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RANDOM TIME CHANGES FOR PROCESSES WITH RANDOM BIRTH AND DEATH[1]

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We study a random time change for stationary Markov processes (Y_t, Q) with random birth and death. We use an increasing process, obtained from a homogeneous random measure (HRM) as our clock. We construct a time change that preserves both the stationarity and the Markov property. The one-dimensional distribution of the time-changed process is the characteristic measure ν of the HRM, and its semigroup $(\tilde{P}_t)$ is a naturally defined time-changed semigroup. Properties of ν as an excessive measure for $(\tilde{P}_t)$ are deduced from the behaviour of the HRM near the birth time. In the last section we apply our results to a simple HRM and connect the study of Y near the birth time to the classical Martin entrance boundary theory.

1. Introduction.

1.1. *The classical case.* Time change provides a powerful tool in the study of Markov processes. Among other things, it is used to compare two Markov processes that traverse the same path, but do so at different speeds, it enables one to restrict the process to a subset of its state space and it provides the clock (local time) for the study of excursions from a set.

In the classical setting one starts from a Markov process (X_t, P) $[t \in [0, \zeta)$, $X_t = \Delta$ for $t \geq \zeta$, where Δ is a cemetery point] with stationary transition function and defines a new process of the same type by the formula $\tilde{X}_t(\omega) = X_{S_t}(\omega)$, where $(S_t)_{t \geq 0}$ is an increasing process.

1.2. *Processes with random birth and death.* During recent years a great deal of work has been done on Markov processes for which both the birth and death times (denoted α and β, respectively) are random. An important class of such processes is stationary processes (Y_t, Q), where the law Q is invariant under the time-shift operator. If (P_t) is the transition semigroup for such a process (Y_t, Q), then the measure

$$m(B) = Q(Y_t \in B, \alpha < t < \beta) \tag{1.1}$$

is (P_t) excessive. It follows from a theorem of Kuznetsov [10] that the converse is also true. Namely, given a transition semigroup (P_t) and an excessive measure m for it, there exists a unique process with random birth and death (i.e., a unique Q) with one-dimensional distribution m and transition semigroup (P_t).

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1.3. *Time change.* To introduce time change in this setting, we consider the canonical realization of such a process on the space of paths W. We consider a time change that preserves both the stationarity and the Markov property. Toward this end, we need to introduce an extra random parameter r that has a uniform distribution on the real line. We then produce an increasing process $C_t(w, r)$ and we put $\tilde{Y}_t(w, r) = Y_{C_t(w,r)}(w)$, where $(Y_t(w))$ is the coordinate process on W. Because of the choice of r, the law of $(\tilde{Y}_t)$ is never σ-finite. Nevertheless, there exists a measure $\tilde{Q}$ on W with the following properties:

(i) $\tilde{Q}$ is σ-finite.

(ii) $\tilde{Q}$ is invariant with respect to the time shifts (σ_s) on W.

(iii) Under $\tilde{Q}$ the coordinate process $t \to w_t$ is Markovian with stationary transition function.

(iv) For every fixed t, the joint law of C_t and $\tilde{Y}$ is σ-finite and factorizes to the product of the Lebesgue measure on $\mathbb{R}$ and $\tilde{Q}$ on $W \cap \{\alpha < t < \beta\}$.

1.4. *Additive functionals and homogeneous random measures.* In the classical case the process (S_t) is obtained as the right-continuous inverse of a continuous additive functional (A_t), i.e.,

$$S_t = \inf\{u\colon A_u > t\}. \tag{1.2}$$

To construct $C_t(w, r)$, we start with a diffuse homogeneous random measure (HRM) $B(dt)$; that is, a random measure $B(w, \cdot)$ on $(\mathbb{R}, \mathscr{R})$ concentrated on (α, β), such that for any $C \in \mathscr{R}$,

(1.3)
(i) $B(\sigma_s w, C) = B(w, C + s)$,
(ii) $w \to B(w, C)$ is measurable with respect to the universal completion of $\sigma\{Y_s\colon s \in C\}$,
(iii) for $C = [a, b] \subset (\alpha, \beta)$, $B(w, C) < \infty$.

The process $C_t(w, r)$ is the right-continuous inverse of the nondecreasing process

$$B_t(w, r) = r + \int_{T(w)}^{t} B(w, ds), \tag{1.4}$$

where T is an arbitrary random variable on W that satisfies $\alpha < T < \beta$.

The stated properties of the time change described previously are proved in the next section. We then use the construction of the time change to establish relations between the behaviour of a HRM B near α and properties of the characteristic measure of B given by

$$\nu_B(f) = Q\int_{[0,1)} f(Y_t)B(dt). \tag{1.5}$$

From [12] it follows that ν_B is σ-finite at least when B is diffuse. It turns out that ν_B is the one-dimensional distribution of $\tilde{Q}$ and it is therefore excessive for the semigroup $(\tilde{P}_t)$ of $(w_t, \tilde{Q})$ [the same semigroup appears in the classical setting if we do a time change for (X_t, P) using an additive functional A naturally related to the HRM B]. It follows from our construction that if Q a.e. $B(\alpha, t] =$

∞ for all $t > \alpha$, then ν_B is invariant for $(\tilde{P}_t)$, and if Q a.e. $B(\alpha, t] \to 0$ as t decreases to α, then ν_B is purely excessive [i.e., $\int \nu_B(dx)\tilde{P}_t f(x) \to 0$ as $t \to \infty$ if $\nu_B(f) < \infty$].

1.5. *Applications to entrance spaces.* The time change introduced here enables one to connect the study of the behaviour of Y near α to the classical theory of Martin entrance boundary, which is based on a Ray–Knight compactification. Using a HRM that charges every subinterval of (α, β) as our time change clock, the behaviour near α of Y is equivalent to the behaviour near 0 of a right process in a Ray topology. Many questions that arise from capacity theory have very simple answers in terms of the time-changed process. We explain the relation and collect some simple results in Section 3. The purpose of that section is only to serve as an example for possible applications of time changes on W. We therefore do not attempt to cover many aspects of capacity theory and most of the (obvious) proofs are omitted.

1.6. *Notation.* We shall work in the setting considered by Fitzsimmons and Maisonneuve [2]. Our notation follows theirs.

$(E, \mathscr{E})$ is a Lusin space. Δ a point not in E, $E_\Delta = E \cup \Delta$, $\mathscr{E}_\Delta = \mathscr{E} \vee \Delta$. Let W be the space of functions w from $\mathbb{R}$ into E_Δ that are E-valued and right-continuous for $t \in (\alpha(w), \beta(w))$ and are equal to Δ for t outside $(\alpha(w), \beta(w))$. Two families of time shifts are introduced on W. The first, (σ_t), is defined by $\sigma_t w(s) = w(t+s)$ $(t, s \in \mathbb{R})$; the second, (τ_t), is related to birthing by

$$(1.6)\qquad \begin{aligned} \tau_t w(s) &= w(t+s), && s > 0,\ t \in \mathbb{R},\\ &= \Delta, && s < 0,\ t \in \mathbb{R}. \end{aligned}$$

Note that $\sigma_t \circ \sigma_s = \sigma_{t+s}$, $\tau_t = \tau_0 \circ \sigma_t$. As before, (Y_t) is the coordinate process on W. Let $\mathscr{G}^0 = \sigma\{Y_s\colon s \in \mathbb{R}\}$ and $\mathscr{G}_t^0 = \sigma\{Y_s\colon s \le t\}$. Let $(P_t)_{t \ge 0}$ be a Borel right semigroup (in the sense of [4]) and $(\nu_t)_{t \in \mathbb{R}}$ be an entrance rule for it. Let Q_ν be the measure on $(W, \mathscr{G}^0)$ with one-dimensional distribution, at time t, equal to ν_t, and transition semigroup equal to (P_t). The existence of such a measure follows from Kuznetsov's work [10], and we shall refer to it as the Kuznetsov measure corresponding to (ν, P_t). In the case $\nu_t \equiv m$, m is excessive for (P_t) and Q_m stands for Q_ν. This is the stationary case considered in [2]. Let h be an excessive function for (P_t) satisfying $\nu_t\{h = \infty\} = 0$ for all t. We shall denote by Q_ν^h the Kuznetsov measure that corresponds to $(\nu \cdot h, P_t^h)$, where $P_t^h f = 1/h\, P_t f \cdot h$ on $E_h = \{0 < h < \infty\}$.

Let $\Omega = \{\alpha = 0,\ Y_{t+}$ exists in E for $t \ge 0\} \cup \{[\Delta]\}$ (where for $x \in E_\Delta$, $[x]$ is the constant path $t \to x$). For $s \ge 0$ let X_s, θ_s, ζ be the restrictions of $Y_{s+}, \tau_s, \beta \vee 0$, respectively to Ω, and let $\mathscr{F}^0 = \mathscr{G}^0|_\Omega$, $\mathscr{F}_s^0 = \mathscr{G}^0|_\Omega$. For $x \in E$, P^x is the measure on $(\Omega, \mathscr{F}^0)$ with one-dimensional distribution at t equal to $P_t(x, \cdot)$ and transition semigroup (P_t). $(\Omega, \mathscr{F}^0, \mathscr{F}_t^0, \theta_t, X_t, P^x\colon x \in E)$ is a Borel right process. We let $\mathscr{F}^\mu$ be the completion of $\mathscr{F}^0$ with respect to $\int \mu(dx) P^x$, and $\mathscr{F} = \bigcap \mathscr{F}^\mu$. $\mathscr{F}_t$ is a similar completion of $\mathscr{F}_t^0$ in $\mathscr{F}$. The resolvent that corresponds to (P_t) is denoted $(U^\gamma)_{\gamma \ge 0}$. For a σ-algebra $\mathscr{K}$, we let $\mathscr{K}$ denote the

$\mathscr{K}$-measurable functions, and $b\mathscr{K}$, $\mathscr{K}_+$, $\mathscr{K}_{++}$ be the bounded, nonnegative, strictly positive measurable functions, respectively. We extend the definition of $f \in \mathscr{E}$ to E_Δ by setting $f(\Delta) = 0$. $g \in \mathscr{R}$ is extended to $\overline{\mathbb{R}}$ by setting $g(\pm\infty) = 0$. For a measure μ on $(E, \mathscr{E})$ and $f \in \mathscr{E}$, $\mu(f)$ stands for $\int \mu(dx) f(x)$, whereas $\mu \cdot f$ is the measure $\mu(dx)f(x)$. By convention, $Y_{\pm\infty} = \Delta$, $X_\infty = \Delta$. We shall introduce additional notation in the sequel as it becomes necessary.

2. The time change.

2.1. *Preliminaries.* In this section we restrict our attention to the stationary case (W, Q_m), where m is excessive for (P_t). Let B be a diffuse HRM.

It follows from [11] that there exists a unique continuous additive functional (A_t) on Ω that satisfies

$$A_t(\tau_s w) = B(w, (s, s+t]), \quad \text{on } \{\alpha < s < \beta\}. \tag{2.1}$$

Let S_t be its right-continuous inverse as defined in (1.2) and

$$\tilde{P}_t f(x) = P^x\big(f(X_{S_t})\big). \tag{2.2}$$

Restricted to the fine support of A, $\tilde{X}_t = X_{S_t}$ is a right process with semigroup $(\tilde{P}_t)$.

2.2. *Time change in a simple case.* Unlike additive functionals on Ω, which are assumed to be equal to 0 at 0, HRMs may accumulate infinite mass around α. Therefore the definition of a clock from a HRM requires some care. More care is needed if one wishes to produce a stationary time-changed process. The case where $B(\alpha, t] \downarrow 0$ as $t \downarrow \alpha$ is very similar to the classical case. It produces a neat formula, as well as some intuition as to why (1.4) works in the general case. We shall treat it here first.

(2.3) THEOREM. *Suppose that Q_m a.e. $\lim_{t\downarrow\alpha} B(\alpha, t] = 0$. Then there exists an entrance law at 0, (η_t), for $(\tilde{P}_t)$ so that $\nu_B(\cdot) = \int_0^\infty \eta_t(\cdot)\,dt$.*

(2.4) REMARK. It follows from [1] and (2.3) that ν_B is purely excessive for $(\tilde{P}_t)$.

PROOF. Let $C_t = \inf\{s: B(\alpha, s] > t\}$. For every positive t, $C_t > \alpha$ a.e. by our assumptions. By the shift invariance of Q_m,

$$\begin{aligned} Q_m\big(C_t \in du, Y_{C_t} \in dx\big) &= Q_m\big(C_t \circ \sigma_s \in du, Y_{C_t} \circ \sigma_s \in dx\big) \\ &= Q_m\big(C_t \in d(u+s), Y_{C_t} \in dx\big). \end{aligned} \tag{2.5}$$

It follows from a result of Getoor [6], that for each $t \in (0, \infty)$, there exists a measure (countable sum of finite measures) η_t on $(E, \mathscr{E})$, so that

$$Q_m\big(C_t \in du, Y_{C_t} \in dx\big) = \eta_t(dx)\,du. \tag{2.6}$$

Note that $C_{t+s} = C_t + S_s \circ \tau_{C_t}$ on $\{C_t < \infty\}$. Since for $t > 0$, $C_t \in (\alpha, \beta)$ Q_m a.e.

on $\{C_t < \infty\}$, it follows from the strong Markov property at C_t ([2]) that for $g \in \mathscr{R}_+$, $f \in \mathscr{E}$,

$$(2.7)\qquad Q_m\big(g(C_{t+s})f(Y_{C_{t+s}})\big) = Q_m\big(C_t < \infty, P^{Y(C_t)}\big(g(C_t + S_s)f(X_{S_s})\big)\big).$$

Applying (2.6) to (2.7), we obtain

$$(2.8)\qquad \begin{aligned} &\int_{\mathbb{R}} g(u)\,du \int_E \eta_{t+s}(dx)f(x) \\ &\quad = \int_{\mathbb{R}} du \int_E \eta_t(dx) \int_{\mathbb{R}}\int_E P^x(S_s \in dv, X_{S_s} \in dy)g(u+v)f(y) \\ &\quad = \int_{\mathbb{R}} g(u)\,du \int_E \eta_t(dx)P^x\big(f(X_{S_s})\big) \\ &\quad = \int_{\mathbb{R}} g(u)\,du \int_E \eta_t(dx)\tilde{P}_s f(x), \end{aligned}$$

which implies that $\eta_{t+s} = \eta_t\tilde{P}_s$, $s \geq 0$, $t > 0$. Defining $\eta_s \equiv 0$ for $s \leq 0$, $(\eta_s)_{s\in\mathbb{R}}$ is an entrance law at 0 for $(\tilde{P}_t)$. It follows from (2.7) and (2.8) that

$$(2.9)\qquad \eta_s(f) = Q_m\big(C_s \in [0,1), f(Y_{C_s})\big).$$

Hence (using Fubini's theorem),

$$\int_0^\infty \eta_s(f)\,ds = Q_m\int_0^\infty 1_{\{C_s \in [0,1)\}} f(Y_{C_s})\,ds,$$

which after the change of variable $u = B(\alpha, t]$ is equal to

$$Q_m\int_{\mathbb{R}} 1_{[0,1)}(u)f(Y_u)B(du) = \nu_B(f).$$

The σ-finiteness of ν_B implies now that η_t is σ-finite for all t and our proof is complete. □

Let $\tilde{Q}_\eta$ be the Kuznetsov measure that corresponds to (η_s) given previously and $\tilde{P}_t$, and define Π: $W \to W$ by $(\Pi w)_t = Y_{C_t}(w)$.

(2.10) THEOREM. *For any* $A \in \mathscr{G}^0|_{\{\alpha=0\}}$ *and* $g \in \mathscr{R}_+$ *that satisfies* $\int g(t)\,dt = 1$,

$$\tilde{Q}_\eta(A \cap \{\beta > t\}) = Q_m\big(\Pi^{-1}(A)g(C_t)\big), \qquad t \geq 0.$$

PROOF. It is enough to prove (2.10) for A of the form

$$A = \big\{w_{t_1} \in A_1, \dots, w_{t_n} \in A_n\big\}, \quad \text{for } 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n.$$

For such A, the proof is a straightforward computation of the kind performed in (2.21), and it is therefore omitted at this stage. □

(2.11) COROLLARY. *Let* $A \in \mathscr{G}^0$, $A \subset \{\alpha = 0\}$. *Then* $Q_m(\Pi^{-1}(A)) = 0$ *if, and only if*, $\tilde{Q}_\eta(A) = 0$.

PROOF. Suppose $\tilde{Q}_\eta(A) = 0$. Then $Q_\eta(A \cap \{\beta > t\}) = 0$ for all t, and for any $g \in \mathscr{R}_{++}$ with $\int g(t)\,dt = 1$, $Q_m(\Pi^{-1}(A)g(C_t)) = 0$ for all t. This implies that $Q_m(\Pi^{-1}(A), C_t \in \mathbb{R}) = 0$ for all t. But $\cup_{r \in Q}\{C_r \in \mathbb{R}\} = \{B(\mathbb{R}) > 0\}$ (Q are the rational numbers), and for $w \in \{B(w,\mathbb{R}) = 0\}$, $\pi(w) = [\Delta]$. Hence for $A \subset \{\alpha = 0\}$, $\pi^{-1}(A) \cap \{B(\mathbb{R}) = 0\} = \varnothing$ and $Q_m(\Pi^{-1}(A)) = 0$. The converse is argued in the same manner. □

Let $\tilde{Q}_{\nu_B}$ be the Kuznetsov measure that corresponds to $(\nu_B, \tilde{P}_t)$, and $\tilde{Q}^r_\eta = \sigma_r(\tilde{Q}_\eta)$. Then (2.3) implies that

$$\tilde{Q}_{\nu_B} = \int_{\mathbb{R}} \tilde{Q}^r_\eta\, dr. \tag{2.12}$$

We may think of $\tilde{Q}_{\nu_B}$ arising from B and Q_m in the following manner. To get a nondecreasing process (B_t) from B, we introduce a parameter r, uniformly distributed in $\mathbb{R}$, and define $B_\alpha = r$. For $t > \alpha$, $B_t = r + B(\alpha, t]$, and for $t \le \alpha$, $B_t = r$. If $C_t(w, r)$ is the right-continuous inverse of (B_t), then $r = \inf\{u\colon C_u > -\infty\}$ is the birth time of the time-changed process. The law governing it, after its birth at r, is $\tilde{Q}^r_\eta$. This procedure works only when B does not accumulate infinite mass near α. Using, however, the translation invariance of the Lebesgue measure, it is easy to see that defining $B_s = r$ at any $s \in (\alpha, \beta)$ will produce the same effect. This is exactly (1.4) and, as we shall see, produces the required time change for all HRMs.

2.3. *The general case.* Let λ be the Lebesgue measure on $\mathbb{R}$ and define

$$(\hat{W}, \hat{\mathscr{G}}, P_m) = (W, \mathscr{G}^m, Q_m) \times (\mathbb{R}, \mathscr{R}, \lambda), \tag{2.13}$$

where $\mathscr{G}^m$ is the Q_m completion of $\mathscr{G}^0$. Let T be an arbitrary $\mathscr{G}^m$ random variable taking values in (α, β). For each $(w, r) \in \hat{W}$ define $B_t(w, r)$ by (1.4) [where we remember that if $t > T(w)$ the integral is equal to $-B(w, [T(w), t])$]. For $t \in \mathbb{R}$, we let

$$C_t(w, r) = \inf\{u\colon B_u(w, r) > t\}, \qquad \inf \varnothing = +\infty, \tag{2.14}$$

$$\tilde{Y}_t(w, r) = Y_{C_t(w, r)}(w). \tag{2.15}$$

(2.16) PROPOSITION. *Let $f \in \mathscr{E}_+$ and $g \in \mathscr{R}_+$ with $\lambda(g) < \infty$. Then*

$$P_m\big(f(\tilde{Y}_t)g(C_t)\big) = \nu_B(f)\lambda(g). \tag{2.17}$$

In particular, the left-hand side of (2.17) is independent of t.

PROOF. For $t > 0$ define on W,

$$\begin{aligned} U_t(w) &= \inf\{u > 0\colon B(w, [0, u]) > t\}, \qquad \inf \varnothing = \infty,\\ U_{-t}(w) &= \sup\{u < 0\colon B(w, [u, 0]) > t\}, \qquad \sup \varnothing = -\infty, \end{aligned} \tag{2.18}$$

and note that for $v \in \mathbb{R}$,

$$(2.19)\qquad \begin{aligned} v + U_t(\sigma_v w) &= \inf\{u > v\colon B(w,[v,u]) > t\},\\ v + U_{-t}(\sigma_v w) &= \sup\{u < v\colon B(w,[u,v]) > t\}. \end{aligned}$$

By the definition of P_m, the left-hand side of (2.17) is equal to

$$\int_W Q_m(dw)\int_{\mathbb{R}} g(C_t(w,r)) f\big(Y_{C_t(w,r)}(w)\big)\,dr.$$

Fix $w \in W$ and let $v = T(w)$,

$$\begin{aligned}
&\int_{\mathbb{R}} g(C_t(w,r)) f\big(Y_{C_t(w,r)}(w)\big)\,dr\\
&\quad = \int_{r\in(-\infty,t)} g(U_{t-r}\circ\sigma_v + v) f\big(Y_{U_{t-r}}\circ\sigma_v\big)\,dr\\
&\qquad + \int_{r\in[t,\infty)} g(U_{t-r}\circ\sigma_v + v) f\big(Y_{U_{t-r}}\circ\sigma_v\big)\,dr\\
&\quad = \int_{u\in(0,\infty)} g(U_u\circ\sigma_v + v) f\big(Y_{U_u}\circ\sigma_v\big)\,du\\
&\qquad + \int_{u\in[0,\infty)} g(U_{-u}\circ\sigma_v + v) f\big(Y_{U_{-u}}\circ\sigma_v\big)\,du.
\end{aligned}$$

In the first term we now use the change of variable $u = B(\sigma_v,[0,t])$, and in the second $u = B(\sigma_v,[-t,0])$. The last expression is equal to

$$\begin{aligned}
&\left[\int_{u\in(0,\beta]} g(u+v) f(Y_u) B(du)\right]\circ\sigma_v + \left[\int_{u\in(\alpha,0]} g(u+v) f(Y_u) B(du)\right]\circ\sigma_v\\
&\quad = \int_{(v,\beta(w))} g(u) f(Y_u) B(du) + \int_{(\alpha(w),v]} g(u) f(Y_u) B(du)\\
&\quad = \int_{\alpha(w)}^{\beta(w)} g(u) f(Y_u(w)) B(w,du).
\end{aligned}$$

Integrating with respect to Q_m, the result follows. □

In particular, we note that for $g = 1_{[0,1)}$,

$$(2.20)\qquad P_m\big(1_{[0,1)}(C_t) f(\tilde{Y}_t)\big) = \nu_B(f).$$

(2.21) **Proposition.** *For $g_1,\dots,g_n \in \mathscr{R}_+$ with $\lambda(g_i) < \infty$, $f_1,\dots,f_n \in \mathscr{E}_+$ and $-\infty < t_1 \le t_2 \le \cdots \le t_n < \infty$,*

$$(2.22)\qquad \begin{aligned}
&P_m\left(\prod_{i=1}^n g_i(C_{t_i}) f_i(\tilde{Y}_{t_i})\right)\\
&\quad = Q_m\int_\alpha^\beta g_1(t) f_1(Y_t) P^{Y_t}\left(\prod_{i=2}^n g_i\big(t + S_{t_i-t_1}\big) f_i\big(X_{S_{t_i-t_1}}\big)\right) B(dt)\\
&\quad = \int_{\mathbb{R}} g_1(t)\,dt \int_E f_1(x)\nu_B(dx) P^x\left(\prod_{i=2}^n g_i\big(t + S_{t_i-t_1}\big) f_i\big(X_{S_{t_i-t_1}}\big)\right).
\end{aligned}$$

PROOF. The second equality follows easily from the first and the definition of ν_B. For the first we note that for $i > 1$,

$$C_{t_i}(w, r) = C_{t_1}(w, r) + S_{t_i - t_1} \circ \tau_{C_{t_1}(w, r)}(w)$$

and

$$\tilde{Y}_{t_i}(w, r) = X_{S_{t_i - t_1}} \circ \tau_{C_{t_1}(w, r)}(w).$$

Repeating now the argument that led to (2.16), we obtain

$$P_m\left(\prod_{i=1}^{n} g_i(C_{t_i}) f_i(\tilde{Y}_{t_i}) \right)$$
$$= Q_m \int_\alpha^\beta g_1(t) f_1(Y_t) \prod_{i=2}^{n} g_i\left(t + S_{t_i - t_1} \circ \tau_t\right) f_i\left(X_{S_{t_i - t_1}} \circ \tau_t\right) B(dt),$$

which, by the strong Markov property, yields the first equality in (2.22). □

(2.23) COROLLARY. *The measure ν_B is excessive for $(\tilde{P}_t)$.*

PROOF. Put $g = 1_{[0,1)}$, $f \in \mathscr{E}_+$ in (2.22). Then for $s > 0$,

$$P_m\left(1_E(\tilde{Y}_t) f(\tilde{Y}_{t+s}) g(C_{t+s})\right) \le P_m\left(f(\tilde{Y}_{t+s}) g(C_{t+s})\right),$$

and the result follows. □

(2.24) COROLLARY. *Let $f_1, \ldots, f_n \in \mathscr{E}_+$ and $g \in \mathscr{R}_+$ with $\lambda(g) < \infty$. Then for all $1 \le i \le n$,*

$$P_m\left(f_1(\tilde{Y}_{t_1}) \cdots f_n(\tilde{Y}_{t_n}) g(C_{t_i})\right) = P_m\left(g(C_{t_1}) f_1(\tilde{Y}_{t_1}) \cdots f_n(\tilde{Y}_{t_n})\right).$$

PROOF. Follows from the second equality of (2.22) using the translation invariance of the Lebesgue measure. □

Let $\tilde{Q}_{\nu_B}$ be the Kuznetsov measure that corresponds to $(\nu_B, \tilde{P}_t)$. Define $\hat{\Pi}$: $\hat{W} \to W$ by $(\hat{\Pi}\hat{w})_t = \tilde{Y}_t(\hat{w})$. Then it follows from (2.22) and (2.24) that

(2.25) THEOREM. *For $A \in \mathscr{G}^0$ and $g \in \mathscr{R}_+$,*

$$P_m\left(\hat{\Pi}^{-1}(A) g(C_t)\right) = \lambda(g) \tilde{Q}_{\nu_B}(A;\ \alpha < t < \beta).$$

(2.26) REMARK. It had been pointed out to me by the referee that the time change given previously is the analog of an old result in the theory of flows ([3] and the references therein). Indeed, let $(\Omega, \mathscr{F}, \theta_t, P)$ be a flow [P σ-finite $\theta_t(P) = P$] and (B_t) a CAF over the flow; that is,

(i) $t \to B_t$ is nondecreasing and continuous,
(ii) $B_{t+s} = B_t + B_s \circ \theta_t$, $t \in \mathbb{R}$, $s \ge 0$, and
(iii) $B_t \to \pm\infty$ as $t \to \pm\infty$, respectively.

Let $\tilde{P}$ be the Palm measure of B,

$$\tilde{P}(A) = P\int_0^1 1_A \circ \theta_t \, dB_t, \qquad A \in \mathscr{F}, \tag{2.27}$$

and (C_t) the right-continuous inverse of B. Then for $\tilde{\theta}_t = \theta_{C_t}$, $\tilde{\theta}_t(\tilde{P}) = \tilde{P}$ so that $(\Omega, \mathscr{F}, \tilde{\theta}_t, \tilde{P})$ is a new flow. In our context (2.25) can be interpreted as

$$\tilde{Q}_{\nu_B}|_{\{\alpha < 0 < \beta\}} = \text{Palm measure of } B \text{ under } Q_m. \tag{2.28}$$

Things are complicated here because we are not assuming (iii), which is the reason for the presence of α and β in (2.28). Since our B is a measure, C_t is only defined on $\hat{W}$, and even there is not necessarily finite. The (σ_s) invariance of $\tilde{Q}_{\nu_B}$ is in essence the analog of $\tilde{\theta}_t(\tilde{P}) = \tilde{P}$. Indeed, by (2.25), this invariance is equivalent to the following: For $A_1, \dots, A_n \in \mathscr{E}$, $g \in \mathscr{R}_+$ with $\lambda(g) < \infty$, $-\infty < t_1 \le t_2 \le \cdots \le t_n < \infty$, $s \ge 0$ and $t \in \mathbb{R}$,

$$P_m\big(\tilde{Y}_{t_1} \in A_1, \dots, \tilde{Y}_{t_n} \in A_n; \, g(C_t)\big) = P_m\big(\tilde{Y}_{t_1+s} \in A_1, \dots, \tilde{Y}_{t_n+s} \in A_n; \, g(C_{t+s})\big).$$

Let $\Lambda = \{w\colon \lim_{t \downarrow \alpha} B(w, (\alpha, t]) = 0\}$. The set Λ is σ_t invariant and belongs to $\mathscr{G}^m_{\alpha+}$ (where $\mathscr{G}^m_{\alpha+} = \{A \in \mathscr{G}^m\colon A \cap \{\alpha < t\} \in \mathscr{G}^m_t$ for all $t \in \mathbb{R}\}$). It was proved by Dynkin [1] that

$$\begin{aligned} m_1(f) &= Q_m(\Lambda; f(X_t)), \\ m_2(f) &= Q_m(\Lambda^c; f(X_t)) \end{aligned} \tag{2.29}$$

are (P_t) excessive measures and that $Q_m = Q_{m_1} + Q_{m_2}$.

For an excessive measure n, we denote by ν_B^n the characteristic measure of B relative to Q_n.

(2.30) **Theorem.** *$\nu_B^{m_i}$, $i = 1, 2$, are the $(\tilde{P}_t)$ purely excessive and invariant parts of ν_B^m, respectively.*

Proof. The fact that $\nu_B^{m_1}$ is purely excessive for $(\tilde{P}_t)$ follows directly from (2.3). We only need to show that $\nu_B^{m_2}$ is invariant, and then use the fact that the decomposition of an excessive measure into its invariant and purely excessive parts is unique.

Let $t \ge 0$ and $f \in \mathscr{E}_+$,

$$\int \nu_B^{m_2}(dx)\tilde{P}_t f(x) = P_{m_2}\big(1_{[0,1)}(C_u) 1_E(\tilde{Y}_u) f(\tilde{Y}_{t+u})\big), \tag{2.31}$$

for any $u \in \mathbb{R}$. But P_{m_2} a.e. $B_\alpha = -\infty$, and therefore P_{m_2} a.e. on $\{\tilde{Y}_{t+u} \in E\}$, $\tilde{Y}_u \in E$. Applying (2.24), the right-hand side of (2.31) is equal to

$$P_{m_2}\big(f(\tilde{Y}_{t+u}) 1_{[0,1)}(C_{t+u})\big) = \nu_B^{m_2}(f),$$

and the result follows. □

(2.32) **Remark.** If A is not a continuous additive functional (or equivalently if B is not diffuse), but the sizes of its jumps are functions of X at the time of the jump, one can replace the jumps by exponential random variables (as was done, for example, in [9]). The time-changed process by the inverse of this modified clock is still Markovian and our results carry to that case with almost no changes.

2.4. *The conservative case.*

(2.33) THEOREM. *If m is a conservative excessive measure, then $Q_m(0 < B(-\infty, t] < \infty) = 0$ for all $t \in \mathbb{R}$ and ν_B is invariant for $(\tilde{P}_t)$.*

The elegant proof that follows is due to the referee. It replaces a longer proof based on ergodic theory.

PROOF. If $Q_m(0 < B(-\infty, t] < \infty) > 0$ for some t, then the intrinsic time ([2]) $S = \inf\{t\colon B(-\infty, t) > \varepsilon\}$ satisfies $Q_m(S \in \mathbb{R}) > 0$ for $\varepsilon > 0$ sufficiently small. This, by (5.8)(ii) of [2], contradicts the fact that m is conservative. Let Λ be as defined in (2.29). Then it follows that $\Lambda = \{w\colon B(w, \mathbb{R}) = 0\}$ and so $\nu_B^{m_1} = 0$ and $\nu_B^{m_2} = \nu_B^m$ is invariant for $(\tilde{P}_t)$. □

3. An application to entrance boundary. In this section we apply our time change to a HRM of the form $B(dt) = g(Y_t)\,dt$, with g strictly positive and such that $B(\alpha, \beta) < \infty$. We are, therefore, in the framework of the simple case of 2.2. Our clock (B_t) is strictly increasing and continuous in (α, β). It enables us to study the behaviour of Y near α via the classical Martin boundary theory, using a Ray–Knight compactification. We shall use the Ray topology defined in [7] for transient processes. We sketch its details here.

3.1. *Ray–Knight compactification and the entrance space.* Let m be a (P_t) dissipative excessive measure. Let $l \in \mathscr{E}_{++}$ with $m(l) < \infty$ and $D = \{Ul < \infty\}$. The set D^c is finely open and $m(D^c) = 0$. Hence it is m-polar. The process X, restricted to D, is a transient Borel right process in the sense of [5]. We may, therefore, assume (without loss of generality) that $D = E$. It follows from [5] that there exists a $q \in \mathscr{E}$ satisfying $m(q) < \infty$, $0 < q < 1$ and $h = Uq \leq 1$. Set $Vf(x) = 1/h(x)Uq \cdot f(x)$. V is the 0 potential of a process Z obtained from X, first by an h-path transform using h and then by a time change by the right-continuous inverse of the additive functional $dA_t = q/h(X_t)\,dt$. We denote by (V^γ) the resolvent corresponding to this process, and by (Q_t) its semigroup. Since $V1_E = 1_E$, the semigroup $R_t = e^t Q_t$ satisfies $R_t 1_E = 1_E$. Denote by $(W^\gamma)_{\gamma \geq 0}$ its resolvent. Let $\mathbf{R}$ be the Ray cone generated by (W^γ), $\overline{E}$ the corresponding Ray–Knight compactification of E and $(\overline{W}^\gamma)$ the corresponding Ray resolvent. E is Borel in $\overline{E}$. Define $\overline{V}^\gamma = \overline{W}^{1+\gamma}$ and let $\overline{X} = (\overline{X}_t, \overline{Q}^x)$ be the Ray process on $\overline{E}$ with resolvent $(\overline{V}^\gamma)$. Denote its semigroup by $(\overline{Q}_t)$. For $x \in E$ and $f \in \mathscr{E}$, $V^\gamma f(x) = \overline{V}^\gamma f(x)$. Let B be the set of branch points of $\overline{X}$ and E^r the points regular for E in $\overline{E}$. Set $F = E^r \cap B^c$. The process $\overline{X}$ restricted to F is a Borel right process in the Ray topology. It was proved by Getoor and Glover [7] that for any dissipative n with $n(q) < \infty$, there exists a finite measure λ on

$(F, \bar{\mathscr{F}})$ ($\bar{\mathscr{F}}$ being the trace of $\bar{\mathscr{E}}$ on F) so that

$$n(f) = \int_F \lambda(dx)\bar{V}\frac{f}{q}(x). \tag{3.1}$$

The following simple observation and (3.1) identify F as the Martin entrance space.

(3.2) PROPOSITION. *Let $m_x(f) = \bar{V}f/q(x)$. Then for $x \in F$, $m_x(f)$ is a minimal excessive measure.*

PROOF. By its construction, $m_x(f)$ is (P_t) excessive and satisfies $m_x(q) = 1$. Suppose $m_x = m_1 + m_2$ where both m_1, m_2 are excessive. Since $m_x(q) < \infty$ both $m_1(q)$ and $m_2(q)$ are finite. It follows that there exist finite measures λ_1, λ_2 so that

$$m_i(f) = \int_F \lambda_i(dx)\bar{V}\frac{f}{q}(x), \qquad i = 1,2.$$

Let $\lambda = \lambda_1 + \lambda_2$. Then for every $f \in \mathscr{E}_+$, $\bar{V}f(x) = \int_F \lambda(dx)\bar{V}f(x)$. Since E is absorbing for $\bar{X}$ and $F \subset E^r$, it follows that the same equality holds for $f \in \bar{\mathscr{F}}$. Since $\bar{X}$ restricted to F is a Borel right process, this is possible only if $\lambda(\cdot) = \varepsilon_x(\cdot)$, where ε_x is the Dirac measure with mass at x, and our assertion is proved. □

3.2. *The time change.* As we have seen, at the base of the definition of the Ray topology, there is a (classical) time change. We shall now perform a similar time change on W.

(3.3) LEMMA. *With q and h as before, the HRM $B(dt) = q/h(Y_t)\,dt$ satisfies $B(\alpha, \beta) < \infty$ Q_m^h a.e.*

PROOF. The proof is a simple computation. For m invariant, it is easy to show that for any $u \in \mathbb{R}$,

$$Q_m^h\left(\int_{-\infty}^{\beta} \frac{q}{h}(Y_s)\,ds\frac{q}{h}(Y_u)\right) = 2m(q) < \infty.$$

This implies that Q_m^h a.e. on $\{\beta > u\}$, $B(\alpha, \beta) < \infty$. Hence

$$Q_m^h\left(\bigcup_{r \in Q} \{\beta > r,\ B(\alpha, \beta) = \infty\}\right) = 0,$$

from which the result follows immediately. If m is purely excessive and (μ_t) is its corresponding entrance law (at 0), then

$$Q_\mu^h\left(\int_0^\infty \frac{q}{h}(Y_t)\,dt\right) = m(q),$$

and again the result is immediate. □

Let (η_t) be the entrance law defined in (2.3), this time relative to Q_m^h and B.

(3.4) THEOREM. *Let* $m = \int_F \lambda(dx) m_x$ *[with* m_x *and* λ *as in* (3.1)]. *Then*

$$\eta_t(f) = \int_F \lambda(dx) \overline{Q}^x(f(\overline{X}_t)).$$

PROOF. The measure $m \cdot q$ is the characteristic measure of B relative to Q_m^h. Hence for $f \in \mathscr{E}_+$,

$$m \cdot q(f) = \int_0^\infty \eta_t(f)\, dt.$$

By our assumptions, it is also equal to

$$\int \lambda(dx) \overline{V} f(x) = \int_0^\infty \left[\int_F \lambda(dx) \overline{Q}^x(f(X_t)) \right] dt.$$

Since both $\eta_t(f)$ and $\int \lambda(dx) \overline{Q}^x(f(\overline{X}_t))$ are entrance laws for the right semigroup $(\tilde{P}_t^h)$ [with $S_t = (A^{-1})_t$, A defined in (3.1)], and they integrate to the same $(\tilde{P}_t^h)$ excessive measure, they must be equal. □

Denote by Ω_+^0 the space of all functions from $(0, \infty)$ into $E \cup \Delta$ that are right-continuous in the original and the Ray topologies. Arguments similar to those used in 11.8 of [4] will prove that Ω_+^0 has full $\tilde{Q}_\eta^h$ measure, and that for $x \in F$, Ω_+^0 has $\overline{Q}^x$ outer measure 1. Theorem (3.4) implies that, when restricted to the events in the natural σ-algebra on Ω_+^0, $Q_\eta^h = \int_F \lambda(dx) \overline{Q}^x$.

3.3. *Applications*. Many results that deal with the behaviour of Y near α are now a simple consequence of the previous discussion. We collect some examples.

(i) If $m(f) < \infty$ and $m(g) < \infty$, then Q_m a.e.

$$Z_t \equiv \frac{Uf(Y_t)}{Ug(Y_t)} \text{ converges as } t \downarrow \alpha \quad \text{(Theorem 7.2 of Dynkin [1])}.$$

We note that

$$Z_t = \frac{Vf/q(Y_t)}{Vg/q(Y_t)},$$

and since for all $l \in \mathscr{E}$ with $m(l) < \infty$, $x \to Vl(x)$ is excessive for $(\overline{Q}_t)$, Z_t converges $\tilde{Q}_\eta^h$ a.e. as $t \downarrow 0$. By the time-change result, the same is true Q_m^h a.e. as $t \downarrow \alpha$. Since $h \in \mathscr{E}_{++}$, the same is true Q_m a.e. (by 5.4 of [1]).

If $m = m_x$ is minimal excessive, then $\tilde{Q}_\eta^h = c(x) \overline{Q}^x$, and Z_t converges to

$$\frac{Vf/q(x)}{Vg/q(x)} = \frac{m_x(f)}{m_x(g)}, \quad \text{as } t \to \alpha,$$

again a result proved in [1], using a different technique. One may also start from Theorem 7.2 of [1], use a time change and obtain the Getoor–Glover representation (3.1).

(ii) The Riesz decomposition of m. m is a potential if, and only if, $\lambda(F - E) = 0$ [7]. Translated to the behaviour near α, m is a potential if, and only if, $\rho - \lim_{t \downarrow \alpha} Y_t \in E$ for a.e. w, where $\rho - \lim$ is the limit in the Ray distance. This condition is, a fortiori, equivalent to the Fitzsimmons and Maisonneuve [2] condition $w \in \Omega_q$ for a.e. w (page 323 of [2]) (a direct proof that the two sets of conditions are equivalent is not too difficult).

(iii) The balayage of m on B, denoted $R_B m$, was defined in [2] by

$$R_B m(f) = Q_m(T_B \le t, f(Y_t)),$$

where $T_B = \inf\{t \in \mathbb{R}: Y_t \in B\}$. It is an excessive measure. It is equal to m if, and only if, $T_B = \alpha$, Q_m a.e. This condition holds if, and only if, λ is concentrated on B^r—the points regular for B with respect to $\overline{X}$.

(iv) The minimal excessive measures $(m_x)_{x \in F}$ are either invariant or purely excessive. The following uses (2.30) to identify them.

(3.5) THEOREM. *The excessive measure $m = m_x$ is purely excessive if, and only if, $\overline{Q}^x$ a.s. $\lim_{t \to 0} \int_0^t h/q(\overline{X}_s)\, ds = 0$, otherwise it is invariant.*

PROOF. We note that $m \cdot h$ is purely excessive (invariant) for (P_t^h) if, and only if, m is purely excessive (invariant) for (P_t). (W, Q_m^h) is obtained from $(W, \tilde{Q}^h_{m \cdot q})$ by the inverse time change via $C(dt) = h/q(Y_t)\, dt$. $m \cdot h$ will therefore be purely excessive if, and only if, $\lim_{t \downarrow \alpha} C(\alpha, t] = 0$, $\tilde{Q}^h_{m \cdot q}$ a.e. This happens if, and only if, $\lim_{t \downarrow 0} C(0, t] = 0$ $\tilde{Q}^h_\eta$ a.e. By (3.4) this is equivalent to $\tilde{Q}^x(\lim_{t \downarrow 0} \int_0^t h/q(\overline{X}_s)\, ds = 0) = 1$. □

Let $A = \{\lim_{t \downarrow 0} \int_0^t h/q(\overline{X}_s)\, ds = 0\}$. For any $x \in F$, $\overline{Q}^x(A) = 0$ or 1. Let

$$F_I = \{x: \overline{Q}^x(A) = 0\}, \qquad F_p = F_I^c. \tag{3.6}$$

Then $m_I = \int_{F_I} \lambda(dx) m_x$ and $m_p = \int_{F_p} \lambda(dx) m_x$ are the invariant and purely excessive parts of m, respectively.

(v) In [8] Getoor and Steffens define a set $B \in \mathscr{E}$ to be m-cotransient if Q_m a.e. $T_B > -\infty$. For $m = \int_F \lambda(dx) m_x$, B is m-cotransient if, and only if, $\lambda(B^r \cap F_I) = 0$.

One may obtain expressions for the co-capacities and co-capacitary measures defined in [8] in terms of the measure λ and $(\overline{Q}^x)_{x \in F}$. The results are what one would expect them to be. Since we do not attempt to expand on capacity theory in this paper, we leave such computations to the interested reader.

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