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COUNTABLE STATE SPACE MARKOV RANDOM FIELDS AND MARKOV CHAINS ON TREES

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Let S and A be countable sets and let $\mathscr{G}(\Pi)$ be the set of Markov random fields on S^A (with the σ-field generated by the finite cylinder sets) corresponding to a specification Π, Markov with respect to a tree-like neighbour relation in A. We define the class $\mathscr{M}(\Pi)$ of Markov chains in $\mathscr{G}(\Pi)$, and generalise results of Spitzer to show that every extreme point of $\mathscr{G}(\Pi)$ belongs to $\mathscr{M}(\Pi)$. We establish a one-to-one correspondence between $\mathscr{M}(\Pi)$ and a set of "entrance laws" associated with Π. These results are applied to homogeneous Markov specifications on regular infinite trees. In particular for the case $|S| = 2$ we obtain a quick derivation of Spitzer's necessary and sufficient condition for $|\mathscr{G}(\Pi)| = 1$, and further show that if $|\mathscr{M}(\Pi)| > 1$ then $|\mathscr{M}(\Pi)| = \infty$.

1. Introduction. Let S and A be countable sets, and $\mathscr{F}$ the σ-field in S^A generated by the finite cylinder sets. Our aim is to characterise those probability measures on $(S^A, \mathscr{F})$ corresponding to a given "conditional probability structure", or *specification* (Föllmer, 1975, Dynkin, 1978), Markov with respect to a "tree-like" neighbour relation in A. This problem is considered by Spitzer (1975a) and Preston (1974, 1976) for the case $S = \{0, 1\}$. The somewhat different approach used here enables results to be obtained for a general countable state space S, and yields additional results for the binary state space.

For any subset B of A let ξ_B be the natural projection function $S^A \to S^B$ and $x_B = \xi_B(x_A)$ the corresponding projection of a generic point $x_A \in S^A$; x_B will also represent a generic point of S^B; let $\mathscr{F}(B)$ be the σ-field in S^A (or S^B) generated by the sets of the form $\{\xi_i = x_i\}$, $x_i \in S$, $i \in B$. Let $\mathscr{V}$ be the set of finite non-empty subsets of A.

Let $\sim$ be a neighbour relation (a symmetric non-reflexive binary relation) in A such that the graph $(A, \sim)$ is a tree, i.e. a connected graph which becomes disconnected when any one of its edges is removed. For any $V \in \mathscr{V}$ let ∂V be the set of all elements of $S \backslash V$ which have neighbours in V; let $\Delta V = V \cup \partial V$. We also require $\sim$ to be such that ∂V is finite for all $V \in \mathscr{V}$. A probability measure P (on $(S^A, \mathscr{F})$) is a *Markov random field* (with respect to $\sim$) if for all $V \in \mathscr{V}$,

$$(1.1) \qquad P\{\xi_V = x_V / \mathscr{F}(A \backslash V)\} \quad \text{is } \mathscr{F}(\partial V)\text{-measurable}, \quad x_V \in S^V.$$

If P is *positive*, i.e. has positive density on the finite cylinder sets (where both here and elsewhere "positive" means "strictly positive" rather than simply "non-negative"), it is known to be sufficient to verify (1.1) for those V such that $V = \{i\}$, $i \in A$.

We define a *Markov specification* to be a collection $\Pi = \{\pi_V\}_{V \in \mathscr{V}}$ of stochastic kernels $\pi_V: S^{\partial V} \times \mathscr{F}(V) \to \mathbb{R}_+$ (where $\mathbb{R}_+$ is the set of non-negative real numbers, and where for each $V \in \mathscr{V}$, $x_{\partial V} \in S^{\partial V}$, $\pi_V(x_{\partial V}, \cdot)$ is a probability measure) which satisfy the natural consistency condition: for each $V, W \in \mathscr{V}$ with $V \subset W$,

$$(1.2) \qquad \pi_W(x_{\partial W}, \xi_W = x_W) = h_{W,V}(x_{\Delta W \backslash V})\pi_V(x_{\partial V}, \xi_V = x_V), \quad x_{\Delta W} \in S^{\Delta W},$$

for some "normalising" function $h_{W,V}: S^{\Delta W \backslash V} \to \mathbb{R}_+$. We say that Π is *positive* if, for each $V \in \mathscr{V}$, $\pi_V(\cdot, \xi_V = \cdot)$ is a positive function. A probability measure P is said to *correspond* to Π if for all $V \in \mathscr{V}$,

$$P\{\xi_V = x_V / \mathscr{F}(A \backslash V)\} = \pi_V(\xi_{\partial V}, \xi_V = x_V) \quad \text{a.s.} \quad P, \quad x_V \in S^V.$$

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Let $\mathcal{G}(\Pi)$ be the set of all such probability measures. (Note that when A is finite $\mathcal{G}(\Pi)$ has exactly one member.) Each member of $\mathcal{G}(\Pi)$ is a Markov random field. (Conversely, at least every positive Markov random field corresponds to some unique Markov specification.)

The set $\mathcal{G}(\Pi)$ is convex, i.e. if $P_1, P_2 \in \mathcal{G}(\Pi)$ and $0 \le k \le 1$, then $kP_1 + (1-k)P_2 \in \mathcal{G}(\Pi)$; indeed $\mathcal{G}(\Pi)$ is a simplex in the sense of Choquet or of Dynkin (1978). Let $\mathcal{E}(\Pi)$ be the set of extreme points of $\mathcal{G}(\Pi)$. It is known that $\mathcal{E}(\Pi)$ is precisely the set of probability measures in $\mathcal{G}(\Pi)$ with respect to which the tail σ-field $\hat{\mathcal{F}} = \cap_{V \in \mathcal{V}} \mathcal{F}(A \setminus V)$ is (almost) trivial, and also that $\mathcal{G}(\Pi)$ may be put into a natural (one-to-one) correspondence with the set of probability measures on $\mathcal{E}(\Pi)$, with an appropriate σ-field; (see, for example, Dynkin, 1978).

In Section 2 we define Markov chains on the tree $(A, \sim)$. These are a natural generalisation of Markov chains in the usual sense, with the integers as parameter set, and were originally introduced for trees by Preston (1974) and Spitzer (1975a) with state space $S = \{0, 1\}$. Markov chains are Markov random fields, though the converse is in general false. However, let $\mathcal{M}(\Pi)$ be the set of Markov chains in $\mathcal{G}(\Pi)$; we generalise results of Spitzer (1975a, 1975b) to show that $\mathcal{E}(\Pi) \subset \mathcal{M}(\Pi)$. It follows in particular that the cardinality $|\mathcal{G}(\Pi)|$ of $\mathcal{G}(\Pi)$ is equal to 0, 1 or ∞ according as $|\mathcal{M}(\Pi)|$ is equal to 0, 1, or is greater than 1. Thus "*phase transition*" occurs if and only if $|\mathcal{M}(\Pi)| > 1$.

In Section 3 we establish the key result that provided Π satisfies certain mild conditions, weaker than the requirement that it be positive, there is a one-to-one correspondence between $\mathcal{M}(\Pi)$ and a set of "entrance laws" associated with Π. This result is applied in the subsequent sections where we take $(A, \sim)$ to be the regular tree of "dimension" d, i.e. each element of A to have exactly $d + 1$ neighbours, and Π to be *homogeneous* in the sense of being invariant under graph isomorphisms of the tree. Let $\mathcal{M}_0(\Pi)$ be the set of corresponding Markov chains which are themselves similarly homogeneous.

In Section 4 we show how to identify the members of $\mathcal{M}_0(\Pi)$. We also consider a (sometimes strictly) wider class of Markov chains lying between $\mathcal{M}_0(\Pi)$ and $\mathcal{M}(\Pi)$.

In Section 5 we consider further the case $S = \{0, 1\}$. In particular we give a quick derivation of Spitzer's necessary and sufficient condition for $|\mathcal{G}(\Pi)| = 1$. We also give a new result: for each Markov specification Π the set $\mathcal{M}(\Pi)$ consists of either exactly one, or else infinitely many, Markov chains. In Section 6 we consider a further example with $|S| = 3$.

2. Markov chains. Given $B \subset A$, let $(B, \sim)$ denote the graph obtained by the restriction of $\sim$ to B. If this graph is connected it is a tree. Let $\mathcal{V}^*$ denote the set of $V \in \mathcal{V}$ such that $(V, \sim)$ is connected.

For any probability measure P on $(S^A, \mathcal{F})$, and for each $V \in \mathcal{V}$, let P_V denote the marginal, or cylinder, probability measure induced by P on $(S^V, \mathcal{F}(V))$; P_V will be said to be a Markov random field if it is so with respect to $(V, \sim)$. We define a *Markov chain* to be a probability measure P on $(S^A, \mathcal{F})$ such that for each $V \in \mathcal{V}^*$, P_V is a Markov random field. Then P is itself a Markov random field: this follows easily by considering any sequence in $\mathcal{V}^*$ which increases to A.

In the important special case where $(A, \sim)$ is the *one-dimensional lattice* $\mathbb{Z}^1$, given by taking A to be the set $\mathbb{Z}$ of integers and defining consecutive pairs of these to be neighbours, a probability measure P on $(S^{\mathbb{Z}}, \mathcal{F})$ is a Markov chain if and only if it satisfies the more usual condition:

$$P[\xi_{n+1} = x/\mathcal{F}(\{\cdots, n-1, n\})] = P[\xi_{n+1} = x/\mathcal{F}(\{n\})], \quad x \in S, \quad n \in \mathbb{Z}. \tag{2.1}$$

For if (2.1) holds then for any $V = \{m, m+1, \cdots, n\}$, $m, n \in \mathbb{Z}$, $m \le n$, P_V is a Markov random field (see for example Kelly, 1979, page 186), and so P is a Markov chain. Conversely, if P is a Markov chain then (2.1) follows straightforwardly.

For the general tree $(A, \sim)$ we have the following theorem.

THEOREM 2.1. *Let the Markov random field P on $(S^A, \mathscr{F})$ have trivial tail σ-field. Then P is a Markov chain.*

PROOF. We must show that for each $V \in \mathscr{V}^*$, $W \subset V$, $x_W \in S^W$,

$$P\{\xi_W = x_W / \mathscr{F}(V \setminus W)\} = P\{\xi_W = x_W / \mathscr{F}(V \cap \partial W)\}. \tag{2.2}$$

Consider any $U \in \mathscr{V}$ such that $V \subset U$. We show first that there is a set $\bar{W}$ (depending on U) such that $W \subset \bar{W} \subset U$ and $\partial\bar{W} \subset (V \cap \partial W) \cup (A \setminus U) \subset (V \setminus W) \cup (A \setminus U)$. Define the sequence $\{W_n\}_{n \ge 0}$ of subsets of U by $W_0 = W$, $W_n = W_{n-1} \cup (\partial W_{n-1} \cap U \setminus V)$. For all $n \ge 1$ we have that $V \cap \partial W_n \subset V \cap \partial W_{n-1}$: for if there exists some $i \in V \cap \partial W_n$, $i \notin V \cap \partial W_{n-1}$, then there is a path in $(U, \sim)$ from $i \in V$ to some $j \in W \subset V$ which does not lie entirely within $(V, \sim)$, in contradiction to the requirement that $(V, \sim)$ be a connected subgraph of the *tree* $(A, \sim)$. Further because U is finite, there is some $\bar{W} \subset U$ and some n_0 such that for all $n \ge n_0$, $W_n = \bar{W}$. Thus $U \cap \partial\bar{W} = V \cap \partial\bar{W} \subset V \cap \partial W$ and $\bar{W}$ is as required. Since P is a Markov random field and $\{\xi_W = x_W\} \in \mathscr{F}(\bar{W})$, it follows that

$$P[\xi_W = x_W / \mathscr{F}\{(V \setminus W) \cup (A \setminus U)\}] = P[\xi_W = x_W / \mathscr{F}\{(V \cap \partial W) \cup (A \setminus U)\}]. \tag{2.3}$$

Now replace U by any sequence $\{U_n\}_{n \ge 0}$ in $\mathscr{V}$ which increases to A. Then $\mathscr{F}\{(V \setminus W) \cup (A \setminus U_n)\} \to \mathscr{F}(V \setminus W) \vee \hat{\mathscr{F}}$, and $\mathscr{F}\{(V \cap \partial W) \cup (A \setminus U_n)\} \to \mathscr{F}(V \cap \partial W) \vee \hat{\mathscr{F}}$. (These results are a consequence of the following observation: let $\mathscr{E}$, $\mathscr{F}_n$, $n \ge 1$, be sub-σ-fields of $\mathscr{F}$ such that $\mathscr{E}$ is generated by the countable collection of *disjoint* sets E_k, $k \ge 1$, and $\mathscr{F}_n$ decreases to $\hat{\mathscr{F}}$; trivially $\mathscr{E} \vee \hat{\mathscr{F}} \subset \mathscr{E} \vee \mathscr{F}_n$ for all n; if, for all n, $G \in \mathscr{E} \vee \mathscr{F}_n$, then $G = \cup_{k \ge 1} (E_k \cap F_{n,k})$ with $F_{n,k} \in \mathscr{F}_n$, and so $G = \cup_{k \ge 1} (E_k \cap \lim\inf_n F_{n,k}) \in \mathscr{E} \vee \hat{\mathscr{F}}$.) Thus from (2.3), the (reversed) martingale convergence theorem, and the hypothesis that $\hat{\mathscr{F}}$ is trivial, we obtain (2.2) as required. □

COROLLARY 1. *When A is finite the classes of Markov chains and Markov random fields on $(S^A, \mathscr{F})$ coincide.*

COROLLARY 2. *Let Π be a Markov specification. Then $\mathscr{E}(\Pi) \subset \mathscr{M}(\Pi)$.*

This last result ensures that the Markov random fields belonging to $\mathscr{E}(\Pi)$ have a particularly simple structure. It has no obvious analogue for Markov specifications associated with graphs more general than trees.

3. Interactions and entrance laws. We now show that any positive Markov specification Π (associated with the tree $(A, \sim)$) may be more simply represented by an "interaction" Φ, and that there is then a one-to-one correspondence between $\mathscr{M}(\Pi)$ and the set of "entrance laws" for Φ. (We shall in fact be able to partially relax the "positivity" requirement.)

Let $\mathscr{I}$ be the set of subsets of A of the form $\{i, j\}$ where $i \sim j$ (i.e. the set of edges of $(A, \sim)$). We define an *interaction* to be a family $\Phi = \{\phi_{\{i,j\}}\}_{\{i,j\} \in \mathscr{I}}$ of *interaction functions* $\phi_{\{i,j\}} : S^{\{i,j\}} \to \mathbb{R}_+$ such that for each $V \in \mathscr{V}$,

$$0 < \textstyle\sum_{x_V \in S^V} \mu_{\Phi,V}(x_{\partial V}, x_V) < \infty, \quad x_{\partial V} \in S^{\partial V}, \tag{3.1}$$

where $\mu_{\Phi,V} : S^{\partial V} \times S^V \to \mathbb{R}_+$ is given by

$$\mu_{\Phi,V}(x_{\partial V}, x_V) = \textstyle\prod_{\{i,j\} \in \mathscr{I}, \{i,j\} \cap V \ne \varnothing} \phi_{\{i,j\}}(x_{\{i,j\}}). \tag{3.2}$$

(Here we are identifying $S^{\partial V} \times S^V$ with $S^{\Delta V}$; we shall continue to make such obvious identifications.)

For any interaction Φ we define the Markov specification $\Pi_\Phi = \{\pi_{\Phi,V}\}_{V \in \mathscr{V}}$ by: for each $V \in \mathscr{V}$,

$$\pi_{\Phi,V}(x_{\partial V}, \xi_V = x_V) = k_{\Phi,V}(x_{\partial V}) \mu_{\Phi,V}(x_{\partial V}, x_V),$$

where $k_{\Phi,V}: S^{\partial V} \to \mathbb{R}_+$ is determined by the requirement that for each $x^{\partial V} \in S^{\partial V}$, $\pi_{\Phi,V}(x_{\partial V}, \cdot)$ be a probability measure. (It is straightforward to check that Π_Φ *is* a Markov specification. Note also that two interactions may define the same specification.) At least every positive Markov specification corresponds to some interaction (with positive component functions). This follows from a well-known result for Markov specifications on more general graphs (see, for example, Spitzer (1973), Preston (1974), Theorem 4.1, Preston (1976), Proposition 5.7).

In practice we both define a Markov specification and study the corresponding Markov random fields via a "generating" interaction. We shall therefore require the following result.

LEMMA 3.1. *Let Φ be an interaction and let P be a probability measure on $(S^A, \mathscr{F})$. If $P \in \mathscr{G}(\Pi_\Phi)$ then for each $V \in \mathscr{V}$,*

$$P(\xi_{\Delta V} = x_{\Delta V}) = \nu_V(x_{\partial V})\mu_{\Phi,V}(x_{\partial V}, x_V), \tag{3.3}$$

for some function $\nu_V: S^{\partial V} \to \mathbb{R}_+$ (in general depending on both P and Φ). Conversely, if $\{V_n\}_{n\geq 0}$ is any sequence in $\mathscr{V}$ increasing to A and (3.3) *is satisfied with $V = V_n$ for each n, then $P \in \mathscr{G}(\Pi_\Phi)$.*

PROOF. If $P \in \mathscr{G}(\Pi_\Phi)$ then for each $V \in \mathscr{V}$,

$$P(\xi_{\Delta V} = x_{\Delta V}) = P(\xi_{\partial V} = x_{\partial V})\pi_{\Phi,V}(x_{\partial V}, \xi_V = x_V).$$

and so (3.3) follows trivially. Conversely, if (3.3) is satisfied for each member V_n of the given sequence, then for any $V \in \mathscr{V}$, $x_V \in S^V$, and for each n such that $V \subset V_n$,

$$P\{\xi_V = x_V / \mathscr{F}(\Delta V_n \backslash V)\} = \pi_{\Phi,V}(\xi_{\partial V}, \xi_V = x_V), \quad \text{a.s. } P,$$

by the consistency condition (1.2). Letting $n \to \infty$, we may replace ΔV_n by A in the above relation, and since V, x_V are arbitrary, it follows that $P \in \mathscr{G}(\Pi_\Phi)$. □

In order to define an entrance law we need some new concepts and some further notation. Let Ψ be the space of equivalence classes of functions $S \to \mathbb{R}_+$, excluding the function which is identically zero, where two such functions ψ, $\hat{\psi}$ are defined to be equivalent if there is some $k > 0$ such that $\psi(x) = k\hat{\psi}(x)$ for all $x \in S$. We identify any element of Ψ with the functions in the corresponding equivalence class, and where necessary we use the proportionality sign $\propto$ (rather than $=$) to denote equality up to a strictly positive multiplicative constant. We define multiplication in Ψ (always denoted by $\prod$) to correspond to pointwise multiplication of the functions in the corresponding equivalence classes. Let $\mathscr{N}$ be the set of *ordered* pairs of neighbouring elements of A, i.e. $\mathscr{N} = \{(i, j)\}_{i\in A, j\in\partial i}$. For any interaction $\Phi = \{\phi_{\{i,j\}}\}_{\{i,j\}\in\mathscr{I}}$ and for each $(i, j) \in \mathscr{N}$ we shall find it convenient to write $\phi_{ij}(x_i, x_j)$ for $\phi_{\{i,j\}}(x_{\{i,j\}})$, $x_{\{i,j\}} \in S^{\{i,j\}}$. (Thus $\phi_{ij}(x_i, x_j) = \phi_{ji}(x_j, x_i)$.) Further, given $\psi \in \Psi$ we define $\phi_{ij}\psi \in \Psi$ (where it exists) by $\phi_{ij}\psi = \sum_{x\in S} \phi_{ij}(\cdot, x)\psi(x)$. (We adhere to the convention that equations involving $\phi_{ij}\psi$ include the assertion that it exists.) An *entrance law* for Φ is a family $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathscr{N}}$ of elements of Ψ such that

$$\lambda_i^j = \textstyle\prod_{k\in\partial i\backslash j} \phi_{ik}\lambda_k^i, \quad (i, j) \in \mathscr{N} \quad \text{(consistency)}, \tag{3.4}$$

$$\textstyle\sum_{x\in S} (\prod_{k\in\partial i} \phi_{ik}\lambda_k^i)(x) < \infty, \quad i \in A \quad \text{(normalisability)}. \tag{3.5}$$

Note that (3.4) asserts equality in the space Ψ. Note also that (3.5) is automatically satisfied when the state space S is finite.

Now recall that $\mathscr{V}^*$ is the set of finite connected subsets of A. For each $V \in \mathscr{V}^*$, $i \in \partial V$, let $V(i)$ represent the unique element of $V \cap \partial i$.

THEOREM 3.2. *Let the interaction $\Phi = \{\phi_{\{i,j\}}\}_{\{i,j\}\in\mathscr{I}}$ be such that for some reference element $s_A \in S^A$,*

$$\phi_{ij}(x, s_j) > 0, \quad x \in S, \quad (i, j) \in \mathscr{N}. \tag{3.6}$$

Then there is a one-to-one correspondence between $\mathscr{M}(\Pi_\Phi)$ *and the set of entrance laws for* Φ, *given by*: *for all* $v \in \mathscr{V}^*$,

$$P(\xi_{\Delta V} = x_{\Delta V}) \propto \mu_{\Phi,V}(x_{\partial V}, x_V) \prod_{i \in \partial V} \lambda_i^{V(i)}(x_i), \tag{3.7}$$

(*the "constant of proportionality" being independent of* $x_{\Delta V}$) *for each* $P \in \mathscr{M}(\Pi_\Phi)$ *and corresponding entrance law* $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathscr{N}}$.

PROOF. Suppose first that $P \in \mathscr{M}(\Pi_\Phi)$. Define the collection $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathscr{N}}$ in Ψ by

$$\lambda_i^j \propto P(\xi_i = \cdot/\xi_j = s_j)/\phi_{ij}(\cdot, s_j). \tag{3.8}$$

We show that (3.7) holds for each $V \in \mathscr{V}^*$ for this, and only this, collection Λ and that Λ is an entrance law for Φ. For any $V \in \mathscr{V}^*$, consider the representation of $P_{\Delta V}$ given by (3.3) and (3.2). Now $\nu_V(x'_{\partial V}) > 0$ for some $x'_{\partial V} \in S^{\partial V}$ and so from (3.6) it follows that $P(\xi_V = s_V) > 0$. We therefore have that

$$P(\xi_{\partial V} = x_{\partial V}/\xi_V = s_V) = a_V \nu_V(x_{\partial V}) \prod_{i\in\partial V} \phi_{i,V(i)}(x_i, s_{V(i)}), \quad x_{\partial V} \in S^{\partial V}, \tag{3.9}$$

for some positive constant a_V. Because V belongs to $\mathscr{V}^*$ no two elements of ∂V are neighbours, and because P is a Markov chain and thus $P_{\Delta V}$ a Markov random field, it follows that (under P) the random variables ξ_i, $i \in \partial V$, are conditionally independent relative to $\mathscr{F}(V)$, and that

$$P(\xi_{\partial V} = x_{\partial V}/\xi_V = s_V) = \prod_{i\in\partial V} P(\xi_i = x_i/\xi_{V(i)} = s_{V(i)}). \tag{3.10}$$

Thus by (3.9) and (3.10), ν_V has a unique representation as a product of elements of Ψ, given by

$$\nu_V(x_{\partial V}) \propto \prod_{i\in\partial V} \lambda_i^{V(i)}(x_i),$$

so that (3.7) holds. Since for every $(i, j) \in \mathscr{N}$, there is some $V \in \mathscr{V}^*$ with $j = V(i)$, it follows that Λ is the only collection in Ψ for which (3.7) is true. To show that Λ is an entrance law, consider each $i \in A$: by (3.7) with $V = \{i\}$,

$$P(\xi_{\Delta i} = x_{\Delta i}) \propto \prod_{k\in\partial i} \phi_{ik}(x_i, x_k)\lambda_k^i(x_k); \tag{3.11}$$

thus for any $j \in \partial i$, $P(\xi_i = \cdot/\xi_j = s_j)/\phi_{ij}(\cdot, s_j) \propto \prod_{k\in\partial i \setminus j} \phi_{ik}\lambda_k^i$, and this, together with (3.8), gives the consistency condition (3.4). Also by (3.11), $P(\xi_i = \cdot) \propto \prod_{k\in\partial i} \phi_{ik}\lambda_k^i$, yielding the normalisability condition (3.5).

We now establish the converse result. Let $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathscr{N}}$ be an entrance law for Φ. We construct (the obviously unique) $P \in \mathscr{M}(\Pi_\Phi)$ such that (3.7) holds for all $V \in \mathscr{V}^*$. Observe first that (3.4) and (3.6) imply that $\lambda_i^j(s_i) > 0$ for all $(i, j) \in \mathscr{N}$. For each $V \in \mathscr{V}^*$ define the function $p_V: S^{\Delta V} \to \mathbb{R}_+$ by

$$p_V(x_{\Delta V}) = \mu_V(x_{\partial V}, x_V) \prod_{i\in\partial V} \{\lambda_i^{V(i)}(x_i)/\lambda_i^{V(i)}(s_i)\}.$$

Fix any $h \in A$ and define the sequence $\{V_n\}_{n\geq 0}$ in $\mathscr{V}^*$ by $V_0 = \{h\}$, $V_n = \Delta V_{n-1}$, $n \geq 1$. It follows easily from (3.4) that for all $n \geq 1$,

$$\sum_{x_{\partial V_n} \in S^{\partial V_n}} p_{V_n}(x_{V_{n-1}}) = c_n p_{V_{n-1}}(x_{V_n}),$$

for some finite positive constant c_n. Further

$$\sum_{x_{\Delta h}\in S^{\Delta h}} p_{\{h\}}(x_{\Delta h}) \propto \sum_{x\in S} (\prod_{k\in\partial h} \phi_{hk}\lambda_k^h)(x) < \infty$$

by (3.5). Thus each function p_{V_n} may be normalised to a probability density on $S^{\Delta V_n}$; such probability densities form a consistent family and so define a probability measure P on $(S^A, \mathscr{F})$, with the property that (3.7) holds at least for all $V = V_n$, $n \geq 0$. By Lemma 3.1, $P \in \mathscr{G}(\Pi_\Phi)$. Further every $V \in \mathscr{V}^*$ is contained in some V_n, $n \geq 0$, and because P_{V_n} is by construction a Markov random field so also is P_V (by Corollary 1 to Theorem 2.1). We therefore have that $P \in \mathscr{M}(\Pi_\Phi)$. It remains to show that (3.7) holds for *all* $V \in \mathscr{V}^*$, i.e.

that if $\hat{\Lambda} = \{\hat{\lambda}_i^j\}_{(i,j)\in\mathcal{N}}$ is the entrance law corresponding to P then $\hat{\Lambda} = \Lambda$. We have already established that $\hat{\lambda}_i^j = \lambda_i^j$ for all $(i, j) \in \mathcal{N}$ such that $i \in \partial V_n$, $j \in V_n$ for some $n \geq 0$. Iterative application of (3.4) now shows that $\hat{\lambda}_i^j = \lambda_i^j$ for all $(i, j) \in \mathcal{N}$ as required. □

4. Homogeneous specifications on regular trees. Henceforth we take $(A, \sim)$ to be the regular tree of "dimension" d, i.e. with $d + 1$ edges meeting at each vertex. We say that a Markov specification, an interaction, or a probability measure is *homogeneous* if it is invariant under graph isomorphisms of the tree. For any homogeneous Markov specification Π let $\mathcal{G}_0(\Pi)$ and $\mathcal{M}_0(\Pi)$ denote the sets of homogeneous elements of $\mathcal{G}(\Pi)$ and $\mathcal{M}(\Pi)$ respectively.

For the special case $d = 1$ (so that $(A, \sim)$ is the one-dimensional lattice $\mathbb{Z}^1$) it is more natural to consider Markov specifications which are simply *stationary*, i.e. translation invariant. Many authors have studied such specifications and their corresponding Markov random fields. (See in particular Dobrushin (1968), Spitzer (1975b), Kesten (1976), Cox (1977), and also Preston (1976), Chapter 5.) Here the theory of the preceding sections has much in common with the approach of Spitzer (1975b); in particular entrance laws are equivalent to the double sequences of functions introduced in his Theorem 6. For the general case $d \geq 1$, and with state space restricted to $S = \{0, 1\}$, homogeneous Markov specifications have been studied by Preston (1974) and Spitzer (1975a); (see also Section 5 of this paper).

Let the function $\phi: S \times S \to \mathbb{R}_+$ satisfy

$$\phi(x, y) = \phi(y, x), \quad x, y \in S, \tag{4.1}$$

$$\phi(x, s) > 0, \quad x \in S, \tag{4.2}$$

$$\textstyle\sum_{x_V \in S^V} \mu_{\phi,V}(x_{\partial V}, x_V) < \infty, \quad V \in \mathcal{V}, \quad x_{\partial V} \in S^{\partial V}, \tag{4.3}$$

where s is some fixed reference element of S and $\mu_{\phi,V}: S^{\partial V} \times S^V \to \mathbb{R}_+$ is defined by

$$\mu_{\phi,V}(x_{\partial V}, x_V) = \textstyle\prod_{\{i,j\}\in\mathcal{E}, \{i,j\}\cap V\neq\varnothing} \phi(x_i, x_j).$$

Then the collection $\{\phi_{\{i,j\}}\}_{\{i,j\}\in\mathcal{E}}$ of functions $\phi_{\{i,j\}}: S^{\{i,j\}} \to \mathbb{R}_+$, defined by $\phi_{\{i,j\}}(x_{\{i,j\}}) = \phi(x_i, x_j)$, is a homogeneous interaction. We identify this interaction with ϕ; then Π_ϕ denotes the corresponding (necessarily homogeneous) Markov specification. Every positive homogeneous Markov specification may be represented by such an interaction (for example, by adapting the representation of Preston (1976), Proposition 5.7); here (4.2) is satisfied for all $s \in S$.

For any member ψ of the space Ψ introduced in the preceding section, define $F_\phi\psi \in \Psi$, where it exists, by $F_\phi\psi \propto \{\sum_{y\in S} \phi(\cdot, y)\psi(y)\}^d$. Now Theorem 3.2 defines a correspondence between $\mathcal{M}(\Pi_\phi)$ and the set of entrance laws for ϕ. An element of $\mathcal{M}(\Pi_\phi)$ belongs to $\mathcal{M}_0(\Pi_\phi)$ if and only if the corresponding entrance law has all its components equal. We thus have immediately from Theorem 3.2 and the defining relations (3.4), (3.5) for an entrance law the result below. (Note that the relation (4.4) is used in obtaining (4.5).)

THEOREM 4.1. *There is a one-to-one correspondence between $\mathcal{M}_0(\Pi_\phi)$ and the set of $\lambda \in \Psi$ which satisfy*

$$\lambda = F_\phi\lambda, \tag{4.4}$$

$$\textstyle\sum_{x\in S} \sum_{y\in S} \lambda(x)\phi(x, y)\lambda(y) < \infty. \tag{4.5}$$

The Markov chains comprising $\mathcal{M}_0(\Pi_\phi)$ are mutually singular: for if $(B, \sim)$ is any subgraph of $(A, \sim)$ isomorphic to $\mathbb{Z}^1$, then the marginal distribution on $(S^B, \mathcal{F}(B))$ of each Markov chain in $\mathcal{M}_0(\Pi_\phi)$ is a stationary (indeed reversible) irreducible one-dimensional Markov chain and is therefore ergodic. In particular no strictly convex combination of any two distinct Markov chains in $\mathcal{M}_0(\Pi_\phi)$ can also belong to $\mathcal{M}_0(\Pi_\phi)$; (see also Theorem 4.4).

Theorem 4.2 below is analogous to a result of Dobrushin (1968) for $\mathbb{Z}^d$.

THEOREM 4.2. *Let S be finite. Then* $|\mathscr{M}_0(\Pi_\phi)| \geq 1$.

PROOF. Fix each element ψ of Ψ by the requirement $\sum_{x\in S}\psi(x) = 1$, and regard Ψ as a subset of the Banach space of all real-valued functions on S with, say, the norm $\|f\| = \sup_{x\in S}|f(x)|$. Then Ψ is convex and compact. The function $F_\phi : \Psi \to \Psi$ is given by

$$(4.6) \qquad (F_\phi\psi)(x) = \frac{\{\sum_{y\in S}\phi(x, y)\psi(y)\}^d}{\sum_{z\in S}\{\sum_{y\in S}\phi(z, y)\psi(y)\}^d}, \quad \psi \in \Psi.$$

For all $\psi \in \Psi$, the denominator of the right side of (4.6) is positive (consider $z = s$). Hence it is easily verified that F_ϕ is continuous and so by the Leray-Schauder-Tychonoff theorem (Reed and Simon, 1980, page 151) F_ϕ has a fixed point. Therefore by Theorem 4.1 $\mathscr{M}_0(\Pi_\phi)$ is non-empty. □

We now consider a further class of Markov chains (first introduced by Spitzer for the case $S = \{0, 1\}$), which arises naturally in the study of "repulsive" interactions. (See also Section 5.) Label the vertices of the tree $(A, \sim)$ alternately *even* and *odd*, denoting by E and O the respective sets of even and odd vertices. (Thus, of every pair of neighbours in A, exactly one belongs to E.) Denote by $\mathscr{M}_1(\Pi_\phi)$ the class of those Markov chains in $\mathscr{M}(\Pi_\phi)$ which are invariant under those graph isomorphisms of the tree mapping E onto E. Trivially $\mathscr{M}_0(\Pi_\phi) \subset \mathscr{M}_1(\Pi_\phi)$. Theorem 4.1 has the following straightforward generalisation.

THEOREM 4.3. *There is a one-to-one correspondence between* $\mathscr{M}_1(\Pi_\phi)$ *and the set of ordered pairs* (λ^e, λ^o) *of elements of* Ψ *which satisfy*

$$\lambda^e = F_\phi\lambda^o, \quad \lambda^o = F_\phi\lambda^e, \quad \sum_{x\in S}\sum_{y\in S}\lambda^e(x)\phi(x, y)\lambda^o(y) < \infty.$$

If $P \in \mathscr{M}_1(\Pi_\phi)$ corresponds to (λ^e, λ^o) we define its *complement* P', say, to be the chain in $\mathscr{M}_1(\Pi_\phi)$ corresponding to (λ^o, λ^e). Then $P = P'$ if and only if $P \in \mathscr{M}_0(\Pi_\phi)$. The following result has two important corollaries.

THEOREM 4.4. *Suppose that* P_1, P_2 *are distinct Markov chains in* $\mathscr{M}_1(\Pi_\phi)$ *and that* $0 < k < 1$. *Then the Markov random field* $P_0 = kP_1 + (1 - k)P_2$ *is not a Markov chain.*

PROOF. Suppose the contrary. Then necessarily $P_0 \in \mathscr{M}_1(\Pi_\phi)$. For each $m = 0, 1, 2$ let $(\lambda_m^e, \lambda_m^o)$ be the ordered pair of elements of Ψ corresponding to P_m as in Theorem 4.3. Since $\lambda_m^e = F_\phi\lambda_m^o$ it follows that $\lambda_m^e(s) > 0$, so that we may for definiteness take $\lambda_m^e(s) = 1$. Then there is a positive constant a_m such that for any $i \in O$,

$$P_m(\xi_{\partial i} = x_{\partial i}) = a_m \sum_{x\in S}\prod_{j\in\partial i}\phi(x, x_j)\lambda_m^e(x_j)$$

(see Theorem 3.2). We thus obtain

$$a_0\prod_{j\in\partial i}\lambda_0^e(x_j) = a_1 k\prod_{j\in\partial i}\lambda_1^e(x_j) + a_2(1 - k)\prod_{j\in\partial i}\lambda_2^e(x_j),$$

valid for all $x_{\partial i} \in S^{\partial i}$. This can only be the case if $\lambda_1^e = \lambda_2^e$, implying by Theorem 4.3 that $P_1 = P_2$ in contradiction to the hypothesis that P_1, P_2 are distinct. □

COROLLARY 1. *If* P, P' *form a complementary pair of distinct Markov chains in* $\mathscr{M}_1(\Pi_\phi)$ *then* $\frac{1}{2}(P + P')$ *belongs to* $\mathscr{G}_0(\Pi_\phi)$ *but not to* $\mathscr{M}_0(\Pi_\phi)$.

(Spitzer's results for the case $S = \{0, 1\}$ show that such complementary pairs of distinct Markov chains do in fact exist.)

COROLLARY 2. *Let* $d = 1$, *so that* $(A, \sim) = \mathbb{Z}^1$, *and let the homogeneous interaction* ϕ *be positive. Then* $\mathscr{M}_1(\Pi_\phi) = \mathscr{M}_0(\Pi_\phi)$ *and contains at most a single element.*

PROOF. This is immediate from Corollary 1 above and the result of Kesten (1976) that here $\mathscr{G}_0(\Pi_\phi) = \mathscr{M}_0(\Pi_\phi)$ and contains at most a single element. □

5. The binary state space. We now consider the case $S = \{0, 1\}$. We continue to take $(A, \sim)$ to be the regular tree of dimension d, and consider the homogeneous Markov specification Π_ϕ defined by $\phi: S \times S \to \mathbb{R}_+$ satisfying (4.1) and (4.2)—since S is finite the condition (4.3) is here redundant. We give a short proof of Spitzer's result that $|\mathcal{G}(\Pi_\phi)| = 1$ if (and only if) $|\mathcal{M}_1(\Pi_\phi)| = 1$. We also establish a further result: that if $|\mathcal{M}_1(\Pi_\phi)| > 1$ then $|\mathcal{M}(\Pi_\phi)| = \infty$.

We take the reference element of the condition (4.2) to be given by $s = 0$. Now for all $k > 0$, $\Pi_{k\phi} = \Pi_\phi$. Thus without loss of generality we may take ϕ to be given by $\phi_{p,q}: S \times S \to \mathbb{R}_+$ where

$$\phi_{p,q}(0, 0) = 1, \quad \phi_{p,q}(0, 1) = \phi_{p,q}(1, 0) = q, \quad \phi_{p,q}(1, 1) = p^2,$$

and $0 < q < \infty$, $0 \le p < \infty$. For simplicity we write $\Pi_{p,q}$ for $\Pi_{\phi_{p,q}}$. The corresponding "single point" kernels $\pi_{(p,q),i}$, $i \in A$, are given by

$$\frac{\pi_{(p,q),i}(x_{\partial i}, \xi_i = 1)}{\pi_{(p,q),i}(x_{\partial i}, \xi_i = 0)} = p^{2r} q^{d+1-2r}, \tag{5.1}$$

where r, $0 \le r \le d + 1$, is the number of neighbours j of i such that $x_j = 1$. It follows in particular that distinct pairs (p, q) define distinct homogeneous Markov specifications. Note also that $\Pi_{p,q}$ is positive if and only if $p > 0$, and that here every positive homogeneous Markov specification has such a representation; (see Section 4).

Throughout this section we represent any element ψ of the space Ψ introduced in Section 3 by the corresponding point in $\mathbb{R}_+ \cup \{\infty\}$ given by $\psi(1)/\psi(0)$ (the element ψ of Ψ with $\psi(0) = 0$ being represented by ∞). An entrance law for the interaction $\phi_{p,q}$ is then a collection $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathcal{N}}$ in $\mathbb{R}_+ \cup \{\infty\}$ such that

$$\lambda_i^j = \textstyle\prod_{k\in\partial i\setminus j} g_{p,q}(\lambda_k^i), \quad (i, j) \in \mathcal{N} \tag{5.2}$$

where $g_{p,q}: \mathbb{R}_+ \cup \{\infty\} \to \mathbb{R}_+$ is given by

$$g_{p,q}(\lambda) = \begin{cases} (q + p^2\lambda)/(1 + q\lambda), & \lambda \in \mathbb{R}_+ \\ p^2/q & \lambda = \infty. \end{cases}$$

Theorem 4.1 then states that there is a one-to-one correspondence between $\mathcal{M}_0(\Pi_{p,q})$ and the set of solutions λ (necessarily in $\mathbb{R}_+$) of the consistency condition: $\lambda = \{g_{p,q}(\lambda)\}^d$. Similarly Theorem 4.3 states that there is a one-to-one correspondence between $\mathcal{M}_1(\Pi_{p,q})$ and the set of ordered pairs (λ^e, λ^o), $\lambda^e, \lambda^o \in \mathbb{R}_+$, satisfying $\lambda^e = \{g_{p,q}(\lambda^o)\}^d$, $\lambda^o = \{g_{p,q}(\lambda^e)\}^d$. These results are essentially those obtained by Spitzer (1975a).

When $p \ge q$, each side of (5.1) is an increasing function of r and the Markov specification $\Pi_{p,q}$ is said to be *attractive*; similarly when $p \le q$ each side of (5.1) is a decreasing function of r, and $\Pi_{p,q}$ is said to be *repulsive*. Spitzer (1975a) deduces the following results. In the attractive case $\mathcal{M}_0(\Pi_{p,q}) = \mathcal{M}_1(\Pi_{p,q})$; further for $d > 1$, $\mathcal{M}_0(\Pi_{p,q})$ may contain 1, 2 or 3 Markov chains according to the precise value of (p, q). In the repulsive case $|\mathcal{M}_0(\Pi_{p,q})| = 1$ always; however for $d > 1$ and suitable (p, q), $|\mathcal{M}_1(\Pi_{p,q})| > 1$.

The first part of Theorem 5.1 below is again due to Spitzer (1975a). His proof uses the general theory of supermodular potentials (see Preston, 1974), while that given here is based on the use of entrance laws.

THEOREM 5.1.

(i) *If* $|\mathcal{M}_1(\Pi_{p,q})| = 1$ *then* $\mathcal{M}_1(\Pi_{p,q}) = \mathcal{M}(\Pi_{p,q}) = \mathcal{G}(\Pi_{p,q})$.
(ii) *If* $|\mathcal{M}_1(\Pi_{p,q})| > 1$ *then* $|\mathcal{M}(\Pi_{p,q})| = \infty$.

PROOF. We consider separately the attractive and repulsive cases.

1. Suppose $p \ge q$. Define the sequences $\{\lambda_n^-\}_{n\ge0}$ and $\{\lambda_n^+\}_{n\ge0}$ in $\mathbb{R}_+ \cup \{\infty\}$ by $\lambda_0^- = 0$, $\lambda_n^- = \{g_{p,q}(\lambda_{n-1}^-)\}^d$ for $n \ge 1$, and $\lambda_0^+ = \infty$, $\lambda_n^+ = \{g_{p,q}(\lambda_{n-1}^+)\}^d$ for $n \ge 1$. Trivially $\lambda_0^- \le \lambda_1^-$ and because $g_{p,q}$ is here an increasing function it follows by induction that $\lambda_n^- \le \lambda_{n+1}^- \le \{g_{p,q}(\infty)\}^d$ for all $n \ge 0$. Hence there exists $\lambda^- \in \mathbb{R}_+$ such that $\lambda_n^- \to \lambda^-$ as $n \to \infty$. Similarly

λ_n^+ decreases in n to a limit $\lambda^+ \in \mathbb{R}_+$. Since $g_{p,q}$ is continuous it follows that

$$\lambda^- = \{g_{p,q}(\lambda^-)\}^d, \quad \lambda^+ = \{g_{p,q}(\lambda^+)\}^d. \tag{5.3}$$

Let P^-, $P^+ \in \mathcal{M}_0(\Pi_{p,q})$ be the homogeneous Markov chains corresponding to λ^-, λ^+ respectively.

Let $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathcal{N}}$ be any entrance law for the interaction $\phi_{p,q}$. Because Λ satisfies (5.2) and $g_{p,q}$ is increasing and (trivially) $\lambda_0^- \le \lambda_i^j \le \lambda_0^+$ for all $(i, j) \in \mathcal{N}$, it follows by induction that $\lambda_n^- \le \lambda_i^j \le \lambda_n^+$ for all $(i, j) \in \mathcal{N}$ and for all $n \ge 0$. Letting $n \to \infty$ gives $\lambda^- \le \lambda_i^j \le \lambda^+$ for all $(i, j) \in \mathcal{N}$.

Now if $|\mathcal{M}_1(\Pi_{p,q})| = 1$ then $P^- = P^+$, $\lambda^- = \lambda^+$, and therefore the only entrance law for $\phi_{p,q}$ is that with all its components equal to λ^-. Thus $P^- = P^+$ is the sole member of $\mathcal{M}(\Pi_{p,q})$ and so also of $\mathcal{G}(\Pi_{p,q})$ (see Section 1).

If $|\mathcal{M}_1(\Pi_{p,q})| > 1$ the argument of the preceding paragraph shows that $P^- \ne P^+$, $\lambda^- \ne \lambda^+$. Fix any $h \in A$. We show how to construct an infinite family of Markov chains in $\mathcal{M}(\Pi_{p,q})$, each (for simplicity) rotationally symmetric about h. Let λ_0 satisfy $\lambda^- \le \lambda_0 \le \lambda^+$; we may then define a sequence $\{\lambda_n\}_{n\ge 0}$ in $\mathbb{R}_+$ satisfying

$$\lambda^- \le \lambda_n \le \lambda^+, \quad n \ge 0 \tag{5.4}$$

$$\lambda_n = \{g_{p,q}(\lambda_{n+1})\}^d, \quad n \ge 0; \tag{5.5}$$

for since $g_{p,q}$ is increasing and continuous, and λ^-, λ^+ satisfy (5.3), we may define the sequence recursively via (5.5) noting at each state the (5.4) is satisfied. Define now the sequence $\{V_n\}_{n\ge 0}$ in $\mathcal{V}^*$ by $V_0 = \{h\}$, $V_n = \Delta V_{n-1}$ for $n \ge 1$. We construct an entrance law $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathcal{N}}$ for $\phi_{p,q}$ as follows: if $(i, j) \in \mathcal{N}$ is such that $i \in \partial V_n$, $j \in V_n$ for some $n \ge 0$, let $\lambda_i^j = \lambda_n$; by (5.5) the components λ_i^j thus defined satisfy the consistency condition (5.2); the remaining components of Λ are then uniquely and consistently determined by the iterative use of (5.2). Distinct values of λ_0 between λ^- and λ^+ define distinct entrance laws and thus distinct Markov chains in $\mathcal{M}(\Pi_{p,q})$. Hence $|\mathcal{M}(\Pi_{p,q})| = \infty$.

2. Now suppose $p < q$. Let the sequence $\{\lambda_n^-\}_{n\ge 0}$ in $\mathbb{R}_+$ be as defined before. The function $g_{p,q}$ is here decreasing and continuous. Thus, starting with the trivial relation $\lambda_0^- \le \lambda_2^-$, we may show by induction that λ_{2n}^- increases with n to a limit $\lambda^e \in \mathbb{R}_+$, while λ_{2n+1}^- decreases to a limit $\lambda^o \in \mathbb{R}_+$; further $\lambda^e = \{g_{p,q}(\lambda^o)\}^d$, $\lambda^o = \{g_{p,q}(\lambda^e)\}^d$, so that the ordered pairs (λ^e, λ^o) and (λ^o, λ^e) define a complementary pair of Markov chains in $\mathcal{M}_1(\Pi_{p,q})$ coincident if and only if $\lambda^e = \lambda^o$. If $\Lambda = \{\lambda_i^j\}_{(i,j)\in\mathcal{N}}$ is any entrance law for $\phi_{p,q}$ then, starting with the relation $\lambda_0^- \le \lambda_i^j$ for all $(i, j) \in \mathcal{N}$, we may show (essentially as in the attractive case) that $\lambda^e \le \lambda_i^j \le \lambda^o$ for all $(i, j) \in \mathcal{N}$. Thus if $|\mathcal{M}_1(\Pi_{p,q})| = 1$, $\lambda^e = \lambda^o$ and we may deduce as before that $\mathcal{M}(\Pi_{p,q}) = \mathcal{M}_1(\Pi_{p,q})$, while if $|\mathcal{M}_1(\Pi_{p,q})| > 1$ then, again essentially as previously, we may construct infinitely many further members of $\mathcal{M}(\Pi_{p,q})$. □

6. An example with $|S| = 3$. Let $S = \{0, 1, 2\}$ and A again be the regular tree of dimension d. Let Π_k denote the positive homogeneous Markov specification defined as in Section 4 by $\phi_k : S \times S \to \mathbb{R}_+$ where $\phi_k(x, x) = 1$, $x \in S$ and $\phi_k(x, y) = k > 0$, $x, y \in S$, $x \ne y$. By Theorem 4.3 $\mathcal{M}_1(\Pi_k)$ is in a one-to-one correspondence with the set of solutions (λ^e, λ^o), $\lambda^e, \lambda^o \in \Psi$, of

$$\lambda^e = \{\textstyle\sum_{x\in S} \phi_k(\cdot, x)\lambda^o(x)\}^d, \quad \lambda^o = \{\textstyle\sum_{x\in S} \phi_k(\cdot, x)\lambda^e(x)\}^d. \tag{6.1}$$

As usual a Markov chain in $\mathcal{M}_1(\Pi_k)$ belongs to $\mathcal{M}_0(\Pi_k)$ if and only if the corresponding solution of (6.1) satisfies $\lambda^e = \lambda^o$. One such solution is given by $\lambda^e(x) =$ constant for all $x \in S$. Identification of all solutions of (6.1) is analytically difficult (though possible by numerical investigation for any given d and k). We look here for solutions with $\lambda^e(1)/\lambda^e(0) = \lambda^e(2)/\lambda^e(0) = \hat{\lambda}^e$, say. (To each such solution, other than that with $\hat{\lambda}^e = 1$, there correspond two further distinct solutions of (6.1), obtained by cyclic permutation of the states of S.) The equations (6.1) then reduce to

$$\hat{\lambda}^e = f_k(\hat{\lambda}^o), \quad \hat{\lambda}^o = f_k(\hat{\lambda}^e), \tag{6.2}$$

where $\hat{\lambda}^o = \lambda^o(1)/\lambda^o(0) = \lambda^o(2)/\lambda^o(0)$ and where $f_k: \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $f_k(\lambda) = [\{k + (1 + k)\lambda\}/\{1 + 2k\lambda\}]^d$. The solutions of (6.2) are most easily obtained by further reference to Spitzer's (1975a) treatment of the case $S = \{0, 1\}$. We obtain (6.2) from the equation considered in Theorem 9 there by putting $x = (\lambda^e)^{1/d}$, $s = 1/(1 + 2k)$, $t = (1 + k)/(1 + 2k)$. The results below then follow immediately from Spitzer's analysis. When $k \leq 1$ any solution of (6.2) necessarily satisfies $\hat{\lambda}^e = \hat{\lambda}^o$ (for the function f_k is here increasing), and depending on d and the precise value of k there may be 1, 2 or 3 such solutions. When $k > 1$, the equations (6.2) have exactly one solution with $\hat{\lambda}^e = \hat{\lambda}^o$ (for the function f_k is here decreasing), but for suitable d and k there is an additional pair of solutions $(\hat{\lambda}^e, \hat{\lambda}^o)$ and $(\hat{\lambda}^o, \hat{\lambda}^e)$ with $\hat{\lambda}^e \neq \hat{\lambda}^o$.

For $d = 2$ we may again use Spitzer's results to classify completely the solutions of (6.2). Here for all $k > 0$, we have $\hat{\lambda}^e = \hat{\lambda}^o$ always. For $k > (2\sqrt{2} - 1)/7 (\doteqdot 0.261)$ the only solution of (6.2) is that with $\hat{\lambda}^e = 1$. For each of the two cases $k = (2\sqrt{2} - 1)/7$ and $k = 1/4$ (6.2) has exactly one additional solution (given by $\hat{\lambda}^e = 1/2$ and $\hat{\lambda}^e = 1/4$ respectively); it follows that for these two cases $|\mathscr{M}_0(\Pi_k)| \geq 4$. For $k < (2\sqrt{2} - 1)/7$, $k \neq 1/4$, (6.2) has exactly two additional solutions—so that $|\mathscr{M}_0(\Pi_k)| \geq 7$.

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