*The Annals of Probability*
1988, Vol. 16, No. 1, 361–374

# EXTREME SOJOURNS OF DIFFUSION PROCESSES[1]

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Let $(X_t)$ be a diffusion process on the interval $(r_1, r_2)$, where $r_2$ is inaccessible. For $r_1 < z < r_2$, let $T_z$ be the first passage time to $z$, and define $L_u = \int_{\{t:\, 0 \le t \le T_z,\, X_t > u\}} J(X_t)\, dt$, where $J$ is a particular function determined by the generator of the diffusion. An explicit asymptotic expression is obtained for the probability $P(L_u > y | X_0 = x)$, for $u \to r_2$, fixed $y > 0$, and $r_1 < x < r_2$. From this the corresponding asymptotic form of the distribution of the sojourn time, mes$(t\colon 0 \le t \le T_z,\ X_t > u)$ is determined when $r_2 = \infty$. Related theorems are given for the distribution of $L_u - L_v$, for $u, v \to \infty$, $u \le v$ and $u/v \to 1$. Finally, the results are extended to the long-term sojourn integral $L_u^* = \int_{\{t:\ 0 \le t \le S(u),\ X_t > u\}} J(X_t)\, dt$, where $S(u) \to \infty$ for $u \to r_2$.

**1. Introduction and summary.** Let $X_t$, $t \ge 0$, be a diffusion process on the real interval $(r_1, r_2)$, where either endpoint may be finite or infinite. The subject of this work is a set of limit theorems for the time spent by $(X_t)$ above the level $u$, for $u \to r_2$. To be specific, let $x$ and $z$ be two points in $(r_1, r_2)$ with $z < x$, and let $T_z$ be the first passage time to $z$. Our main interest is in the asymptotic form of the probability

$$(1.1)\qquad P\left\{\int_0^{T_z} J(X_t) I_{[X_t > u]}\, dt > y | X_0 = x\right\},$$

for $u \to r_2$ and $y > 0$, and where $J(x)$ is an explicit function determined by the infinitesimal generator of the diffusion, and $I$ is the indicator random variable. Under appropriate conditions in the case $r_2 = \infty$, the function $J(X_t)$ in (1.1) may be replaced by $J(u)$ and taken out of the integral, so that the integral in (1.1) becomes

$$(1.2)\qquad J(u)\int_0^{T_z} I_{[X_t > u]}\, dt.$$

The latter integral is precisely the sojourn time above $u$.

The special diffusion, Brownian motion with constant drift $-1$ and variance parameter 2, plays a central role in the formulation and proofs of the results. Indeed, by the well-known method of Volkonskii [6], every integral of a function on the path of $(X_t)$ has a canonical representation as a corresponding integral on the path of the drifting Brownian motion. The latter process is denoted $V_s$, $s \ge 0$. The limiting form of (1.1) is then easily computed in terms of $(V_s)$. In the

Received November 1985; revised July 1986.

[1]This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of National Science Foundation Grant DMS-85-01512, and Army Research Office Contract DAAL 03-86-K-0127.

*AMS* 1980 *subject classifications*. Primary 60J60; secondary 60G17.

*Key words and phrases*. Diffusion process, sojourn time, first passage time, local time, scale function.



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last section, the results for (1.1) are applied to determining the limiting distribution of the long-term sojourn,

$$\int_0^{S(u)} J(X_t) I_{[X_t > u]}\, dt, \tag{1.3}$$

where $S(u)$ is the scale function of the diffusion, and $S(u) \to \infty$ for $u \to r_2$.

Here is a brief summary of the main results. Let $(X_t)$ have drift coefficient $b(x)$ and diffusion coefficient $a(x)$; then the scale function is computed in terms of these in accordance with a well-known formula [(2.3) below]. We assume $S(r_2) = \infty$, and define $J(x) = \frac{1}{2}a(x)(S'(x)/1 + S(x))^2$. Our main results follow.

THEOREM 3.2.

$$\lim_{u \to r_2} S(u) P\left\{ \int_0^{T_z} J(X_t) I_{[X_t > u]}\, dt > y | X_0 = x \right\}$$
$$= (S(x) - S(z)) P\left\{ \int_0^{\infty} I_{[V_s > 0]}\, ds > y | V_0 = 0 \right\}.$$

The distribution in the latter expression is identified by its Laplace–Stieltjes transform.

When $r_2 = \infty$, Theorem 4.1 furnishes conditions under which the factor $J(X_t)$ above may be replaced by $J(u)$ to get the corresponding result about (1.2). These conditions require the rapid growth of $S(u)$ for $u \to \infty$, which is equivalent to a significant downward drift for large values of the diffusion.

THEOREM 5.3. *Assume* $r_2 = \infty$. *If* $S(u)$ *is of regular oscillation for* $u \to \infty$, *then, under the limit operation* $u, v \to \infty$, $u < v$, $u/v \to 1$,

$$\lim S(u) P\left\{ \frac{\int_0^{T_z} J(X_t) I_{[u < X_t < v]}\, dt}{\log(S(v)/S(u))} > y \middle| X_0 = x \right\}$$
$$= e^{-y}(S(x) - S(z)).$$

The exponential density above arises in the following way. The integral $\int JI\, dt$ above is canonically transformed into the corresponding integral for $V_s$. When divided by $\log(S(v)/S(u))$, the integral converges, for $u \to \infty$, to the local time at 0 for the process $V_s$ (Theorem 5.1). The distribution of the local time is then shown to be exponential (Theorem 5.2). Finally, it is shown that under certain conditions $J(X_t)$ may be replaced by $J(u)$ (Theorem 5.4).

The last set of results is about the long-term integral (1.3).

THEOREM 6.1. *Assume*

$$m = \int_{r_1}^{r_2} \frac{dx}{a(x)S'(x)} < \infty, \qquad S(r_2) = -S(r_1) = \infty. \tag{1.4}$$

*Then the integral* (1.3) *has, for* $u \to \infty$, *a limiting distribution with the*

*Laplace–Stieltjes transform*

$$\exp\left\{-\frac{s}{m}\left(\frac{2}{1+(1+4s)^{1/2}}\right)^2\right\}, \qquad s > 0.$$

*This does not depend on the initial state x.*

THEOREM 6.2. *Under* (1.4) *and the hypothesis and limit operation of Theorem* 5.3, *the random variable*

$$\frac{\int_0^{S(u)} J(X_t) I_{[u < X_t < v]}\, dt}{\log(S(v)/S(u))}$$

*has a limiting distribution with the Laplace–Stieltjes transform*

$$\exp\left\{-\frac{s}{m(1+s)}\right\}, \qquad s > 0.$$

Under the conditions of Theorems 4.1 and 5.4, the factor $J(X_t)$ may be replaced by $J(u)$ in the statements of Theorems 6.1 and 6.2, respectively.

This work is related to our previous papers [2] and [3]. In [2] we considered a different version of Theorem 3.2 for a much more restricted class of diffusions. It was assumed that the process is ergodic, and that there is an initial distribution over the state space which is, in fact, the stationary distribution, so that $X_t$ is actually a stationary process. However, the upper limit $T_z$ of the integral in (1.2) was replaced by the fixed number $t > 0$, so that the earlier result is not a consequence of the current Theorem 3.2.

The conditions of [3] are comparable to those of the current Section 6. The main result of [3] now follows as an immediate consequence of Theorem 6.1, and the corresponding version of Theorem 4.1. The other results of Section 6 apply to more general classes of processes than in [3].

**2. A canonical representation.** Let $X_t$ be a separable diffusion process on the interval $(r_1, r_2)$ with the infinitesimal generator

$$\frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}. \tag{2.1}$$

Assume $a(x) > 0$, define

$$w(x) = -2b(x)/a(x), \tag{2.2}$$

and suppose that $w(x)$ is locally integrable. Let $S(x)$ be the scale function of the diffusion,

$$S(x) = \int_{x_0}^{x} \exp\left(\int_{x_0}^{y} w(v)\, dv\right) dy, \tag{2.3}$$

where $x_0$ is a fixed arbitrary point in $(r_1, r_2)$.

Let $P_x(\cdot)$ and $E_x(\cdot)$ denote the probability measure and the expectation operator, respectively, for the process which starts at $x$ and evolves according to the transition distribution determined by the generator (2.1). For $r_1 < y < r_2$ define

(2.4) $$T_y = \inf(t\colon X_t = y);$$

then, it is well known that ([5], page 174)

(2.5) $$P_y(T_z < T_x) = \frac{S(y) - S(x)}{S(z) - S(x)},$$

for $x < y < z$.

Let $M_u(x)$ be a nonnegative measurable function on $(r_1, r_2)$ such that

(2.6) $$M_u(x) = 0, \quad \text{for } x < u,$$

and define

(2.7) $$L_u = \int_0^{T_z} M_u(X_t)\,dt$$

for $r_1 < z < u < r_2$, where $z$ is fixed. Then consider the Laplace–Stieltjes transform of the distribution of $L_u$,

(2.8) $$Q(s; u, x) = E_x(e^{-sL_u}).$$

LEMMA 2.1. *For $r_1 < z < x < u < r_2$, we have the equation*

(2.9) $$Q(s; u, x) = 1 - \frac{S(x) - S(z)}{S(u) - S(z)}[1 - Q(s; u, u)].$$

PROOF. Under the condition $X_0 = x$, the event $T_z < T_u$ implies, by (2.6) and (2.7), that $L_u = 0$; therefore,

(2.10) $$E_x[\exp(-sL_u)] = P_x(T_z < T_u) + E_x\left[I_{[T_u < T_z]}\exp(-sL_u)\right].$$

By (2.6), the last term in (2.10) is equal to

$$E_x\left\{I_{[T_u < T_z]}\exp\left[-s\int_{T_u}^{T_z} M_u(X_t)\,dt\right]\right\}.$$

Conditioning by the value of the process at the stopping time $T_u$, and invoking the strong Markov property, we find that the expectation above is equal to

$$P_x(T_u < T_z)E_u\left\{\exp\left[-s\int_0^{T_z} M_u(X_t)\,dt\right]\right\}.$$

Hence the assertion (2.9) is a consequence of (2.5) and (2.10). □

Let $W_t$, $t \geq 0$, be the standard Brownian motion on the line, and define the process

(2.11) $$V_s = \sqrt{2}\,W_s - s, \qquad s \geq 0.$$

This is a diffusion with the generator

(2.12) $$\frac{d^2}{dx^2} - \frac{d}{dx}.$$

We show, as in [3], that an arbitrary diffusion satisfying

$$S(r_2) = \infty, \tag{2.13}$$

and starting at $x$ and stopping at the first passage time to $z$, with $r_1 < z < x < r_2$, can be canonically transformed into the diffusion $(V_s)$, starting at $v$, and stopping at the first passage time to 0, for some $v > 0$. Let $(X_t)$ have the generator (2.1); and take $x_0 = z$ in (2.3), so that $S(z) = 0$; and define

$$f(x) = \log(1 + S(x)), \qquad z \le x < r_2. \tag{2.14}$$

It is obvious that $f$ is continuous and strictly monotonic with $f(z) = 0$ and $f(r_2) = \infty$ [see (2.13)]. Define

$$J(x) = \tfrac{1}{2}a(x)(f'(x))^2, \tag{2.15}$$

where $f'(x) = S'(x)/1 + S(x)$, and $f^{-1}$ as the inverse of $f$. By a restriction of the statement and proof of [3], Theorem 3.1, to the intervals $(z, r_2)$ and $(0, \infty)$ for the processes $(X_t)$ and $(f(X_t))$, respectively, and with $S$ in the place of $H$, we see that the generator of the process $Y_t = f(X_t)$ is equal to

$$J \circ f^{-1}(y)(d^2/dy^2 - d/dy), \tag{2.16}$$

for $y > 0$. For $t > 0$, let $\tau_t$ be a positive random variable defined as

$$\tau_t = \inf\left\{\sigma\colon \int_0^\sigma \frac{ds}{J \circ f^{-1}(V_s)} = t\right\}. \tag{2.17}$$

Then, by the well-known result of Volkonskii [6], the time substitution $s = \tau_t$ transforms the generator (2.12) of $(V_s)$ into the generator (2.16) of $(V_{\tau_t})$, for $y > 0$. Therefore, the statement leading to (2.16) implies that the generator of $f^{-1}(V_{\tau_t})$ is equal to that of $X_t$, namely (2.1), for $z < x < r_2$. By the uniqueness of the generator, it follows that $X_t$ has the representation

$$X_t = f^{-1}(V_{\tau_t}), \tag{2.18}$$

with the understanding that $X_t$ starts at $x$ and stops at time $T_z$, while the process $V_{\tau_t}$ starts at $f(x)$ and stops at time $\sup(t\colon \tau_t \le T_0^*)$, where

$$T_0^* = \inf(s\colon V_s = 0). \tag{2.19}$$

**Lemma** 2.2. *The random variable $L_u$ in* (2.7) *has the representation*

$$\int_0^{T_0^*} \frac{M_u(f^{-1}(V_s))}{J \circ f^{-1}(V_s)}\, ds. \tag{2.20}$$

**Proof.** $L_u$ may be written as

$$\int_0^\infty I_{[t < T_z]} M_u(X_t)\, dt.$$

By (2.17), (2.18) and (2.19), the event $t < T_z$ is equivalent to $\tau_t < T_0^*$, and the integral above is equal to

$$\int_0^\infty I_{[\tau_t < T_0^*]} M_u(f^{-1}(V_{\tau_t}))\, dt.$$

By the change of variable $s = \tau_t$, and the relation $dt = ds/J \circ f^{-1}(V_s)$ ([3]), the preceding integral is equal to (2.20). □

Our main example is the integral $L_u$ in (2.7) with $M_u(y) = J(y)I_{(u,\infty)}(y)$; then, by Lemma 2.2, $L_u$ has the representation

$$\int_0^{T_0^*} I_{[V_s > f(u)]}\, ds. \tag{2.21}$$

It also follows for $u < v$ that $L_u - L_v$ has the representation

$$\int_0^{T_0^*} I_{[f(u) < V_s < f(v)]}\, ds. \tag{2.22}$$

**3. Limiting distribution for a specific $L_u$.** The first result here is a limit theorem for the conditional distribution of $L_u$, given $X_0 = u$, for the particular $M_u$ mentioned at the end of the last section.

THEOREM 3.1. *Let $V_s$ be the diffusion* (2.11) *starting at the origin, and define*

$$\xi = \int_0^\infty I_{[V_s > 0]}\, ds. \tag{3.1}$$

*For any $z$, the conditional distribution of*

$$L_u = \int_0^{T_z} J(X_t) I_{[X_t > u]}\, dt, \tag{3.2}$$

*given $X_0 = u$, converges, for $u \to r_2$, to the distribution of $\xi$.*

PROOF. The integral (3.2) with $X_0 = u$ is equivalent to the integral (2.21) with $V_0 = f(u)$. Put

$$T_y^* = \inf(s\colon V_s = y); \tag{3.3}$$

then the process $V_s$, starting at $f(u)$ and stopping at time $T_0^*$, is equivalent in distribution to the process $V_s - f(u)$, starting at 0 and stopping at time $T^*_{-f(u)}$ because the Brownian term $\sqrt{2}\, W_s$ is spatially homogeneous. Therefore, the integral (2.21), under the condition $V_0 = f(u)$ has the same distribution as

$$\int_0^{T^*_{-f(u)}} I_{[V_s > 0]}\, ds, \tag{3.4}$$

under the condition $V_0 = 0$. Since, under (2.13), $f(u) \to \infty$ for $u \to r_2$, it follows that $T^*_{-f(u)} \to \infty$ almost surely under $V_0 = 0$. Hence, the integral (3.4) converges to (3.1). The conditional Laplace–Stieltjes transform of $\xi$, given $V_0 = 0$, is identical with the function $Q(0, t)$ in [2], formula (6.2), which has the explicit form ([2], formula (6.7))

$$Ee^{-s\xi} = \frac{2}{1 + (1 + 4s)^{1/2}}. \tag{3.5}$$ □

The next result concerns the distribution of $L_u$ for any starting point $x$ with $z < x < r_2$.

THEOREM 3.2. *For every $z < x$,*

$$\lim_{u \to r_2} S(u) P_x(L_u > y) = (S(x) - S(z)) P(\xi > y), \tag{3.6}$$

*for all $y > 0$ in the continuity set of the latter function.*

PROOF. By Lemma 2.1 and Theorem 3.1,

$$\lim_{u \to r_2} \frac{S(u) - S(z)}{S(x) - S(z)} (1 - Q(s; u, x)) = \lim_{u \to r_2} (1 - Q(s; u, u))$$
$$= 1 - Ee^{-s\xi},$$

which is equivalent to

$$\lim_{u \to r_2} \frac{S(u) - S(z)}{S(x) - S(z)} E_x(1 - e^{-sL_u}) = E(1 - e^{-s\xi}).$$

By integration by parts, this relation is equivalent to

$$\lim_{u \to r_2} \int_0^\infty e^{-sy} \frac{S(u) - S(z)}{S(x) - S(z)} P_x(L_u > y)\, dy = \int_0^\infty e^{-sy} P(\xi > y)\, dy. \tag{3.7}$$

Since $P_x(L_u > y) \le P_x(\max(X_t: 0 \le t \le T_z) > u) = P_x(T_u < T_z)$, the relation (2.5) implies that the coefficient of $e^{-sy}$ in the first integral in (3.7) is bounded by 1. The assertion (3.6) now follows from (3.7) by an application of the continuity theorem for the Laplace transform, and from the assumption (2.13). □

**4. Application to the sojourn time above $u$.** In this section we consider processes for which $r_2 = \infty$. While there are suitable versions of the results for $r_2 < \infty$, the hypothesis has a most natural form in the former case. We show how the limit theorem for $L_u$ in (3.2) can be converted into a corresponding limit theorem for the integral

$$\int_0^{T_z} I_{[X_t > u]}\, dt, \tag{4.1}$$

which represents the sojourn time above $u$. The main idea here is that under appropriate conditions the function $J(X_t)$ in the integrand of (3.2) may be replaced by $J(u)$, and then factored from the integral

$$\int_0^{T_z} J(X_t) I_{[X_t > u]}\, dt \approx J(u) \int_0^{T_z} I_{[X_t > u]}\, dt. \tag{4.2}$$

For this purpose we assume a certain condition defined and used in [2] and [3].

DEFINITION 4.1. Let $f(x)$ be a positive continuous function defined for all large $x > 0$. $f$ is said to be regularly oscillating for $x \to \infty$ if $\log f(e^x)$ is uniformly continuous for all $x > c$ for some $c > 0$.

We recall that $f$ is regularly oscillating if and only if [2]

$$\lim_{u/v \to 1,\, u \geq c,\, v \geq c} f(u)/f(v) = 1, \tag{4.3}$$

for some $c > 0$.

**Theorem 4.1.** *If the function $S$ satisfies*

$$\lim_{u \to \infty} \frac{S(u(1+\varepsilon))}{S(u)} = \infty, \tag{4.4}$$

*for every $\varepsilon > 0$, and $J(u)$ is regularly oscillating for $u \to \infty$, then*

$$\lim_{u \to \infty} S(u) P_x\left( J(u) \int_0^{T_z} I_{[X_t > u]}\, dt > y \right) = (S(x) - S(z)) P(\xi > y), \tag{4.5}$$

*for all $r_1 < z < x < \infty$, and $y > 0$ in the continuity set.*

**Proof.** For $\varepsilon > 0$, the event

$$\int_0^{T_z} J(X_t) I_{[X_t > u(1+\varepsilon)]}\, dt > 0 \tag{4.6}$$

implies $T_{u(1+\varepsilon)} < T_z$; therefore, by (2.5) and (4.4),

$$\begin{aligned} &\lim_{u \to \infty} S(u) P_x\left( \int_0^{T_z} J(X_t) I_{[X_t > u(1+\varepsilon)]}\, dt > 0 \right) \\ &\leq \lim_{u \to \infty} S(u) \frac{S(x) - S(z)}{S(u(1+\varepsilon)) - S(z)} = 0. \end{aligned}$$

Therefore, if we write $L_u$ in (3.2) as the sum $(L_u - L_{u(1+\varepsilon)}) + L_{u(1+\varepsilon)}$, the relation above shows that, in the statement of Theorem 3.2, one may replace $L_u$ by $L_u - L_{u(1+\varepsilon)}$, which is equal to

$$\int_0^{T_z} J(X_t) I_{[u < X_t < u(1+\varepsilon)]}\, dt.$$

By the regular oscillation of $J$, for every $\delta > 0$, there exists $\varepsilon > 0$ such that $|J(v)/J(u) - 1| < \delta$, for all large $u$ and $v$ such that $u < v < u(1+\varepsilon)$. Hence, the integral above is bounded above and below by

$$(1 \pm \delta) J(u) \int_0^{T_z} I_{[u < X_t < u(1+\varepsilon)]}\, dt, \tag{4.7}$$

with $+$ and $-$, respectively. The reasoning following (4.6) also shows that

$$\lim_{u \to \infty} S(u) P_x\left( J(u) \int_0^{T_z} I_{[X_t > u(1+\varepsilon)]}\, dt > 0 \right) = 0, \tag{4.8}$$

for $\varepsilon > 0$. Therefore, in evaluating the lim sup of the expression under the limit in (4.5), we do not decrease its value if we substitute (4.7) with the $+$ sign. Similarly, the lim inf is not increased if we substitute (4.7) with $-$. Since $\varepsilon > 0$ is arbitrary, so is $\delta > 0$. This justifies the replacement of the integral $L_u$ in (3.6) by the random variable in the left-hand member of (4.5). This completes the proof. $\square$

It was shown in [3] that a sufficient condition for the regular oscillation of $J(x)$ is the regular oscillation of $a(x)$ and $-b(x)$. In this case $J$ satisfies

$$J(u) \sim 2b^2(u)/a(u), \quad \text{for } u \to \infty, \tag{4.9}$$

so that the right-hand member may be used in the place of $J(u)$ in (4.5). It was also shown that if the function $w$ in (2.2) satisfies the condition $xw(x) \to \infty$, for $x \to \infty$, then (4.4) holds.

EXAMPLE 4.1. For the Ornstein–Uhlenbeck process we have $a(x) = 2$ and $b(x) = -x$; hence $w(x) = 4x$, and $J(u) \sim u^2$, so that the conditions of Theorem 4.1 are satisfied.

**5. Sojourn time between two high levels.** Theorem 4.1 does not apply unless (4.4) holds; in particular, diffusions in the natural scale, where $S(x) = x$, are excluded from the applications. The intuitive reason for this is that processes of this type lack a sufficient downward drift so that sojourns above $u$, if they occur, tend to be very large. In this section we show that if we consider the sojourn time in an interval of the form $(u, v)$ with $v = v(u)$, a function of $u$, then, under suitable conditions on the growth of $v$ with $u$, the sojourn time does have a limiting distribution in the sense of Theorem 4.1.

Local time arises in the statements of the results of this section. Local time for diffusion is conventionally defined as the derivative of the sojourn time distribution with respect to the speed measure [5], page 139. In line with my previous work on the local time for Gaussian and more general processes, I have defined the local time as the derivative with respect to Lebesgue measure. Since, in this case, the speed measure and Lebesgue measure are mutually absolutely continuous, the two local times differ by a factor equal to the density of one measure with respect to the other. Let $\eta_t(x)$ be the local time at $x$ of the process $V_s$, $0 \le s \le t$, where $V_0 = 0$. As a Radon–Nikodym derivative, it is uniquely defined for each $t$ except for an $x$-set of measure 0 which may depend on $t$. The well-known theorem of Trotter (see [5], page 115) states that, for the Brownian motion process, there is a version of the local time which is valid for all $x$ and $t$ with probability 1 which is jointly continuous in $(x, t)$. This extends easily to the local time of $(V_s)$, and we take $\eta_t(x)$ to be such a version. In this case $\eta_t(x)$ is uniquely determined for each $x$ as the derivative of the sojourn distribution, without exceptional sets of measure 0. Furthermore, $\eta_t(x)$ is nondecreasing for each $x$, so that $\eta(x) = \lim_{t\to\infty}\eta_t(x)$ exists. Since $V_s \to -\infty$, for $s \to \infty$, it also follows that, for each bounded interval $J$, $\eta_t(x) = \eta(x)$, for all $x \in J$, for all large $t$. Therefore, $\eta(x)$ is continuous and represents the local time of $V_s$, $s \ge 0$.

In the following we take $r_2 = \infty$; however, the theorem has a suitable form for $r_2 < \infty$, but we omit it.

THEOREM 5.1. *Let $\eta$ be the local time at* 0 *of the process $(V_s)$, where $V_0 = 0$. If $S(x)$ is of regular oscillation for $x \to \infty$, then*

$$\lim_{u,v\to\infty,\, u<v,\, u/v\to 1} P_u\left\{\frac{L_u - L_v}{\log(S(v)/S(u))} > y\right\} = P(\eta > y). \tag{5.1}$$

PROOF. By formula (2.22) and (3.2), $L_u - L_v$ has the representation

$$\int_0^{T_0^*} I_{[f(u)<V_s<f(v)]}\, ds.$$

The distribution of this random variable under $V_0 = f(u)$ is equal to the distribution of

$$\int_0^{T^*_{-f(u)}} I_{[0<V_s<f(v)-f(u)]}\, ds, \tag{5.2}$$

under $V_0 = 0$. The definition (2.14) and the assumption (2.13) imply

$$f(v) - f(u) \sim \log(S(v)/S(u)), \tag{5.3}$$

for $u, v \to \infty$. Therefore, the random variable (5.2), upon division by $\log(S(v)/S(u))$, is asymptotically equal to

$$\frac{\int_0^{T^*_{-f(u)}} I_{[0<V_s<f(v)-f(u)]}\, ds}{f(v) - f(u)}. \tag{5.4}$$

Under the limit operation $u, v \to \infty$, $u/v \to 1$, we have $f(v) - f(u) \to 0$, by virtue of (5.3) and the regular oscillation of $S$. Furthermore, $T^*_{-f(u)} \to \infty$ for $u \to \infty$ under $V_0 = 0$. It follows that, for every $t > 0$, under the limit operation above for $u$ and $v$, the limiting values of (5.4) are bounded below by those of

$$\frac{\int_0^t I_{[0<V_s<f(v)-f(u)]}\, ds}{f(v) - f(u)},$$

which, by the definition of the local time, is equivalent to

$$\frac{1}{f(v) - f(u)} \int_0^{f(v)-f(u)} \eta_t(x)\, dx. \tag{5.5}$$

The latter converges to $\eta_t(0)$ for $f(v) - f(u) \to 0$. Similarly, the limiting values of (5.4) are bounded above by those of

$$\frac{\int_0^\infty I_{[0<V_s<f(v)-f(u)]}\, ds}{f(v) - f(u)},$$

which, by the definition of the local time, is equivalent to

$$\frac{1}{f(v) - f(u)} \int_0^{f(v)-f(u)} \eta(x)\, dx, \tag{5.6}$$

which converges to $\eta(0)$ for $f(v) - f(u) \to 0$. Since, as noted before the statement of the theorem, $\eta_t(0) = \eta(0)$ for all sufficiently large $t$, it follows that the random variable (5.4) converges almost surely to $\eta = \eta(0)$. Hence, the relation (5.1) follows, and the proof is complete. □

In the following theorem, which is of independent interest, we identify the distribution of $\eta$ as the standard exponential.

THEOREM 5.2. *For* $y > 0$,

$$P(\eta > y) = e^{-y}. \tag{5.7}$$

**Proof.** Let $p(x_1,\dots,x_m;t_1,\dots,t_m)$ be the joint density of the random variables $V_{t_1},\dots,V_{t_m}$ at the point $(x_1,\dots,x_m)$, for arbitrary $m \ge 1$. According to the general formula for the $m$th moment at $x=0$ of the local time of a stochastic process [1, page 90], the moment is given by the formula

$$\int_0^\infty \cdots \int_0^\infty p(0,\dots,0;t_1,\dots,t_m)\,dt_1\cdots dt_m. \tag{5.8}$$

By the invariance of the integrand under permutations of the indices of the $t$'s, the integral is equal to

$$m! \underset{0<t_1<\cdots<t_m<\infty}{\int\cdots\int} p(0,\dots,0;t_1,\dots,t_m)\,dt_1\cdots dt_m. \tag{5.9}$$

Let $\phi(z)$ be the standard normal density function; then, by the definition of the transition density for Brownian motion, we have for the process $V_s$

$$p(0,\dots,0;t_1,\dots,t_m) = \prod_{j=1}^m \left(2(t_j-t_{j-1})\right)^{-1/2}\phi\left(\left(\frac{t_j-t_{j-1}}{2}\right)^{1/2}\right), \tag{5.10}$$

where $t_0=0$. By a standard computation, the integral of the function (5.10) over the domain $0<t_1<\cdots<t_m<\infty$ is equal to 1. Therefore, by (5.9), the $m$th moment (5.8) is equal to $m!$, for $m\ge 1$. This is the $m$th moment of the standard exponential distribution, which is uniquely determined by its moments. □

Our next result is concerned with the distribution of $L_u - L_v$ for any starting point $x > z$.

**Theorem 5.3.** *If $S$ is of regular oscillation, then*

$$\begin{aligned}\lim_{u,v\to\infty,\,u<v,\,u/v\to 1} S(u)P_x\left(\frac{L_u-L_v}{\log(S(v)/S(u))} > y\right)\\ = e^{-y}(S(x)-S(z)),\end{aligned} \tag{5.11}$$

*for $z < x$ and every $y > 0$.*

**Proof.** The proof is similar to that of Theorem 3.2 so we omit the details. In applying Lemma 2.1 we simply use $L_u - L_v$ in the place of $L_u$, and the conclusion is the same. Theorem 5.1 is used in the place of Theorem 3.1, and the limiting exponential distribution is identified by Theorem 5.2. □

Finally, as in Theorem 4.1, we extend the conclusion about $L_u - L_v$ to the sojourn time in $(u,v)$ for the process $(X_t)$.

**Theorem 5.4.** *If $S(x)$ and $J(x)$ are of regular oscillation, then, with the same limit operation for $u$ and $v$,*

$$\begin{aligned}\lim S(u)P_x\left\{\frac{J(u)}{\log(S(v)/S(u))}\int_0^{T_z} I_{[u<X_t<v]}\,dt > y\right\}\\ = e^{-y}(S(x)-S(z)).\end{aligned} \tag{5.12}$$

PROOF. Write $L_u - L_v$ as

$$\int_0^{T_z} J(X_t) I_{[u < X_t < v]}\, dt.$$

Under the limit operation for $u$ and $v$, with $u/v \to 1$, the regular oscillation of $J$ implies that the factor $J(X_t)$ in the integrand above may be replaced by $J(u)$. For this reason the statement (5.12) is a direct consequence of (5.11). □

EXAMPLE 5.1. If $a(x)$, $S(x)$ and $S'(x)$ are of regular oscillation, then so is $J(x) \sim \frac{1}{2}a(x)(S'(x)/S(x))^2$, and the conditions of Theorem 5.4 hold.

**6. The sojourn time over a long interval.** In this section we consider the problem of finding the limiting distribution of the sojourn time above $u$ over an interval whose length tends to $\infty$ with $u$. More exactly, we consider an interval of length $S(u)$, and define

$$L_u^* = \int_0^{S(u)} J(X_t) I_{[X_t > u]}\, dt. \tag{6.1}$$

Here we assume the existence of a stationary probability measure for the diffusion, so that the mean first passage times between points are finite.

THEOREM 6.1. *Assume*

$$m = \int_{r_1}^{r_2} \frac{dx}{a(x)S'(x)} < \infty, \qquad S(r_2) = -S(r_1) = \infty. \tag{6.2}$$

*Then $L_u^*$ has, for any starting point $x$ in $(r_1, r_2)$, a limiting distribution with the Laplace–Stieltjes transform*

$$\exp\left\{ -\frac{s}{m}\left(\frac{2}{1 + (1 + 4s)^{1/2}}\right)^2 \right\}, \qquad s > 0. \tag{6.3}$$

PROOF. Fix $z < x$; then the first part of the proof of Theorem 3.2 asserts that, for $u \to \infty$,

$$S(u)(1 - Q(s; u, x)) \to (S(x) - S(z))E(1 - e^{-s\xi}).$$

By the elementary relation $-\log Q \sim 1 - Q$, this is equivalent to

$$(Q(s; u, x))^{S(u)} \to \exp\{-(S(x) - S(z))E(1 - e^{-s\xi})\}. \tag{6.4}$$

This is equivalent to the statement that the sum of $[S(u)]$ independent random variables with the common transform $Q(s; u, x)$ has the limiting distribution whose transform is the limit in (6.4).

Let $\tau_1$ be the sum of the first passage time from $x$ to $z$ and the subsequent first passage time from $z$ back to $x$; and let $\tau_2, \tau_3, \ldots$ be the succeeding times of passage to $z$ and back to $x$. [These $\tau$'s are different from the values of the function $\tau_t$ used in the canonical transformation (2.16).] It is well known that the parts of the process between these successive roundtrips between $x$ and $z$ are independent and identically distributed. Furthermore, the assumption (6.2) implies that $E\tau_1 < \infty$ [4]. Let $N_u$ be the number of complete roundtrips before time $S(u)$; then $L_u^*$ is approximately equal to the sum of $N_u$ independent random variables with the common distribution given by that of $L_u$ under $X_0 = x$. By a standard argument based on renewal theory, the random number $N_u$ of summands may be asymptotically replaced by $S(u)/E\tau_1$ in the calculation of the distribution of the sum. Therefore, the Laplace–Stieltjes transform of the distribution of $L_u^*$ is asymptotically equal to

$$(Q(s; u, x))^{S(u)/E\tau_1},$$

which, by (6.4), converges to

$$\exp\left\{-\frac{S(x) - S(z)}{E\tau_1} E(1 - e^{-s\xi})\right\}. \tag{6.5}$$

Let us show that

$$E\tau_1 = m(S(x) - S(z)), \tag{6.6}$$

where $m$ is the constant in (6.2). Let $M(x)$ be the speed measure defined by the formula

$$M(dx) = \frac{dx}{a(x)S'(x)}.$$

By an adaptation of the formula in ([5], page 145) the expected first passage time from $x$ to $z$ is

$$\int_z^x (M(y) - M(r_1))S'(y)\,dy,$$

and from $z$ to $x$ is

$$\int_z^x (M(r_2) - M(y))S'(y)\,dy.$$

Hence, since $\tau_1$ is the sum of these two passage times, it follows that $E\tau_1 = (M(r_2) - M(r_1))(S(x) - S(z))$, which is identical with (6.6).

It follows that the function (6.5) is equal to $\exp[-m^{-1}E(1 - e^{-s\xi})]$ which, by (3.5), is equal to (6.3). □

There are analogous versions of Theorems 4.1, 5.3 and 5.4. For example, here is the version of Theorem 5.3:

**THEOREM 6.2.** *Assume* (6.2), $r_2 = \infty$, *and the regular oscillation of S. Then*

$$\frac{L_u^* - L_v^*}{\log(S(v)/S(u))}$$

*has a limiting distribution with the Laplace–Stieltjes transform*

$$\exp\left[-\frac{E(1-e^{-s\eta})}{m}\right] = \exp\left(-\frac{s}{m(1+s)}\right).$$

*This is independent of the initial state $x$.*

We note that the main result of [3] is a special case of the "long-term" version of Theorem 4.1 above.

*Note added in proof.* After this paper was accepted for publication, the explicit forms of the distribution and density functions of the random variable $\xi$ in (3.1) were published by J. P. Imhof (1986), On the time spent above a level by Brownian motion with negative drift, *Adv. in Appl. Probab.* **18** 1017–1018.

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