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LYAPUNOV EXPONENTS FOR MATRICES WITH INVARIANT SUBSPACES

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If $\mathbf{M}$ is a fixed $d \times d$ complex-valued matrix, then the eigenvalues of $\mathbf{M}'$, the conjugate transpose of $\mathbf{M}$, are the complex conjugates of the eigenvalues of $\mathbf{M}$, with the same multiplicities, and if $\mathbf{M}$ is upper block triangular, the eigenvalues of $\mathbf{M}$ are the eigenvalues of the diagonal blocks, and the multiplicities add. We shall show that if $\{\mathbf{M}(j)\}$ is a stationary, ergodic, time-reversible sequence taking values in the $d \times d$ complex matrices, then similar properties hold for the Lyapunov exponents of $\{\mathbf{M}(j)\}$.

1. Introduction. Suppose that $\{\mathbf{M}(j)\}$ is a stationary, ergodic sequence taking values in the $d \times d$ complex matrices. According to Oseledec's multiplicative ergodic theorem, if $E(\log^+\|\mathbf{M}\|) < \infty$, then there are r constants, $r \le d$,

$$-\infty \le l(1) < l(2) < \cdots < l(r),$$

called the Lyapunov exponents of $\{\mathbf{M}(j)\}$, such that with probability 1, for each $\mathbf{v}$ in $C^d/\{\mathbf{0}\}$,

$$\lim_{n \to \infty} n^{-1}\log\|\mathbf{M}(n) \cdots \mathbf{M}(1)\mathbf{v}\| = l(j) \tag{1}$$

and

$$V = \left\{\mathbf{v}\colon \lim_{n \to \infty} n^{-1}\log\|\mathbf{M}(n) \cdots \mathbf{M}(1)\mathbf{v}\| \le l(j)\right\} \tag{2}$$

is a random subspace of C^d with nonrandom dimension. If V_0 is defined to be $\{\mathbf{0}\}$, then $m(j) = \dim V_j - \dim V_{j-1}$ is a nonrandom constant called the multiplicity of the Lyapunov exponent $l(j)$. The information about the Lyapunov exponents and their multiplicities can be summarized as a function

$$m(\mathbf{M}, *)\colon [-\infty, \infty] \to \{0, 1, \ldots, d\},$$

where

$$m(\mathbf{M}, y) = \begin{cases} 0, & \text{if } y \text{ is not a Lyapunov exponent of } \{\mathbf{M}(j)\}, \\ m(j), & \text{if } y \text{ is the Lyapunov exponent } l(j). \end{cases} \tag{3}$$

From (1), (2) and the definition of multiplicity, it is clear that for any x in $[-\infty, \infty]$,

$$\dim\left\{\mathbf{v}\colon \lim_{n \to \infty} n^{-1}\log\|\mathbf{M}(n) \cdots \mathbf{M}(1)\mathbf{v}\| \le x\right\} = \sum_{y \le x} m(\mathbf{M}, y). \tag{4}$$

Let W be a nontrivial proper subspace of C^d with dimension d'. If

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$P(\mathbf{M}(1)W \subset W) = 1$, then W is almost surely invariant for $\{\mathbf{M}(j)\}$, and without loss of generality we may assume that each $\mathbf{M}(j)$ is of the form

$$\begin{bmatrix} \mathbf{A}(j) & \mathbf{B}(j) \\ \mathbf{0} & \mathbf{C}(j) \end{bmatrix}, \tag{5}$$

where $\mathbf{A}(j)$ is $d' \times d'$, $\mathbf{B}(j)$ is $d' \times d''$, $\mathbf{0}$ is the $d' \times d''$ zero matrix, $\mathbf{C}(j)$ is $d'' \times d''$ and $d'' = d - d'$. In this case we shall say that $\{\mathbf{M}(j)\}$ is upper block triangular.

If $\{\mathbf{M}(j)\}$ is a constant sequence, then the Lyapunov exponents are the logarithms of the absolute values of the eigenvalues of $\mathbf{M}(1)$, and the following facts are well known:

(6) If $\mathbf{M}'$ denotes the conjugate transpose of $\mathbf{M}$, then $m(\mathbf{M}', y) = m(\mathbf{M}, y)$.

(7) If $\mathbf{M}(1)$ has the form (5), then $m(\mathbf{M}, y) = m(\mathbf{A}, y) + m(\mathbf{C}, y)$.

(8) If $\mathbf{M}(1)$ is invertible, then $m(\mathbf{M}^{-1}, y)) = m(\mathbf{M}, -y)$.

(8) has been established in general for Lyapunov exponents [see Walters (1982), pages 230–235]. Furstenberg and Kifer (1983) remark that if $\{\mathbf{M}(j)\}$ is an iid sequence, then the largest Lyapunov exponent of $\{\mathbf{M}(j)\}$ equals the largest Lyapunov exponent of $\{\mathbf{M}'(j)\}$. Hennion (1984) has shown that if $P(\mathbf{M}(1)$ is invertible$) = 1$ and $E(\log^+\|\mathbf{M}^{-1}(1)\|) < \infty$, then $m(\mathbf{M}, y) > 0$ iff $m(\mathbf{A}, y) + m(\mathbf{C}, y) > 0$.

This paper will establish (6) and (7) assuming that $\{\mathbf{M}(j)\}$ is stationary, ergodic and time-reversible (Theorems 1 and 5, respectively). This differs from the work of Hennion and Furstenberg and Kifer who assume that $\{\mathbf{M}(j)\}$ takes values in $GL(R, d)$, and either assume that the sequence is iid (Furstenberg and Kifer) or stationary and ergodic (Hennion).

The results will be used in subsequent papers to compute limit laws for random walk in a random environment, generalizing the results of Kesten, Kozlov and Spitzer (1975) and Key (1983), and to examine the connection between the largest Lyapunov exponent and relative entropy in a simple dynamical system.

This investigation was motivated by Pincus (1985), where the largest Lyapunov exponent was computed in the case where the matrices were almost surely upper triangular, and Key (1984), where all the Lyapunov exponents were calculated in the case where there were $(d-1)$-dimensional invariant subspaces.

2. Notational conventions. Throughout, vectors and matrices will be boldface.

$\mathbf{X}'$ denotes the conjugate transpose of $\mathbf{X}$.
$\|\mathbf{X}\|$ refers to any matrix norm that is equivalent to $[\text{trace}(\mathbf{X}\mathbf{X}')]^{1/2}$.
$\text{Mat}(C, d)$ is the set of $d \times d$ complex-valued matrices.
$\mathbf{e}(j)$, $j = 1, 2, \ldots, d$, will denote the standard basis of C^d.

If $\{\mathbf{X}(j)\}$ is a stationary, ergodic sequence taking values in Mat(C, d), we shall refer to its Lyapunov exponents and their multiplicities as being those of $\mathbf{X}$.

3. The Lyapunov exponents of M and M′.

THEOREM 1. *Suppose that* $\{\mathbf{M}(j)\}$ *is a stationary, ergodic, time-reversible sequence taking values in* Mat(C, d). *If* $E(\log^+\|\mathbf{M}(1)\|) < \infty$, *then* $m(\mathbf{M}, *) = m(\mathbf{M}', *)$.

PROOF. Let $V^{(k)}$ be the kth exterior power of C^d. Let $M^{(k)}$ be the action of M on $V^{(k)}$. Raghunathan (1979) and Ledrappier (1984) have shown that the Lyapunov exponents of $\mathbf{M}$ and their multiplicities are determined by the values

$$(9)\qquad L(k) = \lim_{n\to\infty} n^{-1}E\Big(\log\big\|\mathbf{M}(n)^{(k)}\cdots\mathbf{M}(1)^{(k)}\big\|\Big), \qquad k = 1,\dots,d.$$

It is straightforward to show that $(\mathbf{M}')^{(k)} = (\mathbf{M}^{(k)})'$. Taking $\|\mathbf{M}\| = [\mathrm{tr}(\mathbf{M}\mathbf{M}')]^{1/2}$, we have

$$(10)\qquad \begin{aligned} E\Big(\log\big\|(\mathbf{M}(n)')^{(k)}\cdots(\mathbf{M}(1)')^{(k)}\big\|\Big) &= E\Big(\log\big\|\big(\mathbf{M}(n)^{(k)}\big)'\cdots\big(\mathbf{M}(1)^{(k)}\big)'\big\|\Big)\\ &= E\Big(\log\big\|\big(\mathbf{M}(1)^{(k)}\cdots\mathbf{M}(n)^{(k)}\big)'\big\|\Big)\\ &= E\Big(\log\big\|\mathbf{M}(1)^{(k)}\cdots\mathbf{M}(n)^{(k)}\big\|\Big)\\ &= E\Big(\log\big\|\mathbf{M}(n)^{(k)}\cdots\mathbf{M}(1)^{(k)}\big\|\Big). \end{aligned}$$

It follows from (10) that if each $\mathbf{M}(j)^{(k)}$ is replaced by $(\mathbf{M}(j)')^{(k)}$ in (9), the same values for $L(k)$ are obtained, so $m(\mathbf{M}, *) = m(\mathbf{M}', *)$. □

The following example shows that the hypothesis of reversibility cannot be dropped.

EXAMPLE. Let

$$\mathbf{Q} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad\text{and}\quad \mathbf{S} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

and let $\{\mathbf{M}(j)\}$ be the stationary ergodic Markov chain with state space $\{\mathbf{Q}, \mathbf{R}, \mathbf{S}\}$ and transition probabilities

$$\begin{aligned} P(\mathbf{M}(j+1) = \mathbf{R}\,|\,\mathbf{M}(j) = \mathbf{Q}) &= 1,\\ P(\mathbf{M}(j+1) = \mathbf{S}\,|\,\mathbf{M}(j) = \mathbf{R}) &= 1,\\ P(\mathbf{M}(j+1) = \mathbf{Q}\,|\,\mathbf{M}(j) = \mathbf{S}) &= 1/2,\\ P(\mathbf{M}(j+1) = \mathbf{S}\,|\,\mathbf{M}(j) = \mathbf{S}) &= 1/2. \end{aligned}$$

Note that $\mathbf{SRQ} = \mathbf{0}$, $\mathbf{SRQ}' = \mathbf{S}$, $\mathbf{Q}'\mathbf{S} = \mathbf{S}$, and $\mathbf{S}^2 = \mathbf{S}$.

Since a visit to $\mathbf{R}$ always follows a visit to $\mathbf{Q}$ and a visit to $\mathbf{S}$ always follows a visit to $\mathbf{R}$, $\|\mathbf{M}(n)\cdots\mathbf{M}(1)\| = 0$ for some n almost surely, so both of the

Lyapunov exponents of $\mathbf{M}$ are $-\infty$, but $1 \le \|\mathbf{M}'(n) \cdots \mathbf{M}'(1)\| \le 2$, so one of the Lyapunov exponents of $\mathbf{M}'$ is 0 and the other is $-\infty$.

4. $m(\mathbf{M}, *) = m(\mathbf{A}, *) + m(\mathbf{C}, *)$ for $\{\mathbf{M}(j)\}$ upper block triangular.

Let d, d' and d'' be positive integers, with $d = d' + d''$, and suppose that $\{\mathbf{M}(j)\}$ is a stationary, ergodic, time-reversible sequence taking values in $\{\mathbf{U} \in \mathrm{Mat}(C, d)$: $\mathbf{U}_{ij} = 0$ if $d' < i \le d, 1 \le j \le d'\}$. Each $\mathbf{M}(j)$ is upper block triangular, and we write

$$\mathbf{M}(j) = \begin{bmatrix} \mathbf{A}(j) & \mathbf{B}(j) \\ \mathbf{0} & \mathbf{C}(j) \end{bmatrix}.$$

LEMMA 2. *If* $E(\log^+\|\mathbf{M}(1)\|) < \infty$, *then* $E(\log^+\|\mathbf{A}(1)\|) < \infty$, $E(\log^+\|\mathbf{B}(1)\|) < \infty$, $E(\log^+\|\mathbf{C}(1)\|) < \infty$, *and for each* x *in* $[-\infty, \infty]$,

$$\sum_{y \le x} m(\mathbf{M}, y) \le \sum_{y \le x} m(\mathbf{A}, y) + \sum_{y \le x} m(\mathbf{C}, y). \tag{11}$$

PROOF. First, note that

$$\mathrm{trace}(\mathbf{M}\mathbf{M}') = \mathrm{trace}(\mathbf{A}\mathbf{A}') + \mathrm{trace}(\mathbf{B}\mathbf{B}') + \mathrm{trace}(\mathbf{C}\mathbf{C}').$$

Since $\log^+$ is nondecreasing, it follows that if $E(\log^+\|\mathbf{M}(1)\|) < \infty$, then $E(\log^+\|\mathbf{A}(1)\|) < \infty$, $E(\log^+\|\mathbf{B}(1)\|) < \infty$ and $E(\log^+\|\mathbf{C}(1)\|) < \infty$. Therefore, $m(\mathbf{A}, *)$ and $m(\mathbf{C}, *)$ are well defined.

In order to prove (11), we make use of Oseledec's multiplicative ergodic theorem in the following way. According to this theorem, certain limits and the dimensions of certain subspaces exist and are almost surely constant. We have made use of this in defining $m(\mathbf{M}, *)$, $m(\mathbf{A}, *)$ and $m(\mathbf{C}, *)$. Throughout the proof it is to be understood, then, that choices of vectors and subspaces, and the assertions about the convergence of limits can only be made on a set whose probability is 1, but that is sufficient. For example, if we say, "There exists a vector $\mathbf{v}$ in C^d so that

$$\lim_{n\to\infty} n^{-1}\log\|\mathbf{M}(n) \cdots \mathbf{M}(1)\mathbf{v}\| = x,"$$

we mean that for almost every w there is a vector $\mathbf{v}(w)$ in C^d so that

$$\lim_{n\to\infty} n^{-1}\log\|\mathbf{M}(n, w) \cdots \mathbf{M}(1, w)\mathbf{v}(w)\| = x.$$

Let $a = \min\{y$ in $[-\infty, \infty]$: $\max\{m(\mathbf{A}, y), m(\mathbf{C}, y)\} > 0\}$. Suppose that $m(\mathbf{M}, y) > 0$. Then for some vector $\mathbf{v}$ in $C^d/\{\mathbf{0}\}$,

$$\lim_{n\to\infty} n^{-1}\log\|\mathbf{M}(n) \cdots \mathbf{M}(1)\mathbf{v}\| = y.$$

Define $\mathbf{v}''$ in $C^{d''}$ by $(\mathbf{v}'')_i = (\mathbf{v}'')_{i+d''}$. If $\mathbf{v}'' \ne \mathbf{0}$, then

$$\lim_{n\to\infty} n^{-1}\log\|\mathbf{M}(n) \cdots \mathbf{M}(1)\mathbf{v}\| \ge \lim_{n\to\infty} n^{-1}\log\|\mathbf{C}(n) \cdots \mathbf{C}(1)\mathbf{v}''\|$$
$$\ge a,$$

so $y \ge a$.

If $\mathbf{v}'' = \mathbf{0}$, define $\mathbf{v}'$ in $C^{d'}$ by $(\mathbf{v}')_i = (\mathbf{v})_i$. Since $\mathbf{v} \neq 0$, we have $\mathbf{v}' \neq \mathbf{0}$ and

$$\lim_{n\to\infty} n^{-1}\log\|\mathbf{M}(n)\cdots\mathbf{M}(1)\mathbf{v}\| = \lim_{n\to\infty} n^{-1}\log\|\mathbf{A}(n)\cdots\mathbf{A}(1)\mathbf{v}'\| \geq a,$$

so $y \geq a$. Therefore (11) holds if $x < a$, as both sides of the inequality are 0.

Now suppose that $x \geq a$. Let

$$g = \sum_{y\leq x} m(\mathbf{M}, y), \qquad h = \sum_{y\leq x} m(\mathbf{A}, y) \quad \text{and} \quad k = \sum_{y\leq x} m(\mathbf{C}, y),$$

and suppose that $h \geq k$. Since $x \geq a$ we have $g > 0$ and $h > 0$. Let

$$W = \left\{\mathbf{v} \in C^d \colon \lim_{n\to\infty} n^{-1}\log\|\mathbf{M}(n)\cdots\mathbf{M}(1)\mathbf{v}\| \leq x\right\}.$$

$\dim W = g$. Let $V = W \cap \{\mathbf{v} \in C\colon (\mathbf{v})_i = 0 \text{ for } i > d'\}$.

From the upper block triangular form of the $\{\mathbf{M}(j)\}$ it follows that $\dim V = h$. We shall now proceed by contradiction. Suppose that $g > h + k$. Then we may choose a basis for W by first choosing a basis for V, say $\{\mathbf{v}(j)\}$, and then choosing another $(g - h)$ vectors, $\{\mathbf{w}(j)\}$, to complete the basis of W.

Next, let $\mathbf{u}''(j)$ be the vector in $C^{d''}$ defined by the last d'' entries of $\mathbf{w}(j)$. $\{\mathbf{u}''(j)\}$ is a linearly independent set since $\{\mathbf{v}(j)\} \cup \{\mathbf{w}(j)\}$ is a linearly independent set and the last d'' coordinates of each $\mathbf{v}(j)$ are 0.

On the other hand,

$$\{\mathbf{u}''(j)\} \subset \left\{\mathbf{u}'' \text{ in } C^{d''}\colon \lim_{n\to\infty} n^{-1}\log\|\mathbf{C}(n)\cdots\mathbf{C}(1)\mathbf{u}''\| \leq x\right\},$$

a subspace of dimension $k < (g - h)$, so $\{\mathbf{u}''(j)\}$ is a linearly dependent set, which is a contradiction.

If $h < k$, repeat the argument for $\mathbf{M}'$, $\mathbf{A}'$ and $\mathbf{C}'$, and use Theorem 1. □

Lemma 3. *The largest Lyapunov exponent of* $\mathbf{M}$ *is the maximum of the largest Lyapunov exponent of* $\mathbf{A}$ *and the largest Lyapunov exponent of* $\mathbf{C}$.

Proof. If $\{\mathbf{M}(j)\}$ is an iid sequence, then this is Lemma 3.6 of Furstenberg and Kifer (1983). The same proof, using Theorem 1 where necessary, shows that the result remains true in the stationary, ergodic, time-reversible case. □

Lemma 3 generalizes as follows.

Lemma 4. *Let* $1 \leq j \leq \min\{k, d'\}$ *and* $1 \leq k \leq d$. *The sum of the k largest Lyapunov exponents of* $\mathbf{M}$, *including multiplicities, is equal to*

max{*sum of the k largest Lyapunov exponents of* $\mathbf{M}$, *including multiplicities, of which at least j are Lyapunov exponents of* $\mathbf{A}$, *sum of the k largest Lyapunov exponents of* $\mathbf{M}$, *including multiplicities, of which at least* $k - j + 1$ *are Lyapunov exponents of* $\mathbf{C}$}.

PROOF. Let $V(j, k)$ be the subspace of $V^{(k)}$, the kth exterior product of C^d, spanned by $\mathbf{v}(1) \wedge \cdots \wedge \mathbf{v}(k)$ such that at least j of the $\mathbf{v}(i)$ are in $\{\mathbf{e}(i), i \le d'\}$. Let $V'(j, k)$ be the subspace of the kth exterior product of C^d spanned by $\mathbf{v}(1) \wedge \cdots \wedge \mathbf{v}(k)$ such that at least $k - j + 1$ of the $\mathbf{v}(i)$ are in $\{\mathbf{e}(i), d' < i \le d\}$. Then $V^{(k)} = V(j, k) \oplus V'(j, k)$, $V(j, k)$ is $\mathbf{M}^{(k)}$ invariant and $V'(j,k)$ is $\mathbf{M}^{(k)'}$ invariant.

By Lemma 3, the largest Lyapunov exponent of $\mathbf{M}^{(k)}$ is equal to the maximum of the largest Lyapunov exponent of $\mathbf{M}^{(k)'}$ restricted to $V(j, k)$ and the largest Lyapunov exponent of $\mathbf{M}^{(k)'}$ restricted to $V'(j, k)$. By the definitions of $V(j, k)$, $V'(j, k)$ and the action of $\mathbf{M}^{(k)}$ on $V^{(k)}$, the largest Lyapunov exponent of the restriction of $\mathbf{M}^{(k)}$ to $V(j, k)$ is the sum of the k largest Lyapunov exponents of $\mathbf{M}$, of which at least j are Lyapunov exponents of $\mathbf{A}$. Similarly, the largest Lyapunov exponent of the restriction of $\mathbf{M}^{(k)'}$ to $V'(j, k)$ is the sum of the k largest Lyapunov exponents of $\mathbf{M}'$, of which at least $k - j + 1$ are Lyapunov exponents of $\mathbf{C}'$. The lemma now follows from Theorem 1. □

THEOREM 5. *Let* $\{\mathbf{M}(j)\}$ *be a stationary, ergodic, time-reversible sequence taking values in* $\mathrm{Mat}(C, d)$. *If*

$$\mathbf{M}(j) = \begin{bmatrix} \mathbf{A}(j) & \mathbf{B}(j) \\ \mathbf{0} & \mathbf{C}(j) \end{bmatrix}$$

and $E(\log^+\|\mathbf{M}(1)\|) < \infty$, *then* $m(\mathbf{M}, *) = m(\mathbf{A}, *) + m(\mathbf{C}, *)$.

PROOF. It is sufficient to show that for all y in $[-\infty, \infty]$,

$$0 \le m(\mathbf{M}, Y) \le m(\mathbf{A}, y) + m(\mathbf{C}, y),$$

for then

$$\begin{aligned} 0 &\le \sum_y [m(\mathbf{A}, y) + m(\mathbf{C}, y) - m(\mathbf{M}, y)] \\ &= d - d \\ &= 0, \end{aligned} \tag{12}$$

which proves the theorem.

Consider $\{y\colon m(\mathbf{A}, y) + m(\mathbf{C}, y) - m(\mathbf{M}, y) < 0\}$. This set is bounded above, and by Lemma 2, this set does not contain $-\infty$. We need to prove that it is empty. Suppose not. Let b be its largest element. By Lemma 2, we must have

$$\sum_{y \le b} m(\mathbf{M}, y) \le \sum_{y \le b} m(\mathbf{A}, y) + \sum_{y \le b} m(\mathbf{C}, y),$$

which, along with the definition of b, implies

$$m(\mathbf{M}, y) = m(\mathbf{A}, y) + m(\mathbf{C}, y), \quad \text{for } y > b. \tag{13}$$

Write

$$m(\mathbf{M}, b) = m(\mathbf{A}, b) + m(\mathbf{C}, b) + p, \tag{14}$$

where p is a positive integer.

Lemma 4 with $k = \Sigma_{y \geq b} m(\mathbf{M}, y)$, $j = 1 + \Sigma_{y \geq b} m(\mathbf{A}, y)$ and $k - j + 1 = p + \Sigma_{y \geq b} m(\mathbf{C}, y)$, implies

$$\begin{aligned}\sum_{y \geq b} ym(\mathbf{M}, y) = \max\{&\text{sum of } j \text{ Lyapunov exponents of } \mathbf{A} \\ &+\text{sum of } (k - j) \text{ other Lyapunov exponents of } \mathbf{M}, \\ &\text{sum of } k - j + 1 \text{ Lyapunov exponents of } \mathbf{C} \\ &+\text{sum of } (j - 1) \text{ other Lyapunov exponents of } \mathbf{M}\}.\end{aligned} \tag{15}$$

However, it follows from (13) and (14) that

$$\begin{aligned}&\text{sum of } j \text{ Lyapunov exponents of } \mathbf{A} \\ &\quad +\text{sum of } (k - j) \text{ other Lyapunov exponents of } \mathbf{M} \\ &\qquad < b + \sum_{y \geq b} ym(\mathbf{A}, y) \\ &\qquad\quad +\text{sum of } (k - j) \text{ other Lyapunov exponents of } \mathbf{M} \\ &\qquad \leq b + \sum_{y \geq b} ym(\mathbf{A}, y) + (p - 1)b + \sum_{y \geq b} ym(\mathbf{C}, y) \\ &\qquad = \sum_{y \geq b} ym(\mathbf{M}, y).\end{aligned}$$

Similarly,

$$\begin{aligned}&\text{sum of } k - j + 1 \text{ Lyapunov exponents of } \mathbf{C} \\ &\quad +\text{sum of } (j - 1) \text{ other Lyapunov exponents of } \mathbf{M} \\ &\qquad < \sum_{y \geq b} ym(\mathbf{M}, y).\end{aligned}$$

This contradicts (15), which completes the proof of Theorem 5. □

5. A concluding observation and question. $m(\mathbf{M}, *)/d$ defines a probability measure on $[-\infty, \infty]$. If $\{\mathbf{M}(j)\}$ takes values on a compact subset of $GL(C, d)$, all the moments of this measure are finite. The mean of this measure is $E(\log|\det(\mathbf{M}(1))|)/d$. Can we find other useful relationships between this measure and the stationary distribution of $\{\mathbf{M}(j)\}$?

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