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OPTIMAL STOPPING AND BEST CONSTANTS FOR DOOB-LIKE INEQUALITIES I: THE CASE $p = 1$

BY S. D. JACKA

University of Warwick

This paper establishes the best constant c_q appearing in inequalities of the form

$$\mathbb{E}S_\infty \le c_q \sup_{t\ge 0} \|M_t\|_q,$$

where M is an arbitrary nonnegative submartingale and

$$S_t = \sup_{s \le t} M_s.$$

The method of proof is via the Lagrangian for a version of the problem

$$\sup_\tau \mathbb{E}\{\lambda S_t - \lambda^q M_t^q\},$$

where $M \equiv |B|$, B a Brownian motion. More general inequalities of the form

$$\mathbb{E}S_\infty \le C_\Phi \sup_{t\ge 0} \|M_t\|_\Phi$$

and

$$\mathbb{E}S_\infty \le C_\Phi \sup_{t\ge 0} |||\, M_t \,|||_\Phi$$

(where $\|\cdot\|_\Phi$ and $|||\cdot|||_\Phi$ are, respectively, the Luxemburg norm and its dual, the Orlicz norm, associated with a Young function Φ) are established under suitable conditions on Φ. A simple proof of the John–Nirenberg inequality for martingales is given as an application.

1. Introduction. It is, or should be, well known that if X is a $D(0,\infty)$ (cadlag) process, then for all stopping times T and all $q > 1$

$$EX_T^* \le \frac{q}{q-1} \sup_{S\in \mathrm{T}(T)} \|X_S\|_q, \qquad X_t^* = \sup_{s\le t}|X_S|, \tag{1.1}$$

where $\mathrm{T}(T) = \{S\colon S \le T \text{ a.s., } S \text{ a stopping time}\}$.

If X is a nonnegative submartingale and $(X_{t\wedge T})$ is uniformly integrable (ui), then the right-hand side of (1.1) may, by virtue of Doob's submartingale inequality, be replaced by $[q/(q-1)]\|X_T\|_q$.

One aim of this paper is to establish the following theorem.

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THEOREM 1.1. *For any nonnegative, cadlag, ui submartingale X:*

$$\mathbb{E}X_\infty^* \le (\Gamma(1+\tilde{q}))^{1/\tilde{q}} \|X_\infty\|_q, \tag{1.2}$$

where $\tilde{q}$ is the conjugate of q: $\tilde{q} = q/(q-1)$), and the constant appearing in (1.2) *is the best possible.*

In a companion paper we shall characterise the best constant $C_{p,q}$ appearing in

$$\mathbb{E}(X_\infty^*)^p \le C_{p,q} \|X_\infty^p\|_q;$$

this may go some way toward explaining the subtitle of this paper.

As the title suggests, the main tool used in establishing inequality (1.2) is that of optimal stopping. We establish (1.2) [and more general inequalities bounding $\mathbb{E}X_\infty^*$ by suitable multiples of the Luxemburg and Orlicz norms of X_∞ with respect to Φ (a convex Young function)] by explicitly solving the problem of optimally stopping the process $(X_t^* - \Phi(\mu|X_t|))$.

Section 2 is devoted to the groundwork establishing the connection between payoffs of a class of optimal stopping problems and global inequalities like (1.2). In Section 3 we simplify the relevant class of these problems by scaling and by embedding in Brownian motion. Section 4 is devoted to the heuristic considerations which establish a candidate optimal policy and the corresponding payoff. Section 5 is devoted to establishing the validity of this candidate payoff, a task which, unfortunately, necessitates the proof of a whole batch of lemmas, because of the unusual form of the optimal stopping problem for which, apparently, no suitable theory exists.

Throughout the paper we address a larger class of optimal stopping problems than that which is dictated by a desire solely to establish Theorem 1.1. The payoff for this is seen in Section 6, where we find the solution to all optimal stopping problems of the form:

$$\text{maximise over stopping times } T \text{ such that } (B_{t\wedge T}) \text{ is ui:}$$

$$\mathbb{E}B_T^* - f(|B_T|),$$

$$\text{where } f \text{ is an arbitrary } \mathbb{R}\cup\{+\infty\}\text{-valued function.}$$

Section 7 is devoted to some applications of the result in Section 5. First Theorem 1.1 is proved and then a suitable generalisation is given of the form

$$\mathbb{E}X_\infty^* \le \Sigma_\Phi \|X_\infty\|_\Phi \quad \text{and} \quad \mathbb{E}X_\infty^* \le \sigma_\Phi |||X_\infty|||_\Phi,$$

where $\|\ \|_\Phi$ is the Luxemburg norm, and $|||\cdot|||_\Phi$ is the Orlicz norm induced by Φ, an arbitrary Young function and

$$\Sigma_\Phi < \infty \Leftrightarrow \sigma_\Phi < \infty \Leftrightarrow \int_0^\infty e^{-\mu\phi(t)}\,dt < \infty \quad \text{for some } \mu > 0,$$

where ϕ is the left derivative of Φ.

The final application in Section 7 is a proof of a version of the John–Nirenberg inequality for martingales of *bounded mean oscillation*.

Note that inequality (1.2) has been obtained independently by Gilat (1987) by an entirely different method.

2. Some preliminaries. In what follows we shall assume that we are working with a fixed filtered probability space $[\Omega, F, (F_t;\ t \geq 0), \mathbb{P}]$, which is rich enough to carry a Brownian motion; all stopping times are with respect to this filtration. We recall the definition of a Young function: Φ: $\mathbb{R}_+ \to \mathbb{R}_+$ is a *Young function* if Φ is convex, increasing, $\Phi(0) = 0$ and $\Phi(x)/x \to \infty$. It will be convenient in some of what follows to extend the definition: Φ: $[0, \infty) \to [0, \infty]$ is an *extended Young function* if Φ is convex (as an extended real valued function), increasing, $\Phi(0) = 0$ and $\Phi(x)/x \to \infty$.

We may associate with any Young function Φ (indeed with any extended Young function) its convex conjugate (or Fenchel conjugate) $\tilde{\Phi}$ given by $\tilde{\Phi}(x) = \sup_{t \geq 0}[xt - \Phi(t)]$ [see Rockafellar (1970), pages 102–111]. The function $\tilde{\Phi}$ is also a Young function (or an increasing convex function started at 0). There ares two norms associated with Young functions: the Luxemburg norm (slightly redefined)

$$\|X\|_\Phi = \inf\left\{\mu > 0: E\Phi\left(\frac{|X|}{\mu}\right) \leq \Phi(1)\right\}$$

and what is, essentially, the dual norm of $\|\cdot\|_{\tilde{\Phi}}$, the Orlicz norm, given by

$$|||\, X \,|||_\Phi = \inf\left\{\mu > 0: \sup_{\lambda \geq 0}(\mu\lambda - E\Phi(\lambda|X|)) \geq \tilde{\Phi}(1)\right\},$$

at least for Φ: $\tilde{\Phi}(1) > 0$; see Jacka (1991) for details.

The main reason for introducing Φ-norms and the associated L^Φ spaces is that we can "see" finer structure using Luxemburg norms than by simply looking at the L^p-norms for $p > 1$. Note that both L^Φ-norms coincide with the L^p-norm if $\Phi(x) \equiv x^p$.

THEOREM 2.1. *Let $(X_t;\ t \geq 0)$ and $(Y_t;\ t \geq 0)$ be nonnegative adapted processes, let* T *be any collection of F-measurable nonnegative random variables and let Φ be a fixed Young function. Define for each $\mu > 0$,*

$$\mathrm{T}(\mu, \Phi) = \{T \in \mathrm{T}: \mathbb{E}\Phi(\mu Y_T) < \infty\}.$$

Then

$$\mathrm{T}(\Phi) = \bigcup_{\mu > 0} \mathrm{T}(\mu, \Phi) = \{T \in \mathrm{T}: \|Y_T\|_\Phi < \infty\}$$

$$= \{T \in \mathrm{T}: |||\, Y_T \,|||_\Phi < \infty\}.$$

Define

$$L^*(\mu) = \sup_{\lambda > 0}\ \sup_{T \in \mathrm{T}(\lambda, \Phi)} \mathbb{E}\left(\frac{\lambda}{\mu} X_T - \Phi(\lambda Y_T)\right).$$

Then if we define

$$\sigma_\Phi = \sigma_\Phi(X, Y, \mathrm{T}) = \sup_{T \in \mathrm{T}(\Phi)} \frac{\mathbb{E} X_T}{|||\, Y_T \,|||_\Phi}$$

and

$$\Sigma_\Phi = \Sigma_\Phi(X, Y, \mathrm{T}) = \sup_{T \in \mathrm{T}(\Phi)} \frac{\mathbb{E} X_T}{|||\, Y_T \,|||_\Phi},$$

then

$$\begin{aligned}\sigma_\Phi &= \inf\{\mu > 0 \colon L^*(\mu) \le \tilde{\Phi}(1)\} \\ &= \inf\left\{\mu > 0 \colon \sup_{T \in \mathrm{T}(\Phi)} \sup_{\lambda \ge 0 \colon T \in \mathrm{T}(\lambda, \Phi)} \mathbb{E}\left(\frac{\lambda}{\mu} X_T - \Phi(\lambda Y_T)\right) \le \tilde{\Phi}(1)\right\}\end{aligned} \tag{2.1}$$

and

$$\Sigma_\Phi = \inf_{\mu > 0} \{\mu(L^*(\mu) + \Phi(1))\}.$$

This is just Theorem 10 of Jacka (1991).

In particular we have the following corollaries.

COROLLARY 2.2. *Suppose X and Y are nonnegative adapted processes and*

$$\mathrm{T} = \{T \colon T \text{ a stopping time};\ T < \infty\ a.s. \text{ and } (Y_{t \wedge T}) \text{ is ui}\}.$$

Then the best constants C_Φ and c_Φ, appearing in the inequalities

$$\begin{aligned}\mathbb{E} X_T &\le C_\Phi \,|||\, Y_T \,|||_\Phi, \qquad \forall\, T \in \mathrm{T}, \\ \mathbb{E} X_T &\le c_\Phi \| Y_T \|_\Phi, \qquad \forall\, T \in \mathrm{T},\end{aligned} \tag{2.2}$$

are given by

$$C_\Phi = \sigma_\Phi, \qquad c_\Phi = \Sigma_\Phi.$$

COROLLARY 2.3. *If X, Y and T are as in Corollary 2.2, then the best constant appearing in*

$$\mathbb{E} X_T \le C_q \| Y_T \|_q, \qquad \forall\, T \in \mathrm{T}, \tag{2.3}$$

where $q > 1$, is σ_q given by

$$\sigma_q \equiv \tilde{q}^{1/\tilde{q}} q^{1/q} \left(\sup_{\lambda \ge 0} \sup_{T \in \mathrm{T}(X_q)} \mathbb{E}\big(\lambda X_T - (\lambda Y_T)^q\big)\right)^{1/\tilde{q}}, \tag{2.4}$$

where $\tilde{q} = q/(q-1)$.

These are, respectively, an obvious corollary of Theorem 2.1 of this paper and Corollary 11 of Jacka (1991). The interested reader is referred to Jacka (1991) and to Barlow, Jacka and Yor (1986) for further details and applications of these techniques.

We now see the connection between our original problem of finding the best σ_q in (1.2) and a class of optimal stopping problems:

$$\text{find, for each } \lambda, \quad \sup_{T \in \mathrm{T}(x^q)} \mathbb{E}\big(\lambda M_T^* - (\lambda M_T)^q\big), \tag{2.5}$$

where $T(x^q) = \{\text{stopping times } T: T < \infty \text{ a.s., } (M_{t \wedge T}) \text{ is ui and } \mathbb{E} M_T^q < \infty\}$.

We can and will consider the more general problem:

$$\text{find, for each } \lambda, \mu > 0, \quad \sup_{T \in \mathrm{T}(\Phi)} \mathbb{E}\left(\frac{\lambda M_T^*}{\mu} - \phi(\lambda M_T)\right). \tag{2.6}$$

By embedding the positive submartingale M in the modulus of a Brownian motion we shall see, in the next section, that in order to obtain a uniform bound for σ_Φ and Σ_Φ over all cadlag martingales we need only consider problems (2.5) and (2.6) for $M = B$, a Brownian motion. We shall solve this problem in Sections 4 and 5 and then go on to find the solution to the still larger problem:

$$\text{find} \quad \sup_{T \in \mathrm{T}} \mathbb{E}\big(B_T^* - f(|B_T|)\big),$$

where $\mathrm{T} \equiv \mathrm{T}(B, f) = \{\text{finite stopping times } T: (B_{t \wedge T}) \text{ is ui and } \mathbb{E} f(|B_T|)^+ < \infty\}$ and f is an *arbitrary* $\mathbb{R} \cup \{+\infty\}$*-valued function.*

3. Embedding in Brownian motion. Define $\mathfrak{M}_x$ to be the class of nonnegative, cadlag submartingales M adapted to (F_t) with $M_0 = x$, and for each $x \in \mathbb{R}$ and $M \in \mathfrak{M}_x$ set

$$\mathrm{T}(M) = \{\text{stopping times } T: T < \infty \text{ a.s. and } (M_{t \wedge T}) \text{ is ui}\}.$$

Then, for each $f: \mathbb{R}_+ \to \overline{\mathbb{R}}$ define

$$\begin{aligned} \mathrm{T}(M, f) &= \{T \in \mathrm{T}(M): \mathbb{E} f(M_T)^+ < \infty\}, \\ V(M, f) &= \sup_{T \in \mathrm{T}(M, f)} \mathbb{E}\big(M_T^* - f(M_T)\big). \end{aligned} \tag{3.1}$$

Let us now simplify the problem further.

LEMMA 3.1. *The supremum over $M \in \mathfrak{M}_x$ of $V(M, f)$ is given by*

$$\sup_{M \in \mathfrak{M}_x} V(M, f) = V(|B^x|, f), \tag{3.2}$$

where B^x is an (F_t)-adapted Brownian motion with $B_0 = x$.

Moreover, if we define $\mathfrak{M}$ to be the class of nonnegative cadlag submartingales on (F_t), then

$$\sup_{M \in \mathfrak{M}} V(M, f) = V(|B^0|, f). \tag{3.3}$$

PROOF. Clearly, since $|B^x| \in \mathfrak{M}_x$ and $|B^0| \in \mathfrak{M}$, we need only establish that

$$V(|B^x|, f) \geq V(M, f), \qquad \forall\, M \in \mathfrak{M}_x, \tag{3.4}$$

$$V(|B^0|, f) \geq V(M, f), \qquad \forall\, M \in \mathfrak{M}. \tag{3.5}$$

Now we know that (M_t) may be written as $(|N_t|)$, where N is a cadlag martingale [see Barlow (1981)].

Take a $T \in \mathrm{T}(M, f)$. Then $|N_{t\wedge T}|$ is ui and so N_T is integrable, so we may maximally embed the law of N_T in a Brownian motion B started at $x = M_0$; that is, we may find a stopping time τ s.t. (i) $L(B_\tau) = L(N_T)$, (ii) $(B_{t\wedge\tau})$ is ui and (iii) B_T^* is the stochastic maximum of $(N'_\infty)^*$, where N' runs through all cadlag ui martingales with $L(N'_\infty) = L(B_\tau)$ [see Jacka (1988), Theorems 1 and 2]. But $(N_{t\wedge T})$ is ui, so $\mathbb{E}B_\tau^* \geq EN_T^*$, while, since $L(B_\tau) = L(N_T)$, $\mathbb{E}\, f(|B_\tau|)^+ < \infty$, so $\tau \in T(|B|, f)$ and

$$\mathbb{E}\big(M_T^* - f(M_T)\big) = \mathbb{E}\big(N_T^* - f(|N_T|)\big) \leq \mathbb{E}\big(B_\tau^* - f(|B_\tau|)\big),$$

establishing (3.4).

To establish (3.5), by conditioning upon the value of M_0 we see that we need only show that

$$V(|B^0|, f) \geq V(|B^x|, f), \qquad \forall\, x \in \mathbb{R}. \tag{3.6}$$

But if we define $T_x = \inf\{t \geq 0: |B_t^0| = |x|\}$, then $\{|B^0_{T_x+t}|\}$ has the same law as $|B_t^x|$ and we may embed the law of $|B_t^x|$ in $|B^0_{T_x+t}|$ and repeat the argument above, using the fact that $(B^0_{T_x})^* \geq \sup_{0\leq t\leq\tau} |B^0_{T_x+t}|$, thus establishing inequality (3.6). □

We are now confronted with the class of optimal stopping problems: find $V(|B|, f)$ for each real function f, or, restricting attention to extended Young functions, find for each $x \in \mathbb{R}$, $\lambda \geq 0$, $\mu > 0$,

$$\frac{\lambda}{\mu} V\left(|B^x|, \frac{\mu}{\lambda}\Phi_\lambda\right) = \sup_{T\in\mathrm{T}(|B^x|, \Phi_\lambda)} \mathbb{E}\left(\frac{\lambda}{\mu} B_T^* - \Phi_\lambda(|B_T|)\right), \tag{3.7}$$

where $\Phi_\lambda(x) = \Phi(\lambda x)$. Note that by scaling B, (3.7) is the same as finding

$$\sup_{T\in\mathrm{T}(|B^{x'}|, \Phi_\mu)} \mathbb{E}\big(B_T^* - \Phi_\mu(|B_T|)\big), \tag{3.8}$$

where $x' = (\lambda x/\mu)$, and it is this problem which we shall solve in Sections 4 and 5.

4. Solving the optimal stopping problem: heuristic principles. Recalling once more Theorems 1 and 2 of Jacka (1988), we know that if

$T \in \mathrm{T}(|B^0|, \Phi_\mu)$ with $L(B_T^0) = v$, with v symmetric, then, defining

$$\psi(x) = \begin{cases} \int_x^\infty t\,dv/v([x,\infty)), & x > 0, \\ 2\int_0^\infty t\,dv, & x = 0, \end{cases} \tag{4.1}$$

and letting

$$\tau \equiv \tau_v = \inf\{t \ge 0: B_t^{0*} \ge \psi(|B_t^0|)\}, \tag{4.2}$$

then, since B_T is integrable and $(B_{t\wedge T}^0)$ is ui,

$$\mathbb{P}(B_\tau^{0*} \ge \lambda) \ge \mathbb{P}(B_T^{0*} \ge \lambda), \qquad \forall \lambda \ge 0,$$

while $L(B_\tau^0) = v$, at least if v does not charge $\{0\}$.

Now by the symmetry of the optimal stopping problem (3.7) we would expect that if T achieves the supremum, then B_T^0 should have a symmetric law and so T must be of the form given in (4.2) for some law v. We know that (since T must have this form, and since the process $(|B_t|, B_t^*)$ is strong Markov) if we write

$$V(x,y) = \sup_{T \in \mathrm{T}(|B^x|, \Phi_\mu)} \mathbb{E}\left[\left(B_T^* - \Phi_\mu(|B_T|)\right)/B_0^* = y\right],$$

then V must be given by

$$V(x,y) = \begin{cases} V(\psi(0), \psi(0)), & y < \psi(0), \\ y - \Phi_\mu(|x|), & \psi(|x|) \le y, \\ \dfrac{|x| - \psi^{-1}(y)}{y - \psi^{-1}(y)} V(y,y) & \\ \quad + \dfrac{y - |x|}{y - \psi^{-1}(y)} V(\psi^{-1}(y), y), & \psi(0) \le y < \psi(|x|), \end{cases} \tag{4.3}$$

where ψ is the solution of (4.1) corresponding to the optimal law $\hat{v}$. The third expression in (4.3) is obtained by conditioning on the first time that $|B_t|$ leaves the interval $(\psi^{-1}(y), y)$.

In order to find $\hat{v}$ we apply the following heuristic principle:

(4.4) $V(x,y)$ has a continuous first derivative in y.

This sort of requirement often occurs in optimal stopping problems and seems to correspond to the extremal nature of V.

Clearly the constraint (4.4) bites at the stopping boundary $y = \psi(x)$. Applying (4.4) at this boundary we see that (for $x > 0$)

$$\frac{\partial}{\partial z}\left(\frac{x - z}{\psi(z) - z} V(\psi(z), \psi(z)) + \frac{\psi(z) - x}{\psi(z) - z}\left(\psi(z) - \Phi_\mu(z)\right)\right)\bigg|_{z=x} \tag{4.5}$$

should be equal to $(\partial/\partial z)(\psi(z) - \Phi_\mu(x))|_{z=x} = \psi'(x)$. Evaluating the expression in (4.5) gives us

$$-\frac{V(\psi(x),\psi(x)) + \psi(x) - \Phi_\mu(x)}{\psi(x) - x} - \phi_\mu(x) = 0, \tag{4.6}$$

where ϕ_μ is the derivative of Φ_μ.

If we now use the fact that

$$V(\psi(x),\psi(x)) = \mathbb{E}B_\tau^* - \Phi_\mu(|B_\tau|),$$

where $B_0 = \psi(x)$ and $\tau = \inf\{t \geq 0: \psi(|B_\tau|) = B_t^*\}$, we see that $|B_\tau|$ must have the conditional law $\mathbb{P}(|B_\tau| \geq \lambda) = \upsilon([\lambda,\infty))/\upsilon([x,\infty))$, and so we deduce that $V(\psi(x),\psi(x))$ must be given by

$$V(\psi(x),\psi(x)) = \int_x^\infty \big(\psi(t) - \Phi_\mu(t)\big)\, d\upsilon(t)/\bar{\upsilon}(x), \tag{4.7}$$

where $\bar{\upsilon}(x) = \upsilon([x,\infty))$. Substituting (4.7) in (4.6), we obtain

$$\begin{aligned} \bar{\upsilon}(x)(\psi(x) - x)\phi_\mu(x) &= \bar{\upsilon}(x)\big(\psi(x) - \Phi_\mu(x)\big) \\ &\quad - \int_x^\infty \big(\psi(t) - \Phi_\mu(t)\big)\, d\upsilon(t). \end{aligned} \tag{4.8}$$

Rewriting the right-hand side of (4.7) as $-\int_x^\infty \bar{\upsilon}(t)(d\psi(t) - \phi_\mu(t)\,dt)$ and recalling from Jacka (1988) or Azema and Yor (1978) that

$$\bar{\upsilon}\, d\psi = (\psi - x)\, d\upsilon, \tag{4.9}$$

we see, by considering the differential of (4.8), that we must have

$$\begin{aligned} &\bar{\upsilon}(\psi - x)\, d\phi_\mu + \bar{\upsilon}\phi_\mu(d\psi - dx) - (\psi - x)\phi_\mu\, d\upsilon - \bar{\upsilon}(d\psi - \phi_\mu\, dx) \\ &\quad \equiv (\psi - x)\big(\bar{\upsilon}\, d\phi_\mu - d\upsilon\big) = 0. \end{aligned}$$

Applying the symmetry condition on υ we see that we must have

$$\begin{aligned} \bar{\upsilon}(x) &= \tfrac{1}{2}\exp\big(-\phi_\mu(x)\big), \quad \text{for } x > 0, \\ \bar{\upsilon}(0) &= \tfrac{1}{2}. \end{aligned} \tag{4.10}$$

In the preceding argument we have assumed that Φ_μ is differentiable and that $\phi_\mu(0) = 0$. Clearly this is not the case for general extended Young functions, but it also seems clear that for any extended Young function Φ we can approximate Φ uniformly on D_Φ°, the interior of its effective domain ($D_\Phi = \{x: \Phi(x) < \infty\}$), by a sequence of extended Young functions which are C^1 on their effective domains, and deduce that the optimal law υ should be given by

$$\begin{aligned} \bar{\upsilon}(x) &= \tfrac{1}{2}\exp\big(-\phi_\mu(x)\big), \qquad x > 0 \\ \bar{\upsilon}(0) &= 1 - \tfrac{1}{2}\exp\big(-\phi_\mu(0+)\big), \end{aligned} \tag{4.11}$$

where ϕ_μ is the left-hand derivative of Φ_μ.

If we return to our expression for $\bar{v}$ we see that, dropping the μ-dependence of Φ_μ and once more assuming that Φ is differentiable,

(i)
$$\psi(x) = \int_x^\infty t\,dv/\bar{v}(x)$$
$$= x + \int_x^\infty \bar{v}(t)\,dt/\bar{v}(x)$$
$$= x + e^{\phi(x)}\int_x^\infty e^{-\phi(t)}\,dt,$$

(ii)
$$V(\psi(x),\psi(x)) = \mathbb{E}B_\tau^* - \Phi(|B_\tau|),$$

where $|B_\tau|$ has conditional law $\bar{v}_x(t) = [\bar{v}(t)/\bar{v}(x)]$, at least for $t \ge x$, while $B_\tau^* = \psi(|B_\tau|)$; thus

$$V(\psi(x),\psi(x)) = \mathbb{E}\psi(|B_\tau|) - \Phi(|B_\tau|).$$

Now

$$\mathbb{E}\psi(|B_\tau|) = \int_x^\infty \psi(t)\,d\bar{v}_x(t)$$
$$= \int_x^\infty \left\{t + e^{\phi(t)}\int_t^\infty e^{-\phi(u)}\,du\right\} d\bar{v}(t)/\bar{v}(x)$$
$$= \psi(x) + \int_x^\infty \left\{\int_t^\infty e^{-\phi(u)}\,du\right\} d\phi(t)e^{\phi(x)} \tag{4.12}$$
$$= \psi(x) + e^{\phi(x)}\int_x^\infty (\phi(u) - \phi(x))e^{-\phi(u)}\,du \quad \text{(by Fubini)}$$
$$= \psi(x) - \phi(x)(\psi(x) - x) + e^{\phi(x)}\int_x^\infty \phi(u)e^{-\phi(u)}\,du,$$

while

$$\begin{aligned}\mathbb{E}\Phi(|B_\tau|) &= \Phi(x) + \int_x^\infty \phi(t)\bar{v}_x(t)\,dt \\ &= \Phi(x) + e^{\phi(x)}\int_x^\infty \phi(t)e^{-\phi(t)}\,dt.\end{aligned} \tag{4.13}$$

Thus, assuming $\int_x^\infty \phi(t)e^{-\phi(t)}\,dt < \infty$,

$$V(\psi(x),\psi(x)) = \psi(x) - \Phi(x) - \phi(x)(\psi(x) - x)$$

and so for $|x| \ge \psi(0)$,

$$V(x,|x|) = |x| - \Phi(\psi^{-1}(|x|)) - \phi(\psi^{-1}(|x|))(|x| - \psi^{-1}(|x|)). \tag{4.14}$$

If we substitute (4.14) into (4.3) we obtain the expression

$$(4.15)\quad V(x,y) = \begin{cases} V(\psi(0),\psi(0)), & y < \psi(0), \\ y - \Phi(|x|), & \psi(|x|) < y, \\ y - \Phi(\psi^{-1}(y)) - p(y)(|x| - \psi^{-1}(y)), & \\ & \psi(0) \le y \le \psi(|x|), \end{cases}$$

where $p(y) = \phi(\psi^{-1}(y))$ and ψ is given by

$$\psi(x) = x + e^{\phi(x)}\int_x^\infty e^{-\phi(t)}\,dt.$$

Note that by virtue of the representation (4.9) we may write $p(\cdot)$ as

$$p(y) = \int_{\psi(0)}^{y} \frac{dt}{t - \psi^{-1}(t)}$$

and note that ψ^{-1} represents the right inverse of ψ,

$$\psi^{-1}(z) = \inf\{x : \psi(x) \ge z\} \quad \text{for } z \ge x,$$

so, since the continuity of ϕ implies that of ψ, we see that $\psi(\psi^{-1}(z)) = z$ if Φ is differentiable.

5. The optimal stopping problem: statement and proof of the result.

Given an extended Young function (EYF) Φ, which is not identically $+\infty$, let D_Φ be given by

$$D_\Phi = \{x \in \mathbb{R}_+ : \Phi(x) < \infty\}.$$

Since Φ is convex, D_Φ is an interval of the form $[0, d]$ or $[0, d)$, $d \in [0, \infty]$. We shall assume in this section that Φ is a fixed EYF and

(5.1) *either* D_Φ is closed and Φ is continuous on D_Φ
or D_Φ is half-open and $\lim_{x \uparrow d} \Phi(x) = \infty$.

In either case Φ has nonnegative increasing left and right derivatives Φ'_- and Φ'_+. We define $\phi(x)$ as

$$\phi(x) = \begin{cases} 0 & x = 0, \\ \Phi'_-(x), & 0 < x < d, \\ \lim_{t \uparrow d} \Phi'_-(t), & x = d, \\ +\infty, & x > d. \end{cases}$$

We are now in a position to give a statement of our main result.

THEOREM 5.1. *Suppose Φ is an EYF satisfying condition* (5.1). *Then defining*

$$(5.2)\qquad \bar{v}(x) = \exp(-\phi(x)) \quad \textit{for } x \ge 0,$$

$$(5.3)\qquad \psi(x) = \int_x^\infty t\,dv(t)/\bar{v}(x),$$

we have: *either*

(i) $$\int_0^\infty \exp(-\phi(t))\,dt = \infty,$$

$$V(|B^x|,\Phi) = \infty, \qquad \forall\, x \in \mathbb{R}_+,$$

or

(ii) $$\int_0^\infty \exp(-\phi(t))\,dt < \infty,$$

$$V(|B^x|,\Phi) = V_\Phi(|x|,|x|) < \infty, \qquad \forall\, x \in D_\Phi.$$

Here $V_\Phi(x,y)(|x| \le y)$ *is given by*

$$(5.4)\quad V(x,y) \equiv V_\Phi(x,y) = \begin{cases} V(\psi(0),\ \psi(0)) \equiv \psi(0), & y < \psi(0), \\ y - \Phi(|x|), & \psi(|x|) < y, \\ y - p(y)(|x| - \psi^{-1}(y)) & \\ \quad - \Phi(\psi^{-1}(y)), & \psi(0) \le y \le \psi(|x|), \end{cases}$$

where ψ^{-1} *is the left inverse of* ψ *and*

$$(5.5)\qquad p(y) = \int_{\psi(0)}^{y} \frac{dy}{y - \psi^{-1}(y)}.$$

Moreover

(iii) *if* $\int_0^\infty \phi(t)e^{-\phi(t)}\,dt < \infty$ *and* D_Φ *is closed*,

then, *setting* $\tau = \inf\{t \ge 0\colon B^*_\tau \ge \psi(|B_\tau|)\}$,

$$\tau \in T(B^*,\Phi) \quad \textit{and} \quad V(x,x) = \mathbb{E}_x\big(B^*_\tau - \Phi(|B_\tau|)\big) \quad \textit{for } |x| \le d.$$

The standard approach to showing that a proposed solution V to an optimal stopping problem is correct is in four stages: First show that V is a supermartingale; second show that there is a stopping time τ such that $V_t = \mathbb{E}[X_\tau|F_t]$, where X is the optimally stopped process; third establish a suitable version of Snell's criterion (that the optimal payoff is the minimal supermartingale which dominates X); fourth show that V dominates X. The following lemmas execute this procedure for the given problem. We start with a lemma which determines a suitable weak version of Snell's criterion.

LEMMA 5.2. *Given a process* X, *define* $\mathrm{T}(X) \equiv \mathrm{T} = \{$*stopping times* T: $T < \infty$ *a.s. and* $(X_{t\wedge T})$ *is ui*$\}$. *Suppose that* M *is a local supermartingale and that*

(i) *for any* $T \in \mathrm{T}$, $X_T \le M_T$ *a.s. and*
(ii) *there exists a* $\tau \in \mathrm{T}$ *with* $\mathbb{E}X_\tau = M_0$.

Then

$$\sup_{T \in \mathrm{T}} \mathbb{E} X_T = M_0.$$

PROOF. Given a $T \in \mathrm{T}$, take a localising sequence (T_n) for M [so that $(M_{t \wedge T_n})$ is a ui supermartingale for each n]. Then clearly $T \wedge T_n \in \mathrm{T}$, and so, from (i),

$$\mathbb{E} X_{T \wedge T_n} \le \mathbb{E} M_{T \wedge T_n} \le M_0.$$

Now $T_n \uparrow \infty$ a.s. so $X_{T \wedge T_n} \to X_T$ a.s., since $T < \infty$ a.s.; so, since $T \in \mathrm{T}$ [and hence $(X_{T \wedge T_n})$ is ui], $X_{T \wedge T_n} \to_{L^1} X_T$ and we may deduce that $\mathbb{E} X_T \le M_0$. The result now follows from (ii). □

The lemma above maps out our plan of attack on Theorem 5.1. We know that if $B_t = x$ and $B_t^* = y$, then our conditional expected payoff is $V(|x|, y)$ so we want to show that, essentially,

$$(5.6) \qquad \begin{array}{ll} \text{(i)} & V(B_t, B_t^*) \text{ is a local supermartingale,} \\ \text{(ii)} & V_t = V(B_t, B_t^*) \ge B_t^* - \Phi(|B_t|) \equiv X_t, \\ \text{(iii)} & \mathrm{T}(|B|, \Phi) \subset \mathrm{T}(X), \\ \text{(iv)} & \tau = \inf\{t \ge 0 \colon B_t^* \ge \psi(|B_t|)\} \in \mathrm{T}(|B_t|, \Phi), \end{array}$$

where $\mathrm{T}(\cdot, \Phi)$ is as defined in section 3. We shall prove these four requirements in order.

LEMMA 5.3. *Assume $d > 0$ and suppose B is a Brownian motion started at x, where $|x| < d$, the upper limit of the effective domain of Φ. Then, defining*

$$T_a = \begin{cases} \inf\{t \ge 0 \colon |B_t| = a\}, & a < \infty, \\ +\infty, & a = \infty, \end{cases}$$

$$(5.7) \qquad \text{if } d = +\infty, \quad V(B_t, B_t^*) \text{ is a local supermartingale},$$

while if $0 < d < \infty$, either

$$(5.8) \qquad \Phi(d) < \infty \quad \text{and} \quad V\big(B_{t \wedge T_d}, B^*_{t \wedge T_d}\big) \text{ is a supermartingale},$$

or

$$(5.9) \qquad \Phi(d) = \infty \quad \text{and} \quad V\big(B_{t \wedge T_a}, B^*_{t \wedge T_a}\big)$$

$$\text{is a supermartingale for any } x \colon |x| \le a < d.$$

PROOF. We assume first of all that Φ is C^2 with $\Phi'_+(0) = 0$ so that $\Phi(|x|)$ is C^2 and that $\int_\cdot^\infty \phi(t) e^{-\phi(t)}\, dt < \infty$. Note that V is continuous,

$$\frac{\partial V}{\partial x} = \begin{cases} 0, & y < \psi(0), \\ -\phi(|x|)\operatorname{sgn}(x), & \psi(|x|) < y, \\ -\operatorname{sgn}(x) p(y), & \psi(0) \le y \le \psi(|x|), \end{cases}$$

so that V is C^1 in x, while

$$\frac{\partial V}{\partial y} = \begin{cases} 0, & y < \psi(0), \\ 1, & \psi(|x|) < y, \\ \dfrac{y - |x|}{y - \psi^{-1}(y)}, & \psi(0) < y < \psi(|x|), \end{cases}$$

so that V is piecewise differentiable in y, and

$$\frac{\partial^2 V}{\partial x^2} = \begin{cases} 0, & y < \psi(0), \\ -\phi'(|x|), & \psi(|x|) < y, \\ 0, & \psi(0) < y < \psi(|x|). \end{cases} \tag{5.10}$$

In other words, V possesses generalised first and second derivatives in x and a generalised first derivative in y in the sense of Krylov [(1980), page 47, Definition 1].

It follows by a standard smoothing argument that we may approximate V arbitrarily closely by a $C^{2,1}$ function (indeed, by an arbitrarily smooth function) to deduce that $V(B_t, B_t^*)$ satisfies the generalised Itô formula

$$dV(B_t, B_t^*) = \frac{\partial V}{\partial x}(B_t, B_t^*)\, dB_t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(B_t, B_t^*)\, dt + \frac{\partial V}{\partial y}(B_t, B_t^*)\, dB_t^*$$

and it follows, on observing that B_t^* only increases on the set $\{t\colon |B_t| = B_t^*\}$ and that $(\partial V/\partial y)1_{|x|=y} = 0$, that

$$dV(B_t, B_t^*) = -\phi'(|B_t|)1_{(\psi(|B_t|) < B_t^*)}\, dt + \frac{\partial V}{\partial x}(B_t, B_t^*)\, dB_t. \tag{5.11}$$

Since $\partial V/\partial x$ is bounded on compact sets, while $\phi' \geq 0$ (since Φ is convex), it follows from the representation (5.11) that $V(B_t, B_t^*)$ is a local supermartingale and indeed that $V(B_{t\wedge T_a}, B^*_{t\wedge T_a})$ is a supermartingale for any $a \in \mathbb{R}_+$. To establish the result for any EYF satisfying (5.1) we smooth ϕ in a suitable manner.

We now extend the result to the case where Φ is an EYF with $D_\Phi = \mathbb{R}_+$. Given Φ satisfying (5.4) with $D_\Phi = [0, \infty)$, take a family of smooth, that is, C^∞ kernels $\rho_\varepsilon\colon \mathbb{R} \to \mathbb{R}_+$, $\varepsilon > 0$, satisfying

(5.12)
(i) the support of ρ_ε is contained in $[0, \varepsilon]$,
(ii) $\int_{-\infty}^{\infty} \rho_\varepsilon(t)\, dt = 1$,
(iii) $\rho_\varepsilon \geq 0$.

Now extend the definition of ϕ by defining $\phi(t) = 0$ for $t \leq 0$ and define

$$\begin{aligned} \phi_\varepsilon(x) &= \int_{-\infty}^{\infty} \phi(t)\rho_\varepsilon(x - t)\, dt, \\ \Phi_\varepsilon(x) &= \int_0^x \phi_\varepsilon(t)\, dt. \end{aligned} \tag{5.13}$$

Note that (5.12) and (5.13) imply that Φ_ε is a C^2 Young function, $\phi_\varepsilon \le \phi$, $\Phi_\varepsilon \le \Phi$ and because $\phi_\varepsilon(x) \ge \phi(x-\varepsilon)$, $\int^\infty \phi_\varepsilon(t) e^{-\phi_\varepsilon(t)}\,dt < \infty$, since the function te^{-t} is decreasing on $[1,\infty)$.

Since Φ_ε is C^2, we deduce that

$$V_\varepsilon \equiv V_{\Phi_\varepsilon}(B_t, B_t^*) \text{ is a local supermartingale.}$$

Moreover, since ϕ is left-continuous, $\phi_\varepsilon \uparrow \phi$ pointwise, $\Phi_\varepsilon \uparrow \Phi$ and $\psi_\varepsilon \to \psi$; thus $V_\varepsilon(x,y) \to V(x,y)$. It follows immediately that $V(B_t, B_t^*)$ is a local supermartingale and that $V(B_{t\wedge T_a}, B^*_{t\wedge T_a})$ is a supermartingale since V is bounded on compact sets.

Finally, to establish the lemma for an EYF satisfying (5.4) we may define

$$\phi^\eta(x) = \begin{cases} \phi(x), & x \le d, \\ \phi(d) + \eta(x-d), & x > d \end{cases}$$

and deduce the result from the above by letting $\eta \to \infty$, while if Φ is an EYF satisfying (5.1) with D_Φ half open we may define (for $a < d$)

$$\phi^{\eta,a}(x) = \begin{cases} \phi(x), & x \le a, \\ \phi(x) \wedge \eta, & a < x < d, \\ \eta(1 + x - d), & x \ge d, \end{cases}$$

where $\eta \ge \phi(a)$, and again let $\eta \to \infty$. $\square$

We now wish to prove that V_t satisfies the second criterion in (5.6).

LEMMA 5.4. *For* $|x| \le y \le d$,

$$V_\Phi(|x|, y) \ge y - \Phi(|x|). \tag{5.14}$$

PROOF. Clearly we need only establish (5.14) on $D = \{(x,y): \psi(0) \le y \le \psi(|x|)\}$, assuming w.l.o.g. that $x \ge 0$. Now setting $d(x,y) \equiv V_\Phi(x,y) - (y - \Phi(x))$,

$$d(x, \psi(x)) \equiv 0,$$

while as we saw in the proof of Lemma 5.3,

$$\frac{\partial d}{\partial y} = -\frac{x - \psi^{-1}(y)}{y - \psi^{-1}(y)} \quad \text{on } D;$$

thus $d(x,y) \ge 0$ on D. $\square$

We are now left with the task of establishing suitable versions of criteria (iii) and (iv) in (5.6) (and showing that V_t satisfies them). The clue to the right version of (iii) is contained in part (iii) of Theorem 5.1: The point is that if Φ satisfies

$$\int_0^\infty \phi(t) e^{-\phi(t)}\,dt < \infty \quad \text{and} \quad D_\Phi \text{ is closed,} \tag{5.15}$$

then we have the following result.

LEMMA 5.5. *Suppose Φ satisfies* (5.15) *and $x \in D_\Phi$. Then given a Brownian motion B, started at x, if we define*

$$\tau = \inf\{t \ge 0: B_t^* \ge \psi(|B_t|)\}, \tag{5.16}$$

where ψ is given by (5.3), *then*

$$\tau \in T(|B_t|, \Phi)$$

and

$$\mathbb{E}B_\tau^* - \Phi(|B_t|) = V_\Phi(x, x). \tag{5.17}$$

Note this is just part (ii) of Theorem 5.1.

To prove the lemma we need the following standard result.

LEMMA 5.6. *Suppose f is a strictly increasing convex function (possibly taking the value $+\infty$), continuous on its domain, and (X_t) is a nonnegative submartingale with $X_t \to X_\infty$ a.s. Then if $\mathbb{E} f(X_\infty) < \infty$,*

$$f(X_t) \text{ is ui} \quad \text{and} \quad f(X_t) \to f(X_\infty) \text{ in } L^1 \text{ and a.s.}$$

PROOF. It is immediate that $X_t \to_{L^1} X_\infty$ [Chung (1968), Theorem 4.5.4] and so by the conditional version of Jensen's inequality,

$$\mathbb{E}[f(X_\infty)|F_t] \ge f(\mathbb{E}[X_\infty|F_t]) = f(X_t).$$

Thus $f(X_t)$ is a submartingale bounded in L^1 and so is uniformly integrable. But $f(X_t) \to f(X_\infty)$ *a.s.* [f is continuous on D_f and $\mathbb{P}(X_\infty \in D_f) = 1$] and so $f(X_t) \to_{L^1} f(X_\infty)$. □

PROOF OF LEMMA 5.5. We have already established (5.17) in Section 4 in the case where Φ is C^2. Note that for general Φ we first smooth ϕ as in the proof of Lemma 5.3.

Defining τ_ε as in (5.16) with ψ replaced by ψ_ε (the ψ function corresponding to Φ_ε), then as before we may deduce that $V_\varepsilon \to V$, so that

$$\mathbb{E}B_{\tau_\varepsilon}^* - \Phi_\varepsilon(|B_{\tau_\varepsilon}|) \to V(x, |x|).$$

Note that, since $\phi_\varepsilon \to \phi$, $\psi_\varepsilon \to \psi$, and so $\tau_\varepsilon \to \tau$ a.s. This implies, since $\Phi_\varepsilon \to \Phi$ and Φ is continuous, that

$$B_{\tau_\varepsilon}^* \to B_\tau^* \text{ a.s.}$$

and

$$\Phi_\varepsilon(|B_{\tau_\varepsilon}|) \to \Phi(|B_\tau|) \text{ a.s.},$$

so it is sufficient to prove that the collections $\{B_{\tau_\varepsilon}^*\}$ and $\{\Phi_\varepsilon(|B_{\tau_\varepsilon}|)\}$ are uniformly integrable to establish that $V(x, |x|) = \mathbb{E}B_\tau^* - \Phi(|B_\tau|)$.

Now

$$\begin{aligned}\mathbb{E}\left[B_{\tau_\varepsilon}^* 1_{(B_{\tau_\varepsilon}^* \ge a)} | B_0 = x\right] &= \mathbb{E}\left[\psi_\varepsilon(|B_{\tau_\varepsilon}|) 1_{(|B_{\tau_\varepsilon}| \ge \psi_\varepsilon^{-1}(a))} | B_0 = x\right] \\ &\qquad [\text{at least for } a \ge \psi_\varepsilon(0)] \\ &= r_\varepsilon(a, x)\mathbb{E}\left[\psi_\varepsilon(|B_{\tau_\varepsilon}|)/B_0 = x\right],\end{aligned}$$

where

$$r_\varepsilon(a,x) = \frac{\bar{v}_\varepsilon\big(\psi_\varepsilon^{-1}(av\psi_\varepsilon(0))\big)}{\bar{v}_\varepsilon\big(\psi_\varepsilon^{-1}(|x|v\psi_\varepsilon(0))\big)}.$$

We deduce from (4.12) that

$$\mathbb{E}\big(\psi_\varepsilon(|B_{\tau_\varepsilon}|)|B_0 = x\big) = \begin{cases} (|x| - P_\varepsilon(|x|)(|x| - z_\varepsilon)) \\ \quad + e^{\phi_\varepsilon^+(z_\varepsilon)}\displaystyle\int_{z_\varepsilon}^\infty \phi_\varepsilon(t)e^{-\phi_\varepsilon(t)}\,dt, & |x| \geq \psi_\varepsilon(0), \\ \psi_\varepsilon(0) + \displaystyle\int_0^\infty \phi_\varepsilon(t)e^{-\phi_\varepsilon(t)}\,dt, & |x| < \psi_\varepsilon(0), \end{cases}$$

where $z_\varepsilon = \psi_\varepsilon^{-1}(|x|)$.

If we now note that for all $\varepsilon \in (0,1]$ and $t \geq 1$, $\phi(t) \geq \phi_\varepsilon(t) \geq \phi(t-1)$ so that

$$\psi_\varepsilon(t) \leq k(t) = t + e^{\phi(t)}\int_t^\infty e^{-\phi(u-1)}\,du,$$

we see that $\psi_\varepsilon^{-1}(z) \geq k^{-1}(z)$ ($\forall\ \varepsilon \in (0,1]$) so that $\psi_\varepsilon^{-1}(a) \to \infty$ uniformly in $\varepsilon \in (0,1]$ and hence $\bar{v}_\varepsilon(\psi_\varepsilon^{-1}(a)) \to 0$ uniformly in $\varepsilon \in (0,1]$. It follows that $r_\varepsilon(a,x) \to 0$ as $a \to \infty$ uniformly in $\varepsilon \in (0,1]$ and hence that the collection $(\psi_\varepsilon(|B_{\tau_\varepsilon}|))$ is ui. We may establish in a similar fashion that the collection $(\Phi_\varepsilon(|B_{\tau_\varepsilon}|))$ is ui and thus $\tau \in \mathrm{T}(|B|, \Phi)$. □

We need one final lemma.

LEMMA 5.7. *Suppose $T \in \mathrm{T}(|B|, \Phi)$. Define the process X by $X_t \equiv B_\tau^* - \Phi(|B_\tau|)$; then $\mathbb{E}X_T = \lim_{n\to\infty} \mathbb{E}X_{T\wedge T_n}$.*

PROOF. This is immediate from Lemma 5.6 and the dominated convergence theorem. □

PROOF OF THEOREM 5.1. The proof is achieved by first assuming that Φ satisfies condition (5.15) and then by approximating Φ by

$$\Phi_a(x) = \begin{cases} \Phi(x), & x \leq a, \\ \infty, & x > a. \end{cases} \tag{5.18}$$

(i) Assume that Φ satisfies (5.15). Taking B to be a Brownian motion started at x, with $|x| \leq d$, we know from the proof of Lemma 5.5 that

$$\mathbb{E}B_\tau^* < \infty \quad \text{and} \quad \mathbb{E}\Phi(|B_\tau|) < \infty,$$

while from Jacka (1988) we know that $B_{t\wedge\tau}$ is ui and $\tau < \infty$ a.s. [so $\tau \in \mathrm{T}(|B|, \Phi)$]. Thus we see that $\Phi(|B_{t\wedge\tau}|)$ is ui (from Lemma 5.6) and $B_{t\wedge\tau}^*$ is ui (since $B_{t\wedge\tau}^*$ is increasing and B_τ^* is integrable), so $X_{t\wedge\tau}$ is ui and $\tau \in \mathrm{T}(X) \cap \mathrm{T}(|B|, \Phi)$. Moreover, by Lemma 5.6, $\mathbb{E}X_\tau = V_\Phi(|x|, |x|)$ while, setting $V_t \equiv V_\Phi(|B_t|, B_t^*)$, $V_{t\wedge T_d}$ is a local supermartingale (from Lemma 5.3) and $V_{t\wedge T_d} \geq X_{t\wedge T_d}$ (from Lemma 5.4).

We have established that $(X_{t \wedge T_d})$, $(V_{t \wedge T_d})$ and τ satisfy the conditions of Lemma 5.2, so that

$$V_0 \equiv V_\Phi(|X|, |X|) = \sup_{T \in T(X)} \mathbb{E}\left[B_T^* - \Phi(|B_T|)\right]. \tag{5.19}$$

Now take a $T \in \mathrm{T}(|B|, \Phi)$; it is not clear that $T \in \mathrm{T}(X)$, but we know that $T \wedge T_n \in \mathrm{T}(X)$ [by the argument which established that $\tau \in \mathrm{T}(X)$], so from (5.19),

$$\mathbb{E}X_{T \wedge T_n} \le V_\Phi(|x|, |x|)$$

and now from Lemma 5.7 we deduce, letting $n \to \infty$, that $\mathbb{E}X_T \le V_\Phi(|x|, |x|)$. This establishes that

$$V_\Phi(|x|, |x|) = \sup_{T \in \mathrm{T}(|B|, \Phi)} \mathbb{E}X_T,$$

since $\tau \in \mathrm{T}(|B|, \Phi)$.

(ii) Define Φ_a as in (5.18). Note that we may take

$$\phi_a(x) = \begin{cases} \phi(x), & x < a, \\ \lim_{t \uparrow a} \phi(t), & x = a, \\ \infty, & x > a, \end{cases}$$

and Φ_a satisfies (5.15).

Now

$$\begin{aligned} \psi_a(x) &= x + e^{\phi_a(x)} \int_x^\infty e^{-\phi_a(t)}\, dt \\ &= x + e^{\phi_a(x)} \int_x^a e^{-\phi(t)}\, dt, \end{aligned}$$

and so $\psi_a(x) \uparrow \psi(x)$, for $a > x$, provided

$$\int_0^\infty e^{-\phi(t)}\, dt < \infty. \tag{5.20}$$

Assuming that Φ satisfies (5.20) we can easily show that $\psi_a^{-1}(x) \downarrow \psi^{-1}(x)$ $[x \ge \psi(0)]$. Thus we see that $V_{\Phi_a}(|x|, |x|) \to V_\Phi(|x|, |x|)$ as $a \to \infty$ if (5.20) holds. Define

$$\tau_a = \inf\{t \ge 0\colon B_t^* \ge \psi_a(|B_t|)\}. \tag{5.21}$$

Now if $D_\Phi = \mathbb{R}_+$, then for any $a \ge |x|$, $\tau_a \in \mathrm{T}(X_{t \wedge T_a})$, and by the same argument as in Part (i),

$$V_{\Phi_a}(|x|, |x|) = \sup_{\substack{T \le T_a, \\ T \in \mathrm{T}(|B|, \Phi)}} \mathbb{E}X_T$$

and taking an arbitrary $T \in \mathrm{T}(|B|, \Phi)$,

$$\begin{aligned} \mathbb{E}X_T &= \lim_{a \uparrow \infty} \mathbb{E}X_{T \wedge T_a} \le \lim_{a \uparrow \infty} V_{\Phi_a}(|x|, |x|) \\ &= V_\Phi(|x|, |x|). \end{aligned}$$

If, on the other hand, $D_\Phi = [0, d)$, then if $T \in \mathrm{T}(|B|, \Phi)$ it is clear that $\mathbb{P}(T < T_d) = 1$ and so

$$\begin{aligned} \mathbb{E}X_T &= \lim_{a \uparrow\uparrow d} \mathbb{E}X_{T \wedge T_a} \\ &\le \lim_{a \uparrow\uparrow d} V_{\Phi_a}(|x|, |x|) \\ &= V_\Phi(|x|, |x|), \end{aligned}$$

while $\lim_{a \uparrow\uparrow d} \mathbb{E}X_{\tau_a} = V_\Phi(|x|, |x|)$, establishing Part (ii) of the theorem (here $a \uparrow\uparrow x$ means a increases *strictly* to x).

Part (i) is established in the same manner, since if $\int_0^\infty e^{-\phi(t)}\,dt = \infty$, then $D_\Phi = \mathbb{R}_+$ and $\lim_{a \to \infty} V_{\Phi_a}(|x|, |x|) = \infty$ [since $\lim \psi_a(0) = \infty$]. □

6. The optimal stopping problem for arbitrary f. This section is devoted to finding $V(|B|, f)$ for arbitrary functions

$$f\colon [0, \infty) \to (-\infty, \infty].$$

Essentially, we do this by showing that

$$V(|B|, f) \equiv V\big(|B|, \Phi_f\big)$$

for a suitable function Φ_f, where either

(a) $\tilde{\Phi}_f(x) \equiv \Phi_f(x) - \Phi_f(0)$ is an EYF satisfying condition (5.1) so that

$$\begin{aligned} V\big(|B|, \Phi_f\big) &\equiv V\big(|B|, \tilde{\Phi}_f\big) - \Phi_f(0) \\ &\equiv V_{\tilde{\Phi}_f}(|B_0|, |B_0|) - \Phi_f(0); \end{aligned}$$

or

(b) Φ_f is bounded above by a linear function and $V(|B|, \Phi_f) = \infty$.

For any f: $[0, \infty] \to (-\infty, \infty]$ let $\tilde{f}$ be defined by $\tilde{f}(x) = f(|x|)$. Clearly $\tilde{f}$ is a symmetric function. Define $\hat{\Phi}_f$ by

$$\hat{\Phi}_f(x) = \inf\big\{\lambda \tilde{f}(s) + (1 - \lambda)\tilde{f}(t) \colon \lambda \in [0, 1], \lambda s + (1 - \lambda)t = x\big\}. \tag{6.1}$$

$\hat{\Phi}_f$ is the greatest convex function bounded above by $\tilde{f}$ [Rockafellar (1970), page 57, Corollary 17.5]. Note that, by the symmetry of $\tilde{f}$, $\hat{\Phi}_f$ is symmetric increasing on $\mathbb{R}_+$ and, of course, is convex.

Lemma 6.1. *Given a function* f: $[0, \infty) \to (-\infty, \infty]$,

$$V(|B|, f) = V\big(|B|, \hat{\Phi}_f\big).$$

Proof. Given a $T \in \mathrm{T}(|B|, f)$, define

$$\tau \equiv \tau(\varepsilon, T) = \inf\big\{t \ge T \colon B_t = s_\varepsilon(B_T) \text{ or } t_\varepsilon(B_T)\big\},$$

where s_ε and t_ε are functions chosen so that

$$\text{(i)} \qquad s_\varepsilon(x) \le x \le t_\varepsilon(x),$$

$$\text{(ii)} \quad \tilde{f}(s_\varepsilon(x))\frac{t_\varepsilon(x) - x}{t_\varepsilon(x) - s_\varepsilon(x)} + \tilde{f}(t_\varepsilon(x))\frac{x - s_\varepsilon(x)}{t_\varepsilon(x) - s_\varepsilon(x)} \le \hat{\Phi}_f(x) + \varepsilon.$$

Conditional on B_T, $(B_{(t+T)\wedge\tau})$ is clearly ui and so, since $(B_{t\wedge T})$ is ui, $(B_{t\wedge\tau})$ is ui. Moreover

$$\begin{aligned} f(|B_\tau|) &= \tilde{f}(B_\tau) \le \hat{\Phi}_f(B_T) + \varepsilon \\ &\le f(|B_\tau|) + \varepsilon, \end{aligned}$$

so $\mathbb{E} f(|B_\tau|) \le \mathbb{E} f(|B_T|) + \varepsilon$ and so $\tau \in \mathrm{T}(|B|, f)$.

Moreover, since $\tau \ge T$, $B_\tau^* \ge B_T^*$, while $\hat{\Phi}_f(|B_T|) \le f(|B_T|)$ so $T \in \mathrm{T}(|B|, \hat{\Phi}_f)$; thus we see that

$$\mathbb{E} B_T^* - f(|B_T|) \le \mathbb{E} B_T^* - \hat{\Phi}_f(|B_T|) \le \mathbb{E} B_\tau^* - f(|B_\tau|) + \varepsilon,$$

so, since ε is arbitrary, the result follows. □

We now define $\Phi_f \equiv \mathrm{cl}(\hat{\Phi}_f)$, that is, Φ_f is given by

$$(6.2) \qquad \Phi_f(x) = \lim_{t \uparrow\uparrow x} \hat{\Phi}_f(t), \quad \text{for } x > 0 \text{ and } \hat{\Phi}_f \text{ symmetric;}$$

Φ_f is the function whose epigraph is the closure of the epigraph of $\hat{\Phi}_f$ [see Rockafellar (1970) for details]. Clearly Φ_f is symmetric increasing on $\mathbb{R}_+$, and either D_{Φ_f} is closed or $D_{\Phi_f} = (-d, d)$ and $\lim_{x\uparrow d} \Phi_f(x) = \infty$ or $\Phi_f \equiv +\infty$. Clearly, either Φ_f satisfies condition (5.1) or Φ_f is bounded above by an increasing linear function (on $\mathbb{R}_+$) or $\Phi_f \equiv +\infty$.

LEMMA 6.2. *Suppose* Φ_f *is given by* (6.1) *and* (6.2). *Then* $V(|B|, f) = V(|B|, \Phi_f)$.

PROOF. We need only prove that

$$V(|B|, \hat{\Phi}_f) = V(|B|, \Phi_f),$$

thanks to Lemma 6.1.

Take $T \in \mathrm{T}(|B|, \hat{\Phi}_f)$. If $\hat{\Phi}_f \ne \Phi_f$, then since $\hat{\Phi}_f$ is increasing we must have $D \equiv D_{\hat{\Phi}_f} = [-d, d]$ or else $D = (-d, d)$ for some $0 < d < \infty$ and $\hat{\Phi}_f(d) > \lim_{x\uparrow d} \hat{\Phi}_f(x)$. In either case

$$\Phi_f(x) = \begin{cases} \hat{\Phi}_f, & |x| \ne d, \\ \lim_{t\uparrow d} \hat{\Phi}_f(t), & |x| = d. \end{cases}$$

Note that if $0 \le a \le d$, $T \in \mathrm{T}(|B|, \hat{\Phi}_f)$ and $T' \in \mathrm{T}(|B|, \Phi_f)$, then

$$T \in \mathrm{T}\big(|B|, \Phi_f\big) \quad \text{and} \quad T' \wedge T_a \in \mathrm{T}\big(|B|, \hat{\Phi}_f\big).$$

Thus

$$V\big(|B|, \Phi_f\big) \ge V\big(|B|, \hat{\Phi}_f\big).$$

Conversely, given $T \in \mathrm{T}(|B|, \Phi_f)$, let $\tau_a = T \wedge T_a$. Then

$$\begin{aligned}\mathbb{E} B^*_{\tau_a} - \Phi_f\big(|B_{\tau_a}|\big) &= \mathbb{E} B^*_{\tau_a} - \Phi_f\big(|B_{\tau_a}|\big)\\ &\ge \mathbb{E} B^*_{\tau} - \Phi_f\big(|B_{\tau}|\big) - (d-a) \quad \text{(by Lemma 5.7)}.\end{aligned}$$

Thus, letting $a \uparrow d$, we obtain

$$V\big(|B|, \hat{\Phi}_f\big) \ge V\big(|B|, \Phi_f\big). \qquad \square$$

We are now ready to prove the following theorem.

THEOREM 6.3. *Suppose f is an arbitrary function from* $[0, \infty)$ *to* $(-\infty, \infty]$. *Then define*

$$\tilde{\Phi}_f(x) \equiv \Phi_f(x) - \Phi_f(0), \tag{6.3}$$

where Φ_f is given by (6.1) *and* (6.2). *If f is not identically* $+\infty$ *either*:

(i) $\tilde{\Phi}_f$ *is an EYF satisfying condition* (5.1) *and*

$$\begin{aligned}V(|B|, f) &= V\big(|B|, \tilde{\Phi}_f\big) - \Phi_f(0)\\ &= V_{\tilde{\Phi}_f}(|B_0|, |B_0|) - \Phi_f(0);\end{aligned}$$

or

(ii) $\tilde{\Phi}_f$ *is bounded above on* $\mathbb{R}_+$ *by a linear function and*

$$V(|B|, f) = \infty,$$

while if $f \equiv +\infty$,

$$V(|B|, f) = -\infty.$$

PROOF. The case where $f \equiv +\infty$ is obvious. If $f \not\equiv +\infty$, then consider $\tilde{\Phi}_f$: if $\tilde{\Phi}_f$ is not bounded above by a linear function, then clearly $\tilde{\Phi}_f$ is an EYF satisfying (5.1) and so (i) holds by Lemmas 6.1 and 6.2. Otherwise we still have

$$V(|B|, f) = V\big(|B|, \Phi_f\big),$$

but $\Phi_f(x)$ is bounded above by $c + dx$, say.

Clearly such a linear function may be bounded above by $c' + \Phi$, where Φ is an EYF satisfying (5.1) but with $\int_0^\infty e^{-\phi(t)}\,dt = \infty$ so that, by Theorem 5.1

$$V(|B|, f) \ge V(|B|, \Phi) = \infty. \qquad \square$$

7. Some applications of the results. We still have to prove Theorem 1.1.

PROOF OF THEOREM 1.1. We simply apply Corollary 2.3, Lemma 3.1 and (3.8) to see that the best constant σ_q is given by

$$\sigma_q \equiv p^{1/p} q^{1/q} \big[V_q(0,0) \big]^{1/p},$$

where $V_q \equiv V_\Phi$ with $\Phi(x) \equiv x^q$, and $p \equiv \tilde{q} = q/(q-1)$. Now

$$V_q(0,0) = \psi_q(0),$$

where $\psi_q(x) = x + e^{qx^{q-1}} \int_x^\infty e^{-qt^{q-1}}\, dt$ (from Theorem 5.1); so

$$\begin{aligned} \psi_q(0) &= \int_0^\infty e^{-qt^{q-1}}\, dt \\ &= \frac{p-1}{q^{p-1}} \Gamma(p-1) \\ &= \frac{\Gamma(p)}{q^{p-1}}, \end{aligned}$$

where p is the conjugate of q; thus $\sigma_q = [\Gamma(p+1)]^{1/p}$. □

We see the following, more general theorem.

THEOREM 7.1. *If Φ is a Young function with conjugate $\tilde{\Phi}$ such that $\Phi(1) > 0$, then there exist σ_Φ and Σ_Φ such that*

$$\mathbb{E} \sup_{s \geq 0} X_s \leq \begin{cases} \Sigma_\Phi \|X_\infty\|_\Phi, \\ \sigma_\Phi \||\, X_\infty \||_\Phi, \end{cases} \tag{7.1}$$

for all cadlag, nonnegative ui *submartingales X iff*

$$\exists\, \mu > 0\colon \quad \int_0^\infty e^{-\mu\phi(t)}\, dt < \infty, \tag{7.2}$$

and if (7.2) *holds for some $\mu > 0$, then the best constants σ_Φ and Σ_Φ appearing in* (7.1) *are given by*

$$\sigma_\Phi = \inf\left\{ \mu > 0\colon \int_0^\infty e^{-\mu\phi(t)}\, dt/\mu \leq \tilde{\Phi}(1) \right\},$$

$$\Sigma_\Phi = \inf_{\mu > 0} \left(\int_0^\infty e^{-\mu\phi(t)}\, dt + \mu\Phi(1) \right).$$

PROOF. We observe that Corollary 2.2, Lemma 3.1 and (3.8) indicate that

$$\sigma_\Phi \equiv \inf\left\{ \mu \geq 0\colon V_{\Phi_\mu} \leq \tilde{\Phi}(1) \right\},$$

$$\Sigma_\Phi = \inf_{\mu > 0} \left(\mu V_{\Phi_\mu}(0,0) + \mu\Phi(1) \right).$$

But, by Theorem 5.1, $V_{\Phi_\mu}(0,0) = \int_0^\infty e^{-\mu\phi(\mu t)}\, dt \equiv (1/\mu) \int_0^\infty e^{-\mu\phi(u)}\, du$, establishing both the inequalities in (7.1). □

We now use Theorem 7.1 to prove the John–Nirenberg inequality for martingales [see Dellacherie and Meyer (1982) for full details].

THEOREM 7.2 (John–Nirenberg inequality). *Suppose X is a martingale with*

$$\|X\|_{\mathrm{BMO}} < \frac{1}{c}.$$

Then

$$\mathbb{E}e^{X_\infty^*} \le 1 + \frac{4c\|X\|_{\mathrm{BMO}}}{1 - c\|X\|_{\mathrm{BMO}}}, \tag{7.3}$$

where $\|\cdot\|_{\mathrm{BMO}}$ is the norm on BMO *space (the dual of H^1) and c is the best constant appearing in*

$$\mathbb{E}[X_\infty Y_\infty] \le c\|X_\infty^*\|_1\|Y\|_{\mathrm{BMO}}, \tag{7.4}$$

X and Y martingales in H^2 [see Dellacherie and Meyer (1982) *or Garsia* (1973) *for details*].

We need a couple of lemmas.

LEMMA 7.3. *Define $\Phi_t(x) \equiv (1 + tx)\log(1 + tx) - tx$, $x \ge 0$. Then*:

(i) *its conjugate is*

$$\Psi_t(x) = \exp(x/t) - x/t - 1;$$

(ii) *If X is a ui nonnegative submartingale with $\mathbb{E}\Psi_t(X_\infty) < \infty$, then*

$$\mathbb{E}\Psi_t(X_\infty^*) \le 4\mathbb{E}\Psi_t(X_\infty).$$

PROOF. (i) We obtain Ψ_t by using the fact that $\phi_t(x) = \psi_t^{-1}(x) = [\exp(x/t) - 1]/t$ (where ϕ and ψ are the derivatives of Φ and Ψ, respectively).

(ii) Lemma 5.7 tells us that $(\Psi_t(X_s);\ s \ge 0)$ is ui and the result follows by looking at $\Psi_t(X_{T_n})$ where $T_n = \inf\{s \ge 0\colon X_s \ge n\}$, taking a power series expansion for Ψ_t and observing that $\mathbb{E}X_s^{*p} \le [p/(p-1)]^p\mathbb{E}X_s^p \le 4\mathbb{E}X_s^p$ for $p \ge 2$. □

LEMMA 7.4. *If X is a ui martingale, then*

$$c\sigma_t\|X\|_{\mathrm{BMO}} \ge \|X_\infty\|_{\Psi_t}, \tag{7.5}$$

where c is the constant given in (7.4), *$\|\cdot\|_{\Psi_t}$ is the norm defined by*

$$\|X\|_{\Psi_t} = \inf\left\{\mu > 0\colon \mathbb{E}\Psi_t\left(\frac{|X|}{\mu}\right) \le \Psi_t(1)\right\} \tag{7.6}$$

and σ_t is the constant appearing in Theorem 7.1 *with $\Phi \equiv \Phi_t$.*

PROOF. Given $X \in \mathrm{BMO}$ we may take a localising sequence (T_n) such that $(X_{t \wedge T_n}) \in H^2$ and $\forall\ Y \in H^2$,

$$\mathbb{E} X_{T_n} Y_\infty \le c \|X_{T_n}\|_{\mathrm{BMO}} \mathbb{E} Y_\infty^*.$$

But, from Theorem 6 of Jacka (1991) we know that the dual norm of $\|\cdot\|_{\Psi_t}$ is $|||\cdot|||_{\Phi_t}$, so take $\hat{Y}$: $|||\hat{Y}_\infty|||_{\Phi_t} < \infty$ and

$$\mathbb{E} X_{T_n} \hat{Y}_\infty = \|X_{T_n}\|_{\Psi_t} |||\hat{Y}_\infty|||_{\Phi_t};$$

by localising we may assume $\hat{Y} \in H^2$ and

$$\mathbb{E} X_{T_n} \hat{Y}_\infty = \|X_{T_n}\|_{\Psi_t} |||\hat{Y}_\infty|||_{\Phi_t}.$$

Thus

$$\sigma_t \|X_{T_n}\|_{\Psi_t} |||\hat{Y}_\infty|||_{\Phi_t} = \sigma_t \mathbb{E}\left(X_{T_n} \hat{Y}_\infty\right) \le c\sigma_t \|X\|_{\mathrm{BMO}} \mathbb{E} \hat{Y}_\infty^*.$$

But by Theorem 7.1, $\mathbb{E}\hat{Y}_\infty^* \le \sigma_t |||\hat{Y}_\infty|||_{\Phi_t}$ so $\|X_{T_n}\|_{\Psi_t} \le c\sigma_t \|X\|_{\mathrm{BMO}}$ and the result follows by letting $n \to \infty$. □

LEMMA 7.5. *If $X \in \mathrm{BMO}$, X a martingale, then* $\mathbb{E} X_\infty^* \le 4c\|X\|_{\mathrm{BMO}}$.

PROOF. Suppose $X \in H^2$. Then

$$\begin{aligned} \mathbb{E}(X_\infty^*)^2 &\le 4\mathbb{E}(X_\infty)^2 \quad \text{(by Doob's inequality)} \\ &\le 4c\|X\|_{\mathrm{BMO}} \mathbb{E} X_\infty^* \quad [\text{by } (7.4)]; \end{aligned}$$

conversely $\mathbb{E} X_\infty^{*2} \ge (\mathbb{E} X_\infty^*)^2$. The result follows by localisation. □

LEMMA 7.6. *If σ_t is as given in Lemma 7.4, then*

$$t\sigma_t = \tfrac{1}{2} + \left(\tfrac{1}{2} + 1/\Psi_t(1)\right)^{1/2}. \tag{7.7}$$

PROOF. From Theorem 7.1,

$$\begin{aligned} \sigma_t &= \inf\left\{\mu > 0: \frac{1}{\mu}\int_0^\infty e^{-\mu\phi_t(u)}\,du \le \Psi_t(1)\right\} \\ &= \inf\left\{\mu > 0: \frac{1}{\mu}\int_0^\infty \exp(-\mu t \log(1 + tu))\,du \le \Psi_t(1)\right\} \\ &= \inf\left\{\mu > 0: \frac{1}{\mu t}\frac{1}{(\mu t - 1)} \le \Psi_t(1)\right\}. \end{aligned}$$

Thus

$$t\sigma_t = \inf\left\{\mu > 0: \frac{1}{\mu(\mu - 1)} \le \Psi_t(1)\right\}. \qquad \square$$

PROOF OF THEOREM 7.2. From Lemma 7.3,

$$\begin{aligned} \mathbb{E}\exp(X_\infty^*) &= 1 + \mathbb{E} X_\infty^* + \mathbb{E}\Psi_t(tX_\infty^*) \\ &\le 1 + \mathbb{E} X_\infty^* + 4\mathbb{E}\Psi_t(t|X_\infty|) \\ &\le 1 + 4c\|X\|_{\mathrm{BMO}} + 4\mathbb{E}\Psi_t(t|X_\infty|) \quad \text{(from Lemma 7.5)}. \end{aligned} \tag{7.8}$$

If $\|X\|_{\mathrm{BMO}} < 1/c$, choose $\hat{t}$ such that $\hat{t}\sigma_{\hat{t}} = (1/c)\|X\|_{\mathrm{BMO}}$; from (7.7) it follows that

$$\Psi_{\hat{t}}(1) \equiv (c\|X\|_{\mathrm{BMO}})^2/(1 - c\|X\|_{\mathrm{BMO}}).$$

Then from Lemma 7.4,

$$\|X_\infty^*\|_{\Psi_{\hat{t}}} \le \frac{1}{\hat{t}}$$

so $\mathbb{E}\Psi_{\hat{t}}(\hat{t}|X_\infty|) \le \Psi_{\hat{t}}(1)$ and substituting in (7.8) we see that

$$\begin{aligned}\mathbb{E}\exp(X_\infty^*) &\le 1 + 4c\|X\|_{\mathrm{BMO}} + \frac{4c^2(\|X\|_{\mathrm{BMO}})^2}{1 - c\|X\|_{\mathrm{BMO}}} \\ &= 1 + \frac{4c\|X\|_{\mathrm{BMO}}}{1 - c\|X\|_{\mathrm{BMO}}}.\end{aligned}$$ □

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DEPARTMENT OF STATISTICS
UNIVERSITY OF WARWICK
COVENTRY CV4 7AL
ENGLAND