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ON THE AVERAGE NUMBER OF LEVEL CROSSINGS OF A RANDOM TRIGONOMETRIC POLYNOMIAL

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There are many known asymptotic estimates of the number of zeros of the polynomial $T(\theta) = g_1 \cos\theta + g_2 \cos 2\theta + \cdots + g_n \cos n\theta$ for $n \to \infty$, where g_i $(i = 1, 2, \dots, n)$ is a sequence of independent normally distributed random variables with mathematical expectation 0 and variance 1. The present paper provides an estimate of the expected number of times that such a polynomial assumes the real value K. It is shown that the results for $K = 0$ are valid when $K = o(\sqrt{n})$.

1. Introduction.

Let

$$T(\theta) \equiv T_n(\theta, \omega) = \sum_{i=1}^{n} g_i(\omega) \cos i\theta, \tag{1.1}$$

where $g_1(\omega), g_2(\omega), \dots, g_n(\omega)$ is a sequence of independent random variables defined on a probability space $(\Omega, \mathscr{A}, P)$, each normally distributed with mathematical expectation 0 and variance 1. Let $N_{n,K}(\alpha, \beta) \equiv N(\alpha, \beta)$ be the number of real roots of the equation $T(\theta) = K$ in the interval $\alpha \le \theta \le \beta$, where multiple roots are counted once only. We know from the work of Dunnage [2] that in the case of $K = 0$, in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of functions $T(\theta)$ have $(2n)/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}$ zeros, when n is large. The measure of his exceptional set does not exceed $(\log n)^{-1}$. In the case of $E[g_i] \ne 0$, [4] and [8] show that the number of real roots remains the same. This indicates a different behaviour of the trigonometric equation $T(\theta) = 0$ from the algebraic equation $Q(x) = \sum_{i=1}^{n} g_i x^i = 0$ for which [5] proved that having coefficients with nonzero means instead of zero, reduces the number of real roots by half. These works on the random polynomial have been reviewed in the recent book of Bharucha-Reid and Sambandham [1], which constitutes the most complete reference.

In this paper, for the case of the coefficients of (1.1) being independent, standard normal random variables, we prove the following theorem:

Theorem. *For any sequence of constants K_n such that (K^2/n) tends to zero as n tends to infinity, the mathematical expectation of the number of real roots of the equation $T(\theta) = K$ satisfies*

$$EN(0, 2\pi) \sim (2/\sqrt{3})n.$$

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In [3] it is shown that for the algebraic polynomial $Q(x)$ the expected number of K-level crossings satisfies

$$EN(-1,1) \sim (1/\pi)\log(n/K^2),$$
$$EN(-\infty,-1) \sim EN(1,\infty) \sim (2\pi)^{-1}\log n.$$

Comparing this with our theorem shows another difference in the behaviour of the trigonometric polynomial from the algebraic one. That is, the number of crossings of the algebraic polynomial with the level K decreases as K increases, while for the trigonometric case this remains fixed, with probability 1, as long as $(K^2/n) \to 0$ as $n \to \infty$.

2. Extension of the Kac–Rice formula. From [6, page 52] (see also [1, page 95]) we see that the mathematical expectation of the number of real roots of the equation $T(\theta) = K$ in the interval (α, β) satisfies

$$EN(\alpha,\beta) = \int_\alpha^\beta d\theta \int_{-\infty}^{\infty} |y|\Phi(K,y)\,dy, \tag{2.1}$$

where $\Phi(x,y)$ is the density of the joint distribution of $T(\theta)$ and its derivative $T'(\theta)$. Let

$$A^2 = \sum_{i=1}^n \cos^2(i\theta), \qquad B^2 = \sum_{i=1}^n i^2\sin^2(i\theta),$$
$$C = -\sum_{i=1}^n i\sin(i\theta)\cos(i\theta)$$

and

$$\Delta^2 = A^2B^2 - C^2.$$

Then the joint density of (T, T') is

$$\Phi(x,y) = (2\pi\Delta)^{-1}\exp\{-(B^2x^2 - 2Cxy + A^2y^2)/(2\Delta^2)\}. \tag{2.2}$$

Now let $t = Ay/\Delta\sqrt{2}$. From (2.2) we have

$$\begin{aligned}\int_{-\infty}^{\infty} |y|\Phi(K,y)\,dy &= (\Delta/\pi A^2)\exp(-B^2K^2/2\Delta^2)\\ &\quad\times\int_{-\infty}^{\infty} |t|\exp(CKt\sqrt{2}/\Delta A - t^2)\,dt.\end{aligned} \tag{2.3}$$

To evaluate the integral on the right-hand side of (2.3), we let $\lambda = CK\sqrt{2}/A\Delta$. Then the integral becomes

$$\int_0^\infty t\{\exp(\lambda t) + \exp(-\lambda t)\}\exp(-t^2)\,dt = J(\lambda) + J(-\lambda), \tag{2.4}$$

where

$$J(\lambda) = \int_0^\infty t\exp(\lambda t - t^2)\,dt.$$

Using integration by parts, we get

$$J(\lambda) = \tfrac{1}{2} + (\lambda/2)\exp(\lambda^2/4)\int_0^\infty \exp\{-(t-\lambda/2)^2\}\,dt \tag{2.5}$$
$$= \tfrac{1}{2} + (\lambda/2)\exp(\lambda^2/4)\{\sqrt{\pi}/2 + \operatorname{erf}(\lambda/2)\},$$

where

$$\operatorname{erf}(x) = \int_0^x \exp(-t^2)\,dt.$$

Hence, (2.1), (2.3), (2.4) and (2.5) give

$$EN(\alpha,\beta) = \int_\alpha^\beta \Delta/(\pi A^2)\exp\{-B^2K^2/(2\Delta^2)\}\,d\theta$$
$$+ \int_\alpha^\beta (\sqrt{2}/\pi)|KC|A^{-3}\exp(-K^2/2A^2)\operatorname{erf}(|KC|/A\Delta\sqrt{2})\,d\theta \tag{2.6}$$
$$= I_1(\alpha,\beta) + I_2(\alpha,\beta), \quad \text{say}.$$

3. Proof of the theorem. In order to estimate the number of real roots we divide them into two groups: (I) those lying in the intervals $(0,\varepsilon)$, $(\pi-\varepsilon,\pi+\varepsilon)$ and $(2\pi-\varepsilon,2\pi)$ and (II) those lying in the intervals $(\varepsilon,\pi-\varepsilon)$ and $(\pi+\varepsilon,2\pi-\varepsilon)$. For the roots (I) we need some modification to apply Dunnage's [2] approach, which is based on an application of Jensen's theorem [9, page 125] or [7, page 332]. For the roots (II) we use the Kac–Rice formula (2.6). The ε chosen should be small enough to make the zeros of type (I) negligible while it should be large enough to allow the calculation of zeros of type (II) to be possible. We will show that $\varepsilon = n^{-1/2}$ satisfies both requirements.

For proof of the theorem we will need the following lemma.

LEMMA. *For $\varepsilon \le \theta < \pi - \varepsilon$, where ε is any positive constant smaller than π, we have*

$$A^2 = n/2 + O(\varepsilon^{-1}), \qquad B^2 = n^3/6 + O(n^2/\varepsilon),$$
$$C = O(n/\varepsilon) \quad \text{and} \quad \Delta^2 = n^4/12 + O(n^3/\varepsilon).$$

PROOF. Let

$$S(\theta) = \sin(2n+1)\theta/\sin\theta.$$

Then, since for θ in this interval $|S(\theta)| \le 1/\sin\varepsilon$, we obtain

$$S(\theta) = O(\varepsilon^{-1}). \tag{3.1}$$

Also, we have

$$S'(\theta) = (2n+1)\operatorname{cosec}\theta\cos(2n+1)\theta - S(\theta)\cot\theta \tag{3.2}$$
$$= O(n/\varepsilon)$$

and

$$S''(\theta) = -(2n+1)^2 S(\theta) - (2n+1)\cos\theta\cos(2n+1)\theta\sin^{-2}\theta \\ - S'(\theta)\cot\theta + \operatorname{cosec}^2\theta S(\theta) = O(n^2/\varepsilon + n/\varepsilon^2). \tag{3.3}$$

Now

$$\begin{aligned}\sin\theta\left(1 + 2\sum_{i=1}^{n}\cos 2i\theta\right) &= \sin\theta + \sin 3\theta - \sin\theta + \sin 5\theta - \sin 3\theta \\ &\quad + \cdots + \sin(2n+1)\theta - \sin(2n-1)\theta \\ &= \sin(2n+1)\theta = \sin(\theta)S(\theta).\end{aligned}$$

Hence we can write

$$\sum_{i=1}^{n}\cos 2i\theta = (1/2)\{S(\theta) - 1\}, \tag{3.4}$$

which together with (3.1) gives

$$\begin{aligned}A^2 &= (1/2)\sum_{i=1}^{n}\{1 + \cos 2i\theta\} = n/2 + (1/4)\{S(\theta) - 1\} \\ &= n/2 + O(\varepsilon^{-1}).\end{aligned} \tag{3.5}$$

Also, from (3.3) and since, from (3.4),

$$S''(\theta) = -8\sum_{i=1}^{n} i^2\cos 2i\theta = 8\sum_{i=1}^{n} i^2(2\sin^2 i\theta - 1),$$

we have

$$B^2 = (1/2)\sum_{i=1}^{n} i^2 + (1/16)S''(\theta) = n^3/6 + O(n^2/\varepsilon). \tag{3.6}$$

From (3.2) and (3.5) we can obtain

$$C = \frac{1}{2}\frac{d}{d\theta}(A^2) = \frac{1}{8}S'(\theta) = O\left(\frac{n}{\varepsilon}\right). \tag{3.7}$$

Finally, from (3.5), (3.6) and (3.7) we can show

$$\Delta^2 = n^4/12 + O(n^3/\varepsilon),$$

which completes the proof of the lemma. □

To calculate the expected number of real roots of $T(\theta) = K$ in the intervals $(\varepsilon, \pi - \varepsilon)$ and $(\pi + \varepsilon, 2\pi - \varepsilon)$, it is sufficient to consider just the first interval, since

$$T(\theta + \pi) = \sum_{i=1}^{n}(-1)^i g_i \cos i\theta,$$

and g_i and $-g_i$ have the same distribution function. From (2.6) and the lemma we can obtain

$$(3.8)\quad I_1(\varepsilon, \pi-\varepsilon) = \{n/\sqrt{3} + O(\varepsilon^{-1})\}[\exp\{-(K^2/n) + O(K^2/n^2\varepsilon)\}]$$

and

$$(3.9)\qquad I_2(\varepsilon, \pi-\varepsilon) = O\{(K/\varepsilon\sqrt{n})\exp(-K^2/n)\}.$$

Now we show that $T(\theta) = K$ has a negligible expected number of real roots in the intervals $(0, \varepsilon)$, $(\pi-\varepsilon, \pi+\varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$. By periodicity, the expected number of real roots in $(0, \varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$ is the same as the expected number in $(-\varepsilon, \varepsilon)$, and so we shall confine our discussion to this last interval; the interval $(\pi-\varepsilon, \pi+\varepsilon)$ can be treated in exactly the same way to give the same result. To avoid repetition, we only point out the generalization necessary for applying Jensen's theorem to the random integral function of the complex variable z,

$$T(z, \omega) - K = \sum_{i=1}^{n} g_i(\omega)\cos iz - K,$$

which is done for $K = 0$ by Dunnage [2, page 82]. We are seeking an upper bound to the number of real roots in the segment of the real axis joining the points $\pm\varepsilon$, and this certainly does not exceed the number in the circle $|z| < \varepsilon$. Let $N(r) = N(r, \omega, K)$ denote the number of real roots of $T(z, \omega) - K = 0$ in $|z| < r$. Assuming that $T(0) \neq K$, then by Jensen's theorem [9, page 125] or [7, page 332] we have

$$(3.10)\qquad \begin{aligned} N(\varepsilon)\log 2 &\le \int_{\varepsilon}^{2\varepsilon} r^{-1}N(r)\,dr \\ &\le (2\pi)^{-1}\int_0^{2\pi}\log\big|[\{T(2\varepsilon e^{i\theta}, \omega) - K\}/\{T(0) - K\}]\big|\,d\theta. \end{aligned}$$

Now, since the distribution function of $T(0, \omega) = \sum_{i=1}^{n} g_i(\omega)$ is

$$G(x) = (2\pi n)^{-1/2}\int_{-\infty}^{x}\exp(-t^2/2n)\,dt,$$

we can see that, for any positive ν,

$$(3.11)\qquad \begin{aligned} \operatorname{Prob}(-e^{-\nu} \le T(0) - K \le e^{-\nu}) &= (2\pi n)^{-1/2}\int_{K-e^{-\nu}}^{K+e^{-\nu}}\exp(-t^2/2n)\,dt \\ &< \sqrt{2/\pi n}\, e^{-\nu}. \end{aligned}$$

Also, from [2, page 82]

$$(3.12)\qquad T(2\varepsilon e^{i\theta}) \le 2ne^{2n\varepsilon}\max|g_i|,$$

where the maximum is taken over $1 \le i \le n$. The distribution function of $|g_i|$

is

$$F(x) = \begin{cases} \sqrt{2/\pi} \int_0^x e^{-t^2/2}\,dt, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and so, for any positive ν and all sufficiently large n,

$$\begin{aligned} \operatorname{Prob}(\max|g_i| > ne^\nu) &\leq n \operatorname{Prob}(|g_1| > ne^\nu) \\ &= n\sqrt{2/\pi} \int_{ne^\nu}^\infty \exp(-t^2/2)\,dt \qquad (3.13) \\ &\sim \sqrt{2/\pi} \exp(-\nu - n^2 e^{2\nu}/2). \end{aligned}$$

Therefore, from (3.12) and (3.13), except for sample functions in an ω-set of measure not exceeding $\exp(-\nu - n^2e^{2\nu}/2)$,

$$T(2\varepsilon e^{i\theta}) < 2n^2 \exp(2n\varepsilon + \nu). \tag{3.14}$$

Hence, from (3.11) and (3.14) and since

$$|2n^2 \exp(2n\varepsilon + \nu) - K| < 3n^2 \exp(2n\varepsilon + \nu) \quad \text{for } K = o(\sqrt{n}),$$

we obtain

$$\begin{aligned} \left|\{T(2\varepsilon e^{i\theta}, \omega) - K\}/\{T(0, \omega) - K\}\right| &\leq e^\nu\left|2n^2 \exp(2n\varepsilon + \nu) - K\right| \\ &\leq 3n^2 \exp(2n\varepsilon + 2\nu), \end{aligned} \tag{3.15}$$

except for sample functions in an ω-set of measure not exceeding

$$2/\sqrt{n}\, e^{-\nu} + \exp(-\nu - n^2e^{2\nu}/2).$$

Therefore, from (3.10) and (3.15) we can show that, outside the exceptional set,

$$N(\varepsilon) \leq (\log 3 + 2\log n + 2n\varepsilon + 2\nu)/\log 2. \tag{3.16}$$

Now we choose $\varepsilon = n^{-1/2}$. Then from (3.16) and for all sufficiently large n, we have

$$\operatorname{Prob}\{N(\varepsilon) > 3n\varepsilon + 2\nu\} \leq (2/\sqrt{n})e^{-\nu} + \exp(-\nu - n^2e^{2\nu}/2). \tag{3.17}$$

Let $n' = [3\sqrt{n}]$ be the greatest integer less than or equal to $3\sqrt{n}$. Then from (3.17) and for n large enough, we obtain

$$\begin{aligned} EN(\varepsilon) &= \sum_{i>0} \operatorname{Prob}\{N(\varepsilon) \geq i\} \\ &= \sum_{0<i\leq n'} \operatorname{Prob}\{N(\varepsilon) > i\} + \sum_{i\geq 1} \operatorname{Prob}\{N(\varepsilon) > n' + i\} \\ &\leq n'(2/\sqrt{n}) \sum_{i\geq 1} e^{-i/2} + \sum_{i\geq 1} \exp(-i/2 - n^2e^i/2) \\ &= O(\sqrt{n}), \end{aligned} \tag{3.18}$$

since, by dominated convergence, the second sum will tend to zero. Finally, from (2.6), (3.8), (3.9) and (3.18), we have

$$EN(0, \pi) = n/\sqrt{3} + o(n),$$

which completes the proof of the theorem.

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