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BRANCHING PARTICLE SYSTEMS AND SUPERPROCESSES[1]

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We start from a model of a branching particle system with immigration and with death rate and branching mechanism depending on time and location. Then we consider a limit case when the mass of particles and their life times are small and their density is high. This way, we construct a measure-valued process X_t which we call a superprocess. Replacing the underlying Markov process ξ_t by the corresponding "historical process" $\xi_{\le t}$, we construct a measure-valued process M_t in functional spaces which we call a historical superprocess. The moment functions for superprocesses are evaluated. Linear positive additive functionals are studied. They are used to construct a continuous analog of a random tree obtained by stopping every particle at a time depending on its path (say, at the first exit time from a domain). A related special Markov property for superprocesses is proved which is useful for applications to certain nonlinear partial differential equations.

The concluding section is devoted to a survey of the literature, and the terminology on Markov processes used in the paper is explained in the Appendix.

1. Main results.

1.1. *Branching particle systems.* Such a system is determined by three parameters:

(a) a Markov process $\xi = (\xi_t, \mathscr{F}(\Delta), \Pi_{r,x})$;
(b) a positive continuous additive functional K of ξ;
(c) a generating function

$$\varphi^t(x, z) = \sum_{n=0}^{\infty} p_n^t(x) z^n,$$

where $p_n^t(x) \ge 0$ and $\sum_n p_n^t(x) = 1$.

The system is characterized by the following properties:

(i) Each particle has random birth and death times.

(ii) Given that a particle is born at time r at point x, the conditional distribution of its path is determined by $\Pi_{r,x}$.

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(iii) Given the path ξ, the conditional probability of survival during time (s, t) is equal to

$$H(s,t) = e^{-K(s,t)}.$$

(iv) The only interaction between the particles is that the birth time and place of offspring coincide with the death time and place of their parent.

(v) For a particle which dies at time t at point x, the number of offspring is a random variable with the generating function $\varphi^t(x, z)$.

Conditions (i)–(v) describe a branching particle system heuristically. A rigorous construction is given in Section 2.

The state space of the process ξ at time t is an arbitrary measurable space $(E_t, \mathscr{B}_t)$. *The global state space* $\mathscr{E}$ is the set of pairs $t \in \mathbb{R}$, $x \in E_t$. We denote by $\mathscr{B}_{\mathscr{E}}$ the σ-algebra in $\mathscr{E}$ generated by functions $f: \mathscr{E} \to \mathbb{R}$ with the following properties:

(α) For every $t \in \mathbb{R}$, $f^t(x)$ is $\mathscr{B}_t$-measurable.

(β) For every $r < u \in \mathbb{R}$, the restriction of $f^t(\xi_t)$ to (r, u) is measurable relative to $\mathscr{B}(r, u) \times \mathscr{F}(r, u)$, where $\mathscr{B}(r, u)$ is the Borel σ-algebra in (r, u).

The set $\mathscr{E}(\Delta) = \{(r, x): r \in \Delta,\ x \in E_r\}$ belongs to $\mathscr{B}_{\mathscr{E}}$ for every interval Δ. We put $\mathscr{E}_{\le t} = \mathscr{E}(-\infty, t]$.

We say that a function f is *progressive* if it is measurable with respect to $\mathscr{B}_{\mathscr{E}}^*$. (Superscript * indicates the *universal completion* of a σ-algebra.)

We assume that:

1.1.A. The transition probabilities are progressive, that is, the function

$$f^t(x) = 1_{t<u}\Pi_{t,x}\{\xi_u \in B\}$$

is progressive for every $u \in \mathbb{R}$, $B \in \mathscr{B}_u$.

1.1.B. The functions $p_n^t(x)$, $n = 0, 1, \dots,$ are progressive.

In addition, to simplify the presentation, we concentrate on the subcritical and critical case, that is, we suppose that

1.1.C. The expected number of offspring

$$a_1 = \sum_{n=0}^{\infty} np_n^t(x) \le 1$$

for all t, x.

Let $Y_t(B)$ be the number of particles at time t in a set B. Properties (i)–(v) imply that Y_t is a Markov process in the space of integer-valued measures (i.e., measures with values $0, 1, \dots, n, \dots, +\infty$). Its transition probabilities $Q_{r,\nu}$

satisfy the condition

$$Q_{r,\nu}\exp\langle -f, Y_t\rangle = \prod_{i=1}^{\infty} Q_{r,\nu_i}\exp\langle -f, Y_t\rangle \tag{1.1}$$

for $\nu = \sum \nu_i$ (here $\langle f, Y_t\rangle$ means the integral of f with respect to the measure Y_t). Besides, if δ_x is the unit mass concentrated at x, then the function

$$w_t^r(x) = Q_{r,\delta_x}\exp\langle -f, Y_t\rangle \quad \text{for } r \le t, \qquad w_t^r(x) = 0 \quad \text{for } r > t \tag{1.2}$$

is progressive in r, x and satisfies the equation

$$\begin{aligned} w_t^r(x) = \Pi_{r,x}\Bigg[&H(r,t)\exp\{-f(\xi_t)\} \\ &+ \int_r^t H(r,s)K(ds)\varphi^s(\xi_s, w_t^s(\xi_s))\Bigg] \quad \text{for } r \le t. \end{aligned} \tag{1.3}$$

Formula (1.3) has a clear heuristic meaning: The first term in the brackets corresponds to the case when the particle born at time r at point x is still alive at time t, and the second term corresponds to the case when it dies at time $s \in (r, t)$.

We consider the process Y_t on the time interval $\mathbb{R}$ and we assume that

$$Q_{r,\nu}\{Y_t = 0 \text{ for all } t < r\} = 1,$$

which means that there is no particle until time r when a bunch of them "immigrate" at locations described by ν. A more general situation—each particle immigrates at its own time—is described by the stochastic process (Y_t, Q_η), where $\eta = \sum_i \delta_{t_i x_i}$ is a configuration in $\mathscr{E}$ and Q_η is the convolution of the measures $Q_{t_i,\delta_{x_i}}$.

By 1.1.C,

$$\phi^t(x,z) = \varphi^t(x,z) - z \ge 0 \quad \text{for } (t,x) \in \mathscr{E}, 0 \le z \le 1. \tag{1.4}$$

Formula (1.3) implies that

$$w_t^r(x) = \Pi_{r,x}\int_r^t \phi^s[\xi_s, w_t^s(\xi_s)]K(ds) + \Pi_{r,x}\exp\{-f(\xi_t)\} \quad \text{for } r \le t \tag{1.5}$$

(see Section 2.3). It follows from (1.1) that, for every integer-valued measure ν,

$$Q_{r,\nu}\exp\langle -f, Y_t\rangle = \exp\langle \log w_t^r, \nu\rangle, \tag{1.6}$$

and therefore, for every measure π on the space of integer-valued measures,

$$Q_{r,\pi}\exp\langle -f, Y_t\rangle = \int \pi(d\nu)\exp\langle \log w_t^r, \nu\rangle. \tag{1.7}$$

If π_μ is the Poisson measure with intensity μ, then

$$\int \pi_\mu(d\nu)e^{\langle f,\nu\rangle} = \exp\langle e^f - 1, \mu\rangle$$

and, by (1.7) and (1.5),

$$Q_{r,\pi_\mu} \exp\langle -f, Y_t\rangle = \exp\langle -v_t^r, \mu\rangle, \tag{1.8}$$

where $v_t^r = 1 - w_t^r$ satisfies the equation

$$v_t^r(x) + \Pi_{r,x}\int_r^t \Psi^s(\xi_s, v_t^s(\xi_s))K(ds) = \Pi_{r,x}[1 - \exp\{-f(\xi_t)\}] \quad \text{for } r < t \tag{1.9}$$

with

$$\Psi^t(x,z) = \phi^t(x, 1-z). \tag{1.10}$$

More generally,

$$Q_{\pi_\eta} \exp\langle -f, Y_t\rangle = \exp\langle -v_t, \eta\rangle, \tag{1.11}$$

where π_η is a Poisson random measure on $(\mathscr{E}, \mathscr{B}_{\mathscr{E}})$ with intensity η, $v_t^r = 0$ for $r > t$, and $\langle v, \eta\rangle = \int_{\mathscr{E}} v^r(x)\eta(dr, dx)$.

1.2. *Passage to the limit.* Imagine a branching system of small particles, with short life, distributed with high density over the state space. More precisely, consider a branching particle system $(\xi, K/\beta, \varphi_\beta)$ which depends on a small positive parameter β, with an initial ("immigration") distribution η. Let Q_η^β be the probability law of the corresponding process Y_t and let ϕ_β and Ψ_β be defined by (1.4) and (1.10) with φ replaced by φ_β. If every particle has mass β, then βY_t is the mass distribution at time t. By (1.11) and (1.9),

$$Q_{\pi_{\eta/\beta}}^\beta \exp\langle -f, \beta Y_t\rangle = \exp\langle -v_t(\beta), \eta\rangle \tag{1.12}$$

and

$$v_t^r(\beta, x) + \Pi_{r,x}\int_r^t \psi_\beta^s[\xi_s, v_t^s(\beta, \xi_s)]K(ds) = \Pi_{r,x}F_\beta(\xi_t) \quad \text{for } r \le t, \tag{1.13}$$

where

$$\psi_\beta^t(x,z) = \Psi_\beta^t(x, \beta z)\beta^{-2}, \qquad F_\beta(x) = (1 - e^{-\beta f(x)})\beta^{-1}. \tag{1.14}$$

Suppose that

$$0 \le f(x) \le c \quad \text{for all } x. \tag{1.15}$$

Then

$$|F_\beta(x) - f(x)| \le \tfrac{1}{2}\beta c^2. \tag{1.16}$$

Hence $F_\beta(x) \to f(x)$ as $\beta \to 0$.

Let

$$\begin{aligned}\varphi_\beta^t(x,z) &= F_k[b^t(x), z] \\ &\quad + \beta^2\int_0^\beta [e^{-u/\beta}e^{uz/\beta} - 1 + u(1-z)\beta^{-1}]n^t(x, du),\end{aligned} \tag{1.17}$$

where $F_k(b,z) = b/k + [1 - b/(k-1)]z + bz^n/k(k-1)$, $b^t(z)$ is a bounded positive progressive function and $n^t(x, du)$ is a kernel from $(\mathscr{E}, \mathscr{B}^*_{\mathscr{E}})$ to $(0, +\infty)$ such that

$$\int_0^1 u^2 n^t(x, du) \quad \text{and} \quad \int_1^\infty u n^t(x, du) \tag{1.18}$$

are bounded functions on $\mathscr{E}$ and $\int_N^\infty u n^t(x, du) \to 0$ uniformly in (t, x) as $N \to \infty$. For sufficiently large k, formula (1.17) represents a generating function and the corresponding function ψ_β is given by the formula

$$\begin{aligned} \psi_\beta^t(x, z) = {} & \frac{b^t(x)}{k(k-1)}\left[(1-\beta z)^k - 1 + k\beta z\right]\beta^{-2} \\ & + \int_0^{\beta^{-1}} (e^{-uz} - 1 + zu) n^t(x, du), \end{aligned} \tag{1.19}$$

which tends to

$$\psi^t(x, z) = \tfrac{1}{2} b^t(x) z^2 + \int_0^\infty (e^{-uz} - 1 + zu) n^t(x, du) \tag{1.20}$$

as $\beta \to 0$. By (1.18), ψ is differentiable with respect to z and

$$D_z \psi^t(x, z) = b^t(x) z + \int_0^\infty (1 - e^{-uz}) u n^t(x, du); \tag{1.21}$$

in particular,

$$D_z \psi^t(x, 0) = 0. \tag{1.21a}$$

It can be deduced from a result of Kawazu and Watanabe [22] and the results in Section 3 that (1.20) is the most general form of a function ψ which can occur as the limit of ψ_β given by (1.14) under conditions 3.1.A, B and (1.21a). A simple direct proof was given recently by Li [27].

Note that the integrals (1.18) converge for $n(du) = \text{const}\ u^{-2-\alpha}\, du$ with $0 < \alpha < 1$ and that the corresponding $\psi(z) = \text{const}\ z^{1+\alpha}$ [assuming that $b^t(z) = 0$].

According to [4], a *Luzin* (*Radon*) *space* is a measurable space $(E, \mathscr{B})$ which is isomorphic to $(\tilde{E}, \tilde{\mathscr{B}})$ where $\tilde{E}$ is a Borel (correspondingly, a universally measurable) set in a metrizable compact space and $\tilde{\mathscr{B}}$ is the trace of the Borel (universally measurable) σ-algebra on $\tilde{E}$. We write $f \in \mathscr{B}$ if f is a positive $\mathscr{B}$-measurable function. We denote by $\mathscr{M}_t$ the set of all finite measures on $(E_t, \mathscr{B}_t)$ with the measurable structure generated by the functions $F(\mu) = \mu(B)$, $B \in \mathscr{B}_t$. We say that a measure η on $(\mathscr{E}, \mathscr{B}_{\mathscr{E}})$ is admissible and we write $\eta \in \mathfrak{M}(\mathscr{E})$ if $\eta(\mathscr{E}_{\le t}) < \infty$ for all t.

THEOREM 1.1. *Suppose that*:

1.2.A. *For each* t, $(E_t, \mathscr{B}_t)$ *is a Radon space*.

1.2.B. ξ *is a Markov process with progressive transition probabilities*.

1.2.C. *K is an additive functional of ξ with the properties*:

(α) *for every* $q > 0$, $r < t \in \mathbb{R}$ *and* $x \in E_r$, $\Pi_{r,x} e^{qK(r,t)} < \infty$;
(β) *for every* $t_0 < t \in \mathbb{R}$, *there exists a constant k such that* $\Pi_{r,x} K(r,t) \le k$ *for all* $r \in [t_0, t)$, $x \in E_r$.

1.2.D. *ψ is given by formula* (1.20) *with b and n subject to the conditions listed above.*

Then there exists an $\mathscr{M}_t$-valued Markov process $X = (X_t, \mathscr{G}(\Delta), P_{r,\mu})$ *such that, for every* $t \in \mathbb{R}$ *and every bounded* $f \in \mathscr{B}_t$,

$$P_{r,\mu} \exp\langle -f, X_t \rangle = \exp\langle -v^r, \mu \rangle, \tag{1.22}$$

where $v^r(x)$ is a progressive function determined uniquely by the equations

$$\begin{gathered} v^r(x) + \Pi_{r,x} \int_r^t \psi^s[\xi_s, v^s(\xi_s)] K(ds) = \Pi_{r,x} f(\xi_t) \quad \textit{for } r \le t, \\ v^r(x) = 0 \quad \textit{for } r > t. \end{gathered} \tag{1.23}$$

Write v_t for v to indicate explicitly its dependence on t. To every $\eta \in \mathfrak{M}(\mathscr{E})$ *there corresponds an $\mathscr{M}_t$-valued Markov process* (X_t, P_η) *such that*

$$P_\eta \exp\langle -f, X_t \rangle = \exp\langle -v_t, \eta \rangle \quad \textit{for every } t \in \mathbb{R}. \tag{1.24}$$

We call X *the superprocess with parameters* (ξ, K, ψ). Note that X and ξ are defined on unrelated sample spaces. We denote them, respectively, Ω and Ω°.

REMARK 1. If we assume that $\Pi_{r,x}\{\xi_t = \partial_t\} = 1$ for all $t < r$, where ∂_t is an extra state added to E_t, and if we put $f(\partial_t) = 0$ for all $f \in \mathscr{B}_t$, then (1.23) is equivalent to the equation

$$v^r(x) + \Pi_{r,x} \int_r^\infty \psi^s[\xi_s, v^s(\xi_s)] K(ds) = \Pi_{r,x} f(\xi_t) \quad \text{for all } r \in \mathbb{R} \tag{1.25}$$

(cf. Section 0.2 in the Appendix).

REMARK 2. We assume that

$$P_\eta Y = P_\eta P_{r, X_r} Y \tag{1.26}$$

for all $Y \in \mathscr{G}_{\ge r}$ and all $\eta \in \mathfrak{M}(\mathscr{E})$ concentrated on $\mathscr{E}_{\le r}$. If $\mathscr{G}_{\ge r}$ is generated by X_t, $t \ge r$, then it follows from (1.22), (1.23) and (1.24). In general this is a slightly stronger form of condition 0.1.B which is a part of the definition of a Markov process.

REMARK 3. Suppose that $a^r(x)$ is a strictly positive progressive function and let $\tilde{\psi}^s(x,z) = \psi^s(x,z)/a^s(x)$, $\tilde{K}(ds) = a^s(\xi_s)K(ds)$. Then

$$\tilde{\psi}^s(\xi_s, h^s(\xi_s))\tilde{K}(ds) = \psi^s(\xi_s, h^s(\xi_s))K(ds)$$

for any positive progressive function h. Therefore the superprocess with parameters $(\xi, \tilde{K}, \tilde{\psi})$ is identical with the superprocess with parameters (ξ, K, ψ).

Theorem 1.1 can be extended to certain classes of infinite measures. Let ρ be a strictly positive progressive function such that: (a) for every t, ρ^t is bounded; (b) for every $r < t \in \mathbb{R}$, there exists a constant a_t^r such that

$$\Pi_{r,x}\rho^t(\xi_t) \le a_t^r \rho^r(x) \quad \text{for all } x \in E_t. \tag{1.27}$$

We denote by $\mathscr{M}_t^\rho$ the space of all measures μ on $(E_t, \mathscr{B}_t)$ such that $\langle \rho^t, \mu \rangle < \infty$. This implies: For every $\mu \in \mathscr{M}_r^\rho$, the measure

$$\nu(B) = \Pi_{r,\mu}\{\xi_t \in B\}$$

belongs to $\mathscr{M}_t^\rho$. Theorem 1.1 remains true with $\mathscr{M}_t$ replaced by $\mathscr{M}_t^\rho$.

By applying formulas (1.22) and (1.23) to uf and by differentiating with respect to u at $u = 0$, we get the relation

$$P_{r,\mu}\langle f, X_t \rangle = \Pi_{r,\mu} f(\xi_t). \tag{1.28}$$

Analogously (1.24) implies

$$P_\eta \langle f, X_t \rangle = \Pi_\eta f(\xi_t). \tag{1.29}$$

Here (ξ_t, Π_η) is the Markov process with transition probabilities $\Pi_{r,x}$ and the initial distribution η (see Section 0.2).

In Section 5 we investigate the moments of order m under the following assumption:

1.2.E_m. $\int_1^\infty u^m n^t(x, du)$ is a bounded function.

Condition 1.2.E_1 is satisfied automatically because of (1.18). Under condition 1.2.E_2 we prove that, for all $t_1, t_2 \in \mathbb{R}$, $f_1 \in \mathscr{B}_{t_1}$, $f_2 \in \mathscr{B}_{t_2}$, $\eta \in \mathfrak{M}(\mathscr{E})$,

$$\begin{aligned} P_\eta \langle f_1, X_{t_1} \rangle \langle f_2, X_{t_2} \rangle &= \Pi_\eta f_1(\xi_{t_1}) \Pi_\eta f_2(\xi_{t_2}) \\ &\quad + \Pi_\eta \int q_2^s(\xi_s) \Pi_{s,\xi_s} f_1(\xi_{t_1}) \Pi_{s,\xi_s} f_2(\xi_{t_2}) K(ds), \end{aligned} \tag{1.30}$$

where

$$q_2^s(x) = D_z^2 \psi^s(x, 0) = b^s(x) + \int_0^\infty u^2 n^s(x, du).$$

This can be proved in a way similar to the proof of (1.29) for $f_1 = f_2$, $t_1 = t_2$ and then generalized by using (1.26) and the polarization in f. Instead we get (1.30) as a particular case of the general expression for the moment functions of all orders.

1.3. *Branching particle systems as measure-valued processes in functional spaces. Historical superprocesses.* The complete picture of a branching particle system is given not by the process Y_t but by the random tree composed of the paths of all particles. Introduction of historical superprocesses makes it possible to preserve this information under the limit procedure of Section 1.2.

A path with the birth time α is a collection of points w_t such that $w_t = \partial_t$ for $t < \alpha$ and $w_t \in E_t$ for $t \geq \alpha$. We assume that the process ξ is *canonical*, i.e., the sample space is a set W of paths, $\xi_t(w) = w_t$ for each $w \in W$ and $\mathscr{F}(\Delta)$ is generated by $\xi_t(w)$, $t \in \Delta$. We denote by $w(\Delta)$ the restriction of $w \in W$ to Δ and by $W(\Delta)$ the image of W under this mapping. Let $\mathscr{F}^\circ(\Delta)$ stand for the σ-algebra in $W(\Delta)$ generated by ξ_t, $t \in \Delta$. Note that $B \in \mathscr{F}^\circ(\Delta)$ if and only if the inverse image of B under the mapping $w \to w(\Delta)$ belongs to $\mathscr{F}(\Delta)$. We write $W_{\leq t}$ for $W(-\infty, t]$. The notation $\mathscr{F}_{\leq t}, \mathscr{F}^\circ_{\leq t}, W_{\geq t}, w_{\leq t}$ etc. has an analogous meaning.

Let $r < t < u$. Suppose that $w' \in W[r, t]$, $w'' \in W[t, u]$. We write $w = w' \vee w''$ if $w_s = w'_s$ for $s \in [r, t]$ and $w_s = w''_s$ for $s \in [t, u]$ (consequently, $w'_t = w''_t$). We assume that:

1.3.A. If $w' \in W[r, t]$, $w'' \in W[t, u]$ and if $w = w' \vee w''$, then $w \in W[r, u]$.

1.3.B. For every t, $(W_{\leq t}, \mathscr{F}^\circ_{\leq t})$ is a Radon space.

Both conditions are satisfied in the following typical situation: The spaces $(E_t, \mathscr{B}_t)$ can be imbedded isomorphically into a compact metrizable C in such a way that:

(α) E_t is a Borel subset of C.
(β) W consists of all right-continuous functions with left limits such that $w_t \in E_t$ for all t.
(γ) If $w_{t_n} \in E_{t_n}$ and if $w_{t_n} \to w \in W$ as $t_n \downarrow t$, then $w \in E_t$.

Condition 1.3.A holds independently of (γ). Condition 1.3.B follows from a result of Dellacherie and Meyer (see [4], 4-18 and 4-19).

With a canonical process $\xi = (\xi_t, \mathscr{F}(\Delta), \Pi_{r,x})$, another Markov process $\Xi = (\xi_{\leq t}, \mathscr{F}_\Xi(\Delta), \Pi_{r, x_{\leq r}})$ in $(W_{\leq t}, \mathscr{F}^\circ_{\leq t})$ is associated which we call the *historical process for* ξ. It is defined by the formulas

$$(1.31) \qquad \xi_{\leq t}(w) = w_{\leq t}, \qquad \mathscr{F}_\Xi(r, u) = \mathscr{F}_{<u},$$

$$(1.32) \qquad \int_W F(w)\Pi_{r, x_{\leq r}}(dw) = \int_{W'} F(x_{\leq r} \vee w_{\geq r})\Pi_{r, x_r}(dw_{\geq r}).$$

Here W' is the set of all $w \in W$ such that $w_r = x_r$. (Note that $\mathscr{F}_\Xi[r, \infty) = \mathscr{F}(\mathbb{R})$ for all r and that $\Pi_{r, x_{\leq r}}$ is the image of Π_{r, x_r} under the map $j \colon W' \to W$ given by the formula $j(w) = x_{\leq r} \vee w_{\geq r}$.) The global state space for Ξ will be denoted $(\mathscr{W}, \mathscr{B}_{\mathscr{W}})$. There exists an obvious correspondence between measures on $(W_{\leq t}, \mathscr{F}^\circ_{\leq t})$, measures on $(W, \mathscr{F}_{\leq t})$ and measures on $(W, \mathscr{F}^*_{\leq t})$. By applying Theorem 1.1 to Ξ we get the following result.

THEOREM 1.2. *Suppose that a historical process* Ξ, *its additive functional* K *and a function* $\psi^t(x_{\leq t}, z)$ *satisfy conditions* 1.2.A–1.2.D. *Then there exists a*

Markov process $M = (M_t, \mathscr{G}(\Delta), P_{r,N})$ *on the space* $\mathscr{M}_{\leq t}$ *of all finite measures on* $(W, \mathscr{F}^*_{\leq t})$ *such that, for every* $t \in \mathbb{R}$ *and every* $F \in \mathscr{F}^*_{\leq t}$,

$$P_{r,N} \exp\langle -F, M_t\rangle = \exp\langle -V^r, N\rangle, \tag{1.33}$$

where $V^r(x_{\leq r})$ *is a progressive function determined uniquely by the equations*

$$\begin{aligned} V^r(x_{\leq r}) + \Pi_{r, x_{\leq r}} \int_r^t \psi^s[\xi_{\leq s}, V^s(\xi_{\leq s})] K(ds) & \\ = \Pi_{r, x_{\leq r}} F \quad & \textit{for } r \leq t, \\ V^r = 0 \qquad\qquad & \textit{for } r > t. \end{aligned} \tag{1.34}$$

To every $\Gamma \in \mathfrak{M}(\mathscr{W})$ *there corresponds an* $\mathscr{M}_{\leq t}$*-valued stochastic process* (M_t, P_Γ) *such that, for every* $t \in \mathbb{R}$,

$$P_\Gamma \exp\langle -F, M_t\rangle = \exp\langle -V_t, \Gamma\rangle, \tag{1.35}$$

where V_t *is determined by* (1.34).

Suppose that K is an additive functional of ξ and that $\psi = \psi^t(x, z)$. Then *the historical superprocess* M *with parameters* (ξ, K, ψ) can be obtained from the superprocess X with the same parameters by the following direct construction.

First we define the finite-dimensional distributions

$$\begin{aligned} M_t\{w_{t_1} \in A_1, \dots, w_{t_n} \in A_n\}, \qquad & t_1 < \cdots < t_n \leq t, \\ & A_1 \in \mathscr{B}_{t_1}, \dots, A_n \in \mathscr{B}_{t_n}, \end{aligned} \tag{1.36}$$

of the random measure M_t. To this end we replace X_{t_1} by its restriction X'_{t_1} to A_1, run the superprocess during $[t_1, t_2]$ starting from X'_{t_1}, proceed analogously until getting a $Z \in \mathscr{M}_t$ and then take $Z(E_t)$ as the value for (1.36). Then we construct M_t by applying Kolmogorov's theorem to the family (1.36).

The historical superprocess can also be obtained from branching particle systems by the limit procedure of Section 1.2 applied not to the process Y but to the process $\mathscr{Y}$ which is defined as follows.

Pick up a particle P at time t at a point z. Its genealogy can be represented by a scheme

$$(r, x) \to (s_1, y_1) \to \cdots \to (s_k, y_k) \to (t, z). \tag{1.37}$$

The labels (s_i, y_i) indicate the birth time and place of P and its ancestors, and the label (r, x) refers to the immigration time and place of the first member of the family. An arrow a from (s, y) to (s', y') corresponds to a path $w^a \in W[s, s']$ such that $w^a_s = y$, $w^a_{s'} = y'$. Combined together, w^a determine a path $w \in W_{\leq t}$ which we call *the historical path for* P (we set $w_t = \partial$ for $t < r$). The historical paths of all particles which are alive at time t form a configuration in $W_{\leq t}$ which can also be described by an integer-valued measure $\mathscr{Y}_t$ on $W_{\leq t}$. As a function of t, $\mathscr{Y}_t$ is a measure-valued process in functional spaces $W_{\leq t}$.

A natural question is if condition 1.1.A for ξ implies an analogous condition for Ξ. Note that, if $B = \{w\colon w_{t_1} \in A_1, \dots, w_{t_n} \in A_n\}$ where $t_1 < t_2 < \cdots < t_n = u$, then

$$
\begin{aligned}
1_{t<u}\Pi_{t,x_{\le t}}\{\xi_{\le u} \in B\} &= \Pi_{t,x_t}\{w_{t_1} \in A_1, \dots, w_{t_n} \in A_n\} \quad \text{for } t < t_1 \\
&= 1_{A_1}(x_{t_1}), \dots, 1_{A_i}(x_{t_i})\Pi_{t_i, x_{t_i}}\{w_{t_{i+1}} \in A_{i+1}, \dots, w_{t_n} \in A_n\} \\
&\qquad\qquad \text{for } t \in [t_i, t_{i+1}) \\
&= 0 \quad \text{for } t \ge u.
\end{aligned} \tag{1.38}
$$

Therefore 1.1.A is satisfied for Ξ if it holds for ξ.

1.4. *Linear additive functionals.* Denote by $\mathscr{R}$ the set of *left-continuous monotone increasing stochastic processes A_t adapted to the filtration $\mathscr{F}^*_{<t}$ of* W and such that $A_\alpha = 0$. The measure $A(dt)$ on $\mathbb{R}$ determined by the condition $A(-\infty, t) = A_t$ is an additive functional of the historical process Ξ. If $A \in \mathscr{R}$, then $A[s,t) = A_t - A_s \in \mathscr{F}^*_{<t}$ for $s \le t$, and the sum

$$
J_A(\Lambda; r, u) = \sum_{i=1}^{n} \langle A[t_{i-1}, t_i), M_{t_i}\rangle \tag{1.39}
$$

is well defined for every $\Lambda = \{r = t_0 < t_1 < \cdots < t_n = u\}$ and represents a $\mathscr{G}[r,u)^*$-measurable function.

THEOREM 1.3. *Under condition* 1.2.E$_2$, *for every $A \in \mathscr{R}$, there exists a unique (up to equivalence) positive functional J_A of the historical superprocess M such that*:

1.4.A. *If*

$$
\sup_{x_{\le r}} \Pi_{r, x_{\le r}} A_u < \infty \tag{1.40}
$$

for all finite $r < u$, then

$$
J_A[r,u) = \lim J_A(\Lambda_n; r, u) \tag{1.41}
$$

in $L^2(P_{r,N})$ for all finite $r < u$, $N \in \mathscr{M}_{\le r}$ and every monotone increasing sequence Λ_n with the union everywhere dense in $[r,u]$.

1.4.B. *If $A_k \uparrow A$, then $J_{A_k} \uparrow J_A$.*

For every $u \le +\infty$, we have

$$
P_\Gamma \exp\{-J_A(-\infty, u)\} = \exp\langle -V, \Gamma\rangle, \tag{1.42}
$$

where

$$
\begin{aligned}
V^r(x_{\le r}) + \Pi_{r,x_{\le r}} \int_r^u \psi^s[\xi_{\le s}, V^s(\xi_{\le s})]K(ds) \\
= \Pi_{r, x_{\le r}} A[r,u) \quad &\textit{for } r < u, \\
V^r = 0 \qquad\qquad &\textit{for } r \ge u.
\end{aligned} \tag{1.43}
$$

In particular,

$$P_{r,N} \exp\{-J_A[r,u)\} = \exp\langle -V^r, N\rangle. \tag{1.44}$$

For every $B, B_1, B_2 \in \mathscr{B}(\mathbb{R})$ and every $\Gamma \in \mathfrak{M}(\mathscr{W})$,

$$P_\Gamma J_A(B) = \Pi_\Gamma A(B), \tag{1.45}$$

$$\begin{aligned} &P_\Gamma J_{A_1}(B_1) J_{A_2}(B_2) \\ &\quad = \Pi_\Gamma A_1(B_1) \Pi_\Gamma A_2(B_2) \\ &\qquad + \Pi_\Gamma \int q_2^s(\xi_{\le s}) \left[\Pi_{s,\xi_{\le s}} A_1(B_1)\right]\left[\Pi_{s,\xi_{\le s}} A_2(B_2)\right] K(ds). \end{aligned} \tag{1.46}$$

We write

$$J_A(B) = \int_B \langle A(dt), M_t\rangle = \int_B \langle dA_t, M_t\rangle. \tag{1.47}$$

We put $Y \in L$ if $Y \in \mathscr{S}^*(\mathbb{R})$ and if $P_\Gamma Y = \int P_{r,\delta_{x_{\le r}}} Y \Gamma(dr, dx_{\le r})$ for every $\Gamma \in \mathfrak{M}(\mathscr{W})$. Obviously, L is a cone which contains all functions $Y = \langle F, M_t\rangle$, $t \in \mathbb{R}$, $F \in \mathscr{F}^*_{\le t}$. An additive functional J of M is called *linear* if $J[r,u) \in L$ for every $r < u \in \mathbb{R}$. Obviously, all functionals J_A have this property.

We call the function

$$H^r(x_{\le r}) = P_{r,\delta_{x_{\le r}}} J(r,\infty) \tag{1.48}$$

the characteristic of a linear additive functional J.

THEOREM 1.4. *Suppose Ξ is a right process. Then every linear positive additive functional J of M with a bounded characteristic corresponds to some $A \in \mathscr{R}$.*

1.5. *Measures M_τ. Special Markov property.* Let $\mathscr{A}_t$ stand for the intersection of $\mathscr{F}^*_{<u}$ over all $u > t$. Note that τ is a stopping time relative to the filtration $\mathscr{A}_t$ if and only if $\{\tau < t\} \in \mathscr{F}^*_{<t}$ for every $t \in \mathbb{R}$. We assume that $\tau \ge \alpha$. Put $C \in \mathscr{A}_\tau$ if $C \cap \{\tau \le t\} \in \mathscr{A}_t$ for all $t \in \mathbb{R}$ which is equivalent to the condition: $C \cap \{\tau < t\} \in \mathscr{F}^*_{<t}$ for all $t \in \mathbb{R}$. Denote by $w_{\le\tau}$ the restriction of $w \in W$ to $(-\infty, \tau(w)]$. Note that $w \to w_{\le \tau}$ is a measurable mapping from $(W, \mathscr{A}_\tau)$ to the global state space $(\mathscr{W}, \mathscr{B}_{\mathscr{W}})$ of the historical process Ξ (see [6], Lemma 5.2, or [4], 4-64). To every $F \in \mathscr{B}_{\mathscr{W}}$ and every stopping time τ there corresponds $F^\tau \in \mathscr{R}$ defined by the formula

$$F_t^\tau(w) = F(w_{\le\tau}) 1_{\tau < t}. \tag{1.49}$$

THEOREM 1.5. *Suppose that a sub-σ-algebra $\hat{\mathscr{B}}$ of $\mathscr{B}_{\mathscr{W}}$ contains $\mathscr{W}_{\le t}$ for all t and that $(\mathscr{W}, \hat{\mathscr{B}})$ is a Radon space. Then there exists a kernel M_τ from $(\Omega, \mathscr{S}(\mathbb{R})^*)$ to $(\mathscr{W}, \hat{\mathscr{B}})$ such that*

$$M_\tau(F) = \int_{\mathbb{R}} \langle dF_t^\tau, M_t\rangle \quad P_\Gamma\text{-}a.s. \tag{1.50}$$

for all $F \in \hat{\mathscr{B}}$ *and all admissible measures* Γ. *We have*

$$P_\Gamma M_\tau(F) = \Pi_\Gamma F(w_{\le\tau}) 1_{\tau<\infty}. \tag{1.50a}$$

If Γ *is admissible, then* $M_\tau(\omega, \cdot)$ *is admissible for* P_Γ*-almost all* ω.

For every stopping time τ we denote by $\mathscr{G}_{\ge\tau}$ the σ-algebra in Ω generated by $\int_{\mathbb{R}}\langle A(dt), M_t\rangle$ with A subject to the conditions: $A_\tau = 0$ and $A\{t\}$ is a reconstructable function (see Section 0.4). We introduce $\mathscr{G}_{\le\tau}$ as the σ-algebra generated by $M_\tau(F)$, $F \in \hat{\mathscr{B}}$.

THEOREM 1.6. *Suppose that the historical process* Ξ *is strong Markov. If* τ *is a stopping time, then*

$$P_\Gamma YZ = P_\Gamma(Y P_{M_\tau} Z) \tag{1.51}$$

for all admissible measures Γ *and all* $Y \in \mathscr{G}_{\le\tau}$, $Z \in \mathscr{G}_{\ge\tau}$.

1.6. *Spaces of historical paths.* A historical path of a particle P observed at time t is represented by a scheme (1.37). Historical paths $\omega^1, \dots, \omega^n$ for particles $P_1, \dots, P_n$ overlap. They can be represented by a scheme like the following:

$$\begin{array}{llllll} (1.52)\quad (r_1, x_1) \to & (s_1, y_1) \to & (s_2, y_2) \to & (t_1, z_1) & \qquad (r_2, x_2) \to (t_2, z_2). \\ & \quad\downarrow & \quad\downarrow & & \\ & (t_3, z_3) & (t_4, z_4) & & \end{array}$$

The paths $\omega^1, \dots, \omega^n$ can be decomposed into elements w^a corresponding to the arrows; they can also be combined into one historical path ω for the family $P_1, \dots, P_n$.

The combinatorial skeleton of the scheme (1.52) is the directed graph

$$\begin{array}{llll} \to & \to & \to & \qquad \to \\ \downarrow & \downarrow & & \end{array} . \tag{1.53}$$

In general, we deal with *rooted trees* and their disjoint unions which we call *groves*. A rooted tree is a family of vertices and arrows such that each vertex, except one, is the end of one arrow, and the exceptional vertex—called the entrance—is the beginning of just one arrow—called the root. (For typographical reasons, we depict trees which lie, not stand up, which, of course, makes no difference since isomorphic graphs are considered as indistinguishable.) A special role is played by the vertices from which no arrow begins—we call them exits.

A *diagram* is a grove with marked exits. For instance, there exist 12 distinguishable diagrams corresponding to the grove (1.53); one of them is

$$\begin{array}{llll} \to & \to & \to 1 & \qquad \to 2. \\ \downarrow & \downarrow & & \\ 3 & 4 & & \end{array} \tag{1.54}$$

[The diagram (1.54) is indistinguishable from one obtained by interchanging the labels 1 and 4.] We denote by $\mathbb{D}_\Lambda$ the family of all diagrams D with exits marked by the elements of a finite set Λ. For every $D \in \mathbb{D}_\Lambda$, there is a standard marking of all vertices. For diagram (1.54) it looks as follows:

$$\begin{array}{ccccccc} 134* & \to & 134 & \to & 14 & \to & 1 \qquad 2* \to 2. \\ & & \downarrow & & \downarrow & & \\ & & 3 & & 4 & & \end{array} \tag{1.55}$$

In general, a vertex v is marked by the list of all exits which can be reached from v in the direction of arrows. In addition $*$ is included in the mark of every entrance.

The marks which do not contain $*$ form a class of $\Gamma \subset \Lambda$ with the properties: Γ contains all singletons and every two elements of Γ are either disjoint or one of them contains the other. We call a class with these properties an $\mathscr{L}$-class. There is a one-to-one correspondence between $\mathbb{D}_\Lambda$ and all $\mathscr{L}$-classes in Λ. [Diagram (1.55) corresponds to $\Gamma = \{1, 2, 3, 4, 14, 134\}$.]

We write $D = \{D_1, \dots, D_m\}$ if $D_1, \dots, D_m$ are the connected components of D (the corresponding $\mathscr{L}$-classes satisfy the relation $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_m$).

Another construction allows us to reduce a connected diagram D to simpler connected diagrams (assuming D contains more than one arrow). If $D \in \mathbb{D}_\Lambda$ is connected, then Λ is an element of the corresponding $\mathscr{L}$-class Γ. By eliminating this element we get a new $\mathscr{L}$-class Γ'. We write $D = D_1 \vee \cdots \vee D_m$ if the diagram corresponding to Γ' is $\{D_1, \dots, D_m\}$. This is equivalent to the following recipe: To get $D_1, \dots, D_m$, drop the entrance $\Lambda *$ from D and replace the vertex Λ by the new entrances $\Lambda_1 *, \dots, \Lambda_m *$ if $\Lambda \to \Lambda_1, \dots, \Lambda \to \Lambda_m$ is the complete list of arrows beginning at Λ.

Let $\Lambda_1 *, \dots, \Lambda_m *$ be the list of all entrances of $D \in \mathbb{D}_\Lambda$. Note that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_m$ is a partition of Λ into disjoint sets. Put $\mathscr{w} \in W_D(r_1, \dots, r_m; t_\Lambda)$ if $\mathscr{w}$ is a historical path for $|\Lambda|$ particles whose scheme coincides with D labeled by elements e of $\mathscr{E}$ in such way that:

(i) $e(\Lambda_i *) \in E_{r_i}$ for every entrance $\Lambda_i *$;
(ii) $e(j) \in E_{t_j}$ for every exit j.

[For instance, the historical path (1.52) belongs to $W_D(r_1, r_2; t_1, \dots, t_4)$ with D shown on (1.54).] Note that

$$W_D(r_1, \dots, r_m; t_\Lambda) = \prod_{i=1}^{m} W_{D_i}(r_i; t_{\Lambda_i}) \tag{1.56}$$

if $D = \{D_1, \dots, D_m\}$.

To every sequence of positive progressive functions q_m, $m = 0, 1, \dots,$ there corresponds a family of measures $L_D(r_1, x_1; \dots; r_m, x_m)$ on $W_D(r_1, \dots, r_m; t_\Lambda)$, $x_1 \in E_{r_1}, \dots, x_m \in E_{r_m}$ defined by the following inductive rules:

1.6.A. If D consists of a single arrow, then $L_D(r, x)$ is the image of $\Pi_{r,x}$ under the natural mapping $w_{\ge r} \to w[r, t]$ from $W^r = W_{\ge r}$ to $W[r, t]$.

1.6.B. If D is connected and if $D = D_1 \vee \cdots \vee D_m$, then

$$\begin{aligned} L_D(r, x)(d\mathscr{w}) &= \Pi_{r,x}(dw) q_m^s(w_s) K(ds) \\ &\quad \times L_{D_1}(s, w_s)(d\mathscr{w}_1) \cdots L_{D_m}(s, w_s)(d\mathscr{w}_m) \end{aligned}$$

[we parameterize elements ϖ of $W_D(r, x; t, \Lambda)$ by $s \in \mathbb{R}$, by a path w corresponding to the entrance arrow and by a family of paths ϖ_i corresponding to the diagrams D_i].

1.6.C. If $D = \{D_1, \dots, D_m\}$, then

$$L_D(r_1, x_1; \dots; r_m, x_m) = \prod_{i=1}^{m} L_{D_i}(r_i, x_i).$$

If $q_m = p_m$ in 1.6.B, then $L_D(r_1, \dots, r_m; t_\Lambda)$ is a subprobability measure. It can be interpreted as describing a certain version of the evolution of a finite branching particle system observed between times $r_1, \dots, r_m$ and $t_1, \dots, t_n$.

1.7. *Moment functions.* Now we consider the measures L_D corresponding to

$$q_m^t(x) = D_z^m \psi^t(x, 0). \tag{1.57}$$

By (1.21),

$$\begin{aligned} D_z^2 \psi^t(x, z) &= b^t(x) + \int_0^\infty u^2 e^{-uz} n^t(x, du) \le q_2^t(x), \\ D_z^m \psi^t(x, z) &= \int_0^\infty u^m e^{-uz} n^t(x, du) \le q_m^t(x) \quad \text{for } m > 2. \end{aligned} \tag{1.58}$$

THEOREM 1.7. *Suppose that conditions* 1.2.A–1.2.D *and* 1.2.E$_n$ *are satisfied. Then, for every* $\eta \in \mathfrak{M}(\mathscr{E})$, $f_1 \in \mathscr{B}_{t_1}, \dots, f_n \in \mathscr{B}_{t_n}$,

$$\begin{aligned} &P_\eta \langle f, X_{t_1} \rangle \cdots \langle f_n, X_{t_n} \rangle \\ &= \sum_{D \in \mathbb{D}_\Lambda} \int \eta(dr_1, dx_1) \cdots \eta(dr_m, dx_m) \\ &\quad \times L_D(r_1, x_1; \dots; r_m, x_m; d\varpi) f_1\big(\varpi_{t_1}^1\big) \cdots f_n\big(\varpi_{t_n}^n\big). \end{aligned} \tag{1.59}$$

Here $\Lambda = \{1, \dots, n\}$ *and* $\varpi^1, \dots, \varpi^n$ *are historical paths of particles* $P_1, \dots, P_n$ *considered as functions of* $\varpi \in W_D(r_1, \dots, r_m; t_1, \dots, t_n)$.

REMARK 1. For $n = 1$, (1.59) is identical to (1.29). For $n = 2$, (1.59) coincides with (1.30). Two terms in (1.30) correspond to the diagrams

$$\begin{array}{l} \longrightarrow 1 \\ \longrightarrow 2 \end{array} \quad \text{and} \quad \longrightarrow\!\!\begin{array}{l} \nearrow 1 \\ \searrow 2 \end{array}.$$

REMARK 2. It follows from (1.18) that 1.2.E$_n$ implies 1.2.E$_m$ for all $m \le n$.

2. Construction of branching particle systems.

2.1. We start from a single particle at point $(r, x) \in \mathscr{E}$ and we define the joint probability distribution $\hat{\Pi}_{r,x}$ for its life path ω and the number m of its offspring. It is convenient to describe a life path ω by a triplet (w, α, β). Here $w \in W$ is a collection of $w_t \in E_t$ for all $t \in \mathbb{R}$ and $-\infty \le \alpha < \beta \le +\infty$ are the birth and death times of the particle. Let $\mathscr{W}$ stand for the space of all such triplets and let $\mathscr{I}$ be the σ-algebra in $\mathscr{W}$ generated by the functions

$$F(w, \alpha, \beta) = f^{\alpha}(w_{\alpha})1_{\alpha < t_1 < t_2 < \beta} Y g^{\beta}(w_{\beta}), \tag{2.1}$$

where $f, g \in \mathscr{B}_{\mathscr{E}}^*$, $Y \in \mathscr{F}(t_1, t_2)^*$. [$\mathscr{F}(\Delta)$ means the σ-algebra in W generated by the coordinate mappings w_t, $t \in \Delta$. We put $f^{\alpha}(w_{\alpha}) = 1$ if $\alpha = -\infty$ and $g^{\beta}(w_{\beta}) = 1$ if $\beta = +\infty$.] Note that, if $w = w'$ on $[\alpha, \beta]$, then $F(w, \alpha, \beta) = F(w', \alpha, \beta)$ for functions (2.1) and therefore for all $F \in \mathscr{I}$. We call $(\mathscr{W}, \mathscr{I})$ the single particle path space, and we say that (α, w_{α}) is the birth point and (β, w_{β}) is the death point of the particle.

LEMMA 2.1. *For every additive functional K, $F(w, \alpha, \beta) = K(\alpha, \beta)$ is $\mathscr{I}$-measurable.*

PROOF. Put

$$\alpha_n = i2^{-n} \quad \text{for } (i-1)2^{-n} \le \alpha < i2^{-n},$$
$$\beta_n = j2^{-n} \quad \text{for } j2^{-n} < \beta \le (j+1)2^{-n}.$$

It is easy to see that $1_{\alpha_n < \beta_n} K(\alpha_n, \beta_n)$ is $\mathscr{I}$-measurable and $(\alpha_n, \beta_n) \uparrow (\alpha, \beta)$ and therefore $K(\alpha_n, \beta_n) \uparrow K(\alpha, \beta)$. □

LEMMA 2.2. *Let $F \in \mathscr{I}$. Then, for every r,*

(a) *$F(w, r, \infty)$ is $\mathscr{F}_{\ge r}$-measurable and $F(w, r, s)$ is $F_{\ge r} \times \mathscr{B}_{\ge r}$-measurable;*
(b) *the functions*

$$\Phi^r(x) = \Pi_{r,x} F(r, \infty)$$

and

$$\Psi^r(x) = \Pi_{r,x} \int_r^{\infty} F(r, s) K(ds)$$

are progressive.

PROOF. By the multiplicative systems theorem, it is sufficient to check this for F given by (2.1), in which case (a) is trivial and (b) follows from 0.6.C, 0.5.α and 0.5.β since

$$\Phi^r(x) = f^r(x) 1_{r < t_1} \Pi_{r,x} Y \quad \text{and} \quad \Psi^r(x) = f^r(x) 1_{r < t_1} \Pi_{r,s} Y \tilde{K}(t_2, \infty),$$

where $\tilde{K}(dt) = g^t(\xi_t) K(dt)$. □

Let $\mathbb{Z}_+ = \{0, 1, \dots, m, \dots\}$. A measurable function on $(\mathscr{W}, \mathscr{S}) \times \mathbb{Z}_+$ is just a sequence of $\mathscr{S}$-measurable functions F^m. It follows from Lemmas 2.1 and 2.2 that the formula

$$\hat{\Pi}_{r,x} F = \sum_{m=0}^{\infty} \Pi_{r,x}\bigg[H(r,\infty) F^0(w,r,\infty) + \int_r^\infty H(r,s) p_m^s(w_s) F^m(w,r,s) K(ds) \bigg], \qquad F^m \in \mathscr{S}, \tag{2.2}$$

defines a kernel from $(\mathscr{E}, \mathscr{B}^*_{\mathscr{E}})$ to $(\mathscr{W}, \mathscr{S}) \times \mathbb{Z}_+$. The intuitive meaning of (2.2) was explained in Section 1.1. For every measurable positive F, $\hat{\Pi}_{r,x} F$ is a progressive function. Note that $\hat{\Pi}_{r,x}\{\alpha = r\} = 1$.

2.2. The construction of a general branching particle system can be easily reduced to constructing a system generated by a single particle P. Let us consider the set $\mathscr{A}$ which consists of an element $\emptyset$ representing P and sequences of strictly positive integers $\{i_1, i_2, \dots, i_n\}$ representing descendants of P. We say that $a = \{i_1, \dots, i_k\}$ is an ancestor of $b = \{j_1, \dots, j_n\}$ and we write $b \prec a$ if $k < n$ and $i_1 = j_1, \dots, i_k = j_k$. If, in addition, $k = n - 1$, then we say that a is the parent of b.

Let $\mathscr{W}$ be the single particle path space. An element ω of $\Omega = (\mathscr{W} \times \mathbb{Z}_+)^{\mathscr{A}}$ describes an eventual family history: The value $\omega(a) = (\mathscr{w}(a), m(a))$ indicates the life path $\mathscr{w}$ and the number of offspring for the particle a. By applying the Ionescu–Tulcéa theorem, we construct a probability measure Q_{r,δ_x} on Ω with the following properties:

2.2.A. The probability distribution of $\omega(\emptyset)$ is $\hat{\Pi}_{r,x}$;

2.2.B. If $a \neq \emptyset$ and if (s, y) is the death point of the parent of a, then the conditional probability distribution of $\omega(a)$ given $\{\omega(b)$ for all $b \prec a\}$ is equal to $\hat{\Pi}_{s,y}$; moreover, the evolutions of the siblings are independent.

Fix $\omega \in \Omega$. We call $a = (i_1, \dots, i_n) \in \mathscr{A}$ "a dream child" if $i_1 > m(\emptyset)$ or if $i_k > m(i_1, \dots, i_{k-1})$ for some $k = 2, \dots, n$. By eliminating all the "dream children" from $\mathscr{A}$, we get the family tree $G(\omega)$. The family path $\mathscr{w}(\omega)$ is the collection w_a, $a \in G(\omega)$. It is legitimate to assume that, for all $a \neq \emptyset$, the birth point of a coincides with the death point of its parent (because this happens with probability 1 with respect to all Q_{r,δ_x}).

By 1.1.A and 1.1.B, the function $w_t^r(x)$ given by (1.2) is progressive in r, x (we put it equal to 0 for $r \geq t$). To prove (1.3), we consider the life path $\mathscr{w} = (w, r, \beta)$ of $\emptyset$ and we note that $\langle f, Y_t \rangle = f(w_t)$ if $t \in (r, \beta)$ and $Y_t = Y_t^1 + \cdots + Y_t^m$ if $t \geq \beta$, where Y_t^i describes the posterity at time t of the child $\{i\}$.

2.3. Now we prove (1.5).

LEMMA 2.3. *Let $F \in \mathscr{B}_t$ and let a function $h^r(x) \geq 0$ be progressive. Then*

$$g^r(x) = 1_{r<t}\Pi_{r,x}\left[\int_r^t H(r,s)h^s(\xi_s)K(ds) + H(r,t)F(\xi_t)\right] \tag{2.3}$$

is progressive and

$$g^r(x) + \Pi_{r,x}\int_r^t g^s(\xi_s)K(ds) = \Pi_{r,x}\left[F(\xi_t) + \int_r^t h^s(\xi_s)K(ds)\right] \tag{2.4}$$

for $r < t$.

PROOF. Functions

$$Y_r = 1_{r<t}\int_r^t H(r,s)h^s(\xi_s)K(ds)$$

and

$$Z_r = 1_{r<t}H(r,t)F(\xi_t)$$

are right continuous and $\mathscr{F}^*_{>r}$-adapted. Therefore $g^r(x)$ is progressive by 0.6.C.

Let $r < t$. By (2.3),

$$\begin{aligned} \Pi_{r,x}\int_r^t g^s(\xi_s)K(ds) &= \Pi_{r,x}\int_r^t K(ds)\Pi_{s,\xi_s}(Y_s + Z_s) \\ &= \Pi_{r,x}\int_r^t K(ds)(Y_s + Z_s). \end{aligned} \tag{2.5}$$

Note that

$$\int_r^t K(ds)H(s,t) = 1 - H(r,t) \tag{2.6}$$

and therefore

$$\begin{aligned} \Pi_{r,x}\int_r^t K(ds)Z_s &= \Pi_{r,x}\int_r^t K(ds)H(s,t)F(\xi_t) \\ &= \Pi_{r,x}(1 - H(r,t))F(\xi_t). \end{aligned} \tag{2.7}$$

By Fubini's theorem and (2.6),

$$\begin{aligned} \Pi_{r,x}\int_r^t K(ds)Y_s &= \Pi_{r,x}\int_r^t K(ds)\int_s^t H(s,u)h^u(\xi_u)K(du) \\ &= \Pi_{r,x}\int_r^t K(du)h^u(\xi_u)\int_r^u K(ds)H(s,u) \\ &= \Pi_{r,x}\int_r^t K(du)h^u(\xi_u)(1 - H(r,u)), \end{aligned} \tag{2.8}$$

and (2.4) follows from (2.5), (2.7), (2.8) and (2.3). □

COROLLARY. *If w_t^r is progressive in r, x and if it satisfies* (1.3), *then it also satisfies* (1.5).

Indeed, it is sufficient to apply Lemma 2.3 to $g^r = w_t^r$, $h^r = \varphi(w_t^r)$ and $F(x) = e^{-f(x)}$.

3. From branching particle systems to superprocesses.

3.1. The proof of Theorem 1.1 is based on a general lemma regarding Laplace functionals and on the fact that, if ψ_β and ψ are given by (1.19) and (1.20) and if $v_t^r(\beta, x)$ satisfies (1.13), then $v_t^r(\beta, x)$ converges, as $\beta \to 0$, to a solution of (1.23). It is easy to see that:

3.1.A. $\psi_\beta^t(x, z)$ converges to $\psi^t(x, z)$ uniformly on the set $(t, x) \in \mathscr{E}$, $z \in [0, c]$ for every $c \in (0, \infty)$.

3.1.B. $\psi^t(x, z)$ is locally Lipschitz in z uniformly in (t, x), that is, for every $c \in (0, \infty)$, there exists a constant $q(c)$ such that

$$\left|\psi^t(x, z_1) - \psi^t(x, z_2)\right| \le q(c)|z_1 - z_2|$$

for all $z_1, z_2 \in [0, c]$, $(t, x) \in \mathscr{E}$.

We use only properties 3.1.A, B but not the concrete form of ψ_β and ψ.

3.2. Let $\mathscr{M}$ be the set of all finite measures on a measurable space $(E, \mathscr{B})$. We consider the σ-algebra $\mathscr{B}_{\mathscr{M}}$ in $\mathscr{M}$ generated by the functions $F_B(\mu) = \mu(B)$, $B \in \mathscr{B}$ and the cone $\mathbb{H}$ of all bounded functions $f \in \mathscr{B}$ with the topology of bounded convergence ($f_n \to f$ boundedly if $f_n \to f$ pointwise and if f_n are uniformly bounded). For every probability measure M on $(\mathscr{M}, \mathscr{B}_{\mathscr{M}})$ the formula

$$L_M(f) = \int_{\mathscr{M}} M(d\nu) e^{-\langle f, \nu\rangle}, \qquad f \in \mathbb{H}, \tag{3.1}$$

defines a continuous functional on $\mathbb{H}$ which is called *the Laplace functional* of M.

LEMMA 3.1. *Let $(E, \mathscr{B})$ be a Radon space. Suppose that M_n are probability measures on $(\mathscr{M}, \mathscr{B}_{\mathscr{M}})$. If the Laplace functionals L_{M_n} converge to a continuous functional on $\mathbb{H}$, then the limit is also the Laplace functional of a probability measure on $(\mathscr{M}, \mathscr{B}_{\mathscr{M}})$.*

PROOF. In the case of a Luzin space $(E, \mathscr{B})$ this was proved in [10]. The proof can be extended to the case of a Radon space by using the following facts:

(a) The space of all probability measures on a Radon space is again a Radon space (see 0.7).

(b) If M_n is a sequence of probability measures on a Radon space $(S, \mathscr{B}_S)$, then there is a set $\tilde{S} \in \mathscr{B}_S$ which supports all measures M_n and such that $(\tilde{S}, \mathscr{B}_{\tilde{S}})$ (where $\mathscr{B}_{\tilde{S}}$ is the trace of $\mathscr{B}_S$ on $\tilde{S}$) is a Luzin space. (This follows directly from the definition of a Radon space.) □

COROLLARY. *If $L_{M_n}(f) \to L(f)$ uniformly on each set $\mathbb{H}_c = \{f: 0 \le f \le c\}$, then L is the Laplace functional of a probability measure.*

3.3. LEMMA 3.2 (Generalized Gronwall inequality). *If a bounded function $h^r(x)$ is progressive and if*

$$h^r(x) \le a + q\Pi_{r,x} \int_r^t h^s(\xi_s) K(ds) \quad \text{for all } r \in [t_0, t), \tag{3.2}$$

then

$$h^r(x) \le a\Pi_{r,x} e^{qK(r,t)} \quad \text{for all } r \in [t_0, t). \tag{3.3}$$

PROOF. By induction,

$$\begin{aligned} h^r(x) &\le a \sum_{k=0}^{n} q^k \Pi_{r,x} \int 1_{r<s_1<\cdots<s_k<t} K(ds_1) \cdots K(ds_k) \\ &\quad + q^{n+1}\Pi_{r,x} \int 1_{r<s_1<\cdots<s_{n+1}<t} K(ds_1) \cdots K(ds_{n+1}) h^{s_{n+1}}(\xi_{s_{n+1}}) \\ &= a \sum_{k=0}^{n} q^k \Pi_{r,x} K(r,t)^k/k! + R_{n+1}, \end{aligned}$$

where

$$R_{n+1} < \text{const}\, q^{n+1} \Pi_{r,x} K(r,t)^{n+1}/(n+1)!. \qquad \square$$

LEMMA 3.3. *Under conditions* 1.2.C *and* 3.1.A, B, *$v_t^r(\beta, x)$ converges uniformly on every set*

$$r \in [t_0, t), \qquad f \in \mathbb{H}_c, \tag{3.4}$$

to the unique solution of the integral equation (1.23).

PROOF. For all $f \in \mathbb{H}_c$,

$$0 \le v_t^r(\beta, x) \le \Pi_{r,x} f(\xi_t) \le c. \tag{3.5}$$

By 3.1.A, for every $\varepsilon > 0$, there exists a $\beta_0 > 0$ such that

$$|\psi_\beta^s(x,z) - \psi^s(x,z)| \le \varepsilon$$

for all $\beta < \beta_0$, $(s, x) \in \mathscr{E}$, $z \in [0, c]$. By 3.1.B,

$$|\psi_\beta^s[x, v_t^s(\beta, y)] - \psi_{\beta_1}^s[x, v_t^s(\beta_1, y)]| \le 2\varepsilon + q(c)|v_t^s(\beta, y) - v_t^s(\beta_1, y)|$$

for all $\beta, \beta_1 \in (0, \beta_0)$, $f \in \mathbb{H}_c$ and all $(s, x), (s, y) \in \mathscr{E}$. We conclude from (3.5), (1.13) and 1.2.C that

$$h^r(x) = \left| v_t^r(\beta, x) - v_t^r(\beta_1, x) \right|$$

satisfies (3.2) with $a = \|F_\beta - F_{\beta_1}\| + 2\varepsilon k$. By Lemma 3.2,

$$h^r(x) \le a \Pi_{r,x} e^{qK(r,t)}$$

and, by an elementary inequality,

$$\|F_\beta - F_{\beta_1}\| \le \|F_\beta - f\| + \|F_{\beta_1} - f\| \le (\beta + \beta_1)\|f\|^2/2.$$

Passage to the limit in (1.13) is legitimate since $\psi_\beta^s(x, v_t^s(\beta, x)) \to \psi^s(x, v^s(x))$ uniformly on the set $s \in [r, t]$, $x \in E_s$. The uniqueness of the solution for (1.23) follows from Lemma 3.2. □

REMARK. The function v is progressive as the limit of progressive functions $v_t(\beta)$.

3.4. PROOF OF THEOREM 1.1. Formula (1.12) means that

$$L_{M_\beta}(f) = \exp\langle -v_t(\beta), \eta\rangle, \tag{3.6}$$

where

$$M_\beta(C) = Q^\beta_{\pi_\eta/\beta}\{\beta Y_t \in C\}, \qquad C \in \mathscr{B}_{\mathscr{M}_t}.$$

Let $\eta \in \mathfrak{M}(\mathscr{E})$ be concentrated on $\mathscr{E}_{\ge t_0}$. Then, by Lemma 3.3, $\langle v_t(\beta), \eta\rangle \to \langle v_t, \eta\rangle$ uniformly on each set $\mathbb{H}_c$ and, by the corollary to Lemma 3.1, there exists a measure $\mathscr{P}(\eta; t, \cdot\,)$ on $(\mathscr{M}_t, \mathscr{B}_{\mathscr{M}_t})$ such that

$$\int \mathscr{P}(\eta; t, d\nu) e^{-\langle f, \nu\rangle} = e^{-\langle v, \eta\rangle}. \tag{3.7}$$

For an arbitrary $\eta \in \mathfrak{M}(\mathscr{E})$, we consider restrictions η_n of η to $\mathscr{E}[n, n+1)$ and we define $\mathscr{P}(\eta; t, \cdot\,)$ as the convolution of measures $\mathscr{P}(\eta_n; t, \cdot\,)$. Since (3.7) holds for η_n, it is true also for η. By Fatou's lemma,

$$\int \mathscr{P}(\eta; t, d\nu)\langle 1, \nu\rangle \le \lim_{\lambda \downarrow 0} \int \mathscr{P}(\eta; t, d\nu)(1 - e^{-\lambda\langle 1, \nu\rangle})/\lambda \le \eta(\mathscr{E}_{\le t}) < \infty$$

and therefore $\mathscr{P}(\eta; t, \cdot\,)$ is concentrated on $\mathscr{M}_t$.

Let $\mu \in \mathscr{M}_r$ and let η_r be the image of μ under the mapping $x \to (r, x)$ from E_r to $(\mathscr{E}, \mathscr{B}_{\mathscr{E}})$. Formula $\mathscr{P}(r, \mu; t, \cdot\,) = \mathscr{P}(\eta_r; t, \cdot\,)$ determines a Markov transition function (see the proof of Theorem 3.1 in [9]). Theorem 1.1 holds for any Markov process X with this transition function. □

The version of Theorem 1.1 stated at the end of Section 1.2 can be proved by the arguments used in [10] to prove Theorem 3.1.

4. Linear additive functionals. Special Markov property.

4.1. LEMMA 4.1. *Let Λ be a finite subset of $\mathbb{R}$ and let $f^t \in \mathscr{B}_t$ for every $t \in \Lambda$. Put*

$$A^r = \sum_{t \in \Lambda_r} f^t(\xi_t), \qquad J^r = \sum_{t \in \Lambda_r} \langle f^t, X_t\rangle, \tag{4.1}$$

where $\Lambda_r = \Lambda \cap (r, \infty)$. *We have*

$$P_{r,\mu} e^{-J^r} = e^{-\langle v^r, \mu \rangle}, \tag{4.2}$$

where

$$v^r(x) + \Pi_{r,x} \int_r^\infty \psi^s[\xi_s, v^s(\xi_s)] K(ds) = \Pi_{r,x} A^r. \tag{4.3}$$

For every $\eta \in \mathfrak{M}(\mathscr{E})$,

$$P_\eta \exp\left\{ - \sum_{t \in \Lambda} \langle f^t, X_t \rangle \right\} = e^{-\langle v, \eta \rangle}. \tag{4.4}$$

PROOF. If the cardinality $|\Lambda|$ of Λ is equal to 1, then (4.2) and (4.3) coincide with (1.22) and (1.23). The general statement is proved by induction in $|\Lambda|$. Suppose that $\Lambda = \{t_1 < \cdots < t_n\}$ and put $\tilde{\Lambda} = \{t_2 < \cdots < t_n\}$. For $r \ge t_1$, we have $\tilde{\Lambda}_r = \Lambda_r$, $\tilde{A}^r = A^r$, $\tilde{J}^r = J^r$ and, by the induction hypothesis, (4.2) and (4.3) hold for $r \ge t_1$. If $r < t_1$, then

$$\begin{aligned} P_{r,\mu} e^{-J^r} &= P_{r,\mu} \exp\left[\langle -f^{t_1}, X_{t_1} \rangle - J^{t_1}\right] \\ &= P_{r,\mu}\left[\exp\langle -f^{t_1}, X_{t_1} \rangle P_{t_1, X_{t_1}} e^{-J^{t_1}}\right] = P_{r,\mu} \exp\langle -(f+v)^{t_1}, X_{t_1} \rangle \end{aligned}$$

and, by (1.22),

$$P_{r,\mu} e^{-J^r} = \exp\langle -\tilde{v}^r, \mu \rangle, \tag{4.5}$$

where

$$\tilde{v}^r(x) + \Pi_{r,x} \int_r^{t_1} \psi^s[\xi_s, \tilde{v}^s(\xi_s)] K(ds) = \Pi_{r,x} (f+v)^{t_1}(\xi_{t_1}). \tag{4.6}$$

Using (4.3) for t_1, 0.1.B and (4.6), we get that, for $r < t_1$,

$$\begin{aligned} \Pi_{r,x} A^r &= \Pi_{r,x}\left[f^{t_1}(\xi_{t_1}) + A^{t_1} \right] \\ &= \Pi_{r,x} f^{t_1}(\xi_{t_1}) + \Pi_{r,x} v^{t_1}(\xi_{t_1}) + \Pi_{r,x} \int_{t_1}^\infty \psi^s[\xi_s, v^s(\xi_s)] K(ds) \\ &= \tilde{v}^r(x) + \Pi_{r,x} \int_r^{t_1} \psi^s[\xi_s, \tilde{v}^s(\xi_s)] K(ds) + \Pi_{r,x} \int_{t_1}^\infty \psi^s[\xi_s, v^s(\xi_s)] K(ds) \end{aligned}$$

and therefore (4.2) and (4.3) hold if we put $v^r = \tilde{v}^r$ for $r < t_1$.

Since $P_{\eta+\eta'} = P_\eta * P_{\eta'}$, formula (4.4) holds for $\eta + \eta'$ if it is true for η and η'. Therefore it is sufficient to prove (4.4) for η_i concentrated on $\mathscr{E}(\Delta_i)$ where $\Delta_1 = (-\infty, t_1]$, $\Delta_i = (t_{i-1}, t_i]$ for $i = 2, \ldots, n$, $\Delta_{n+1} = (t_n, +\infty)$. Since $X_t = 0$, P_{η_i}-a.s. for $t \le t_{i-1}$, formula (4.4) follows from (4.2) and (1.26). □

4.2. Suppose that A, J are defined by (4.1). It follows from (1.28) that

$$P_{r,\mu} J^r = \Pi_{r,\mu} A^r. \tag{4.7}$$

If $\tilde{A}$, $\tilde{J}$ are defined by analogous formulas with f, Λ replaced by $\tilde{f}, \tilde{\Lambda}$, then, by (1.30),

$$(4.8) \quad P_{r,\mu} J^r \tilde{J}^r = \Pi_{r,\mu} A^r \Pi_{r,\mu} \tilde{A}^r + \Pi_{r,\mu} \int_r^\infty q_2^s(\xi_s) \Pi_{s,\xi_s} A^s \Pi_{s,\xi_s} \tilde{A}^s K(ds).$$

Now let A be an arbitrary element of $\mathscr{R}$. If $J_A(\Lambda; r, u)$ is defined by (1.39), then

$$(4.9) \qquad J_A(\Lambda; r, u) = \sum_{t \in \Lambda} \langle F^t, M_t \rangle,$$

where

$$(4.10) \qquad F^{t_i} = \begin{cases} A[t_{i-1}, t_i) & \text{for } i = 1, \dots, n, \\ 0 & \text{for } i = 0 \end{cases}$$

and, by applying (4.7) to the historical superprocesses, we get

$$(4.11) \qquad P_{r,N} J_A(\Lambda; r, u) = \Pi_{r,N} A[r, u).$$

Analogously, (4.8) implies that

$$(4.12) \qquad \begin{aligned} &P_{r,N} J_A(\Lambda; r, u) J_A(\tilde{\Lambda}; r, u) \\ &\quad = [\Pi_{r,N} A[r, u)]^2 \\ &\quad + \Pi_{r,N} \int_r^u q_2^s(\xi_s) \Pi_{s,\xi_{\le s}} A_\Lambda[s, u) \Pi_{s,\xi_{\le s}} A_{\tilde{\Lambda}}[s, u) K(ds), \end{aligned}$$

where

$$A_\Lambda[s, u) = A[\gamma_\Lambda(s), u)$$

and

$$\gamma_\Lambda(s) = \begin{cases} t_{i-1} & \text{for } t_{i-1} \le s < t_i,\ i = 1, \dots, n, \\ t_0 & \text{for } s < t_0. \end{cases}$$

4.3. Proof of Theorem 1.3. *First assume that A satisfies* (1.40). Let Λ_n be the set of diadic fractions $k2^{-n}$ where $k = 0, \pm 1, \pm 2, \dots$. For every $\Delta = [r, u)$ we denote by Λ_n^Δ the intersection of Λ_n with (r, u) augmented by the endpoints r, u. The right side in (4.12) tends to a finite limit as $\Lambda, \tilde{\Lambda}$ run independently over the sequence Λ_n^Δ. Therefore $J_A(\Lambda_n^\Delta; r, u)$ converges in $L^2(P_{r,N})$. We construct an additive functional J which satisfies (1.41) by using a simplified version of arguments in the Appendix to [14]. For every $\Delta = [r, u)$ we consider the Mokobodzki medial limit

$$j(\omega, \Delta) = \operatorname{lim\,med} J_A(\Lambda_n^\Delta; \omega, r, u)$$

(see, e.g., [5], 10-56 and 10-57). Put $t \in G(\omega)$ if there exists an interval $\Delta = [r,u)$ such that $t \in \Delta$ and $j(\omega, \Delta) < \infty$. Each connected component of $G(\omega)$ is either an open interval or an interval of the form $[\alpha, \beta)$, and in the second case $j[\alpha, t] < \infty$ for all $t \in [\alpha, \beta)$. There exists a unique measure $J(\omega, \cdot)$ concentrated on $G(\omega)$ and such that $J(\omega, \Delta) = j(\omega, \Delta)$ for every $\Delta \subset G(\omega)$. This measure is σ-finite. Fundamental properties of medial limits imply that $J(\omega, \Delta) \in \mathscr{G}(\Delta)^*$ and that (1.41) holds (see [14] for details).

For an arbitrary $A \in \mathscr{R}$ we consider $A_t(k) = A_t \wedge k$ which belong to $\mathscr{R}$ and satisfy (1.40). Clearly additive functionals $J_{A(k)}$ increase and we put $J_A = \lim_{k\to\infty} J_{A(k)}$. Obviously, 1.4.B holds.

By applying Lemma 4.1 to the historical superprocess and by using (4.9), we get that, for every $t < u$ and every $\Lambda = \{t = t_0 < t_1 < \cdots < t_n = u\}$,

$$P_{t,\delta_{x_{\le t}}} \exp\{-J_A(\Lambda; t, u)\} = \exp\{-V_\Lambda^t(x_{\le t})\}, \tag{4.13}$$

where

$$\begin{aligned} V_\Lambda^r(x_{\le r}) + \Pi_{r,x_{\le r}} \int_r^u \psi^s(\xi_{\le s}, V_\Lambda^s(\xi_{\le s}))K(ds) & \\ = \Pi_{r,x_{\le r}} A_\Lambda[r,u) \quad & \text{for } r < u, \\ V_\Lambda^r = 0 \quad & \text{for } r \ge u. \end{aligned} \tag{4.14}$$

Let Λ_n be a monotone increasing sequence with the union everywhere dense in $[t,u)$. Since $|e^{-a} - e^{-b}| \le |a-b|$ for $a, b \ge 0$, we get from (4.13) and the Schwarz inequality that

$$P_{t,\delta_{x_{\le t}}} \exp\{-J_A[t,u)\} = \exp\{-V^t(x_{\le t})\},$$

where $V^t(x_{\le t}) = \lim_{n\to\infty} V_{\Lambda_n}^t(x_{\le t})$. It follows from (4.14) that $V_{\Lambda_n}^r(x_{\le r})$ are uniformly bounded, and we get from (4.14) and the dominated convergence theorem

$$\begin{aligned} V^r(x_{\le r}) + \Pi_{r,x_{\le r}} \int_r^u \psi^s(\xi_{\le s}, V^s(\xi_{\le s}))K(ds) & \\ = \Pi_{r,x_{\le r}} A[r,u) \quad & \text{for } r < u, \\ V^r = 0 \quad & \text{for } r \ge u. \end{aligned} \tag{4.14a}$$

By (4.4),

$$P_\Gamma \exp\{-J_A(\Lambda; r, u)\} = \exp\{-\langle V_{\Lambda^\Delta}, \Gamma\rangle\}.$$

Therefore

$$P_\Gamma \exp\{-J_A[r,u)\} = \exp\{-\langle V, \Gamma_\Delta\rangle\},$$

where Γ_Δ is the restriction of Γ to $\mathscr{W}(\Delta)$ and V is determined by (4.14a). Passing to the limit as $t \downarrow -\infty$, we get (1.42) and (1.43). Formulas (1.45) and (1.46) follow from (1.42) and (1.43).

If A is an arbitrary element of $\mathscr{R}$, then $A_u(n) = A_u \wedge n$ satisfies the condition (1.40). We define J_A as the limit of $J_{A(n)}$ and we leave it to the reader to check that with this definition 1.4.B and (1.42) and (1.43) hold for all $A \in \mathscr{R}$. □

4.4. Proof of Theorem 1.4. According to [9], two linear additive functionals of M are equivalent if they have the identical finite characteristic functions. By (1.45), the characteristic of the functional (1.41) is equal to $\Pi_{r,x_{\le r}} A(r,\infty)$. To prove Theorem 1.4, it is sufficient to show that, if a bounded function H is the characteristic of a linear additive functional J of M, then

$$(4.15) \qquad H^r(x_{\le r}) = \Pi_{r, x_{\le r}} A(r, \infty)$$

for some additive functional A of the process Ξ. It is easy to see that, for every $r < s$,

$$(4.16) \qquad \Pi_{r, x_{\le r}} H^s(\xi_{\le s}) = P_{r, \delta_{x_{\le r}}} J(s, \infty),$$

which implies that the left side does not exceed $H^r(x_{\le r})$, tends to $H^r(x_{\le r})$ as $s \downarrow r$ and tends to 0 as $s \to +\infty$. If Ξ is a right process, then these conditions are sufficient for a bounded function H to have the representation (4.15). The proof can be obtained by a slight modification of arguments in Sections 33 and 34 of [33]. (The main point is that every bounded right-continuous supermartingale is generated by an adapted increasing process.) □

4.5. LEMMA 4.2. *Let $Q_n(\omega, B)$ be kernels from a measurable space $(\Omega, \mathscr{F})$ to a measurable Radon space $(E, \mathscr{B})$. Put $P \in \mathscr{P}$ if P is a probability measure on $(\Omega, \mathscr{F})$ such that, for every bounded function $f \in \mathscr{B}$, there exists a finite limit*

$$(4.17) \qquad Q_n(\omega, f) = \int_E Q_n(\omega, dx) f(x) \to l_P(\omega, f) \quad \text{in } P\text{-measure}$$

and, moreover, $l_P(\cdot, f_n) \to l_P(\cdot, f)$ in P-measure if $f_n \to f$ boundedly. Then there exists a kernel $Q(\omega, B)$ from $(\Omega, \mathscr{F}^*)$ to $(E, \mathscr{B})$ such that

$$(4.18) \qquad Q(\omega, f) = l_P(\omega, f) \quad P\text{-a.s.}$$

for every bounded $f \in \mathscr{B}$ and every $P \in \mathscr{P}$.

PROOF. Put $l(\omega, f) = \lim \operatorname{med} Q_n(\omega, f)$, $\Omega_1 = \{\omega: l(\omega, 1) < \infty\}$. Note that

$$(4.19) \qquad l(\omega, f) = l_P(\omega, f) \quad P\text{-a.s.}$$

for every $P \in \mathscr{P}$ and every bounded $f \in \mathscr{B}$. Therefore $P(\Omega_1) = 1$ for all $P \in \mathscr{P}$. We assume that E is a universally measurable subset in a compact metric space K. For every Borel function F on K, the restriction $f = F_E$ to E is $\mathscr{B}$-measurable. If $\omega \in \Omega_1$, then $L(F) = l(\omega, F_E)$ is a positive linear functional on the space $C(K)$ of all continuous functions and, by the Riesz theorem,

$$(4.20) \qquad l(\omega, F_E) = \int R(\omega, dx) F(x), \qquad F \in C(K),$$

where $R(\omega, \cdot)$ is a finite measure on the Borel σ-algebra $\mathscr{B}(K)$. Put $F \in \mathscr{H}$ if

$$(4.21) \qquad \int_K R(\omega, dx) F(x) = l_P(\omega, F_E) \quad P\text{-a.s. for all } P \in \mathscr{P}.$$

Note that $\mathscr{H}$ is a linear space closed under bounded convergence. By (4.19) and (4.20), $\mathscr{H}$ contains $C(K)$ and therefore it contains all bounded Borel functions. Let $\mu_P(\cdot) = \int P(d\omega) R(\omega, \cdot)$. There exists a Borel set $B \supset E$ such that $\mu_P(B \setminus E) = 0$ and therefore $R(\omega, B \setminus E) = 0$ P-a.s. By (4.21), $R(\omega, K \setminus B) = l_P(\omega, 0) = 0$ P-a.s. Hence the P-measure of the set $\Omega_2 = \{\omega: R(\omega, K \setminus E) = 0\}$ is equal to 1 for all $P \in \mathscr{P}$, and $Q(\omega, \cdot) = R(\omega, \cdot)$ for $\omega \in \Omega_1 \cap \Omega_2$, $Q(\omega, \cdot) = 0$ for $\omega \notin \Omega_1 \cap \Omega_2$ satisfies all conditions of Lemma 4.2. □

4.6. Proof of Theorem 1.5. Fix $\Delta = [r, u)$. To every finite set $\Lambda = \{r = t_0 < \cdots < t_n = u\}$ there corresponds a kernel Q_Λ from $(\Omega, \mathscr{G}(\Delta))$ to $(\mathscr{W}, \hat{\mathscr{B}})$ such that, for every $F \in \hat{\mathscr{B}}$,

$$Q_\Lambda(\omega, F) = \sum_{i=1}^{n} \langle F^\tau[t_{i-1}, t_i), M_{t_i} \rangle, \tag{4.22}$$

where $F^\tau[r, u) = F_u^\tau - F_r^\tau$ [see (1.49)]. Note that $Q_\Lambda(\omega, F) = J_A(\Lambda; r, u)$ if we set $A_t = F_t^\tau$ in (1.39). By (1.41),

$$\int_\Delta \langle dY_F^t, M_t \rangle = \lim Q_{\Lambda_n}(\cdot, F) \quad \text{in } L^2(P_\Gamma) \tag{4.23}$$

for every $\Gamma \in \mathfrak{M}(\mathscr{W})$ and every bounded $F \in \hat{\mathscr{B}}$. By (1.46) and Lemma 4.2, there exists a kernel $Q_\Delta(\omega, B)$ from $(\Omega, \mathscr{G}(\Delta)^*)$ to $(\mathscr{W}, \hat{\mathscr{B}})$ such that

$$Q_\Delta(F) = \int_\Delta \langle dY_F^t, M_t \rangle \quad P_\Gamma\text{-a.s.}$$

for all $\Gamma \in \mathfrak{M}(\mathscr{W})$. Let $\Delta_n = (n, n+1]$. The sum M_τ of Q_{Δ_n} over all integers n satisfies (1.50).

Formula (1.50a) follows from (1.45). By applying (1.50a) to $F(w_{\le t}) = 1_{t \le u}$, we get

$$P_\Gamma M_\tau(\mathscr{W}_{\le u}) = \Pi_\Gamma\{\tau \le u\} \le \Pi_\Gamma\{\alpha \le u\} = \Gamma(\mathscr{W}_{\le u}).$$

Therefore $M_\tau(\mathscr{W}_{\le u}) < \infty$ P_Γ-a.s. if $\Gamma \in \mathfrak{M}(\mathscr{W})$. □

4.7. Lemma 4.3. *Let $A \in \mathscr{R}$ and let $A\{t\} = A_{t+} - A_t$ be a reconstructable function. For every $A \in \mathscr{R}$ put*

$$V_A^r(x_{\le r}) = -\log P_{r, \delta_{x_{\le r}}} \exp\left\{-\int \langle dA_t, M_t \rangle\right\}. \tag{4.24}$$

If Ξ is strong Markov, then, for every stopping time τ,

$$\begin{aligned}\Pi_{r, x_{\le r}} V_A^\tau(\xi_{\le\tau}) 1_{\tau < \infty} + \Pi_{r, x_{\le r}} \int_\tau^\infty \psi^s[\xi_{\le s}, V_A^s(\xi_{\le s})] K(ds) \\ = \Pi_{r, x_{\le r}} A[\tau, \infty).\end{aligned} \tag{4.25}$$

Let $A, \tilde{A} \in \mathscr{R}$ and $V = V_A$, $\tilde{V} = V_{\tilde{A}}$. If $\tilde{A} \ge A$ and if $\Pi_{r, x_{\le r}} \tilde{A}[\tau, \infty) < \infty$, then

$$\begin{aligned}\Pi_{r, x_{\le r}} \int_\tau^\infty \left\{\psi^s\left[\xi_{\le s}, \tilde{V}^s(\xi_{\le s})\right] - \psi^s\left[\xi_{\le s}, V^s(\xi_{\le s})\right]\right\} K(ds) \\ \le \Pi_{r, x_{\le r}}\left\{\tilde{A}[\tau, \infty) - A[\tau, \infty)\right\}.\end{aligned} \tag{4.26}$$

If, in addition,

$$\tilde{A}(s,t) = A(s,t) \quad \text{for all } t > s > \tau, \tag{4.27}$$

then

$$\begin{aligned}&\Pi_{r,x_{\le r}} \int_\tau^\infty \psi^s[\xi_{\le s}, V^s(\xi_{\le s})] K(ds)\\ &\qquad = \Pi_{r,x_{\le r}} \int_\tau^\infty \psi^s\big[\xi_{\le s}, \tilde{V}^s(\xi_{\le s})\big] K(ds).\end{aligned} \tag{4.28}$$

In particular, if $A(\tau, t) = 0$ *for all* $t > \tau$, *then*

$$\Pi_{r,x_{\le r}} \int_\tau^\infty \psi^s[\xi_{\le s}, V_A^s(\xi_{\le s})] K(ds) = 0. \tag{4.29}$$

PROOF. It follows from (1.42) and (1.43) that

$$V_A^r(x_{\le r}) + K_A^r(x_{\le r}) = H_A^r(x_{\le r}), \tag{4.30}$$

where

$$K_A^r(x_{\le r}) = \Pi_{r,x_{\le r}} \int_r^\infty \psi^s[\xi_{\le s}, V_A^s(\xi_{\le s})] K(ds),$$

$$H_A^r(x_{\le r}) = \Pi_{r,x_{\le r}} A[r,\infty).$$

By (0.7) and 0.5.β, the second and third terms in (4.25) are equal to $\Pi_{r,x_{\le r}} K_A^\tau(\xi_{\le \tau})$ and $\Pi_{r,x_{\le r}} H_A^\tau(\xi_{\le \tau})$. (Here we use the assumption that $A\{t\}$ is reconstructable.) Therefore (4.25) follows from (4.30).

If $A \le \tilde{A}$, then, by (4.24), $V^r(x_{\le r}) \le \tilde{V}^r(x_{\le r})$ and (4.25) implies (4.26).

For every stopping time τ, there exists a sequence of stopping times $\tau_n > \tau$ such that $\tau_n \downarrow \tau$. Under condition (4.27), $\tilde{A}[\tau_n, t) = A[\tau_n, t)$ for all n and all $t > \tau_n$, and we get (4.28) by applying (4.26) to τ_n, by passing to the limit and by taking into account that $\psi^s(x, z)$ is monotone increasing in z for $z \ge 0$ by (1.21).

We obtain (4.29) by taking $\tilde{A} = 0$ in (4.28). □

COROLLARY. *If* Ξ *is strong Markov, then, for every stopping time* τ,

$$P_\Gamma \exp\{-M_\tau(F)\} = e^{-\langle V, \Gamma\rangle} \tag{4.31}$$

for all $\Gamma \in \mathfrak{M}(\mathscr{W})$, $F \in \hat{\mathscr{B}}$ *where*

$$V^r(x_{\le r}) + \Pi_{r,x_{\le r}} \int_r^\tau \psi^s[\xi_{\le s}, V^s(\xi_{\le s})] K(ds) = \Pi_{r,x_{\le r}} F(\xi_{\le \tau}) 1_{\tau < \infty}. \tag{4.32}$$

This follows from (1.42), (1.43), (1.49) and (4.29).

4.8. PROOF OF THEOREM 1.6. First we consider

$$Y = \exp M_\tau(-F), \tag{4.33}$$

where $F \in \hat{\mathscr{B}}$ is bounded and vanishes outside $\mathscr{W}(\Delta)$ for some finite interval Δ and

$$Z = \exp\left\{-\int \langle dA_t, M_t\rangle\right\}, \tag{4.34}$$

where $A \in \mathscr{R}$ is constant outside a finite (nonrandom) interval, is bounded and $A_\tau = 0$. By Theorem 1.5, $M_\tau \in \mathfrak{M}(\mathscr{W})$ P_Γ-a.s. and, by Theorem 1.3,

$$P_{M_\tau} Z = \exp\langle -V, M_\tau\rangle, \tag{4.35}$$

where $V = V_A$ is defined by (4.24). By (4.35), (4.31) and (4.32),

$$P_\Gamma Y P_{M_\tau} Z = P_\Gamma \exp\langle -V - F, M_\tau\rangle = \exp\langle -\hat{V}, \Gamma\rangle, \tag{4.36}$$

where

$$\begin{aligned} &\hat{V}^r(x_{\le r}) + \Pi_{r, x_{\le r}} \int_r^\tau \psi^s(\xi_{\le s}, \hat{V}^s(\xi_{\le s})) K(ds) \\ &\quad = \Pi_{r, x_{\le r}} (F + V)(\xi_{\le \tau}) 1_{\tau < \infty}. \end{aligned} \tag{4.37}$$

On the other hand, by (1.50) and Theorem 1.3,

$$P_\Gamma YZ = e^{-\langle \tilde{V}, \Gamma\rangle},$$

where

$$\begin{aligned} &\tilde{V}^r(x_{\le r}) + \Pi_{r, x_{\le r}} \int_r^\infty \psi^s(\xi_{\le s}, \tilde{V}^s(\xi_{\le s})) K(ds) \\ &\quad = \Pi_{r, x_{\le r}} \{F(\xi_{\le \tau}) 1_{\tau < \infty} + A[r, \infty)\}. \end{aligned} \tag{4.38}$$

By (4.25),

$$\begin{aligned} &\Pi_{r, x_{\le r}} V^\tau(\xi_{\le \tau}) 1_{\tau < \infty} + \Pi_{r, x_{\le r}} \int_\tau^\infty \psi^s(\xi_{\le s}, V^s(\xi_{\le s})) K(ds) \\ &\quad = \Pi_{r, x_{\le r}} A[\tau, \infty) = \Pi_{r, x_{\le r}} A[r, \infty), \end{aligned} \tag{4.39}$$

since $A_\tau = 0$. Equations (4.38) and (4.39) imply that

$$\begin{aligned} &\tilde{V}^r(x_{\le r}) + \Pi_{r, x_{\le r}} \int_r^\infty \psi^s(\xi_{\le s}, \tilde{V}^s(\xi_{\le s})) K(ds) \\ &\quad = \Pi_{r, x_{\le r}} (F + V)(\xi_{\le t}) 1_{\tau < \infty} + \Pi_{r, x_{\le r}} \int_\tau^\infty \psi^s(\xi_{\le s}, V^s(\xi_{\le s})) K(ds). \end{aligned} \tag{4.40}$$

Note that $\tilde{V} = V_{\tilde{A}}$ where $\tilde{A}_t = F_t^\tau + A_t$. Since $\tilde{A}_t \ge A_t$ and $\tilde{A}_\tau = A_\tau$, it follows from (4.28) and (4.40) that $\tilde{V}$ satisfies the same equation (4.37) as $\hat{V}$. Using Lemma 3.2, it is easy to show that $\tilde{V} = \hat{V}$. Hence (1.51) holds for Y and Z given by (4.33) and (4.34). Since both families are closed under multiplication and generate, respectively, $\mathscr{G}_{\le \tau}$ and $\mathscr{G}_{\ge \tau}$, the general formula (1.51) follows from the multiplicative systems theorem. □

5. Moment functions.

5.1. It is sufficient to prove Theorem 1.7 for measures η concentrated on $\mathscr{E}(\Delta)$ for finite intervals Δ. This is an immediate implication of the following lemma.

LEMMA 5.1. *Suppose that $\eta_k \uparrow \eta \in \mathfrak{M}(\mathscr{E})$ and let $F(s_1, \dots, s_n)$ be a Borel function on $[0, \infty)^n$ monotone increasing in each argument. Put $Z_i = \langle f_i, X_{t_i} \rangle$, $i = 1, \dots, n$, where $f_i \in \mathscr{B}_{t_i}$ are bounded. We have*

$$(5.1) \qquad P_{\eta_k} F(Z_1, \dots, Z_n) \uparrow P_\eta F(Z_1, \dots, Z_n) \quad \text{as } k \to \infty.$$

PROOF. Consider a sequence of independent stochastic processes (X_t^k, P_{ν_k}), where $\nu_1 = \eta_1$, $\nu_k = \eta_k - \eta_{k-1}$ for $k \geq 2$. Put $Z_i^k = \langle f_i, X_{t_i}^k \rangle$, $S_i^k = Z_i^1 + \cdots + Z_i^k$, $S_i = Z_i^1 + \cdots + Z_i^k + \cdots$ and note that

$$P_\eta F(Z_1, \dots, Z_n) = EF(S_1, \dots, S_n), \qquad P_{\eta_k} F(Z_1, \dots, Z_n) = EF(S_1^k, \dots, S_n^k).$$

Formula (5.1) follows from the monotone convergence theorem. □

5.2. Fix a finite interval Δ and a positive integer n and consider the class G of bounded progressive functions $h^r(u, x)$, $u \in [0, 1]^n$, $r \in \Delta$, $x \in E_r$. Put $h \in H$ if h is a polynomial in $u_1, \dots, u_n$ with coefficients in G. Denote by $\mathscr{R}$ the class of bounded functions $h^r(u, x)$, $u \in [0, 1]^n$, $r \in \Delta$, $x \in E_r$ such that $h^r(u, x) \to 0$ as $u \to 0$. Note that:

(a) G, H and $\mathscr{R}$ are algebras and $hR \in \mathscr{R}$ if $h \in H$, $R \in \mathscr{R}$;
(b) the mapping $h \to \hat{h}$ defined by the formula

$$(5.2) \qquad \hat{h}^r(u, x) = \Pi_{r,x} \int_r^t h^s(u, \xi_s) K(ds)$$

preserves spaces H, G and $\mathscr{R}$.

Put

$$u^m = u_1^{m_1} \cdots u_n^{m_n}, \qquad |u| = u_1 + \cdots + u_n, \qquad |m| = m_1 + \cdots + m_n$$

for $u = (u_1, \dots, u_n)$ and $m = (m_1, \dots, m_n)$. Denote by $\mathscr{A}_l$ the class of functions v which admit representation of the form

$$(5.3) \qquad v = \sum_{|m|=1}^{l} g_m u^m + |u|^l R_l,$$

where $g_m \in G$ and $R_l \in \mathscr{R}$. Clearly,

(c) $\mathscr{A}_1 \supset \mathscr{A}_2 \supset \cdots \supset \mathscr{A}_n$;
(d) if $v \in \mathscr{A}_l$, then $gv \in \mathscr{A}_l$ for every $g \in G$;
(e) if $l > 2$ and if $v_1, v_2 \in \mathscr{A}_{l-1}$, then $v_1 v_2 \in \mathscr{A}_l$.

LEMMA 5.2. *Let* $t_i \in \Delta$, $f_i \in \mathscr{B}_{t_i}$ *for* $i = 1, \dots, n$. *Suppose that* f_i *are bounded and that*

$$(5.4)\quad v^r(u,x) + \Pi_{r,x}\int_r^\infty \psi^s[\xi_s, v^s(u,\xi_s)]K(ds) = \sum_{i=1}^n u_i \Pi_{r,x} f_i(\xi_{t_i})$$

[$v^r(u,x) = 0$ *for* $r > \max t_i$]. *Under condition* 1.2.E$_n$, $v \in \mathscr{A}_n$.

PROOF. Let $0 \le f \le c$. Since $\psi \ge 0$, we have

$$(5.5)\qquad v^r(u,x) \le c|u|.$$

By (1.58), there exists a constant a such that

$$(5.6)\qquad \psi^r(x,z) \le az^2 \quad \text{for all } (r,x) \in \mathscr{E},\ z \ge 0.$$

By (5.4), (5.5), (5.6) and 1.2.C, (β)

$$(5.7)\qquad v = \sum g_i u_i + F|u|^2,$$

where $g_i^r(x) = \Pi_{r,x} f_i(\xi_{t_i})$ belong to G and F belongs to H. Hence the lemma holds for $l = 1$. By Taylor's formula, for every $l \le n$,

$$(5.8)\qquad \psi^s(x,z) = \sum_{k=2}^l q_k^s(x) z^k/k! - \phi_l^s(x,z)z^l,$$

where q_k are defined by (1.57) and

$$(5.9)\qquad \begin{aligned} \phi_l^s(x,z) &= \frac{1}{l!}\left[D_z^l\psi^s(x,0) - D_z^l\psi^s(x,\hat{z})\right] \\ &= \frac{1}{l!}\int_0^\infty u^l(1 - e^{-u\hat{z}})n^s(x,du) \end{aligned}$$

with $0 < \hat{z} < z$. Let $t = \max\{t_1, \dots, t_n\}$. By (5.4), (5.8) and (5.5), $v^r(u,x) = 0$ for $r > t$ and

$$(5.10)\qquad \begin{aligned} v^r(u,x) = \sum g_i u_i - \sum_{k=2}^l \Pi_{r,x}\int_r^t q_k^s(\xi_s) v^s(u,\xi_s)^k/k!\, K(ds) \\ + S_l^r(u,x) \quad \text{for } r \le t, \end{aligned}$$

where

$$(5.11)\qquad 0 \le S_l^r(u,x) \le c^l |u|^l \Pi_{r,x}\int_r^t \phi_l^s[\xi_s, v^s(u,\xi_s)]K(ds).$$

By (5.9), (5.5), (5.11) and (b), $|u|^{-l}S_l$ belongs to $\mathscr{R}$. By formula (5.10) with $l = 2$,

$$(5.12)\qquad \begin{aligned} v^r(u,x) &= \sum g_i^r(x) u_i \\ &\quad - \Pi_{r,x}\int_r^t \tfrac{1}{2} q_2^s(\xi_s) v^s(u,\xi_s)^2 K(ds) + S_2^r(u,x). \end{aligned}$$

It follows from (5.7) that v^2 belongs to $\mathscr{A}_2$ and, by (b) and (d), $v \in \mathscr{A}_2$. We conclude from (5.10), (e), (c) and (b) that the lemma holds for $l > 2$ if it is true for $l-1$. □

5.3. Consider the class S of functions

$$h = \sum_{\Lambda} g_\Lambda u_\Lambda + \sum_{i=1}^{n} \gamma_i u_i^2 + |u|^n R, \tag{5.13}$$

where Λ runs over all nonempty subsets of the set $\{1,\dots,n\}$, $g_\Lambda \in G$, $\gamma_i \in H$, $R \in \mathscr{R}$ and

$$u_\Lambda = \prod_{i\in\Lambda} u_i.$$

We establish by induction in n that g_Λ are uniquely determined by h. We write $h \approx 0$ if $h \in S$ and if $g_\Lambda = 0$ in (5.13) for all Λ. Writing $h_1 \approx h_2$ means that $h_1 - h_2 \approx 0$. Clearly, $\mathscr{A}_n \subset S$ and $h\tilde{h} \approx 0$ if $h, \tilde{h} \in S$ and $h \approx 0$.

Note that

$$\exp(g_\Lambda u_\Lambda) \approx 1 + g_\Lambda u_\Lambda, \qquad \exp(\gamma_i u_i^2) \approx 1, \qquad \exp(|u|^n R) \approx 1$$

and therefore

$$e^h \approx 1 + \sum_{\Lambda_1,\dots,\Lambda_k} g_{\Lambda_1}\cdots g_{\Lambda_k} u_{\Lambda_1}\cdots u_{\Lambda_k} = 1 + \sum_{\Lambda} u_\Lambda G_\Lambda, \tag{5.14}$$

where

$$G_\Lambda = \sum_{\Lambda_1,\dots,\Lambda_k} g_{\Lambda_1}\cdots g_{\Lambda_k} \tag{5.15}$$

with the sum taken over all partitions of Λ into disjoint subsets $\Lambda_1,\dots,\Lambda_k$.

By Lemma 4.1,

$$P_\eta \exp \sum_{i=1}^{n} \langle -u_i f_i, X_{t_i}\rangle = e^{-\langle v,\eta\rangle}, \tag{5.16}$$

where v is a solution of (5.4). By Lemma 5.2, $v \in \mathscr{A}_n \subset S$ and therefore

$$v \approx \sum_{\Lambda} (-1)^{|\Lambda|-1} u_\Lambda V_\Lambda \tag{5.17}$$

for some $V_\Lambda \in G$.

Suppose that η is concentrated on Δ. Then (5.17) implies

$$-\langle v,\eta\rangle \approx \sum_{\Lambda} (-1)^{|\Lambda|} u_\Lambda \langle V_\Lambda, \eta\rangle \tag{5.18}$$

and, by (5.16), (5.14) and (5.15),

$$P_\eta \exp \sum_{i=1}^{n} \langle -u_i f_i, X_{t_i}\rangle \approx 1 + \sum_{\Lambda} (-1)^{|\Lambda|} u_\Lambda T_\Lambda, \tag{5.19}$$

where

$$T_\Lambda = \sum_{\Lambda_1,\dots,\Lambda_k} \langle V_{\Lambda_1},\eta\rangle \cdots \langle V_{\Lambda_k},\eta\rangle. \tag{5.20}$$

This implies

$$P_\eta \prod_{i=1}^{n} \langle f_i, X_{t_i} \rangle = \sum_{\Lambda_1, \dots, \Lambda_k} \langle V_{\Lambda_1}, \eta \rangle \cdots \langle V_{\Lambda_k}, \eta \rangle, \tag{5.21}$$

where the sum is taken over all partitions of $\{1, \dots, n\}$ into disjoint sets $\Lambda_1, \dots, \Lambda_k$.

The next step is to evaluate V_Λ.

By (5.8) and (5.17),

$$\begin{aligned} &\psi^r[x, v^r(u,x)] \\ &\quad \approx \sum_{k=2}^{n} q_k^r(x)/k! \sum_{\Lambda_1, \dots, \Lambda_k} (-1)^{|\Lambda_1| + \cdots + |\Lambda_k|} u_{\Lambda_1} \cdots u_{\Lambda_k} \\ &\qquad \times V_{\Lambda_1}^r(x) \cdots V_{\Lambda_k}^r(x) + F^r(u,x) \end{aligned} \tag{5.22}$$

where the second sum is taken over all ordered families of disjoint subsets $\Lambda_1, \dots, \Lambda_k$ and $F^r(u,x) = v^r(u,x)^n \phi_n^r(x, v^r(u,x))$. By (5.9) and (5.5), $F \approx 0$. It follows from (5.4), (5.17), (5.22) and 5.2(b) that, for every $\Lambda \subset \{1, \dots, n\}$,

$$\begin{aligned} &(-1)^{|\Lambda|-1} V_\Lambda^r(x) \\ &\quad + \sum_{\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k} \Pi_{r,x} \int_r^\infty q_k^s(\xi_s) K(ds) (-1)^{|\Lambda|} V_{\Lambda_1}^s(\xi_s) \cdots V_{\Lambda_k}^s(\xi_s) \\ &\quad = \Pi_{r,x} f_i(\xi_{t_i}) \quad \text{if } \Lambda = \{i\}, r \le t_i, \\ &\quad = 0 \quad \text{otherwise}, \end{aligned}$$

where the sum is taken over all partitions of Λ into $k \ge 2$ disjoint nonempty subsets $\Lambda_1, \dots, \Lambda_k$ (disregarding the order among the subsets). Hence

$$V_{\{i\}}^r(x) = 1_{r \le t_i} \Pi_{r,x} f_i(\xi_{t_i}), \tag{5.23}$$

$$V_\Lambda^r(x) = \sum_{\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k} \Pi_{r,x} \int_r^\infty q_k^s(\xi_s) K(ds) V_{\Lambda_1}^s(\xi_s) \cdots V_{\Lambda_k}^s(\xi_s) \tag{5.24}$$

if $|\Lambda| > 1$.

Denote by $\mathbb{D}_\Lambda^\circ$ the set of all connected diagrams with exits enumerated by Λ. We claim that, if $r \le t_i$ for all $i \in \Lambda$, then

$$V_\Lambda^r(x) = \sum_{D \in \mathbb{D}_\Lambda^\circ} \int L_D(r, x; d\omega) \prod_{i \in \Lambda} f_i(\omega_{t_i}^i). \tag{5.25}$$

Indeed, (5.25) follows from (5.23) and 1.6.A if $\Lambda = \{i\}$, $r \le t_i$. If (5.25) is true for all $|\Lambda_i| < |\Lambda|$, then it holds also for Λ by 1.6.B and (5.24).

Finally, (1.59) follows from (5.21), (5.24) and 1.6.C.

6. Bibliographical notes.

6.1. Branching particle systems corresponding to a diffusion ξ and an additive functional $K(dt) = c(t, \xi_t)\, dt$ were studied in [34]. Even earlier special classes of such systems were investigated in [32]. A general theory of branching was developed in [18]. The construction presented in Section 2 is similar to the construction of age-dependent branching systems in Chapter 6 of the monograph [17].

6.2. Passage to the limit from branching particle systems to measure-valued processes was done by Watanabe [35] and Dawson [1]. More general theorems were proved by Ethier and Kurtz [15], Chapter 9 (see also [31]). All these authors considered only the case when ξ is a process with a stationary Feller transition function. This restriction was dropped in [9] and [10]. Another construction of a superprocess over a non-Feller (but time homogeneous) process ξ was given in [16].

In the present paper, in contrast to [9], [10] and [16], we use the passage to the limit not only heuristically but to prove the existence of superprocesses. Introducing a general killing additive functional K seems to be a novelty.

6.3. Historical paths have been considered already by Kallenberg [21] in his "backward tree formula." Various ways of constructing a historical superprocess—outlined in Section 1.3—were discussed at two conferences—at the University of California, San Diego and at Cornell University—in the spring of 1989. An idea for getting a historical superprocess as the superprocess over the historical process, due to Perkins, has been systematically implemented in a recent paper of Dawson and Perkins [3]. Introductory parts of [3] and the present paper are parallel but the terminology is not identical. Our "historical processes" are called "path processes" in [3] and the name "historical processes" is reserved there for what we call "historical superprocesses." The enriched model is used in [3] to get new strong results on path properties for the super Brownian motion and some other superprocesses. (In earlier publications [2], [29] and [30] the authors studied paths by means of nonstandard analysis.)

The primary objective of the present paper consists of developing new probabilistic tools for applications to classical analysis. The next step in this direction is made in [12] and [13]. In [12] a technique was developed to deal with the general branching mechanism. This technique is applicable, in particular, to functions $\psi(z) = \gamma z^{\alpha}$ with $1 < \alpha < 2$ which do not satisfy condition $1.2.\mathrm{E}_2$ and therefore cannot be investigated using moments. The results of [12] are applied in [13] to establish connections between an analytic theory of the equation $-Lv + \psi(v) = \rho$ and path properties of the superdiffusion with the generator L.

6.4. Recent papers [24], [25], [26] are devoted to pathwise construction of superprocesses. Le Jan obtains them as projective limits of branching particle systems. Le Gall constructs superdiffusions and historical superdiffusions (with $\psi = \gamma z^2$) using excursions and local times of one-dimensional Brownian motion.

6.5. A special class of linear additive functionals

$$J(r, u) = \int_r^u \langle f^t, X_t \rangle \, dt$$

have been studied, first, by Iscoe [19] who called them "weighted occupation times." In particular, Iscoe proved for them a form of equations (1.43) and (1.44). A general concept of a linear additive functional was introduced in [9] where all square-integrable fair functionals were described and applied to construction of positive functionals [an additive functional J is called fair if $P_{r,\mu}J(r,u) = 0$ for all r, u and μ]. The relationship between additive functionals of a process ξ and the corresponding superprocess X discussed in Section 1.4 seems to be new.

Linear additive functionals play the central role in [12] and [13]. In [12] a theory of integration relative to a measure-functional is developed and used to introduce random measures X_τ, Y_τ corresponding to the first hitting time τ of any analytic set. The random measures X_τ and Y_τ are used in [13] to study nonlinear PDE in a manner which recalls the classical application of functionals $f(\xi_\tau)$, $\int_0^\tau \rho(\xi_s)\,ds$ to linear elliptic differential equations.

6.6. An analog of the special Markov property for branching particle systems with discrete time parameter was studied by Jagers [20]. This property is useful for probabilistic theory of a class of nonlinear partial differential equations—a subject we shall investigate at another place.

Moment functions of order $n \ge 3$ were evaluated, first, in [8] for the case $K(dt) = dt$, $\psi(z) = z^2$.

6.7. Equilibrium measures for superprocesses were studied in [10]. Analogous problems for branching particle systems with discrete time have been investigated in [28].

APPENDIX

Markov Processes and Their Additive Functionals.

0.1. In this paper we deal with inhomogeneous Markov processes on the time interval $\mathbb{R}$. For a general theory of such processes, we refer to [6], [7] and [23]. The model used in this paper is very close to those presented in [11]. Suppose that we are given:

(a) an arbitrary set W;
(b) for every $t \in \mathbb{R}$, a measurable space $(E_t, \mathscr{B}_t)$ and a mapping ξ_t from W to E_t;
(c) for every open interval Δ, a σ-algebra $\mathscr{F}(\Delta)$ in W;
(d) for every $r \in \mathbb{R}$, $x \in E_r$, a probability measure $\Pi_{r,x}$ on $\mathscr{F}_{>r} = \mathscr{F}(r,\infty)$.

We say that $\xi = (\xi_t, \mathscr{F}(\Delta), \Pi_{r,x})$ is a Markov process if:

0.1.A. $\mathscr{F}(\Delta) \subset \mathscr{F}(\tilde{\Delta})$ for $\Delta \subset \tilde{\Delta}$; ξ_t is adapted to $\mathscr{F}(\Delta)$, that is, $\{\xi_t \in B\} \in \mathscr{F}(\Delta)$ for $t \in \Delta$, $B \in \mathscr{B}_t$.

0.1.B. For each $Y \in \mathscr{F}_{>t}$, $\Pi_{t,x}Y$ is $\mathscr{B}_t$-measurable. For every $r, t \in \mathbb{R}$ and every $Y \in \mathscr{F}(r, t)$, $Z \in \mathscr{F}_{>t}$,

$$\Pi_{r,x}YZ = \Pi_{r,x}(Y\Pi_{t,\xi_t}Z).$$

We denote by $\mathscr{F}_{\geq t}$ the minimal σ-algebra which contains $\mathscr{F}_{>t}$ and $\{\xi_t \in B\}$ for all $B \in \mathscr{B}_t$. The notation $\mathscr{F}_{\leq t}$, $\mathscr{F}(r, u]$ etc. is defined similarly.

In addition to 0.1.A, B, we assume that every measure $\Pi_{r,x}$ can be continued to $\mathscr{F}_{\geq r}$ in such a way that $\Pi_{r,x}\{\xi_r = x\} = 1$. Under this assumption the σ-algebras $\mathscr{F}_{>t}$ and $\mathscr{F}(r, t)$ in 0.1.B can be replaced by $\mathscr{F}_{\geq t}$ and $\mathscr{F}[r, t]$.

0.2. A Markov process ξ is *canonical* if:

(a) W is a subset of the Cartesian product of E_t over all $t \in \mathbb{R}$;
(b) $\xi_t(w) = w_t$ are the coordinate functions;
(c) $\mathscr{F}(\Delta)$ is generated by $\{w: w_t \in B\}$, $t \in \Delta$, $B \in \mathscr{B}_t$.

A canonical process can be modified to introduce a *random birth time*. To this end we extend $(E_t, \mathscr{B}_t)$, W and $\mathscr{F}(\Delta)$ and we continue ξ_t and $\Pi_{r,x}$ in the following way.

$\hat{E}_t$ is obtained from E_t by adding an extra element ∂_t; $\hat{\mathscr{B}}_t$ is generated by $\mathscr{B}_t$ and ∂_t. $\hat{W}$ consists of paths of the form: $w_t = \partial_t$ on $(-\infty, \alpha)$, $w_t \in E_t$ on $[\alpha, +\infty)$ for some $\alpha \in [-\infty, +\infty]$ (α is called the birth time). $\hat{\xi}_t(w) = w_t$ for $w \in \hat{W}$, and $\hat{\mathscr{F}}(\Delta)$ is generated by $\hat{\xi}_t$, $t \in \Delta$. Finally, $\hat{\Pi}_{r,x}$ is the image of $\Pi_{r,x}$ under the measurable mapping $w \to w^r$ from $(W, \mathscr{F}_{\geq r})$ to $(\hat{W}, \hat{\mathscr{F}}(\mathbb{R}))$ defined by the formula: $w_t^r = \partial_t$ for $t < r$; $w_t^r = w_t$ for $t \geq r$. Note that W is a subset of $\hat{W}$ (determined by condition $\alpha = -\infty$), $\hat{\xi}_t$ coincides with ξ_t on W, the trace of $\hat{\mathscr{F}}(\Delta)$ on W is equal to $\mathscr{F}(\Delta)$ and $\hat{\Pi}_{r,x}$ coincides with $\Pi_{r,x}$ on $\mathscr{F}_{\geq r}$. Therefore without any confusion, we can drop all "carets" and use the same notation for the extended spaces and functions.

Note that

$$\Pi_{r,x}\{\alpha = r, \xi_\alpha = x\} = 1. \tag{0.1}$$

Consider the global state space $(\mathscr{E}, \mathscr{B}_{\mathscr{E}})$ defined in Section 1.1. If ξ is progressive, then $\Pi_{r,x}(C)$ is $\mathscr{B}_{\mathscr{E}}$-measurable for every $C \in \mathscr{F}(\mathbb{R})$ and we set

$$\Pi_\eta(C) = \int \Pi_{r,x}(C)\eta(dr, dx) \tag{0.2}$$

for an arbitrary measure η on $(\mathscr{E}, \mathscr{B}_{\mathscr{E}})$. Obviously,

$$\eta(B) = \Pi_\eta\{(\alpha, \xi_\alpha) \in B\} \quad \text{for all } B \in \mathscr{B}_{\mathscr{E}}. \tag{0.3}$$

WARNING. The superprocess $X = (X_t, \mathscr{G}(\Delta), P_{r,\mu})$ constructed in Theorem 1.1 does not need to be canonical but all measures $P_{r,\mu}$ are defined on $\mathscr{G}(\mathbb{R})$ by construction. We can assume that $P_{r,\mu}\{X_t = 0 \text{ for all } t < r\} = 1$ and therefore the role of ∂_t is played by the null of $\mathscr{M}_t$.

0.3. Denote by $\mathscr{A}_{\mathscr{E}}$ the σ-algebra in $\mathscr{E}$ generated by the functions

$$f^t(x) = 1_{t<u}\Pi_{t,x}\{\xi_u \in B\}, \qquad u \in \mathbb{R},\, B \in \mathscr{B}_u. \tag{0.4}$$

The transition probabilities of ξ are progressive if and only if $\mathscr{A}_{\mathscr{E}} \subset \mathscr{B}^*_{\mathscr{E}}$.

A process ξ is called *right* if:

0.3.A. $\mathscr{A}$ is generated by a countable family of functions $g^t(x)$ such that $g^t(\xi_t)$ is right continuous with left limits.

0.3.B. If f is given by (0.4), then, for every $r \in \mathbb{R}$, $x \in E_r$, $f^t(\xi_t)$ is right continuous for $t \geq r$, $\Pi_{r,x}$-a.s.

Clearly, *the transition probabilities of a right process are progressive.*

We say that ξ is *regular* if it satisfies 0.3.A and the following stronger version of 0.3.B.

0.3.B′. If Π is a measure on $\mathscr{F}_{\geq r}$ such that

$$\Pi YZ = \Pi\big(Y\Pi_{t,\xi_t}Z\big) \quad \text{for all } t > r,\, Y \in \mathscr{F}(r,t),\, Z \in \mathscr{F}_{\geq t}, \tag{0.5}$$

and if f is given by (0.4), then $f^t(\xi_t)$ is right continuous for $t \geq r$ Π-a.s. [By 0.1.B, (0.5) holds for all measures $\Pi_{r,x}$.]

PROPOSITION. *Suppose that*:

(α) *ξ is regular*;

(β) *each space $(E_t, \mathscr{B}_t)$ is Luzin*;

(γ) *$\Pi_{r,x}$ separate states, i.e., $\Pi_{r,x}(C) \neq \Pi_{r,y}(C)$ for some $C \in \mathscr{F}_{\geq r}$ if $x \neq y$.*

Then the superprocess X with parameters (ξ, K, ψ) can be chosen to be regular.

For the case $K(dt) = \gamma\, dt$, $\psi(z) = z^2/2$ this has been proved in [11], Theorem 1.2. The proof is applicable to the general case. (Later the same result was proved in [16] by a different method for more general ψ (but only in the time homogeneous case).)

0.4. We say that a function on $\mathbb{R} \times \Omega$ is *reconstructable* if it is measurable with respect to the σ-algebra generated by the sets $(-\infty, u) \times C$, $u \in \mathbb{R}$, $C \in \mathscr{F}^*_{>u}$. It is well known that a function $Y_t(\omega)$ is reconstructable if it is adapted to $\mathscr{F}^*_{>t}$ and right continuous in t for each ω.

A Markov process ξ is called *strong Markov* if

$$\Pi_{r,x}\{f(\xi_u) | \mathscr{F}_{\leq \tau}\} = \Pi_{\tau,\xi_\tau} f(\xi_u) \quad \Pi_{r,x}\text{-a.s. on } \{\tau \leq u\} \tag{0.6}$$

for all $r < u$, $f \in \mathscr{B}_u$ and each stopping time τ relative to the filtration $\mathscr{F}[r,t]$.

If ξ is a strong Markov process with random birth time α and if $\tau \geq \alpha$ is a stopping time relative to the filtration $\mathscr{F}_{\leq t}$, then

$$\Pi_\eta\{Y^\tau | \mathscr{F}_{\leq \tau}\} = F^\tau(\xi_\tau) \quad \text{with } F^r(x) = \Pi_{r,x}Y^r \tag{0.7}$$

for every reconstructable $Y \geq 0$.

Every right Markov process is strong Markov (see, e.g., [7] or [23]).

0.5. We say that a measure is *Σ-finite* if it can be represented as the sum of a countable set of finite measures.

A positive additive functional K of a Markov process ξ is a function K: $\Omega \times \mathscr{B}(\mathbb{R}) \to [0, \infty]$ with the properties:

0.5.A. For every $\omega \in \Omega$, $K(\omega, \cdot\,)$ is a Σ-finite measure on $\mathscr{B}(\mathbb{R})$.

0.5.B. For every open interval Δ, $K(\cdot, \Delta)$ is $\mathscr{F}(\Delta)^*$-measurable.

An additive functional K is called *continuous* if $K\{t\} = 0$ for every singleton t. (Often stronger finiteness conditions on K are imposed. We refer to [14].)

Note that:

0.5.$\mathscr{A}$. If K is an additive functional and if f is progressive, then $\tilde{K}(dt) = f^t(\xi_t)K(dt)$ is also an additive functional.

0.5.$\mathscr{B}$. If K is a positive additional functional, then $Y_r = 1_{r<u}K(r, u)$ is reconstructable.

0.6. We list some simple properties of progressive functions.

0.6.A. $f^t(x) = 1_{t=u}h(x)$ is progressive for each $h \in \mathscr{B}_u^*$.

0.6.B. $f^t(x) = b(t)$ is progressive for every $b \in \mathscr{B}(\mathbb{R})^*$.

0.6.C. If the transition probabilities of ξ are progressive and if $Y_t \geq 0$ is reconstructable, then $f^t(x) = \Pi_{t,x}Y_t$ is progressive.

The first two statements are obvious. To prove 0.6.C, note that, if $Y \in \mathscr{F}_{>u}^*$ and $F(y) = \Pi_{u,y}Y$, then $1_{t<u}\Pi_{t,x}Y = 1_{t<u}\Pi_{t,x}F(\xi_u)$ is progressive. By the multiplicative systems theorem, $\Pi_{t,x}Y_t$ is progressive for every reconstructable $Y \geq 0$.

0.7. The following useful lemma (and its proof) was communicated to me by R. Getoor.

LEMMA. *If $(E, \mathscr{B})$ is a Radon space, then $(\mathscr{M}, \mathscr{B}_{\mathscr{M}})$ is also a Radon space. Here $\mathscr{M}$ is the set of all probability measures on $(E, \mathscr{B})$ and $\mathscr{B}_{\mathscr{M}}$ is the σ-algebra in $\mathscr{M}$ generated by the functions $f_B(\mu) = \mu(B)$, $B \in \mathscr{B}$.*

PROOF. We may assume that E is a universally measurable subset of a compact metric space $\overline{E}$ and that $\mathscr{B}$ is the trace of $\overline{\mathscr{B}}^*$ on E where $\overline{\mathscr{B}}$ is the Borel σ-algebra in $\overline{E}$. The set $\overline{\mathscr{M}}$ of all probabilities on $\overline{\mathscr{B}}^*$ can be identified with the set of all probabilities on $\overline{\mathscr{B}}$ and we can consider it as compact metric space. Besides we can identify $\mathscr{M}$ with $\{\mu: \mu \in \overline{\mathscr{M}}, \mu(E) = 1\}$. Then $\mathscr{B}_{\mathscr{M}}$ coincides with the trace of $\mathscr{B}_{\overline{\mathscr{M}}}$ on $\mathscr{M}$, where $\mathscr{B}_{\overline{\mathscr{M}}}$ is the Borel σ-algebra on $\overline{\mathscr{M}}$ [which is generated by the functions $f_B(\mu) = \mu(B)$, $B \in \overline{\mathscr{B}}$]. It remains to show that $\{\mu: \mu(E) = 1\} \in \mathscr{B}^*_{\overline{\mathscr{M}}}$. To this end, it is sufficient to prove that $f_B(\mu) = \mu(B)$ is measurable with respect to $\mathscr{B}^*_{\overline{\mathscr{M}}}$ for every $B \in \overline{\mathscr{B}}^*$. Let N be an arbitrary probability measure on $(\overline{\mathscr{M}}, \mathscr{B}_{\overline{\mathscr{M}}})$ and let $\nu = \int_{\overline{\mathscr{M}}} N(d\mu)\mu$. If $B \in \overline{\mathscr{B}}^*$, then there exist $B_1 \subset B \subset B_2$ such that $B_1, B_2 \in \overline{\mathscr{B}}$ and $\nu(B_2 \setminus B_1) = 0$. We have $f_{B_1} \le f_B \le f_{B_2}$ and $N(f_{B_1} \setminus f_{B_2}) = \nu(B_2 \setminus B_1) = 0$. Since $f_{B_1}, f_{B_2} \in \mathscr{B}_{\overline{\mathscr{M}}}$, we conclude that $f_B \in \mathscr{B}^*_{\overline{\mathscr{M}}}$. $\square$

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REFERENCES

[1] DAWSON, D. A. (1975). Stochastic equations and related measure processes. *J. Multivariate Anal.* **3** 1–52.

[2] DAWSON, D. A., ISCOE, I. and PERKINS, E. A. (1989). Super-Brownian motion: Path properties and hitting probabilities, *Probab. Theory Related Fields* **83** 135–206.

[3] DAWSON, D. A. and PERKINS, E. A. (1990). Historical processes. Preprint.

[4] DELLACHERIE, C. and MEYER, P.-A. (1978). *Probabilities and Potentials.* North-Holland, Amsterdam.

[5] DELLACHERIE, C. and MEYER, P.-A. (1983). *Probabilités et potentiel: Théorie discrète du potentiel.* Hermann, Paris.

[6] DYNKIN, E. B. (1960). *Theory of Markov Processes.* Pergamon, Oxford.

[7] DYNKIN, E. B. (1973). Regular Markov processes. *Uspehi. Mat. Nauk* **28** 35–64. [Reprinted in *London Math. Soc. Lecture Notes* **54** 187–218. Cambridge Univ. Press.]

[8] DYNKIN, E. B. (1988). Representation for functionals of superprocesses by multiple stochastic integrals, with applications to self-intersection local times. In *Colloque Paul Lévy sur le processus stochastiques. Astérisque* **157–158** 147–171.

[9] DYNKIN, E. B. (1989). Superprocesses and their linear additive functionals. *Trans. Amer. Math. Soc.* **314** 255–282.

[10] DYNKIN, E. B. (1989). Three classes of infinite dimensional diffusions. *J. Funct. Anal.* **86** 75–110.

[11] DYNKIN, E. B. (1989). Regular transition functions and regular superprocesses. *Trans. Amer. Math. Soc.* **316** 623–634.

[12] DYNKIN, E. B. (1990). Path processes and historical superprocesses. *Probab. Theory Related Fields.* To appear.

[13] DYNKIN, E. B. (1990). A probabilistic approach to one class of nonlinear differential equations. Unpublished manuscript.

[14] DYNKIN, E. B. and GETOOR, R. K. (1985). Additive functionals and entrance laws. *J. Funct. Anal.* **62** 221–265.

[15] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes. Characterization and Convergence.* Wiley, New York.

[16] FITZSIMMONS, P. J. (1988). Construction and regularity of measure-valued Markov branching processes. *Israel J. Math.* **64** 337–361.

[17] HARRIS, T. E. (1963). *The Theory of Branching Processes.* Springer, Berlin.

[18a] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968). Branching Markov processes. I. *J. Math. Kyoto Univ.* **8** 233–278.

[18b] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1968). Branching Markov processes. II. *J. Math. Kyoto Univ.* **8** 365–410.

[18c] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1969). Branching Markov processes. III. *J. Math. Kyoto Univ.* **9** 95–160.

[19] ISCOE, I. (1986). A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Related Fields* **71** 85–116.

[20] JAGERS, P. (1988). General branching processes as Markov fields. Preprint.

[21] KALLENBERG, O. (1987). Stability of critical cluster fields. *Math. Nachr.* **77** 7–43.

[22] KAWAZU, K. and WATANABE, S. (1971). Branching processes with immigration and related limit theorems. *Probab. Theory Appl.* **16** 34–51.

[23] KUZNETSOV, S. E. (1984). Nonhomogeneous Markov processes. *J. Soviet Math.* **25** 1380–1498.

[24] LE GALL, J. F. (1989). Un construction trajectorielle de certains processus de Markov à valeurs mesures. *C. R. Acad. Sci. Paris Sér. I Math.* **308** 533–538.

[25] LE GALL, J. F. (1989). Brownian excursions, trees and measure-valued branching processes. Preprint.

[26] LE JAN, Y. (1989). Limites projectives de processus de branchement markovien. *C. R. Acad. Sci. Paris Sér. I Math.* **309** 377–381.

[27] LI, Z. (1990). On the branching mechanisms of superprocesses. *Adv. in Math. (Beijing)* **19** 117–118.

[28] LIEMANT, A., MATTHES, K. and WALKOBINGER, A. (1989). *Equilibrium Distributions of Branching Processes*. Kluwer, Dordrecht.

[29] PERKINS, E. (1988). Polar sets and multiple points for super-Brownian motion. Preprint.

[30] PERKINS, E. (1989). The Hausdorff measure of the closed support of super-Brownian motion. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **25** 205–224.

[31] ROELLY-COPPOLETTA, S. (1986). A criterion of convergence of measure-valued processes: Application to measure branching processes. *Stochastics* **17** 43–65.

[32] SEVASTYANOV, B. A. (1958). Branching stochastic processes for particles diffusing in a restricted domain with absorbing boundaries. *Theory Probab. Appl.* **3** 121–136.

[33] SHARPE, M. J. (1988). *General Theory of Markov Processes*. Academic, San Diego.

[34] SKOROHOD, A. V. (1964). Branching diffusion processes. *Theory Probab. Appl.* **9** 445–449.

[35] WATANABE, S. (1968). A limit theorem of branching processes and continuous state branching processes. *J. Math. Kyoto Univ.* **8** 141–167.

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