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INEQUALITIES FOR MULTIVARIATE INFINITELY DIVISIBLE PROCESSES

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We describe a general class of multivariate infinitely divisible distributions and their related stochastic processes. Then we prove inequalities which are the analogs of Slepian's inequality for these distributions. These inequalities are applied to the distributions of $M/G/\infty$ queues and of sample cumulative distribution functions for independent multivariate random variables.

1. Introduction. In this paper we study conditions under which one or both of the inequalities

$$P(\mathbf{X} \geq \mathbf{c}) \leq P(\mathbf{Y} \geq \mathbf{c}), \tag{1.1a}$$

$$P(\mathbf{X} \leq \mathbf{c}) \leq P(\mathbf{Y} \leq \mathbf{c}) \tag{1.1b}$$

hold for all $\mathbf{c} \in \mathbb{R}^n$, where $\mathbf{X} = (X_1, \dots, X_n)$, $\mathbf{Y} = (Y_1, \dots, Y_n)$ are random vectors in $\mathbb{R}^n$ whose distributions belong to a class of multivariate infinitely divisible distributions, which we will describe.

Approximating monotone functions on $\mathbb{R}$ by linear combinations of indicator functions of semiinfinite intervals, we see that (1.1a, b) are equivalent to

$$E\left\{\prod_{i=1}^{n} \phi_i(X_i)\right\} \leq E\left\{\prod_{i=1}^{n} \phi_i(Y_i)\right\} \tag{1.1c}$$

for any set of nondecreasing (nonincreasing) nonnegative functions ϕ_i on $\mathbb{R}$, provided the expectations exist. When $X_i \sim Y_i$ (i.e., X_i and Y_i have the same marginal distribution) for $i = 1, \dots, n$, then each of the conditions (1.1a, b) implies $\mathrm{Cov}(X_i, X_j) \leq \mathrm{Cov}(Y_i, Y_j)$. Note that if both (1.1a, b) hold, then $X_i \sim Y_i$, $i = 1, \dots, n$, and it is natural to regard the components of $\mathbf{Y}$ as being *more positively dependent* than the components of $\mathbf{X}$.

For the normal distribution, Slepian's (1962) inequality provides conditions for both (1.1a) and (1.1b) to hold. For related inequalities and concepts of dependence see, e.g., Lehmann (1966), Shaked (1982) and references therein. Some notions of orderings by positive dependence can be found in Kimeldorf and Sampson (1987) and Shaked and Tong (1985).

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We now proceed to discuss multivariate infinitely divisible distributions.

Consider a one-parameter family Q_t, $t \geq 0$, of infinitely divisible probability distributions, that is, a family whose characteristic functions $\varphi_t(s) = \int e^{isx}\, dQ_t(x)$ can be expressed as $\varphi_t(s) = [\kappa(s)]^t$ for some characteristic function κ.

Examples of such families include the normal family $N(\mu t, \sigma^2 t)$ with $\varphi_t(s) = [e^{\mu s - \sigma^2 s^2/2}]^t$, the Poisson($\lambda t$) family with $\varphi_t(s) = [e^{\lambda(e^{is}-1)}]^t$, the negative binomial family with $\varphi_t(s) = [p/(1 - qe^{is})]^t$ [see Feller (1968), page 291], the gamma family with $\varphi_t(s) = [1/(1 - i\beta s)]^t$, the Cauchy distribution with $\varphi_t(s) = [e^{-|s|}]^t$ and any other one-parameter family of stable distributions. The *compound Poisson* family is defined by

$$Q_t = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} F_k, \tag{1.2}$$

where F_k denotes the k-fold convolution of some distribution F. Q_t represents the distribution of the random sums $U_1 + \cdots + U_{N(t)}$, where $\{U_i\}$ are independent with common distribution F and $N(t)$ is a Poisson(λt) variable independent of $\{U_i\}$. The corresponding characteristic function is $[e^{\lambda(\mu(s)-1)}]^t$, where $\mu(s)$ is the characteristic function associated with F. The importance of this family stems from the following classical result [the Lévy–Khintchine formula, e.g., Doob (1953) and Feller (1971)].

FACT 1.0. The class of limits in distribution of sequences of compound Poisson distributions and the class of infinitely divisible distributions are identical.

We consider the following construction of multivariate infinitely divisible distributions. Let

$$\mathscr{A} = \{A\colon A \subseteq \{1,\dots,n\}\} = 2^{\{1,\dots,n\}} \text{ and } \mathbf{t} = \{t(A)\colon \phi \neq A \in \mathscr{A}\},$$

where $t(A) \geq 0$ for all $\phi \neq A \in \mathscr{A}$. Let Q_t, $t \geq 0$, be a family of infinitely divisible distributions and let $\{Z_A,\ A \in \mathscr{A}\}$ be independent random variables with Z_A distributed according to $Q_{t(A)}$. Define $\mathbf{X} = (X_1,\dots,X_n)$ by

$$X_i = \sum_{A\colon i \in A} Z_A, \qquad i = 1,\dots,n. \tag{1.3}$$

(The sum extends over $A \in \mathscr{A}$ such that $i \in A$.) We shall say that $\mathbf{X}$ has a *multivariate infinitely divisible* distribution with parameter $\mathbf{t} = \{t(A),\ \phi \neq A \in \mathscr{A}\}$ based on the family Q_t. Note that for each $i = 1,\dots,n$, the distribution of X_i belongs to the family Q_t.

The *multivariate Poisson distribution* is an example of special interest. It arises when in (1.3) the Z_A are taken to be Poisson($t(A)$) variables, that is,

$$P(Z_A = k) = e^{-t(A)}[t(A)]^k/k!, \qquad k = 0,1,\dots.$$

In this case X_i is a Poisson($\Sigma_{A\colon i \in A} t(A)$) variable. This construction of multivariate Poisson and infinitely divisible distributions is discussed in Dwass and Teicher (1957).

If $X(t)$ is a stochastic process with independent stationary increments distributed according to Q_t, then $(X(t_1), \dots, X(t_n))$ has a multivariate infinitely divisible distribution with $Z_{\{j, \dots, n\}} = X(t_j) - X(t_{j-1})$, $j = 1, \dots, n$, $t_0 = 0$.

Note that the components of $\mathbf{X}$ defined by (1.3) are *associated*, that is, $E\{f(\mathbf{X})g(\mathbf{X})\} \geq Ef(\mathbf{X})Eg(\mathbf{X})$ for any pair of coordinatewise nondecreasing (or nonincreasing) functions f and g on $\mathbb{R}^n$ [Harris (1960) and Esary, Proschan and Walkup (1967)]. In particular

$$E\left\{\prod_{i=1}^{n} \phi_i(X_i)\right\} \geq \prod_{i=1}^{n} E\{\phi_i(X_i)\} \tag{1.4}$$

provided all ϕ_i are nonnegative and nondecreasing (or nonincreasing) functions on $\mathbb{R}$.

Our first result provides conditions for positive dependence ordering.

THEOREM 1.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *have multivariate infinitely divisible distributions with parameters* $\mathbf{t}$ *and* $\mathbf{t}^*$, *respectively, based on the same family* Q_t, *and suppose that for all* $B \neq \phi$, $B \in \mathscr{A}$, *the following two conditions hold*:

$$\sum_{A:\, A \supseteq B} t(A) \leq \sum_{A:\, A \supseteq B} t^*(A), \tag{1.5a}$$

$$\sum_{A:\, A \cap B \neq \phi} t(A) \geq \sum_{A:\, A \cap B \neq \phi} t^*(A). \tag{1.5b}$$

Then for all $\mathbf{c} \in \mathbb{R}^n$,

$$P(\mathbf{X} \geq \mathbf{c}) \leq P(\mathbf{Y} \geq \mathbf{c}), \tag{1.6a}$$

$$P(\mathbf{X} \leq \mathbf{c}) \leq P(\mathbf{Y} \leq \mathbf{c}). \tag{1.6b}$$

The following one-sided stochastic comparison holds for *nonnegative* random vectors.

THEOREM 1.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be as in Theorem* 1.1, *based on a family* Q_t *having support on* $[0, \infty)$.

(i) *Condition* (1.5a) $\Rightarrow$ *condition* (1.6a).
(ii) *Conditions* (1.5b) *and* (1.6b) *are equivalent.*

The following converse to (i) holds.

PROPOSITION 1.3. *For the compound Poisson family* Q_t *of* (1.2), *if* $P(U_1 > 0) > 0$ *and* $E\{e^{\alpha U_1}\} < \infty$ *for all* $\alpha > 0$, *then* (1.6a) $\Rightarrow$ (1.5a).

A proof of Proposition 1.3 for the multivariate Poisson distribution was contained in an earlier version of this paper. A simpler proof which covers the compound Poisson case was subsequently given by Ellis (1988).

REMARK 1.4. Theorems 1.1 and 1.2 clearly apply to the multivariate Poisson case where $P(Z_A = k) = e^{-t(A)}[t(A)]^k/k!$, $k = 0, 1, \ldots$. The relevance of infinite divisibility becomes apparent when one attempts to extend these results. For example, consider Z_A having a binomial distribution,

$$P(Z_A = k) = \binom{t(A)}{k} p^k (1-p)^{t(A)-k}, \; k = 0, \ldots, t(A),$$

where $t(A)$ are nonnegative integers, and construct $\mathbf{X}$ as in (1.3). The binomial distribution is not infinitely divisible and indeed in this case we have constructed counterexamples to both theorems with $n \geq 4$. It can be verified by direct calculation that for $n \leq 3$, Theorem 1.1 holds for binomial Z_A's and in fact it holds for $n \leq 3$ whenever $t(A)$ is an integer and Z_A is distributed as $U_1 + \cdots + U_{t(A)}$ for all $A \in \mathscr{A}$, where $\{U_i\}$ are any iid variables. The same is true for Theorem 1.2 provided the U_i's are nonnegative.

REMARK 1.5 (An alternate representation). There are several alternate and equivalent ways of representing multivariate infinitely divisible processes. Here is one.

Let λ be a (nonnegative) measure on $\mathbb{R}^d$ and consider a Poisson point process in $\mathbb{R}^d$ with intensity parameter λ, that is, for any measurable set Γ in $\mathbb{R}^d$ the random variable N_Γ, the number of process points in Γ, is Poisson with parameter $\lambda(\Gamma)$. Let Γ_i, $i = 1, \ldots, n$, be measurable sets in $\mathbb{R}^d$ and define $X_i = N_{\Gamma_i}$. Then $\mathbf{X} = (X_1, \ldots, X_n)$ has a joint multivariate Poisson distribution as previously defined. The parameter $t(A)$ previously defined is related to $\lambda(\cdot)$ by

$$t(A) = \lambda\Big(\bigcap_{i \in A} \Gamma_i - \bigcup_{i \notin A} \Gamma_i\Big). \tag{1.7}$$

Let $Y_i = N_{\Gamma_i^*}$ be similarly defined with respect to a Poisson process with parameter λ^*. Then (1.5a) becomes

$$\lambda\Big(\bigcap_{i \in B} \Gamma_i\Big) \leq \lambda^*\Big(\bigcap_{i \in B} \Gamma_i^*\Big), \qquad B \neq \phi, \tag{1.8}$$

and (1.5b) becomes

$$\lambda\Big(\bigcup_{i \in B} \Gamma_i\Big) \geq \lambda^*\Big(\bigcup_{i \in B} \Gamma_i^*\Big). \tag{1.9}$$

[Use (1.7) to prove (1.8) and (1.9).] Conditions (1.8) and (1.9) together imply $\lambda(\Gamma_i) = \lambda^*(\Gamma_i)$, $i = 1, \ldots, n$.

REMARK 1.6. Suppose the marginal distributions of X_i and Y_i are equal for $i = 1, \ldots, n$. In the case $n = 2$ it then follows that conditions (1.5a) and (1.5b) are equivalent. For the multivariate Poisson distribution one then sees from Theorem 1.2 and Proposition 1.3 that the conditions (1.5a, b) and (1.6a, b) are all equivalent. This pleasant state of affairs does not hold for $n \geq 3$. For example, let $t(\{1,2\}) = t(\{1,3\}) = t(\{2,3\}) = 1$ with $t(\cdot) = 0$ otherwise, and $t^*(1) = t^*(2) = t^*(3) = t^*(\{1,2,3\}) = 1$ with $t^*(\cdot) = 0$ otherwise. Then the marginal distributions are equal and (1.5a) and (1.6a) are true but not (1.5b) or (1.6b). Reversing the roles of t and t^* yields an example where (1.5b) and (1.6b) are true but not (1.5a) and (1.6a).

In Section 2 we prove Theorems 1.1 and 1.2 and some related results.

In Section 3 we extend the results to infinitely divisible processes and obtain inequalities for first passage times of such processes.

In Sections 4 and 5 we discuss applications. In Section 4 we show first that if $X(\tau_i)$ counts the number of customers in service at time τ_i in a $M/G/\infty$ queue, then $\mathbf{X} = (X(\tau_1), \dots, X(\tau_n))$ has a multivariate Poisson distribution. The implications of our results to this model are then studied. Related results regarding circle covering probabilities and inequalities for $M/G/\infty$ queues were recently obtained by Huffer (1987).

In Section 5 we examine the distribution of a bivariate sample cumulative distribution function (C.D.F.). An application of our main results shows that—in an appropriate sense—the least associated C.D.F. is that corresponding to a sample from the distribution uniformly distributed along the negative diagonal of the unit square. A related asymptotic result appears in Adler and Brown (1988).

2. Proofs and further results.

The steps of the proofs of Theorems 1.1 and 1.2 are divided into several lemmas and remarks.

LEMMA 2.1. *Let $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ be a pair of independent vectors on $\mathbb{R}^n$ and let $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ be another such pair. If for all $\mathbf{c} \in \mathbb{R}^n$, $\nu = 1, 2$,*

$$P(\mathbf{X}^{(\nu)} \geq \mathbf{c}) \leq P(\mathbf{Y}^{(\nu)} \geq \mathbf{c}), \qquad P(\mathbf{X}^{(\nu)} \leq \mathbf{c}) \leq P(\mathbf{Y}^{(\nu)} \leq \mathbf{c}), \tag{2.1}$$

then for all $\mathbf{c} \in \mathbb{R}^n$,

$$P(\mathbf{X}^{(1)} + \mathbf{X}^{(2)} \geq \mathbf{c}) \leq P(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)} \geq \mathbf{c}), \qquad P(\mathbf{X}^{(1)} + \mathbf{X}^{(2)} \leq \mathbf{c}) \leq P(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)} \leq \mathbf{c}). \tag{2.2}$$

PROOF. We can assume that $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}$ are independent vectors defined on the same probability space. Then,

$$\begin{aligned} P(\mathbf{X}^{(1)} + \mathbf{X}^{(2)} \geq \mathbf{c}) &= E\{P(\mathbf{X}^{(1)} \geq \mathbf{c} - \mathbf{X}^{(2)} | \mathbf{X}^{(2)})\} \\ &\leq E\{P(\mathbf{Y}^{(1)} \geq \mathbf{c} - \mathbf{X}^{(2)} | \mathbf{X}^{(2)})\} = P(\mathbf{Y}^{(1)} + \mathbf{X}^{(2)} \geq \mathbf{c}) \\ &= E\{P(\mathbf{X}^{(2)} \geq \mathbf{c} - \mathbf{Y}^{(1)} | \mathbf{Y}^{(1)})\} \leq E\{P(\mathbf{Y}^{(2)} \geq \mathbf{c} - \mathbf{Y}^{(1)} | \mathbf{Y}^{(1)})\} \\ &= P(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)} \geq \mathbf{c}). \qquad \square \end{aligned}$$

REMARK 2.2. In view of Fact 1.0 it suffices to prove Theorem 1.1 for the case that the distributions of the variables $\{Z_A,\ A \in \mathscr{A}\}$ appearing in the construction of (1.3) belong to a *compound Poisson family*, i.e., for all $A \in \mathscr{A}$,

$$Z_A = U_1 + \cdots + U_{N(t(A))}, \qquad N(t(A)) \sim \text{Poisson}(\lambda t(A)). \tag{2.3}$$

$\{U_i\}$ have a common distribution F, and $\{U_i\}$ and $\{N(t(A))\}$ are all independent. Indeed if the distributions of $\{Z_A\}$ in (1.3) belong to any family Q_t, they are, by Fact 1.0, limits in distribution of variables constructed as in (2.3) [limits taken over suitable sequences of F's and λ's, with $\{t(A)\}$ fixed]. Thus any multivariate infinitely divisible vector $\mathbf{X}$ is a limit in distribution of a sequence $\mathbf{X}^{(n)}$ of multivariate infinitely divisible vectors based on compound Poisson families and hence it suffices to consider this case.

LEMMA 2.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *satisfy the conditions of Theorem* 1.1 *and suppose both* $\mathbf{X}$ *and* $\mathbf{Y}$ *are based on a compound Poisson family* Q_t, *i.e.,* $\mathbf{X}$ *is defined by* (1.3) *where* Z_A *are as in* (2.3), $\mathbf{Y}$ *is defined similarly with* $Z_A^* = U_1 + \cdots + U_{N^*(t^*(A))}$, $A \in \mathscr{A}$, *and* (1.5a, b) *hold. Define* $t(\phi) = 0$, $t^*(\phi) = \Sigma[t(A) - t^*(A)]$ *and* $N^*(t^*(\phi)) \sim \text{Poisson}(\lambda t^*(\phi))$. *Then for all* $\mathbf{c} \in \mathbb{R}^n$,

$$P\Big(\mathbf{X} \geq \mathbf{c} \,\Big|\, \sum_{A \in \mathscr{A}} N(t(A)) = 1\Big) \leq P\Big(\mathbf{Y} \geq \mathbf{c} \,\Big|\, \sum_{A \in \mathscr{A}} N^*(t^*(A)) = 1\Big). \tag{2.4}$$

PROOF. Condition (1.5b) with $B = \{1, \dots, n\}$ implies $t^*(\phi) \geq 0$. Given $N(t) \equiv \Sigma_{A \in \mathscr{A}} N(t(A)) = 1$, we must have $N(t(A)) = 1$ and $Z_A = U_1$ for some set $A \in \mathscr{A}$ and $Z_C = 0$ for any $C \in \mathscr{A}$, $C \neq A$. In view of (1.3), $X_i = U_1$ if $i \in A$, zero otherwise. Consider now $P(\mathbf{X} \geq \mathbf{c})$, where $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$ and define $B = \{i\colon c_i > 0\} \in \mathscr{A}$. First assume $B \neq \phi$. Then $\mathbf{X} \geq \mathbf{c}$ iff $A \supseteq B$ and $U_1 \geq \max\{c_i, i \in B\} \equiv \gamma$, and therefore

$$\begin{aligned} &P(\mathbf{X} \geq \mathbf{c} | N(t) = 1) \\ &\quad = \sum_{A\colon A \supseteq B} P(N(t(A)) = 1 | N(t) = 1)(1 - F(\gamma^-)), \end{aligned} \tag{2.5}$$

where F denotes the distribution of U_1.

The conditional distribution of $\{N(t(A))\colon A \in \mathscr{A}\}$ given $N(t) = 1$ is multinomial involving a single experiment with 2^n cells corresponding to the elements of $\mathscr{A}$, having cell probabilities $\{\rho(A), A \in \mathscr{A}\}$, where

$$\rho(A) \equiv t(A) \Big/ \sum_{A \in \mathscr{A}} t(A) = P(N(t(A)) = 1 | N(t) = 1).$$

Therefore, by (2.5),

$$\begin{aligned} P(\mathbf{X} \geq \mathbf{c} | N(t) = 1) &= \sum_{A\colon A \supseteq B} \rho(A)(1 - F(\gamma^-)) \\ &= \sum_{A\colon A \supseteq B} t(A)(1 - F(\gamma^-)) \Big/ \sum_{A \in \mathscr{A}} t(A). \end{aligned} \tag{2.6}$$

The definition of $t^*(\phi)$ guarantees $\Sigma_{A \in \mathscr{A}} t(A) = \Sigma_{A \in \mathscr{A}} t^*(A)$. Therefore the inequality (2.4) now follows from (2.6) and (1.5a).

In the case where $B = \phi$ we have $c_i \leq 0$ for all $i = 1, \dots, n$, and $\mathbf{X} \geq \mathbf{c}$ holds iff $U_1 \geq \max\{c_i,\ i \in A\} \equiv \delta(A)$, where as before A is the set where $N(t(A)) = 1$. If $A = \phi$ then $\mathbf{X} = \mathbf{0} \geq \mathbf{c}$ and we define $\delta(\phi) = -\infty$. Thus

$$P(\mathbf{X} \geq \mathbf{c} | N(t) = 1) = \sum_{A \in \mathscr{A}} t(A) P(U_1 \geq \delta(A)) \Big/ \sum_{A \in \mathscr{A}} t(A). \tag{2.7}$$

To complete the proof of Lemma 2.3 assume with no loss of generality that $c_1 \leq c_2 \leq \cdots \leq c_n \leq 0$. Then

$$\begin{aligned}
&\sum_{A \in \mathscr{A}} t(A) P(U_1 \geq \delta(A)) \\
&\quad = \sum_{A \in \mathscr{A}} t(A) P(U_1 \geq c_n) + \sum_{A \in \mathscr{A}:\, A \subseteq \{1, \dots, n-1\}} t(A) P(c_{n-1} \leq U_1 < c_n) \\
&\qquad + \cdots + \sum_{A \in \mathscr{A}:\, A \subseteq \{1\}} t(A) P(c_1 \leq U_1 < c_2) + t(\phi) P(U_1 < c_1).
\end{aligned}$$

Clearly, (2.7) and the preceding decomposition are also valid for $\mathbf{Y}$ and t^*.

This decomposition and (2.7) imply (2.4) since $\sum_{A \in \mathscr{A}} t(A) = \sum_{A \in \mathscr{A}} t^*(A)$ and $\sum_{A \subseteq B} t(A) \leq \sum_{A \subseteq B} t^*(A)$. [To obtain this, note that it is equivalent to $\sum_{\phi \neq A \subseteq B} t(A) \leq \sum_{\phi \neq A \subseteq B} t^*(A) + \sum_{A:\ A \neq \phi} [t(A) - t^*(A)]$, which reduces to $\sum_{A:\ A \cap C \neq \phi} t(A) \geq \sum_{A:\ A \cap C \neq \phi} t^*(A)$, where C denotes the complement of B. The latter inequality holds by (1.5b).] □

REMARK 2.4. Note that both conditions (1.5a, b) were used. If the U_i's are nonnegative, however, the case $B = \phi$ need not be considered and (2.4) would follow from (1.5a) alone. [When (1.5b) is not assumed we may have $\sum_{A:\ A \neq \phi} [t^*(A) - t(A)] \geq 0$. In this case we define $t(\phi)$ to be the latter quantity and $t^*(\phi) = 0$.]

LEMMA 2.5. *Under the conditions of Lemma* 2.3,

$$P\Big(\mathbf{X} \leq \mathbf{c} \Big| \sum_{A \in \mathscr{A}} N(t(A)) = 1\Big) \leq P\Big(\mathbf{Y} \leq \mathbf{c} \Big| \sum_{A \in \mathscr{A}} N^*(t^*(A)) = 1\Big) \tag{2.8}$$

for all $\mathbf{c} \in \mathbb{R}^n$.

PROOF. Lemma 2.5 follows from Lemma 2.3 since

$$\begin{aligned}
P(\mathbf{X} \leq \mathbf{c} | N(t) = 1) &= P(-\mathbf{X} \geq -\mathbf{c} | N(t) = 1) \leq P(-\mathbf{Y} \geq -\mathbf{c} | N(t) = 1) \\
&= P(\mathbf{Y} \leq \mathbf{c} | N(t) = 1).
\end{aligned}$$ □

LEMMA 2.6. *Under the conditions of Lemma* 2.3,

$$P\Big(\mathbf{X} \geq \mathbf{c} \Big| \sum_{A \in \mathscr{A}} N(t(A)) = N\Big) \leq P\Big(\mathbf{Y} \geq \mathbf{c} \Big| \sum_{A \in \mathscr{A}} N^*(t^*(A)) = N\Big), \tag{2.9}$$

$$P\Big(\mathbf{X} \leq \mathbf{c} \Big| \sum_{A \in \mathscr{A}} N(t(A)) = N\Big) \leq P\Big(\mathbf{Y} \leq \mathbf{c} \Big| \sum_{A \in \mathscr{A}} N^*(t^*(A)) = N\Big) \tag{2.10}$$

for any $\mathbf{c} \in \mathbb{R}^n$ *and* $N = 1, 2, \dots$.

PROOF. We proceed by induction on N. For $N = 1$ Lemmas 2.3 and 2.5 apply. Given $\Sigma_{A \in \mathscr{A}} N(t(A)) = N(t) = N$ the distribution of $\{N(t(A))\}$ is multinomial with N experiments. The conditional distribution of $\mathbf{X}|N(t) = N$ can be obtained as the convolution of two such conditional distributions with N replaced by $N - 1$ and 1, respectively. The lemma now follows by induction and Lemma 2.1. □

PROOF OF THEOREM 1.1. In view of Remark 2.2, Theorem 1.1 follows from Lemma 2.6 by unconditioning. □

PROOF OF THEOREM 1.2. The proof that (1.5a) ⇒ (1.6a) and (1.5b) ⇒ (1.6b) is similar to that of Theorem 1.1, taking Remark 2.4 into consideration.

Next we show that (1.6b) ⇒ (1.5b). Let $\phi \neq B \in \mathscr{A}$ and define $\mathbf{c} = (c_1, \dots, c_n)$ by $c_i = 0$ if $i \in B$ and ∞ if $i \notin B$. (One may either interpret $\mathbb{R}^n$ as $[-\infty, \infty]^n$ or use an elementary limiting argument.) Then by (1.3),

$$P(\mathbf{X} \le \mathbf{c}) = P(Z_A = 0 \text{ for all } A \text{ such that } A \cap B \neq \phi).$$

We can assume that the representation (2.3) holds. Replacing the parameter λ in (2.3) by $\lambda[1 - P(U_1 = 0)]$, we can assume $P(U_1 = 0) = 0$. The previous probability then becomes $P(N(t(A)) = 0$ for all A such that $A \cap B \neq \phi) = \exp\{-\lambda \Sigma_{A:\ A \cap B \neq \phi} t(A)\}$. It is now clear that (1.6b) ⇒ (1.5b), except in the trivial case that $P(U_1 = 0) = 1$ for the compound Poisson case. The argument extends to infinitely divisible distributions by the approximation procedure of Remark 2.2. □

Lemma 2.6 and simple modifications of the preceding and Ellis' arguments actually yield a stronger assertion, summarized in the following corollary.

COROLLARY 2.7. *Let* $\{N(t(A)),\ A \in \mathscr{A}\}$ *be multinomial with* N *experiments and cell probabilities* $t(A)$, *i.e.*, $P(\bigcap_{A \in \mathscr{A}} \{N(t(A)) = k_A\}) = N! \prod_{A \in \mathscr{A}} [t(A)]^{k_A} / k_A!$, $k_A \ge 0$ *integers*, $\Sigma_{A \in \mathscr{A}} k_A = N$, *and define* $Z_A = U_1 + \cdots + U_{N(t(A))}$ *with* U_i *iid independent of* $\{N(t(A))\}$, *and* $\mathbf{X}$ *as in* (1.3). *Let* $\mathbf{Y}$ *be constructed similarly with parameters* $t^*(A)$, $\Sigma_{A \in \mathscr{A}} t(A) = \Sigma_{A \in \mathscr{A}} t^*(A) = 1$. *If* (1.5a, b) *are satisfied, then* (1.6a, b) *are satisfied. If both* $\mathbf{X}$ *and* $\mathbf{Y}$ *are nonnegative variables, then* (1.5a) ⇒ (1.6a) *and* (1.5b) ⇔ (1.6b); *if* $E\{e^{\alpha U_1}\} < \infty$ *for all* $\alpha > 0$, *then* (1.6a) ⇒ (1.5a) [*we assume* $P(U_1 = 0) < 1$)].

REMARK 2.8. A slight modification of the proof of Theorem 1.1 shows that we can replace (1.6a, b) in Theorem 1.1 by strict inequalities, i.e., $P(\mathbf{X} > \mathbf{c}) \le P(\mathbf{Y} > \mathbf{c})$ and $P(\mathbf{X} < \mathbf{c}) \le P(\mathbf{Y} < \mathbf{c})$, where $\mathbf{X} > \mathbf{c}$ ($\mathbf{X} < \mathbf{c}$) means $X_i > c_i$ ($X_i < c_i$), $i = 1, \dots, n$.

3. Infinitely divisible processes. A stochastic process $\mathbf{X}(\tau) \in \mathbb{R}^n$, $\tau \ge 0$, with stationary independent increments, such that for each fixed τ, $\mathbf{X}(\tau)$ has a multivariate infinitely divisible distribution with parameter $\tau \mathbf{t} = \{\tau t(A),\ \phi \neq A \in \mathscr{A}\}$ based on a family Q_t, will be called a *multivariate infinitely divisible* *process* (with parameter $\mathbf{t}$ based on Q_t). When $\mathbf{X}(\tau)$ is multivariate Poisson with parameter $\tau\mathbf{t}$ for each fixed τ, the process will be called a *multivariate Poisson process.*

We first prove a preliminary result which is useful also for ordinary multivariate infinitely divisible random variables.

LEMMA 3.1. *Let* $\mathbf{X}, \mathbf{Y}$ *have multivariate infinitely divisible distributions with parameters* $\mathbf{t}$ *and* $\mathbf{t}^*$ *respectively, based on the same family* Q_t. *Let* $D = \{i_1, \dots, i_m\} \in \mathscr{A}$. *Then* $\tilde{\mathbf{X}} = (X_{i_1} \dots X_{i_m})$ *and* $\tilde{\mathbf{Y}} = (Y_{i_1}, \dots, Y_{i_m})$ *have multivariate infinitely divisible distributions based on* Q_t *with parameters* $\tilde{\mathbf{t}}$ *and* $\tilde{\mathbf{t}}^*$, *defined in* (3.1). *If* $\mathbf{t}, \mathbf{t}^*$ *satisfy* (1.5a) [*respectively,* (1.5b)], *then so do* $\tilde{\mathbf{t}}$ *and* $\tilde{\mathbf{t}}^*$.

PROOF. For $A \subset \{1, \dots, m\}$ define $c(A) = \{i_j\colon j \in A\}$. Let

$$\tilde{Z}_A = \sum_{\substack{B \supseteq c(A) \\ (B - c(A)) \cap D = \phi}} Z_B.$$

The family $\{\tilde{Z}_A\colon A \subset \{1, \dots, m\}\}$ of independent infinitely divisible random variables based on Q_t defines $\tilde{X}_j$ according to (1.3) for $j = 1, \dots, m$. Hence $\{\tilde{X}_j\colon j = 1, \dots, m\}$ is multivariate infinitely divisible with parameters

$$\tilde{t}(A) = \sum_{\substack{B \supseteq c(A) \\ (B - c(A)) \cap D = \phi}} t(B). \tag{3.1}$$

$\tilde{t}^*(\cdot)$ is similarly defined, and the remainder of the lemma follows easily. □

LEMMA 3.2. *Let* $\mathbf{X}(\tau)$ *and* $\mathbf{Y}(\tau)$ *be multivariate infinitely divisible processes with parameters* $\mathbf{t}$ *and* $\mathbf{t}^*$, *respectively, based on the same family* Q_t. *Then for any* m, $0 < \tau_j < \infty$ *and integers* $1 \le i_j \le n$, $j = 1, \dots, m$, *the vectors* $\tilde{\mathbf{X}} = (X_{i_1}(\tau_1), \dots, X_{i_m}(\tau_m))$, $\tilde{\mathbf{Y}} = (Y_{i_1}(\tau_1), \dots, Y_{i_m}(\tau_m))$ *have multivariate infinitely divisible distributions with parameters denoted by* $\tilde{\mathbf{s}} = \{\tilde{s}(A)\colon \phi \ne A \subseteq \{1, \dots, m\}\}$ *and* $\tilde{\mathbf{s}}^* = \{\tilde{s}^*(A)\colon \phi \ne A \subseteq \{1, \dots, m\}\}$, *which will be determined later.*

The following implications obtain:

Condition (1.5a) *implies*

$$\sum_{A \supseteq B} \tilde{s}(A) \le \sum_{A \supseteq B} \tilde{s}^*(A). \tag{3.2a}$$

Condition (1.5b) *implies*

$$\sum_{A \cap B \ne \phi} \tilde{s}(A) \ge \sum_{A \cap B \ne \phi} \tilde{s}^*(A) \tag{3.2b}$$

for all $B \subseteq \{1, \dots, m\}$, $B \ne \phi$. [(1.5a) *or* (1.5b) *is assumed for all* $B \subseteq \{1, \dots, n\}$, $B \ne \phi$].

PROOF. Consider the random vector $\{X_{ij}\}$ in R^{nm} defined by $\{X_{ij}\} = \{X_i(\tau_j)\colon 1 \le i \le n, 1 \le j \le m\}$, where $0 = \tau_0 < \tau_1 \le \cdots \le \tau_m$. Note that $\{X_{ij}\}$ contains all the components of $\tilde{\mathbf{X}}$. For sets $A \subseteq \{(i, j)\colon 1 \le i \le n, 1 \le j \le m\}$ of the form

$A = \bar{A} \times \{j, j+1, \ldots, m\}$, where $\phi \neq \bar{A} \in \mathscr{A}$, let

$$s(A) = (\tau_j - \tau_{j-1}) t(\bar{A}); \tag{3.3}$$

otherwise let $s(A) = 0$. It is then easy to check that $\{X_{ij}\}$ is multivariate infinitely divisible with parameter $s(\cdot)$. It follows from Lemma 3.1 that $\tilde{\mathbf{X}}$ is also multivariate infinitely divisible with parameter $\tilde{\mathbf{s}}$ which can be determined from (3.1) and (3.3). $\{Y_{ij}\}$ and $\tilde{\mathbf{Y}}$ are similarly determined with parameters $s^*(\cdot)$ and $\tilde{\mathbf{s}}^*$, respectively.

Suppose Condition (1.5a) holds. Let $B \subset \{(i, j)\colon 1 \leq i \leq n, 1 \leq j \leq m\}$, $B \neq \phi$ and define $C_B = \{i\colon\ 1 \leq i \leq n,\ \exists\ j,\ 1 \leq j \leq m,\ (i, j) \in B\}$ and $j_B = \inf\{j\colon 1 \leq j \leq m;\ \exists\ i, 1 \leq i \leq n, (i, j) \in B\}$. Then, under (1.5a),

$$\sum_{A \supseteq B} s(A) = \tau_{j_B} \sum_{C \supseteq C_B} t(C) \leq \tau_{j_B} \sum_{C \supseteq C_B} t^*(C) = \sum_{A \supseteq B} s^*(A). \tag{3.4}$$

Similarly, let $C_{B,j} = \{i\colon 1 \leq i \leq n, \exists\ k \geq j, (i, k) \in B\}$. Then, under (1.5b),

$$\begin{aligned} \sum_{A \cap B \neq \phi} s(A) &= \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \sum_{C \cap C_{B,j} \neq \phi} t(C) \\ &\geq \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \sum_{C \cap C_{B,j} \neq \phi} t^*(C) = \sum_{A \cap B \neq \phi} s^*(A). \end{aligned} \tag{3.5}$$

Lemma 3.1 together with (3.4) and (3.5) verifies (3.2a) and (3.2b). □

The next result follows from Theorems 1.1 and 1.2 and Lemma 3.2.

THEOREM 3.3. *Under the conditions of Lemma* 3.2, *suppose*

$$\sum_{A\colon A \supseteq B} t(A) \leq \sum_{A\colon A \supseteq B} t^*(A) \quad \textit{for all } \phi \neq B \in \mathscr{A}, \tag{3.6}$$

$$\sum_{A\colon A \cap B \neq \phi} t(A) \geq \sum_{A\colon A \supset B \neq \phi} t^*(A) \quad \textit{for all } B \in \mathscr{A}. \tag{3.7}$$

Then (*with* $\tilde{X}, \tilde{Y}$ *as defined in Lemma* 3.2)

$$P(\tilde{\mathbf{X}} \geq \mathbf{c}) \leq P(\tilde{\mathbf{Y}} \geq \mathbf{c}), \tag{3.8}$$

$$P(\tilde{\mathbf{X}} \leq \mathbf{c}) \leq P(\tilde{\mathbf{Y}} \leq \mathbf{c}). \tag{3.9}$$

If $\mathbf{X}(\tau)$ *and* $\mathbf{Y}(\tau)$ *are nonnegative processes, then* (3.6) $\Rightarrow$ (3.8) *and* (3.7) $\Rightarrow$ (3.9).

Inequalities for first passage times are contained in the following result.

THEOREM 3.4. *Let* $\mathbf{X}(\tau)$ *be a nonnegative multivariate infinitely divisible process with parameter* $\mathbf{t}$ *based on* Q_t. *Define the first passage time* $\mathbf{T}^{(X)} = (T_1^{(X)}, \ldots, T_n^{(X)})$ *by* $T_i^{(X)} = \inf\{\tau\colon X_i(\tau) \geq b_i\}$, $b_1, \ldots, b_n$ *fixed constants.*

Let $\mathbf{Y}(\tau)$ be another such process with parameter $\mathbf{t}^*$. Then

$$(3.10)\qquad \sum_{A:\, A \supseteq B} t(A) \le \sum_{A:\, A\supseteq B} t^*(A) \quad \text{for all } B \ne \phi,\ B \in \mathscr{A}$$

implies

$$(3.11)\qquad P(\mathbf{T}^{(X)} \le \mathbf{c}) \le P(\mathbf{T}^{(Y)} \le \mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{R}^n$$

and

$$(3.12)\qquad \sum_{A:\, A\cap B \ne \phi} t(A) \ge \sum_{A:\, A\cap B\ne\phi} t^*(A) \quad \text{for all } B \in \mathscr{A}$$

implies

$$(3.13)\qquad P(\mathbf{T}^{(X)} > \mathbf{c}) \le P(\mathbf{T}^{(Y)} > \mathbf{c}) \quad \text{for all } \mathbf{c} \in \mathbb{R}^n,$$

where $\mathbf{T} > \mathbf{c}$ *means* $T_i > c_i$, $i = 1, \dots, n$.

If $\mathbf{X}(\tau)$ *and* $\mathbf{Y}(\tau)$ *are multivariate Poisson processes and* $\mathbf{T}^{(X)}, \mathbf{T}^{(Y)}$ *are first passage times for* $b_1 = \cdots = b_n = 1$, *then* (3.12) *and* (3.13) *are equivalent.*

PROOF. (3.10) $\Rightarrow$ (3.11):

$$P(\mathbf{T}^{(X)} \le \mathbf{c}) = P\big(X_i(c_i) \ge b_i,\ i = 1, \dots, n\big) \le P\big(Y_i(c_i) \ge b_i,\ i = 1, \dots, n\big) \qquad \text{(by Theorem 3.3)}$$
$$= P(\mathbf{T}^{(Y)} \le \mathbf{c}).$$

(3.12) $\Rightarrow$ (3.13) by a similar argument.

(3.13) $\Rightarrow$ (3.12) for $\mathbf{X}, \mathbf{Y}$ multivariate Poisson because for the choice $c_i = 1$ if $i \in B$ and 0 otherwise, we have

$$P(\mathbf{T}^{(X)} > \mathbf{c}) = \exp\left\{ - \sum_{A:\, A\cap B\ne\phi} t(A) \right\}. \qquad \square$$

Note that when $\mathbf{X}(\tau)$ is a multivariate Poisson process, then for $b_1 = \cdots = b_n = 1$ we have that $T_i^{(X)}$ is distributed as $\min_{A:\, i \in A} V_A$, where V_A are independent exponential variables with expectation $1/t(A)$, $A \in \mathscr{A}$. The distribution of $\mathbf{T}^{(X)}$ then corresponds to the multivariate exponential distribution of Marshall and Olkin (1967); see also Esary and Marshall (1974).

4. The $M/G/\infty$ queue. In the $M/G/\infty$ queue, customers arrive according to a Poisson process with parameter λ, the service time is distributed according to some general distribution denoted by F and there is an infinite number of servers. Let $X(\tau)$ denote the number of customers in service at time τ.

PROPOSITION 4.1. *The joint distribution of* $(X(\tau_1), \dots, X(\tau_n))$, $0 < \tau_1 < \cdots < \tau_n$, *is multivariate Poisson with parameter* $\{t(A)\}$ *defined for sets of the form* $A = \{k, k+1, \dots, l\}$ *by*

$$(4.1)\qquad t(\{k, \dots, l\}) = \lambda \int_{\tau_{k-1}}^{\tau_k} \left[F(\tau_{l+1} - u) - F(\tau_l - u)\right] du, \qquad k \le l,$$

and $t(A) = 0$ *for sets* A *that do not consist of consecutive numbers.* [*In* (4.1) *observe the natural conventions* $\tau_0 = 0$, $\tau_{n+1} = \infty$ *and* $F(\infty - u) = F(\infty) = 1$.]

PROOF. For $A = \{k, k+1, \dots, l\}$ let Z_A denote the number of customers who entered the system during the time interval (τ_{k-1}, τ_k) and whose service terminated during (τ_l, τ_{l+1}). Then Z_A has the Poisson distribution with parameter equal to $\lambda(\tau_k - \tau_{k-1})\rho$, where ρ denotes the conditional probability, given that an individual customer arrives during (τ_{k-1}, τ_k), that his service terminates during (τ_l, τ_{l+1}). We have

$$\rho = (\tau_k - \tau_{k-1})^{-1} \int_{\tau_{k-1}}^{\tau_k} [F(\tau_{l+1} - u) - F(\tau_l - u)]\, du, \qquad k = 1, \dots, n,\ l = 1, \dots, n$$

and

$$X(\tau_k) = \sum_{1 \le i \le k} \sum_{k \le j \le n} Z_{\{i \cdots j\}}.$$

In this representation of $X(\tau_k)$, $\sum_{k \le j \le n} Z_{\{i \cdots j\}}$ represents customers entering during (τ_{i-1}, τ_i). Proposition 4.1 follows by combining the preceding calculations. □

PROPOSITION 4.2. *Let $X(\tau)$ be as before and let $Y(\tau)$ be another such process with arrival rate γ and service time distribution G. Fix $\tau_1 < \cdots < \tau_n$. Then*

$$(4.2) \quad P(X(\tau_1) \ge c_1, \dots, X(\tau_n) \ge c_n) \le P(Y(\rho_1) \ge c_1, \dots, Y(\tau_n) \ge c_n)$$

for all $\mathbf{c} = (c_1, \dots, c_n)$ if and only if

$$(4.3) \quad \lambda \int_{\tau_j - \tau_i}^{\tau_j} [1 - F(u)]\, du \le \gamma \int_{\tau_j - \tau_i}^{\tau_j} [1 - G(u)]\, du \quad \textit{for all } 1 \le i \le j \le n.$$

On the other hand,

$$(4.4) \quad P(X(\tau_1) \le c_1, \dots, X(\tau_n) \le c_n) \le P(Y(\tau_1) \le c_1, \dots, Y(\tau_n) \le c_1)$$

for all $\mathbf{c} = (c_1, \dots, c_n)$ if

$$(4.5) \quad \lambda \int_0^{\tau_j - \tau_{i-1}} (1 - F(u))\, du \ge \gamma \int_0^{\tau_j - \tau_{i-1}} (1 - G(u))\, du \quad \textit{for all } 1 \le i \le j \le n.$$

Conversely, if (4.4) holds for all $\mathbf{c} = (c_1, \dots, c_n)$, then

$$(4.6) \quad \lambda \int_0^{\tau_j} (1 - F(u))\, du \ge \gamma \int_0^{\tau_j} (1 - G(u))\, du \quad \textit{for all } 1 \le j \le n.$$

PROOF. (4.2) ⇔ (4.3). For $t(A)$ of (4.1) and $B \in \mathscr{A}$ whose smallest and largest elements are i and j, respectively, we have

$$\sum_{A: A \supseteq B} t(A) = \sum_{\nu=1}^{i} \sum_{\mu=j}^{n} t(\nu, \dots, \mu) = \sum_{\nu=1}^{i} \sum_{\mu=j}^{n} \lambda \int_{\tau_{\nu-1}}^{\tau_\nu} [F(\tau_{\mu+1} - u) - F(\tau_\mu - u)]\, du$$
$$= \lambda \int_0^{\tau_i} [1 - F(\tau_j - u)]\, du = \lambda \int_{\tau_j - \tau_i}^{\tau_j} [1 - F(u)]\, du.$$

The first assertion of the proposition now follows from Theorem 1.2 and Proposition 1.3.

(4.5) ⇒ (4.4) and (4.4) ⇒ (4.6). Let

$$B = \{j_1, \ldots, j_m\}, \qquad 1 \le j_1 < \cdots < j_m \le n,\ j_0 = 0.$$

Then

$$\begin{aligned} \sum_{A:\, A \cap B \neq \phi} t(A) &= \sum_{k=1}^{m} \sum_{j_{k-1} < \nu \le j_k \le \mu} t(\nu, \ldots, \mu) \\ &= \sum_{k=1}^{m} \lambda \int_{\tau_{j_{k-1}}}^{\tau_{j_k}} \bigl(1 - F(\tau_{j_k} - u)\bigr)\, du \\ &= \sum_{k=1}^{m} \lambda \int_0^{\tau_{j_k} - \tau_{j_{k-1}}} (1 - F(u))\, du. \end{aligned} \tag{4.7}$$

That (4.5) ⇒ (4.4) now follows from Theorem 1.2. That (4.4) ⇒ (4.6) also follows from Theorem 1.2 and (4.7) with $m = 1$. □

The queue $X(\cdot)$ converges to a stationary process if (and only if) $\infty > \int_0^\infty u\, dF(u) = \int_0^\infty (1 - F(u))\, du$. Let $\overline{X}(\cdot)$ denote this process. Then for any fixed $0 \le \tau_1 < \cdots < \tau_n$, $(X(\tau_1 + s), \ldots, X(\tau_n + s)) \to (\overline{X}(\tau_1), \ldots, \overline{X}(\tau_n))$ in distribution as $s \to \infty$. Consequently $(\overline{X}(\tau_1), \ldots, \overline{X}(\tau_n))$ is a multivariate Poisson variable. The following is an immediate consequence of Proposition 4.2.

COROLLARY 4.3. *Let $\overline{X}(\tau)$ and $\overline{Y}(\tau)$ be the stationary versions of the processes in Proposition* 4.2. *Fix $0 \le \tau_1 < \cdots < \tau_n$. Then $\overline{X}$ and $\overline{Y}$ satisfy* (4.2) *for all* **c** *if and only if*

$$\lambda \int_{\tau_j - \tau_i}^{\infty} [1 - F(u)]\, du \le \gamma \int_{\tau_j - \tau_i}^{\infty} [1 - F(u)]\, du \quad \textit{for all } 1 \le i < j \le n. \tag{4.8}$$

If

$$\lambda \int_0^\infty (1 - F(u))\, du \ge \gamma \int_0^\infty (1 - G(u))\, du \tag{4.9}$$

and

$$\lambda \int_0^{\tau_j - \tau_i} (1 - F(u))\, du \ge \gamma \int_0^{\tau_j - \tau_i} (1 - G(u))\, du, \qquad 1 \le i < j \le n, \tag{4.10}$$

then they satisfy (4.4). *Conversely,* (4.4) ⇒ (4.9) *and if equality holds in* (4.9), *then* (4.4) ⇒ (4.10).

REMARK 4.4. (a) It is easy to conclude from Proposition 4.2 that (4.2) holds for all **c** and all $0 < \tau_1 < \cdots < \tau_n$ if and only if

$$\lambda(1 - F(u)) \le \gamma(1 - G(u)) \quad \text{for all } u \ge 0. \tag{4.11}$$

On the other hand (4.4) holds for all **c** and all $0 < \tau_1 < \cdots < \tau_n$ if and only if

$$\lambda \int_0^\tau (1 - F(u))\, du \ge \gamma \int_0^\tau (1 - G(u))\, du \quad \text{for all } \tau > 0. \tag{4.12}$$

(b) Note that (4.11) and (4.12) cannot simultaneously be valid unless $\lambda(1 - F(u)) = \gamma(1 - G(u))$ for all $u \ge 0$. In this case the two processes are identical;

hence there is no nontrivial situation where both (4.2) and (4.4) hold for all $\mathbf{c}$ and all $0 < \tau_1 < \cdots < \tau_n$.

(c) For the stationary processes $\overline{X}$ and $\overline{Y}$, (4.2) holds for all $\mathbf{c}$ and all $0 \le \tau_1 < \cdots < \tau_n$ if and only if

$$\lambda \int_\tau^\infty (1 - F(u))\, du \le \gamma \int_\tau^\infty (1 - G(u))\, du. \tag{4.13}$$

(4.4) holds for $\overline{X}, \overline{Y}$ for all $\mathbf{c}$ and all $0 \le \tau_1 < \cdots < \tau_n$ if (4.9) and (4.12) are valid. If equality holds in (4.9), then (4.12) is also a necessary condition for (4.4) to hold for all $\mathbf{c}$ and all $0 \le \tau_1 < \cdots < \tau_n$.

(d) When $\lambda = \gamma$ and $\int_0^\infty u\, dF(u) = \int_0^\infty u\, dG(u) < \infty$, then (4.12) and (4.13) are equivalent, familiar conditions. Consequently, there are many pairs of processes X and Y whose stationary versions $\overline{X}$ and $\overline{Y}$ are not identical and satisfy both (4.2) and (4.4).

(e) Passing to the limit (as $n \to \infty$) shows that (4.11) $\Rightarrow P(X(\tau) \ge c(\tau)) \le P(Y(\tau) \ge c(\tau))$ for any function c: $(0, \infty) \to [0, \infty)$ and (4.12) $\Rightarrow P(X(\tau) \le c(\tau)) \le P(Y(\tau) \le c(\tau))$ for any function $c(\cdot)$ with analogous results for $\overline{X}$ and $\overline{Y}$.

Closely related results on stationary $M/G/\infty$ queues and related processes were obtained recently by Huffer (1987). For example, Huffer's Theorem 1.1d is equivalent to the assertion concerning (4.11) for the processes $\overline{X}$ and $\overline{Y}$ in our Remark 4.4e.

5. Bivariate cumulative distribution functions. Let $V_1, \dots, V_N$ be independent identically distributed random variables in $\mathbb{R}^d$ with cumulative distribution function $F(\nu) = P(V \le \nu)$. Their sample cumulative distribution function is $F_N(\nu) = N^{-1}\{\text{number of } i \ni V_i \le \nu\}$, $\nu \in \mathbb{R}^d$.

Note that for $\nu_1, \dots, \nu_n \in R^d$, $N(F_N(\nu_1), \dots, F_N(\nu_n))$ is multivariate multinomial as in Corollary 2.7 [with $U_1 \equiv 1$ in (2.3)]. Its parameters are described by (1.7) with $\Gamma_i = \{\nu\colon \nu \le \nu_i\}$ and λ the distribution of V. Let $W_1, \dots, W_N$ be a second set of independent variables in $\mathbb{R}^d$ with cumulative distribution function G and sample cumulative distribution function G_N.

PROPOSITION 5.1. *Let $S \subset \mathbb{R}^d$ and m: $S \to \mathbb{R}^d$. Suppose that $F(\nu) = G(m(\nu))$.*

(i) *If for every $\{\nu_1, \dots, \nu_n\} \subset S$, $n = 1, 2, \dots$,*

$$P\big(W \le m(\nu_j),\ j = 1, \dots, n\big) \le P\big(V \le \nu_j,\ j = 1, \dots, n\big), \tag{5.1}$$

then for any c: $[0,1] \to [0,1]$,

$$P(G_N(\theta) \ge c(G(\theta))\ \forall\ \theta \in m(S)) \le P\big(F_N(\nu) \ge c(F(\nu))\ \forall\ \nu \in S\big). \tag{5.2}$$

(ii) *If for every $\{\nu_1, \dots, \nu_n\} \subset \mathbb{R}^d$,*

$$P\big(W \le m(\nu_j) \text{ for some } j = 1, \dots, n\big) \ge P\big(V \le \nu_j \text{ for some } j = 1, \dots, n\big), \tag{5.3}$$

then for any c: $[0,1] \to [0,1]$,

$$P(G_N(\theta) \le c(G(\theta))\ \forall\ \theta \in m(S)) \le P\big(F_N(\nu) \le c(F(\nu))\ \forall\ \nu \in S\big). \tag{5.4}$$

PROOF. Fix $\nu_1, \dots, \nu_n \in \mathbb{R}^d$. Then $(F_N(\nu_1), \dots, F_N(\nu_n))$ is multivariate multinomial as mentioned earlier and so is $(G_N(m(\nu_1)), \dots, G_N(m(\nu_n)))$. It then follows from (1.8) and Corollary 2.7 that (5.1) implies

$$P\big(G_N\big(m(\nu_j)\big) \geq c_j;\ j = 1, \dots, n\big) \leq P\big(F_N(\nu_j) \geq c_j,\ j = 1, \dots, n\big).$$

Letting $n \to \infty$ and choosing an appropriate dense sequence $\nu_1, \nu_2, \dots$ and $c_j = c(F(\nu_j))$ yields (5.2) since $F(\nu) = G(m(\nu))$. The same arguments show that (5.3) $\Rightarrow$ (5.4). □

REMARK 5.2. Note that

$$P\big(G_N(\theta) \geq c(G(\theta))\ \forall\ \theta \in \mathbb{R}^d\big) \leq P\big(G_N(\theta) \geq c(G(\theta))\ \forall\ \theta \in m(S)\big)$$

and, if $S = \{\nu\colon 0 < F(\nu) < 1\}$, then either

$$P\big(F_N(\nu) \geq c(F(\nu))\ \forall\ \nu \in S\big) = P\big(F_N(\nu) \geq c(F(\nu))\ \forall\ \nu \in \mathbb{R}^d\big) > 0$$

or

$$P\big(G_N(\theta) \geq c(G(\theta))\ \forall\ \theta \in \mathbb{R}^d\big) = P\big(F_N(\nu) \geq c(F(\nu))\ \forall\ \nu \in \mathbb{R}^d\big) = 0.$$

There are analogous assertions for the quantities appearing in (5.4).

Let $d = 2$. Let F be any given continuous cumulative distribution function. Define G by

$$G((\theta_1, \theta_2)) = (\theta_1 + \theta_2 - 1)^+, \qquad 0 \leq \theta_1, \theta_2 \leq 1. \tag{5.5}$$

(W, whose distribution is G, is uniformly distributed along the counterdiagonal line in $[0,1]^2$, $\{(\theta_1, \theta_2)\colon \theta_1 + \theta_2 = 1,\ 0 \leq \theta_1, \theta_2 \leq 1\}$.) It is shown in Adler and Brown [(1986), equation (3.7)] that there is a map $m\colon \{\nu\colon 0 < F(\nu) < 1\} \to \{\theta\colon 0 < G(\theta) < 1\}$ (onto) such that $G(m(\nu)) = F(\nu)$ and for any $\nu_1, \nu_2 \in \mathbb{R}^2$,

$$P\big(W \leq m(\nu_j),\ j = 1,2\big) \leq P\big(V \leq \nu_j,\ j = 1,2\big). \tag{5.6}$$

PROPOSITION 5.3. *For any distribution F on $\mathbb{R}^2$ and any $c\colon [0,1] \to [0,1]$,*

$$P\big(G_N(\theta) \geq c(G(\theta))\ \forall\ \theta \in \mathbb{R}^2\big) \leq P\big(F_N(\nu) \geq c(F(\nu))\ \forall\ \nu \in \mathbb{R}^2\big), \tag{5.7}$$

$$P\big(G_N(\theta) \leq c(G(\theta))\ \forall\ \theta \in \mathbb{R}^2\big) \leq P\big(F_N(\nu) \leq c(F(\nu))\ \forall\ \nu \in \mathbb{R}^2\big). \tag{5.8}$$

PROOF. Assume that F is continuous. Let $\nu_1, \dots, \nu_n \in \mathbb{R}^2$ with $0 < F(\nu_j) < 1$, $j = 1, \dots, n$ to avoid trivialities. Suppose, without loss of generality, $\nu_{11} = \min_j\{\nu_{j1}\}$ and $\nu_{n2} = \min_j\{\nu_{j2}\}$. Then

$$\{\nu\colon \nu \leq \nu_j,\ j = 1, \dots, n\} = \{\nu\colon \nu \leq \nu_1,\ \nu \leq \nu_n\}.$$

Hence, (5.6) $\Rightarrow$ (5.1) $\Rightarrow$ (5.2) $\Rightarrow$ (5.7) because of Remark 5.2. A similar argument shows that (5.3) holds, implying (5.4) and (5.8).

If F is not continuous, then there is a continuous cumulative distribution function F', say, and a map $m'\colon \mathbb{R}^2 \to \mathbb{R}^2$ (into) such that $F(\nu) = F'(m'(\nu))$. Hence $F_N(\nu) = F'_N(m'(\nu))$ and the validity of (5.7) for F follows from the previously established validity of (5.7) for F'. □

REMARK 5.4. Proposition 5.3 implies in particular that

$$P\left(\sup_{t\in\mathbb{R}^2}\left\{\sqrt{N}\left(F_N(t)-F(t)\right)\right\}>c_N\right)\le P\left(\sup_{t\in\mathbb{R}^2}\left\{\sqrt{N}\left(G_N(t)-G(t)\right)\right\}>c_N\right).$$

Adler and Brown (1986) proved a variation of this result involving the limiting Gaussian process $W(t)=\lim_{N\to\infty}\sqrt{N}\left(F_N(t)-F(t)\right)$.

Proposition 5.3 is applied in Adler and Brown (1988) to derive hypothesis tests and confidence bands for bivariate cumulative distribution functions.

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