{Nm} \le \tau_{Na}$, where, for every (fixed) $K(<\infty)$, $\{N^{-1}\tau_{Na} > K\} \Leftrightarrow \{Z_N(n/N) > N^{1/2}(aN^{-1}n - (1-(1-N^{-1})^n)),\ \forall\ n \le K\}$ and, by (3.10), for every $a > 0$, there exists a finite $K_a(t_a < K_a < \infty)$, such that $N^{1/2}(aK_a - (1-(1-N^{-1})^{NK_a}) \to +\infty$ as $N \to \infty$. Consequently, by Theorem 2.1,

$$\begin{aligned} P\{\sup_{m\ge a} N^{-1}\tau_{Nm} > K_a\} &= P\{N^{-1}\tau_{Na} > K_a\} \\ &\le P\{Z_N(K_a) > N^{1/2}(aK_a - (1-(1-N^{-1})^{NK_a}))\} \to 0, \quad \text{as} \quad N \to \infty. \end{aligned} \tag{3.14}$$

Similarly, for every $b < 1$, there exists a K_b $(0 < K_b < t_b)$, such that

$$P\{\inf_{m\le b} N^{-1}\tau_{Nm} < K_b\} = P\{N^{-1}\tau_{Nb} < K_b\} \to 0, \quad \text{as} \quad N \to \infty. \tag{3.15}$$

From (3.12), (3.14) and (3.15), we conclude that

$$\sup\{|Z_N(N^{-1}\tau_{Nm}) - Z_N(N^{-1}(\tau_{Nm} - 1))| : a \le m \le b\} = O_p(N^{-1/2}). \tag{3.16}$$

Further, note that for every $C > 0$, $\{N^{-1}\tau_{Nm} > t_m + N^{-1/2}C\} \Leftrightarrow \{Z_N(n/N) > N^{1/2}(mN^{-1}n - (1-(1-N^{-1})^n),\ \forall\ n \le N(t_m + N^{-1/2}C)\}$, where, by (3.10), for $n = [N(t_m + N^{-1/2}C)]$, $N^{1/2}(mN^{-1}n - (1-(1-N^{-1})^n)$ can be made greater than $C^* + O(N^{-1/2})$, where $C^* = Cg_{10}^*$, for every $m \in [a, b]$. A similar case holds for $\{N^{-1}\tau_{Nm} < t_m - N^{-1/2}C\}$. Hence, for every $C > 0$,

$$\begin{aligned} P\{\sup_{a\le m\le b} |N^{-1}\tau_{Nm} - t_m| > CN^{-1/2}\} &\le P\{N^{-1}\tau_{Nb} < K_b \quad \text{or} \quad N^{-1}\tau_{Na} > K_a\} \\ &+ P\{\sup_{K_b\le t\le K_a} |Z_N(t)| > C^* + O(N^{-1/2})\}. \end{aligned} \tag{3.17}$$

By (3.14) and (3.15), the first term on the right hand side of (3.17) converges to 0, while, by (3.11), the second term can be made arbitrarily small by choosing C adequately large. Thus, we obtain from (3.17) that

$$\sup\{|N^{-1}\tau_{Nm} - t_m| : a \le m \le b\} = O_p(N^{-1/2}). \tag{3.18}$$

Let $\tilde{Z}_N = \{\tilde{Z}_N(t),\ t \in [0, K]\}$ be defined by letting $\tilde{Z}_N(t) = (k + 1 - Nt)Z_N(k/N) + (Nt - k)Z_N((k + 1)/N)$, for $k/N \le t \le (k + 1)/N$, $k = 0, 1, \cdots, [K]$. Then, by (3.11) and (3.12),

$$\rho(Z_N, \tilde{Z}_N) = \sup\{|Z_N(t) - \tilde{Z}_N(t)| : 0 \le t \le K\} \to_p 0, \quad \text{as} \quad N \to \infty. \tag{3.19}$$

Since the process $\tilde{Z}_N$ belongs to the $C[0, K]$ space, (3.5) actually ensures the tightness of the process with respect to the uniform topology. Hence, if we define the modulus of continuity $\omega_\delta(Z_N)$ as $\sup\{|\tilde{Z}_N(t) - \tilde{Z}_N(s)| : 0 \le s < t < s + \delta \le K\}$, $\delta > 0$, then, for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta(> 0)$ and an N_0, such that

$$P\{\omega_\delta(Z_N) > \varepsilon\} < \eta, \qquad \forall\, N \ge N_0. \tag{3.20}$$

Further,

$$\begin{aligned} &P\{\sup_{a\le m\le b} |\tilde{Z}_N(N^{-1}\tau_{Nm}) - \tilde{Z}_N(t_m)| > \varepsilon\} \\ &\le P\{\sup_{a\le m\le b} |N^{-1}\tau_{Nm} - t_m| > \delta\} \\ &\quad + P\{\sup_{a\le m\le b} |N^{-1}\tau_{Nm} - t_m| \le \delta,\ \sup_{a\le m\le b} |\tilde{Z}_N(N^{-1}\tau_{Nm}) - \tilde{Z}_N(t_m)| > \varepsilon\} \\ &\le P\{\sup_{a\le m\le b} |N^{-1}\tau_{Nm} - t_m| > \delta\} + P\{\omega_\delta(Z_N) > \varepsilon\}. \end{aligned} \tag{3.21}$$

Hence, by (3.18), (3.20) and (3.21), we conclude that as $N \to \infty$,

$$\sup_{a\le m\le b} |\tilde{Z}_N(N^{-1}\tau_{Nm}) - \tilde{Z}_N(t_m)| \to 0, \quad \text{in probability.} \tag{3.22}$$

By virtue of (3.16), (3.19) and (3.22), we have

$$\sup\{|Z_N(N^{-1}\tau_{Nm}) - Z_N(t_m) : a \le m \le b\} \to 0, \quad \text{in probability.} \tag{3.23}$$

On the other hand, by (2.5), (3.11), (3.12) and (3.13), we obtain by expanding $(1 - N^{-1})^{\tau_{Nm}}$ as $(1 - N^{-1})^{Nt_m} + (\tau_{Nm} - Nt_m)(1 - N^{-1})^{Nt_m} \cdot \log(1 - N^{-1})$ plus higher order terms that for every $m : a \le m \le b$,

$$\begin{aligned} Z_N(N^{-1}\tau_{Nm}) &= N^{1/2}\{N^{-1}W_{\tau_{Nm}} - (1 - (1 - N^{-1})^{\tau_{Nm}}\} \\ &= N^{1/2}\{(m - e^{-t_m})(N^{-1}\tau_{Nm} - t_m)\} + \xi_{mN}, \end{aligned} \tag{3.24}$$

where, by (3.12) and (3.18),

$$\sup\{|\xi_{mN}| : a \le m \le b\} \to 0, \quad \text{in probability, as} \quad N \to \infty. \tag{3.25}$$

Thus, by (2.14), (3.23) and (3.24), we obtain that as $N \to \infty$,

$$\sup\{|X_N(m) - (m - e^{-t_m})^{-1}Z_N(t_m)| : a \le m \le b\} \to 0, \quad \text{in probability,} \tag{3.26}$$

and hence, (2.16) follows from (3.26) and Theorem 2.1. □

4. Some general remarks. Samuel's (1968) conjecture (5.10) about the asymptotic normality of the standardized form of t_c in (1.1) follows directly as a corollary to our Theorem 2.2. She has also made a second conjecture (5.11) on the asymptotic normality of the maximum likelihood estimator of N for the randomly stopped sample size t_c. Invariance principles for partial likelihood ratio statistics relating to sequential sampling tagging were studied by Sen (1982) and in view of our (3.18) and these invariance principles, (5.11) of Samuel (1968) also holds. We conclude this section with the remark that technically we have limited ourselves to $m \le b$ where $b < 1$. Note that for $m = 1$, τ_m is equal to one, with probability 1, so that we have a degenerate case, while, for m very close to 1, the Poisson distribution considered in Samuel (1968) works out well.

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