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VARIANCE FUNCTIONS WITH MEROMORPHIC MEANS

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A natural exponential family $\mathscr{F}$ is characterized by the pair (V, Ω), called the variance function (VF), where Ω is the mean domain and V is the variance of $\mathscr{F}$ expressed in terms of the mean. Any VF can be used to construct an exponential dispersion model, thus providing a potential generalized linear model. A problem of increasing interest in the literature is the following: Given an open interval Ω and a function V defined on Ω, is the pair (V, Ω) a VF of a natural exponential family? In this paper, we develop a complex analytic approach to this question and focus on VF's having meromorphic mean functions; that is, if T is the Laplace transform of an element of the family, then T'/T is extendable to a meromorphic function on $\mathbb{C}$. We derive properties of such VF's and characterize a class of VF's (V, Ω), where V admits a unique analytic continuation in $\mathbb{C}$, except for isolated singularities. (Included in this class are VF's having V's that admit meromorphic continuation to $\mathbb{C}$.) We show that this class equals the set of VF's which are at most second degree polynomials. We also investigate the class in which V has the form $P + Q\sqrt{R}$, where P and Q are arbitrary rational functions and R is a polynomial of at most second degree. We characterize all VF's in this class for which the mean function is meromorphic and show that $P = kR$ for some constant k and Q is a polynomial of at most first degree. Throughout the paper, we demonstrate the wide applicability of our results by showing that many classes of simple-form pairs (V, Ω) can be excluded from being VF's.

1. Introduction. A natural exponential family (NEF) $\mathscr{F}$ is characterized by the pair (V, Ω), called the variance function (VF), where Ω (an open interval) is the mean domain of $\mathscr{F}$ and V is the variance of $\mathscr{F}$ expressed in terms of the mean. (Precise definitions will be given in Section 2.) This characterization was established in the seminal work of Morris (1982). A problem of increasing attention and interest in the literature is the following: Given an open interval $\Omega \subseteq \mathbb{R} \equiv (-\infty, \infty)$ and a function V defined on Ω, is the pair (V, Ω) a VF of an NEF? (Henceforth, whenever reference is made to a VF, it will be understood that it is a VF of some NEF.) A rigorous approach to the investigation of this problem can be found in Letac and Mora (1990) and Jørgensen (1986).

Any VF can be used to construct an exponential dispersion model, thus providing a potential generalized linear model [cf. Jørgensen (1987) and the references cited therein]. This results in a huge class of possible models. In

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order to reduce this class to a reasonable size, it is useful to consider as a criterion the simplicity of the variance function [Letac (1987a)]; that is, to restrict consideration to VF's having simple functional forms, such as polynomials, rational functions, exponential functions and other elementary functions.

Let $\mathscr{P}_n$ denote the set of polynomials with degree less than or equal to n. Morris (1982) identified all VF's for which $V \in \mathscr{P}_2$ and identified the corresponding families. Mora (1986) did the same for $V \in \mathscr{P}_3 \setminus \mathscr{P}_2$. [Details can be found in Letac and Mora (1990).] Tweedie (1984), Bar-Lev and Enis (1986), Burridge (1986), Jørgensen (1987) and Letac (1987b) considered the case where V is a multiple of a real power of the mean.

Bar-Lev and Bshouty (1989) considered the case where Ω is bounded and V is a rational function vanishing at the endpoints of Ω. They showed that, up to an affine transformation, the only VF among such rational functions is the one corresponding to the binomial family. Thus, their result implies that if Ω is bounded and $p \in \mathscr{P}_n$, $n \geq 3$, then (p, Ω) is not a VF. Bar-Lev and Bshouty (1989) noted that a referee of their paper indicated that their result can be generalized by replacing the condition that V is rational by the less restrictive condition that V is the quotient of two entire functions (i.e., V is meromorphic), vanishing at the endpoints of Ω. (This generalization is presented in Example 3.1.)

For $\Omega \subseteq \mathbb{R}^+ \equiv (0, \infty)$, Jørgensen (1984) gave necessary and sufficient conditions for (V, Ω) to be a VF. These conditions, however, are frequently very difficult to check in specific examples. Based on Jørgensen's result, Bar-Lev (1987) showed that, if $\Omega \subseteq \mathbb{R}^+$ and V is absolutely monotone on Ω, then (V, Ω) is the VF corresponding to an infinitely divisible NEF. For example, if V is any polynomial having nonnegative coefficients and vanishing at the origin, then $(V, \mathbb{R}^+)$ is such a VF. Letac and Mora [(1990), Corollary 3.3] pointed out that Bar-Lev's (1987) result implies that if $\Omega = (0, b)$, $b \leq \infty$, and V is a power series having nonnegative coefficients and vanishing at the origin, then (V, Ω) is the VF corresponding to an infinitely divisible family.

Let (V, Ω) be a VF. Letac and Mora (1990) showed that Ω is the largest open interval on which V is positive and real analytic. This implies that V is the restriction to Ω of an analytic function on some domain of the complex plane $\mathbb{C}$ containing Ω. This fact motivates the development of the complex-analytic methods used in this paper. Indeed, the results presented here are entirely in the realm of complex variable theory. These results, however, enable us to obtain conditions when a pair (V, Ω) can serve as a VF. Such conditions will exclude a host of pairs (V, Ω) from being VF's.

For an NEF $\mathscr{F}$ with VF (V, Ω), let Θ denote the interior of its canonical parameter space and $\mu = \mu(\theta)$, $\theta \in \Theta$, its mean function. Many VF's possess a mean μ that is meromorphic (i.e., μ admits a meromorphic continuation to $\mathbb{C}$). For example, VF's whose canonical parameter space is $\mathbb{R}$ (such as VF's having bounded support and VF's having $V \in \mathscr{P}_1$) possess meromorphic mean functions. These, however, are not the only cases. For instance, $(\mu^2, \mathbb{R}^+)$ is a VF with meromorphic mean, yet its corresponding canonical parameter space is a ray (see Remark 2.1).

The purpose of this paper is to study the class of VF's with meromorphic means. There are two principal reasons for studying this class. First, as is clear from the preceding paragraph, this class is large. Second, a well-established theory on meromorphic functions is available which facilitates the analysis of this case. After reviewing (in Section 2) some general analytic properties of VF's, we derive in Section 3 several properties concerning the complex analytic behavior of VF's with meromorphic means and thereby provide necessary conditions for VF's to have meromorphic means. These conditions are particularly useful in excluding certain pairs (V, Ω) from being VF's. If the pair (V, Ω) was a VF with $\Theta = \mathbb{R}$, then it must have had a meromorphic mean and thus the above conditions are applicable. Consequently, if such a pair does not satisfy these conditions, it is not a VF. We present here several examples of a general nature, where each example considers a class of pairs (V, Ω). For instance, in Example 3.4, we consider the class $(V, \Omega) = (\mu^{\alpha}(1-\mu)^{\beta}, (0,1))$, $\alpha \geq 1$, $\beta \geq 1$, and show that the only VF in this class is the one with $\alpha = \beta = 1$. (This VF corresponds to the binomial family.)

The results of Section 3 are used in Section 4 to characterize a class of VF's having meromorphic means. This characterization is given in Theorem 4.2. In this theorem, we consider the class of VF's (V, Ω) in which the mean μ admits a meromorphic continuation to $\mathbb{C}$ and V admits a unique analytic continuation to $\mathbb{C}$, except for isolated singularities. It is shown that this class equals the set of VF's having $V \in \mathscr{P}_2$. Essential steps toward the proof of Theorem 4.2 are Theorem 4.1 and Corollary 4.1, in which we consider the special case where V admits a meromorphic extension to $\mathbb{C}$. In this section, we also demonstrate the applicability of the two theorems by excluding several classes of simple-form pairs from being VF's. It should be noted that Theorem 4.1 extends a similar result of Bar-Lev and Bshouty (1989), by removing their conditions that Ω should be bounded and that V vanish at the endpoints of Ω.

Section 5 is devoted to the study of the class of VF's (V, Ω) with V having the form $V = P + Q\sqrt{R}$, where P and Q are arbitrary rational functions and $R \in \mathscr{P}_2$. We identify all VF's in this class for which the mean is meromorphic in $\mathbb{C}$ and show that $P = kR$ for some constant k and $Q \in \mathscr{P}_1$. This identification makes use of a suitable transformation which transforms V to a meromorphic function in $\mathbb{C}$, thus permitting the use of Theorem 4.1. Included in this class is one suggested by Letac (1987a) and Letac and Mora (1990), where $P \in \mathscr{P}_3$, $Q \in \mathscr{P}_2$ and $R \in \mathscr{P}_2$. This latter class includes third degree polynomials as a special case and preserves the property of reciprocity [cf. Letac (1987a)].

2. Preliminary notions and basic properties of VF's. We first recall some definitions and properties of NEF's and their VF's.

Let ν be a positive Radon measure on $\mathbb{R}$, which is not concentrated on one point. The Laplace transform and effective domain of ν are given, respectively, by

$$T_{\nu}(\theta) = \int_{\mathbb{R}} \exp(\theta x)\nu(dx) \tag{2.1}$$

and

$$D_\nu = \{\theta \in \mathbb{R}: T_\nu(\theta) < \infty\}.$$

Let $\Theta_\nu = \operatorname{int} D_\nu$ and assume that Θ_ν is nonempty. For $\theta \in \Theta_\nu$, define

$$F_\theta(dx) = (T_\nu(\theta))^{-1} \exp(\theta x)\nu(dx).$$

The family of probability distributions $\mathscr{F} \equiv \mathscr{F}_\nu = \{F_\theta\colon\ \theta \in \Theta_\nu\}$ is called a natural exponential family generated by ν. The set D_ν is called the canonical parameter space of $\mathscr{F}$ and the measure ν is also said to be a basis of $\mathscr{F}$. [Note that NEF's, as defined by Barndorff–Nielsen (1978), include also those F_θ with $\theta \in D_\nu \setminus \Theta_\nu$.] A basis of $\mathscr{F}$ is not unique [Letac and Mora (1990)]. In fact, for any $\theta_0 \in \Theta_\nu$, the measure defined by $\nu^*(dx) = [T_\nu(\theta_0)]^{-1} \exp\{\theta_0 x\}\nu(dx)$ is another basis of $\mathscr{F}$ (i.e., $\mathscr{F}_\nu = \mathscr{F}_{\nu^*}$) for which $0 \in \Theta_{\nu^*}$ and $T_{\nu^*}(0) = 1$; that is, ν^* is a probability measure. Accordingly, without loss of generality, we assume that ν is a probability measure (i.e., $0 \in \Theta_\nu$ and $T_\nu(0) = 1$). For simplicity, we suppress the dependence on ν and write T, D and Θ for T_ν, D_ν and Θ_ν, respectively.

The mean function of $\mathscr{F}$ is the mapping defined on Θ by

$$\mu(\theta) = \int_{\mathbb{R}} x F_\theta(dx),$$

and the mean domain of $\mathscr{F}$ is $\Omega = \mu(\Theta)$. μ is a one-to-one continuously differentiable mapping and hence Ω is an open interval. Denote by $\theta = \theta(\mu)$ the inverse function of μ and let V on Ω be defined by

$$V(\mu) = \int_{\mathbb{R}} (x - \mu)^2 F_{\theta(\mu)}(dx).$$

The pair (V, Ω), called the VF of $\mathscr{F}$, determines $\mathscr{F}$ uniquely within the class of NEF's. [For further details, see Morris (1982) and Letac and Mora (1990).] Define $\mu_0 = \mu(0)$. The following relations among μ, T, V, Θ and Ω hold for $\theta \in \Theta$ and $\mu \in \Omega$:

$$T'(\theta)/T(\theta) = \mu(\theta); \tag{2.2}$$

$$V(\mu(\theta)) = \mu'(\theta); \tag{2.3}$$

$$\theta = \int_{\mu_0}^{\mu} dt/V(t);$$

$$T(\theta) = \exp\left\{\int_{\mu_0}^{\mu} t\, dt/V(t)\right\}.$$

Accordingly, given that (V, Ω) is a VF with $\Omega = (a, b)$, finite or not, the corresponding Θ is the open interval determined by

$$\left(\lim_{\mu \to a+} \int_{\mu_0}^{\mu} dt/V(t),\ \lim_{\mu \to b-} \int_{\mu_0}^{\mu} dt/V(t)\right). \tag{2.4}$$

Note that the Laplace transform T, given by (2.1), is the restriction to Θ of a unique analytic function on $\Theta \times \mathbb{R} \equiv S_\Theta \subseteq \mathbb{C}$. We shall use the symbol T to denote the extended function as well. From (2.2), we conclude that μ admits a meromorphic extension to S_Θ with at most first order poles. This extension will also be denoted by μ.

The following lemma summarizes several basic properties of VF's (some of which are mentioned above), which will be used in the sequel.

LEMMA 2.1. *Let (V, Ω) be a VF. Then*:

(i) *Ω is the largest open interval on which V is positive real analytic*;
(ii) *the differential equation*

$$T'/T = \mu(z), \qquad z = \theta + i\eta \in S_\Theta,$$

has an analytic solution $T(z)$ in S_Θ having singularities at finite endpoints of Θ;
(iii) *the differential equation*

$$V(\mu) = \mu'$$

admits a meromorphic solution $\mu(z)$ in S_Θ, with at most first order poles and has singularities at finite endpoints of Θ.

PROOF. (i) This was proved by Letac and Mora [(1986), Theorem 2.3].
(ii) This follows from Kawata [(1972), Theorem 8.4.1, page 299].
(iii) This is an immediate consequence of (ii). Let $z_0 \in S_\Theta$. If $T(z_0) \neq 0$, then $\mu(z)$ is analytic in a neighborhood of z_0. If, on the other hand, $T(z_0) = 0$, then $T(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$ near z_0, and thus $\mu(z) = na_n(z - z_0)^{-1} + c_0 + c_1(z - z_0) + \cdots$, near z_0. Therefore, $\mu(z)$ is meromorphic in S_Θ with at most first order poles. Finally, since, by (ii), $T(z)$ is singular at finite endpoints of Θ, so is $\mu(z) = T'(z)/T(z)$. □

REMARK 2.1. If (V, Ω) is a VF, then μ admits a meromorphic continuation to S_Θ. For $\Theta = \mathbb{R}$, μ is meromorphic in $S_\Theta = \mathbb{C}$. However, even when Θ is a proper subset of $\mathbb{R}$ (i.e., S_Θ is a proper subset of $\mathbb{C}$), there exist cases where μ admits a meromorphic continuation to $\mathbb{C}$. For example, if $(V, \Omega) = (\mu^2, \mathbb{R}^+)$, then $\Theta = (-\infty, \alpha)$ for some $\alpha \in \mathbb{R}^+$ and $\mu(z) = (\alpha - z)^{-1}$ admits a meromorphic continuation to $\mathbb{C}$.

Finally, we make the following remark which will be useful in subsequent sections.

REMARK 2.2. Let (V, Ω) be a VF with finite Ω and $\Theta = \mathbb{R}$. In this case, the corresponding family $\mathscr{F}$ is regular (i.e., $D = \Theta$) and thus is steep. The steepness of $\mathscr{F}$ implies that the interior of its convex support equals Ω [see

Barndorff-Nielsen (1978), Theorem 9.2, page 142]. Two conclusions now follow:

(i) Since T is entire, $\mu = T'/T$ is meromorphic in $\mathbb{C}$;
(ii) T has infinitely many zeros [see Lukacs (1970), Theorem 7.2.3, page 202], and if z_0 is a zero of T, then, by Lemma 2.1(iii), z_0 is a first order pole of μ and a second order pole of μ'.

3. Properties of VF's with meromorphic means. In this section, we provide some results concerning properties of VF's with meromorphic means. We then exemplify the utility and application of these properties in excluding many pairs (V, Ω) from being VF's.

We have already noted that if (V, Ω) is a VF, then μ admits a meromorphic continuation to S_Θ. In cases where S_Θ is a proper subset of $\mathbb{C}$ and μ admits a meromorphic continuation to $\mathbb{C}$, it will be useful to know what can be concluded about the behavior of V on $\mu(\mathbb{C})$ and not only on $\mu(S_\Theta)$. Such a conclusion is achieved as a corollary to Lemma 3.1.

In the remainder of this paper, we use the following conventions: Let (V, Ω) be a VF with mean function μ. If μ admits a meromorphic (analytic) continuation to $S \supset S_\Theta$, then we simply refer to μ as meromorphic (analytic) in S. Also, V, which is the restriction to Ω of an analytic function on some domain $D \supset \Omega$ of $\mathbb{C}$, will be referred to as analytic on D. Hereafter, we use the terminology of Ahlfors [(1966), Chapter 8]. In accordance with Ahlfors, the analytic function V on D constitutes a function element denoted by (V, D). Two function elements (V_1, D_1) and (V_2, D_2) are said to be direct continuations of each other if $D_1 \cap D_2$ is nonempty and $V_1(z) = V_2(z)$ in $D_1 \cap D_2$. The function elements $(V_1, D_1), (V_2, D_2), \dots, (V_k, D_k)$ form a chain if (V_k, D_k) is a direct analytic continuation of (V_{k-1}, D_{k-1}). The elements of such a chain are said to be analytic continuations of each other. A complete analytic function is the collection of all function elements such that any two elements in this collection are analytic continuations of each other. We consider the complete analytic function of all function elements which are analytic continuations of (V, D). By the uniqueness of the analytic continuation, there exists a function element of such a complete analytic function which continues to satisfy the differential equation (2.3), in any neighborhood of a point $\mu_0 \in \mu(S_\Theta)$. We shall continue to use the symbol V to denote the function in this function element.

LEMMA 3.1. *Let D be a domain, $z_0 \in D$ and m be a meromorphic function on D such that, in a disk $H \subset D$ around z_0, m' admits the representation*

$$m'(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k,$$

where $a_n \neq 0$ and $a_{-1} = 0$. Let $(W, m(D))$ be a function element such that

$W(m) = m'$ on D. If:

(i) $n = 0$, *then W is analytic at $m(z_0)$;*
(ii) $n > 0$, *then W has an $(n+1)$-fold algebraic singularity at $m(z_0)$ and admits the representation*

$$W(m) = c_0(m - m_0)^{n/(n+1)} + \cdots, \qquad m_0 = m(z_0), c_0 \neq 0;$$

(iii) $n = -2$, *then W is analytic at $m(z_0) = \infty$ and, near infinity, admits the representation*

$$W(m) = c_{-2}m^2 + c_{-1}m + \cdots, \qquad c_{-2} \neq 0;$$

(iv) $n < -2$, *then W has a $(-n-1)$-fold algebraic pole at $m(z_0) = \infty$ and admits the representation*

$$W(m) = c_0 m^{n/(n+1)} + \cdots, \qquad c_0 \neq 0.$$

PROOF. We have

$$m(z) = m_0 + \sum_{k=n}^{\infty} a_k(z - z_0)^{k+1}/(k+1),$$

and therefore $m(z)$ has the form $m(z) = m_0 + [g(z)]^{n+1}$, where g is univalent in a neighborhood of z_0. Hence, $W(m_0 + [g(z)]^{n+1}) = (n+1)[g(z)]^n g'(z)$, $g'(z_0) \neq 0$. Since n and $n+1$ are relatively prime, the desired results follow from Ahlfors [(1966), pages 289–290]. □

COROLLARY 3.1. *Let (V, Ω) be a VF and assume that μ admits a meromorphic continuation to $S \supset S_\Theta$. Then*

(i) *V is the restriction to Ω of a function element of a finitely-sheeted Riemann surface over $\mu(S)$;*
(ii) *if $S = \mathbb{C}$, then V is the restriction to Ω of a function element of a finitely-sheeted Riemann surface over $\mu(\mathbb{C})$, which is $\overline{\mathbb{C}}$ (the closure of $\mathbb{C}$) except perhaps for two points.*

PROOF. (i) Fix $z_0 \in S$. If $\mu(z)$ is analytic at z_0, then, by parts (i) and (ii) of Lemma 3.1, V is either analytic or has an algebraic singularity at $\mu(z_0)$. If $\mu(z)$ has a pole at z_0, then, by parts (iii) and (iv) of Lemma 3.1, V is either analytic or algebraic near $\mu(z_0) = \infty$.
(ii) This follows from part (i) and the Picard theorem [see Nevanlinna (1970), page 1]. □

COROLLARY 3.2. *Let (V, Ω) be a VF. Assume that μ admits a meromorphic continuation to $\mathbb{C}$ having a pole of order s at some point $z_0 \in \mathbb{C}$ (i.e., $\mu(z_0) = \infty$). Then there exists a function element of the Riemann surface of the extension of V over $\mu(\mathbb{C})$ on which $V \sim \mu^{(s+1)/s}$ near infinity.*

PROOF. This follows from parts (iii) and (iv) of Lemma 3.1. □

REMARK 3.1. Note that by Lemma 2.1, μ has at most first order poles in S_Θ. However, if μ can be extended to $\mathbb{C}$, then μ may have poles of any order on $\mathbb{C} \setminus S_\Theta$.

LEMMA 3.2. *Let (V, Ω) be a VF. Assume that μ admits a meromorphic continuation to $\mathbb{C}$ and that V admits a meromorphic continuation to $\mu(\mathbb{C})$. Then V does not vanish on $\mu(\mathbb{C})$.*

PROOF. Assume to the contrary that there exists a $z_0 \in \mathbb{C}$ such that $V(\mu(z_0)) = 0$, and thus $\mu'(z_0) = V(\mu(z_0)) = 0$. Differentiating (2.3), we get

$$\mu''(z_0) = V'(\mu(z_0))\mu'(z_0) = 0.$$

By successive differentiation, we get $\mu^{(k)}(z_0) = 0$, $k = 3, 4, \dots$. This implies that $\mu \equiv$ constant in $\mathbb{C}$, a contradiction. □

EXAMPLE 3.1. Assume that (V, Ω) is a VF such that $\Omega = (a, b)$ is finite and V admits an analytic continuation to $\mathbb{C}$, with $V(a) = V(b) = 0$. Then, by (2.4) and Remark 2.2, $\Theta = \mathbb{R}$, μ admits a meromorphic continuation to $\mathbb{C}$ and there exists a $z_0 \in \mathbb{C}$ such that $\mu(z_0) = \infty$. Using these and Lemma 3.1(iii), it follows that $V \sim \mu^2$ near infinity and therefore $V \in \mathscr{P}_2 \setminus \mathscr{P}_1$.

EXAMPLE 3.2. For even $n \geq 4$, consider $p \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$ and assume that p has no real roots but possesses at least three distinct zeros. Then the pair $(V, \Omega) = (e^{1/p}, \mathbb{R})$ is not a VF. To show this, assume that (V, Ω) is a VF. It follows from (2.4) that $\Theta = \mathbb{R}$ and thus μ is meromorphic in $\mathbb{C}$. By Lemma 3.1 and Corollary 3.1, V is either meromorphic or has algebraic singularities on $\mu(\mathbb{C})$, where $\mu(\mathbb{C})$ is $\overline{\mathbb{C}}$ except perhaps for two points. But $V = e^{1/p}$ has essential singularities at zeros of p, hence these zeros do not belong to $\mu(\mathbb{C})$. This is a contradiction, since, by assumption, p has at least three distinct zeros.

EXAMPLE 3.3. Let $\Omega = (a, b)$ be a finite interval, $n \geq 3$ a positive integer, $p \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, $q \in \mathscr{P}_{n-2} \setminus \mathscr{P}_{n-3}$ and $V(\mu) = p(\mu)e^{1/\mu}/q(\mu)$, where p has at least three distinct zeros, two of which are a and b, $p(\Omega) \subset \mathbb{R}^+$, $q(\overline{\Omega}) \subset \mathbb{R}^+$ and p and q are relatively prime. Then the pair (V, Ω) is not a VF. To prove this, assume that (V, Ω) is a VF, then $\Theta = \mathbb{R}$ and μ is meromorphic on $\mathbb{C}$. By Lemma 3.2, V does not vanish on $\mu(\mathbb{C})$, which, by Corollary 3.1(ii), is $\overline{\mathbb{C}}$ except possibly for two points. We get a contradiction, since, by assumption, p has at least three distinct zeros and V vanishes at these zeros.

EXAMPLE 3.4. Let $(V, \Omega) = (\mu^\alpha(1 - \mu)^\beta, (0, 1))$, where $\alpha \geq 1$, $\beta \geq 1$. By an application of Corollary 3.2, we show that (V, Ω) is a VF iff $\alpha = \beta = 1$ (i.e., the VF corresponding to the binomial family). For this, we assume that (V, Ω) is a VF. It follows from (2.4), Remark 2.2 and Lemmas 2.1 and 3.1 that $\Theta = \mathbb{R}$, μ

is meromorphic in $\mathbb{C}$ with first order poles and V is a function element of a finitely-sheeted Riemann surface over $\mu(\mathbb{C})$. Thus, by Corollary 3.2, there exists a function element of the Riemann surface of V over $\mu(\mathbb{C})$ on which $V \sim \mu^2$ near infinity. We shall prove that such a function element exists only if $\alpha = \beta = 1$. Indeed, on all elements of the Riemann surface of V over $\mu(\mathbb{C})$, $|V(\mu)| \sim |\mu|^{\alpha+\beta}$ near infinity. This implies that $\alpha + \beta = 2$ and hence $\alpha = \beta = 1$.

4. A characterization of a class of VF's with meromorphic means. The main result of this section is presented in Theorem 4.2. In this theorem, we consider the class of VF's (V, Ω) in which μ admits a meromorphic continuation to $\mathbb{C}$ and V admits a unique analytic continuation to $\mathbb{C}$, except for isolated singularities [i.e., V is extendable to an analytic function on $\mu(\mathbb{C})$, as a single-sheeted Riemann surface there]. We show that this class equals the set of VF's with $V \in \mathscr{P}_2$. An essential step for establishing Theorem 4.2 is Theorem 4.1, in which we consider the special case that V admits a meromorphic continuation to $\mathbb{C}$. Theorem 4.1 is also used in the proof of some results in Section 5.

The results of this section provide easily utilizable conditions for excluding many pairs (V, Ω) having simple forms from being VF's. In order to use these conditions for a given pair (V, Ω), it is, of course, necessary to determine whether the corresponding mean μ is meromorphic in $\mathbb{C}$. Such a determination might be difficult, since in most cases $\mu = \mu(z)$ cannot be expressed explicitly in terms of z. However, by assuming that (V, Ω) is a VF, the answer to this question is immediate, if the corresponding Θ, determined by (2.4), is $\mathbb{R}$, since, in such a case, μ is meromorphic in $\mathbb{C}$. This latter conclusion is used in all of the examples provided in this section.

THEOREM 4.1. *Let W be a meromorphic function on $\mathbb{C}$, which is not identically zero. Then the differential equation*

$$W(m) = m' \tag{4.1}$$

admits a nonconstant meromorphic solution $m(z)$ in $\mathbb{C}$ if and only if $W \in \mathscr{P}_2$.

Before presenting the proof of Theorem 4.1, we note that the differential equation (4.1) has a long history. Rellich (1940) [see also Wittich (1968), page 63] considered the case where W is a nonconstant entire function and showed that (4.1) has no nonconstant entire solutions unless W is affine. Accordingly, if (V, Ω) is a VF where both V and μ are extendable to entire functions, then the corresponding NEF is either normal or Poisson. Wittich [(1968), page 63], using Nevanlinna theory, generalized the result of Rellich to higher order differential equations. Our approach for proving Theorem 4.1 follows the idea of Wittich.

PROOF. We first introduce some basic notation [see Hayman (1964), page 4]. Let f be a meromorphic function and for $r > 0$, let

$$m(r, f) \equiv (2\pi)^{-1} \int_0^{2\pi} \log^+ |f(re^{i\varphi})|\, d\varphi,$$

where $\log^+ x = \max(0, \log x)$, $x > 0$. Let $n(r, f)$ be the number of poles of $f(z)$ in $|z| < r$, with poles of order p being counted p times. Also, let

$$N(r, f) \equiv \int_0^r t^{-1} n(t, f)\, dt.$$

The function $T(r, f) \equiv m(r, f) + N(r, f)$ is called the (Nevanlinna) characteristic function of f.

Let W be meromorphic in $\mathbb{C}$ and $m(z)$ be a meromorphic solution of (4.1). Let $c \in \mathbb{C}$ be such that $R(c) \equiv \{p: W(p) = c\} \neq \phi$. [By the Picard theorem, there exist at most two values of c for which $R(c) = \phi$.] Now, for $m_j \in R(c)$, $j = 1, \dots, k$, we have

$$\begin{aligned}\sum_{j=1}^{k} N\left(r, \frac{1}{m - m_j}\right) &\le N\left(r, \frac{1}{W \circ m - c}\right) \le T(r, W \circ m) + O(1)\\ &= T(r, m') + O(1)\\ &\le 2T(r, m) + o(T(r, m)), \quad \text{as } r \to \infty,\end{aligned}$$

where the first inequality is obvious and the second and third inequalities follow from the first fundamental theorem of Nevanlinna [see Hayman (1964), Theorem 1.2, page 7] and the Milloux theorem [see Hayman (1964), page 65], respectively.

By Wittich [(1968), page 63], we have for all $d \in \mathbb{C}$, except for a set Δ of small capacity,

$$\lim_{r \to \infty} N\left(r, \frac{1}{m - d}\right) \Big/ T(r, m) = 1.$$

Let $\varepsilon > 0$, then there exists $r(k, \varepsilon)$ such that for $r > r(k, \varepsilon)$,

$$N\left(r, \frac{1}{m - m_j}\right) > (1 - \varepsilon) T(r, m), \qquad j = 1, \dots, k.$$

There are infinitely many values c such that $R(c) \cap \Delta = \phi$ and therefore

$$k(1 - \varepsilon) T(r, m) \le 2T(r, m) + o(T(r, m)), \quad \text{as } r \to \infty.$$

This implies that $k \le 2$. By the Picard theorem, this could happen for at most two values of c, unless W is a rational function of degree 2. Therefore, W is a rational function of degree 2. Put $\nu = 1/m$, then $\nu' = -\nu^2 W(1/\nu)$ is a differential equation of the form (4.1) admitting a meromorphic solution. Hence, $\nu^2 W(1/\nu)$ is a rational function of degree two so that $W \in \mathscr{P}_2$. $\square$

The following corollary follows immediately from Theorem 4.1 and the results of Morris (1982).

COROLLARY 4.1. *Let (V, Ω) be a VF such that both V and μ admit a meromorphic continuation to $\mathbb{C}$. Then, up to an affine transformation, the corresponding family of distributions is either normal, Poisson, binomial, negative-binomial, gamma or generalized hyperbolic secant.* □

In the following remark, we provide easily utilizable conditions for a pair (V, Ω) not to be a VF.

REMARK 4.1. Consider a pair (V, Ω) such that:

(i) $\Omega = (a, b)$, $-\infty \le a < b \le \infty$, is the largest open interval on which V is positive real analytic;

(ii) $V \notin \mathscr{P}_2$ and admits a meromorphic continuation to $\mathbb{C}$; and

(iii) for $m_0 \in \Omega$,

$$\lim_{m \to a+} \int_m^{m_0} dt/V(t) = \lim_{m \to b-} \int_{m_0}^m dt/V(t) = \infty.$$

Then, by (2.4) and Corollary 4.1, (V, Ω) is not a VF.

There are many pairs (V, Ω) satisfying conditions (i) and (ii) but which violate condition (iii). In such cases, Remark 4.1 cannot be used to conclude that (V, Ω) is not a VF. For instance, if Ω is unbounded and V is a polynomial of degree ≥ 3, then condition (iii) is not satisfied. Nevertheless, many pairs can be excluded from being VF's by use of Remark 4.1. The following examples illustrate this.

EXAMPLES. Assume that (V, Ω) satisfies the following condition:

(iv) If $-\infty < a$, then $V(a+) = 0$, otherwise $V(m) = O(m)$, $m \to -\infty$; and if $b < \infty$, then $V(b-) = 0$, otherwise $V(m) = O(m)$, $m \to \infty$.

Since (iv) $\Rightarrow$ (iii) (i.e., $\Theta = \mathbb{R}$), no pair (V, Ω) satisfying (i), (ii) and (iv) can be a VF. For instance, the following pairs are easily seen not to be VF's:

(a) $\Omega = (0, \pi)$, $V(m) = \sin m$.

(b) $\Omega = (-\pi/2, \pi/2)$, $V(m) = \cos m$.

(c) $\Omega = \mathbb{R}^+$, $V(m) = \tanh m$.

(d) $\Omega = \mathbb{R}$, $p \in \mathscr{P}_n$ with $p(x) = \sum_{j=0}^n a_j x^j$ and $V(m) = \exp\{p(m)\}$; for V to satisfy condition (iv), it is necessary and sufficient that n is even and $a_n < 0$.

(e) $\Omega = \mathbb{R}^+$, $p \in \mathscr{P}_n$ with $p(x) = \sum_{j=1}^n a_j x^j$, $a_j \le 0$ for $j = 1, \dots, n$, and $V(m) = 1 - \exp\{p(m)\}$.

(f) V is a rational function, say $V = p/q$, with $p \in \mathscr{P}_l$ and $q \in \mathscr{P}_k$, $l \le k + 1$, being relatively prime polynomials and where either: (i) $\Omega = \mathbb{R}^+$, $V(\mathbb{R}^+) \subset \mathbb{R}^+$, $p(0) = 0$, $0 \notin q(\mathbb{R}^+)$; or (ii) $\Omega = \mathbb{R}$, $V(\mathbb{R}) \subset \mathbb{R}^+$, $0 \notin q(\mathbb{R})$.

THEOREM 4.2. *Let (V, Ω) be a VF such that μ admits a meromorphic continuation to $\mathbb{C}$. If V admits a unique analytic continuation to $\mathbb{C}$, except for isolated singularities, then $V \in \mathscr{P}_2$.*

PROOF. Let (V, Ω) be a VF such that μ is meromorphic in $\mathbb{C}$. By Corollary 3.1(i) and by our assumption on V, V is an analytic function on $\mu(\mathbb{C})$ satisfying the differential equation

$$\mu'(z) = V(\mu(z)), \qquad z \in \mathbb{C}. \tag{4.2}$$

By Corollary 3.1(ii), $\mu(\mathbb{C})$ is $\overline{\mathbb{C}}$ except possibly for two points. For proving the statement of the theorem, we act as follows. By making a suitable change of variable in (4.2), we define in $\mathbb{C}$ a meromorphic function V_1 which satisfies a differential equation of the same form as (4.2), and hence, by Theorem 4.1, V_1 reduces to a second degree polynomial. This, in turn, will lead us to the desired result. We consider two cases concerning the structure of $\mu(\mathbb{C})$. In the first case, $\mu(\mathbb{C})$ is $\overline{\mathbb{C}}$ except for two points and in the second case, $\mu(\mathbb{C})$ is $\overline{\mathbb{C}}$ except for one point. [Note that the case $\mu(\mathbb{C}) = \overline{\mathbb{C}}$ is covered by Theorem 4.1.]

CASE 1. $\mu(\mathbb{C}) = \overline{\mathbb{C}} \setminus \{\mu_0, \mu_1\}$, $\mu_0 \neq \mu_1$. Define

$$t = \begin{cases} (\mu - \mu_0)/(\mu - \mu_1) & \text{if } \mu_1 \neq \infty, \\ \mu - \mu_0 & \text{if } \mu_1 = \infty. \end{cases}$$

Then $t(\mathbb{C}) = \overline{\mathbb{C}} \setminus \{0, \infty\}$ and therefore there exists an entire function f such that $t(z) = e^{f(z)}$. Substituting for μ in terms of $t = e^f$ in (4.2), we get $f' = V_1(f)$, where

$$V_1(f) = \begin{cases} \dfrac{1}{\mu_0 - \mu_1} e^{-f}(e^f - 1)^2 V\left(\dfrac{e^f \mu_1 - \mu_0}{e^f - 1}\right) & \text{if } \mu_1 \neq \infty, \\ e^{-f} V(e^f + \mu_0) & \text{if } \mu_1 = \infty, \end{cases}$$

is meromorphic in $\mathbb{C}$. Since f is meromorphic, we conclude from Theorem 4.1 that $V_1(f) = af^2 + bf + c$, which, by resubstituting for f in terms of μ, yields

$$V(\mu) = \begin{cases} \dfrac{(\mu - \mu_0)(\mu - \mu_1)}{(\mu_0 - \mu_1)} \left[a \log^2\left(\dfrac{\mu - \mu_0}{\mu_1 - \mu_0}\right) + b \log\left(\dfrac{\mu - \mu_0}{\mu_1 - \mu_0}\right) + c \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mu_1 \neq \infty, \\ (\mu - \mu_0)\left[a \log^2(\mu - \mu_0) + b \log(\mu - \mu_0) + c\right] \qquad \text{if } \mu_1 = \infty. \end{cases}$$

But since V is analytic in $\mathbb{C}$, except for isolated singularities, we must have $a = b = 0$ and thus the desired result.

CASE 2. $\mu(\mathbb{C}) = \overline{\mathbb{C}} \setminus \{\mu_0\}$. If $\mu_0 = \infty$, then V is meromorphic in $\mathbb{C}$ and the desired result follows from Theorem 4.1. Accordingly, we assume $\mu_0 \neq \infty$. By Lemma 3.2, V does not vanish in $\mu(\mathbb{C})$. Therefore, $1/V$ is analytic in $\overline{\mathbb{C}} \setminus \{\mu_0\}$

and thus admits a Laurent expansion around μ_0 that converges in $\overline{\mathbb{C}} \setminus \{\mu_0\}$; that is, we can express $1/V$ as

$$1/V(\mu) = \sum_{n=-\infty}^{\infty} w_n/(\mu - \mu_0)^n, \qquad \mu \in \overline{\mathbb{C}} \setminus \{\mu_0\}.$$

Now, since $\infty \in \mu(\mathbb{C})$, there exists a $z_0 \in \mathbb{C}$ which is a pole of μ. Since V is analytic in $\overline{\mathbb{C}} \setminus \{\mu_0\}$, it follows from Lemma 3.1(iii) and (iv) that z_0 is a first order pole. Consider the positively oriented circle $\mathscr{C}_\varepsilon = \{z\colon |z - z_0| = \varepsilon\}$. For small ε, the image of $\mathscr{C}_\varepsilon$ in the μ-plane is a closed curve Γ_μ that is large enough to encompass any finite point (say, μ_0) only once and has a negative orientation. Thus, for $j < 0$, we have

$$(4.3) \qquad \oint_{\mathscr{C}_\varepsilon} (\mu - \mu_0)^j \, dz = \oint_{\Gamma_\mu} (\mu - \mu_0)^j \, d\mu/V(\mu) = -2\pi i w_{j+1}.$$

However, since $1/(\mu - \mu_0)$ is analytic in $\mathbb{C}$, it follows that the integral on the left-hand side of (4.3) vanishes. Thus, we conclude that

$$1/V(\mu) = \sum_{n=1}^{\infty} w_n/(\mu - \mu_0)^n.$$

Let $V_1(\mu) \equiv V(1/(\mu - \mu_0))$, then V_1 is meromorphic in $\mathbb{C}$. Define $t = (1 - \mu_0\mu)/\mu$; then t is meromorphic in $\mathbb{C}$ and satisfies

$$t' = -(t + \mu_0)^2 V_1(t) \equiv V_2(t),$$

where V_2 is meromorphic in $\mathbb{C}$. Thus, by Theorem 4.1, $V_1 \equiv$ constant and this leads to $V \equiv$ constant, a contradiction. This concludes the proof of the theorem. □

EXAMPLE. Let $(V, \Omega) = (R_n e^{1/S_k}, \Omega)$, where R_n and S_k are rational functions of degree n and k, respectively, $\Omega = (a, b)$ is an arbitrary open interval (finite or infinite), R_n is positive on Ω, S_k does not vanish on $\overline{\Omega}$ and $V \notin \mathscr{P}_2$. (Recall that the degree of a rational function $s = p/q$, $p \in \mathscr{P}_l \setminus \mathscr{P}_{l-1}$, $q \in \mathscr{P}_r \setminus \mathscr{P}_{r-1}$, where p and q are relatively prime, is $\max\{l, r\}$.) For such a pair (V, Ω), assume also that the interval Θ determined by (2.4) is $\mathbb{R}$. Then, since V is analytic in $\mathbb{C}$, except for essential singularities at zeros of S_k, Theorem 4.2 implies that (V, Ω) is not a VF. Note that Examples 3.1 through 3.3 are special cases of the present example.

5. Algebraic VF's with meromorphic means. In Section 1, we mentioned that Letac (1987a) suggested the study of the class of algebraic VF's of the form $P + Q\sqrt{R}$, where $P \in \mathscr{P}_3$, $Q \in \mathscr{P}_2$ and $R \in \mathscr{P}_2$. This class includes cubic VF's as a special case and its members possess the reciprocity property. Letac (1990) identified several NEF's with VF's of this form, some of which have meromorphic means.

In this section, we consider a more general class with the same form that includes the one suggested by Letac (1987a) as a special case, and we characterize those members having meromorphic means. After giving a general result (Theorem 5.1), we consider, in Corollary 5.1, VF's of the form $V = P + Q\sqrt{R}$, where P and Q are rational functions of arbitrary degree and $R \in \mathscr{P}_2$. We show that such VF's have meromorphic means if and only if $P = kR$ for some constant k and $Q \in \mathscr{P}_1$. In Corollary 5.2, we treat a class of VF's of the same form, with P and Q, however, being meromorphic functions. We show that if such VF's have entire mean functions, then, again, $P = kR$ and $Q \in \mathscr{P}_1$.

The proofs of the results of this section utilize Theorem 4.1. Although the functions V considered here are not meromorphic, by suitable transformations, we can define suitable meromorphic functions which permit the use of Theorem 4.1.

THEOREM 5.1. *Let (V, Ω) be a VF such that $V = P + Q\sqrt{R}$, where both P and Q admit meromorphic extensions to $\mathbb{C}$ and $R \in \mathscr{P}_2 \setminus \mathscr{P}_0$. Let the differential equation*

$$\mu' = V(\mu) = \left(P + Q\sqrt{R}\,\right)(\mu) \tag{5.1}$$

admit a meromorphic solution $\mu(z)$ on $\mathbb{C}$. Then, the following two conditions are equivalent:

(i) *The function*

$$(\mu' - P \circ \mu)/(Q \circ \mu) = (R \circ \mu)^{1/2}$$

is meromorphic.

(ii) *$P = kR$ for some constant k and $Q \in \mathscr{P}_1$.*

PROOF. (i) $\Rightarrow$ (ii): First, let $R \in \mathscr{P}_1 \setminus \mathscr{P}_0$ giving $R = a_1\mu + b_1$, where $a_1 \in \mathbb{R}^+$. By substituting $m = a_1\mu + b_1$ in (5.1), we get

$$m' = P^*(m) + m^{1/2}Q^*(m), \tag{5.2}$$

where P^* and Q^* are meromorphic functions. Put

$$x = (\mu' - P \circ \mu)/(Q \circ \mu) = (m' - P^* \circ m)/(Q^* \circ m).$$

Then x is meromorphic and satisfies $x^2 = m$. From (5.2), we get

$$x' = \left[P^*(x^2) + xQ^*(x^2)\right]/(2x). \tag{5.3}$$

Since the right-hand side of (5.3) is meromorphic and admits a meromorphic solution x, it follows from Theorem 4.1 that

$$x' = ax^2 + bx + c. \tag{5.4}$$

Equating the right-hand sides of (5.3) and (5.4) and substituting $x^2 = m$, we obtain

$$P^*(m) + m^{1/2}Q^*(m) = 2bm + 2(am + c)m^{1/2},$$

which implies that $P^*(m) = 2bm$ and $Q^*(m) = 2(am + c)$. Hence, $P(\mu) = 2b(a_1\mu + b_1) = 2bR$ and $Q = 2(aa_1\mu + ab_1 + c) \in \mathscr{P}_1$.

Next, let $R \in \mathscr{P}_2 \setminus \mathscr{P}_1$ and assume that $R = \xi_1 a_1^2(\mu - b_1)^2 + \xi_2 c_1^2$, where $a_1, c_1 \in \mathbb{R}^+$, $b_1 \in \mathbb{R}$ and $\xi_1, \xi_2 \in \{-1, 1\}$. By substituting $m = a_1(\mu - b_1)/c_1$ in (5.1), we get

$$m' = P^*(m) + \left(\xi_1 m^2 + \xi_2\right)^{1/2} Q^*(m), \tag{5.5}$$

where P^* and Q^* are meromorphic functions. Put

$$\begin{aligned} x &= \xi_1^{1/2} m + (m' - P^* \circ m)/(Q^* \circ m) \\ &= \xi_1^{1/2} a_1(\mu - b_1)/c_1 + (\mu' - P \circ \mu)/(c_1(Q \circ \mu)), \end{aligned} \tag{5.6}$$

where $0 \leq \arg \xi_1^{1/2} < \pi$. Then x is a meromorphic function which satisfies

$$x^2 - 2\xi_1^{1/2} m x - \xi_2 = 0.$$

Solving for m, we get $m = (x^2 - \xi_2)/(2\xi_1^{1/2}x)$, so that $m' = x'(x^2 + \xi_2)/(2\xi_1^{1/2}x^2)$. By substituting these in (5.5) and using (5.6), we get

$$x' = \frac{2\xi_1^{1/2}x^2}{x^2 + \xi_2}\left[P^*\left(\frac{x^2 - \xi_2}{2\xi_1^{1/2}x}\right) + \frac{x^2 + \xi_2}{2x} Q^*\left(\frac{x^2 - \xi_2}{2\xi_1^{1/2}x}\right)\right] \equiv W(x). \tag{5.7}$$

Since P^* and Q^* are meromorphic, W is meromorphic in $\mathbb{C} \setminus \{0\}$. We shall prove that W is meromorphic in all of $\mathbb{C}$. For this, we give an indirect proof. Assume that W is not meromorphic at $x = 0$. Then W must have an essential singularity there. We show that this implies that x does not attain the values zero and infinity in $\mathbb{C}$. Indeed, if x attains the value zero (infinity) at some point z_0, then the order of this zero (infinity) is necessarily finite, since x is meromorphic. In this case, $x' = W \circ x$ has an essential singularity at z_0. But this contradicts the fact that x' is meromorphic at z_0. Therefore, x is a nonvanishing entire function. Accordingly, let $f = \log x$, then f is an entire function and

$$f' = x'/x = W(x)/x = W(e^f)/e^f \equiv W_1(f),$$

where W_1 is meromorphic in $\mathbb{C}$. We therefore obtain, by Theorem 4.1, that $W_1(f) = af^2 + bf + c$ and therefore

$$W(x) = x\left[a(\log x)^2 + b \log x + c\right].$$

Since W is meromorphic in $\mathbb{C} \setminus \{0\}$, it follows that $a = b = 0$ and therefore $W(x) = cx$, a contradiction to the fact that W has an essential singularity at $x = 0$. This implies that W is meromorphic in $\mathbb{C}$. Hence, by Theorem 4.1, $W \in \mathscr{P}_2$ and thus (5.7) reduces to

$$x' = ax^2 + bx + c. \tag{5.8}$$

Equating the right-hand sides of (5.7) and (5.8) and using (5.5) and (5.6), we obtain

$$\xi_1 mP^*(m) + \xi_1^{1/2}Q^*(m)R^*(m) + \left(\xi_1 mQ^*(m) + \xi_1^{1/2}P^*(m)\right)[R^*(m)]^{1/2}$$
$$= \left(2a\xi_1 m^2 + b\xi_1^{1/2}m + a\xi_2 + c\right)[R^*(m)]^{1/2} + \left(2a\xi_1^{1/2}m + b\right)R^*(m),$$

where $R^*(m) = \xi_1 m^2 + \xi_2$. This latter relation implies that

$$\xi_1 mP^*(m) + \xi_1^{1/2}Q^*(m)R^*(m) = \left(2a\xi_1^{1/2}m + b\right)R^*(m) \tag{5.9}$$

and

$$\xi_1^{1/2}P^*(m) + \xi_1 mQ^*(m) = 2a\xi_1 m^2 + b\xi_1^{1/2}m + a\xi_2 + c. \tag{5.10}$$

Relation (5.9) implies that P^* divides R^* and hence $P^*(m) = K(m)R^*(m)$, where K is a polynomial. Putting this in (5.9) and (5.10) and solving for K and Q^* yields

$$K(m) \equiv k = a/\xi_1^{1/2} + c/\left(\xi_2\xi_1^{1/2}\right), \quad Q^*(m) = [(a\xi_2 - c)/\xi_2]m + b/\xi_1^{1/2}.$$

Finally, by substituting $m = a_1(\mu - b_1)/c_1$, we obtain the desired result.

(ii) $\Rightarrow$ (i): By using affinity, we can assume without loss of generality that $R = m$ if $R \in \mathscr{P}_1 \setminus \mathscr{P}_0$ and $R = \xi_1 m^2 + \xi_2$, $\xi_1, \xi_2 \in \{-1, 1\}$ if $R \in \mathscr{P}_2 \setminus \mathscr{P}_1$.

Let $R \in \mathscr{P}_1 \setminus \mathscr{P}_0$, then $V = km + (\alpha m + \beta)m^{1/2}$, where k, α and β are constants. Put $b = k/2$, $a = \alpha/2$ and $c = \beta/2$, then

$$V = 2bm + 2(am + c)m^{1/2}.$$

Let $x^2 = m$, then $x' = ax^2 + bx + c$. The solutions $x(z)$ of the latter differential equation have the forms: $(\alpha_1 z + \beta_1)/(\alpha_2 z + \beta_2)$ or $(\alpha_1 e^{\gamma z} + \beta_1)/(\alpha_2 e^{\gamma z} + \beta_2)$, where α_1, α_2, β_1, β_2 and γ are constants and hence $x(z)$ is meromorphic. Thus, $m = x^2(z)$ and $(m' - P \circ m)/(Q \circ m) = x(z)$ are also meromorphic.

Let $R \in \mathscr{P}_2 \setminus \mathscr{P}_1$, then $V = k(\xi_1 m^2 + \xi_2) + (\alpha m + \beta)(\xi_1 m^2 + \xi_2)^{1/2}$. Put $b = \xi_1^{1/2}\beta$ and define a and c as the solutions of the linear system: $\alpha = (a\xi_2 - c)/\xi_2$, $\beta = (a\xi_2 + c)/(\xi_1^{1/2}\xi_2)$. For arbitrary α and β, this system defines a and c uniquely. Define x by

$$x^2 - 2\xi_1^{1/2}mx - \xi_2 = 0,$$

then $x' = ax^2 + bx + c$. Thus x is meromorphic and so is $m = (x^2 - \xi_2)/(2\xi_1^{1/2}x)$. Hence

$$(m' - P \circ m)/(Q \circ m) = x - \xi_1^{1/2}m$$

is meromorphic. □

Corollary 5.1. *Let (V, Ω) be a VF such that $V(\mu) = (P + Q\sqrt{R})(\mu)$, where P and Q are rational and $R \in \mathscr{P}_2 \setminus \mathscr{P}_0$. Then, the following two conditions are equivalent*:

(i) *(V, Ω) possesses a mean function μ, which admits a meromorphic extension to $\mathbb{C}$.*

(ii) *$V = kR + Q\sqrt{R}$ for some constant k, where $Q \in \mathscr{P}_1$.*

In either case, under a suitable transformation, V has the form

$$(5.11)\quad V(m)=\begin{cases}2bm+2(am+c)m^{1/2} & \text{if } R\in\mathscr{P}_1\setminus\mathscr{P}_0,\\ \left(\dfrac{a\xi_2+c}{\xi_1^{1/2}\xi_2}\right)(\xi_1 m^2+\xi_2) & \\ +\left[\left(\dfrac{a\xi_2-c}{\xi_2}\right)m+\dfrac{b}{\xi_1^{1/2}}\right](\xi_1 m^2+\xi_2)^{1/2} & \text{if } R\in\mathscr{P}_2\setminus\mathscr{P}_1,\end{cases}$$

where $\xi_1,\xi_2\in\{-1,1\}$, *and* $a,b,c\in\mathbb{C}$. *Moreover,* $m(z)$ *is given by*

$$m(z)=\begin{cases}x^2(z) & \text{if } R\in\mathscr{P}_1\setminus\mathscr{P}_0,\\ (x^2(z)-\xi_2)/(2\xi_1^{1/2}x(z)) & \text{if } R\in\mathscr{P}_2\setminus\mathscr{P}_1,\end{cases}$$

where $x(z)$ *is the solution of the differential equation*

$$x'=ax^2+bx+c.$$

PROOF. (i) $\Rightarrow$ (ii): Since P and Q are rational and μ is meromorphic, it follows that μ', $P\circ\mu$, and $Q\circ\mu$ are meromorphic and, hence, so is $(\mu'-P\circ\mu)/(Q\circ\mu)$. Thus, by Theorem 5.1, (ii) holds and, under affinity, V has the form (5.11).

(ii) $\Rightarrow$ (i): The proof is an immediate consequence of Theorem 5.1. □

COROLLARY 5.2. *Let* (V,Ω) *be a VF such that* $V(\mu)=(P+Q\sqrt{R})(\mu)$, *where both P and Q admit meromorphic extensions to* $\mathbb{C}$ *and* $R\in\mathscr{P}_2\setminus\mathscr{P}_0$. *If* (V,Ω) *possesses a mean function which admits an entire extension to* $\mathbb{C}$, *then* $V=kR+Q\sqrt{R}$ *for some constant k and* $Q\in\mathscr{P}_1$.

PROOF. Since P and Q are meromorphic and μ is entire, it follows that μ', $P\circ\mu$ and $Q\circ\mu$ are meromorphic. Applying Theorem 5.1 completes the proof. □

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REFERENCES

AHLFORS, L. V. (1966). *Complex Analysis*, 2nd ed. McGraw-Hill, New York.

BAR-LEV, S. K. (1987). Contribution to discussion of paper by B. Jørgensen: Exponential dispersion models. *J. Roy. Statist. Soc. Ser. B* **49** 153–154.

BAR-LEV, S. K. and BSHOUTY, D. (1989). Rational variance functions. *Ann. Statist.* **17** 741–748.

BAR-LEV, S. K. and ENIS, P. (1986). Reproducibility and natural exponential families with power variance functions. *Ann. Statist.* **14** 1507–1522.

BARNDORFF-NIELSEN, O. (1978). *Information and Exponential Families in Statistical Theory.* Wiley, New York.

BURRIDGE, J. (1986). Mean-variance relationships, generalized linear models and GLIM. Unpublished manuscript.

HAYMAN, W. K. (1964). *Meromorphic Functions*. Oxford Univ. Press.

JØRGENSEN, B. (1984). Exponential dispersion models, Reprint 9, Dept. Mathematics, Odense University.

JØRGENSEN, B. (1986). Some properties of exponential dispersion models. *Scand. J. Statist.* **13** 187–198.

JØRGENSEN, B. (1987). Exponential dispersion models (with discussion). *J. Roy. Statist. Soc. Ser. B* **49** 127–162.

KAWATA, T. (1972). *Fourier Analysis in Probability Theory*. Academic, New York.

LETAC, G. (1987a). Contribution to discussion of paper by B. Jørgensen: Exponential dispersion models. *J. Roy. Statist. Soc. Ser. B* **49** 154.

LETAC, G. (1987b). Cours sur les familles exponentielles. Mimeographed Notes, Laboratoire de Statistique et Probabilités, Université Paul Sabatier.

LETAC, G. (1990). Classification of natural exponential families with variance functions $(am + c)\Delta + b\sqrt{\Delta}$. Unpublished manuscript.

LETAC, G. and MORA, M. (1986). On the variance functions of natural exponential families on $\mathbb{R}$. *C. R. Acad. Sci. Paris Ser. A–B* **302** 551–554.

LETAC, G. and MORA, M. (1990). Natural real exponential families with cubic variance functions. *Ann. Statist.* **18** 1–37.

LUKACS, E. (1970). *Characteristic Functions*, 2nd ed. Hafner, New York.

MORA, M. (1986). A classification of cubic variance functions of exponential families on $\mathbb{R}$. *C. R. Acad. Sci. Paris Ser. A–B* **302** 587–590.

MORRIS, C. N. (1982). Natural exponential families with quadratic variance functions. *Ann. Statist.* **10** 65–80.

NEVANLINNA, R. (1970). *Analytic Functions*. Springer, Berlin.

RELLICH, F. (1940). Über die ganzen Lösungen einer gewöhnlichen Differentialgleichung erster Ordnung. *Math. Ann.* **117** 587–589.

TWEEDIE, M. C. K. (1984). An index which distinguishes between some important exponential families. In *Statistics: Applications and New Directions. Proceedings of the Indian Statistical Institute Golden Jubilee International Conference* (J. K. Ghosh and J. Roy, eds.) 579–604. Indian Statistical Institute, Calcutta.

WITTICH, H. (1968). *Neuere Untersuchungen über eindeutige analytische Funktionen*. Springer, Berlin.

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