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CLUSTERING IN THE ONE-DIMENSIONAL THREE-COLOR CYCLIC CELLULAR AUTOMATON[1]

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This paper investigates the dynamics of the one-dimensional three-color cyclic cellular automaton. The author has previously shown that this process fluctuates, meaning that each lattice site changes color infinitely often, so that there is no "final state" for the system. The focus of the current work is on the clustering properties of this system. This paper demonstrates that the one-dimensional three-color cyclic cellular automaton clusters, and the mean cluster size, as a function of time t, is asymptotic to $ct^{1/2}$, where c is an explicitly calculable constant. The method of proof also allows us to compute asymptotic estimates of the mean interparticle distance for a one-dimensional system of particles which undergo deterministic motion and which annihilate upon collision.

No clustering results are known about the four-color process, but evidence is presented to suggest that the mean cluster size of such systems grows at a rate different from $t^{1/2}$.

1. Introduction. A stochastic process of recent interest is the cyclic particle system introduced by Bramson and Griffeath [5]. This is a multitype interacting particle system defined on a d-dimensional lattice $\mathbb{Z}^d$ where each lattice site may take on one of a finite number of values, which are called colors because of the way the values are displayed in computer simulations. The colors are given a structure of their own in order to specify how they may interact with each other. This structure may be described as a cyclic hierarchy (hence the name cyclic particle system). If N is the number of colors, then represent the colors as $0, 1, \ldots, N-1$. We say that color i can *eat* color j if $i = j + 1 \pmod N$; if $|i - j| \neq 1 \pmod N$, then we say that i and j form an *inert pair*.

A cyclic particle system is a continuous time Markov chain. The intuitive idea of the evolution is as follows. Each lattice site waits a random (exponentially distributed) amount of time before choosing a random neighbor. (The neighborhood of a lattice site x is the set of lattice sites y such that $|x - y| = 1$.) If the color of the chosen neighbor can eat the color of the site in question, then it does so, meaning that the color of the site changes to the color of the chosen neighbor. Otherwise, the color of the site in question remains un-

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changed. This procedure occurs repeatedly and independently at each lattice site, so that a dynamic is defined on the entire lattice.

The fundamental question to ask about a cyclic particle system is whether each site of the process changes color at arbitrarily large times, known as fluctuation, or whether each site changes color only finitely many times, known as fixation. Fluctuation means that there is continuing change in the process, whereas fixation means that the process "gets stuck". In [5], Bramson and Griffeath show that for the one-dimensional cyclic particle system, fluctuation occurs a.s. if there are four or fewer colors, and fixation occurs a.s. if there are five or more colors. In two or more dimensions, this question is unsettled, although computer evidence suggests that the two-dimensional cyclic particle system may fluctuate regardless of the number of colors. See [8] for an account of the investigation of Bramson, Griffeath, and the author into these two-dimensional systems.

In the remainder of this paper, we shall restrict ourselves to one-dimensional systems. For those systems that fluctuate, the next task is to examine how the evolution proceeds. If the number of colors is two, then the resulting process is actually the voter model of Clifford and Sudbury [6] and Holley and Liggett [10]. For this process, it is known [4] that clustering occurs, which means that the evolution produces long intervals whose sites all have the same color. Furthermore, the mean cluster size of the voter model grows at a rate proportional to the square root of time. For the three- and four-color cyclic particle systems, one would like to produce results along these same lines. Computer simulations suggest that clustering does indeed occur in these systems, but no progress has been made to date in rigorously computing the rate of growth of the mean cluster size.

This investigation has led to the study of a system known as the cyclic cellular automaton. This process, which is introduced in [7], can be thought of as the deterministic analogue of the cyclic particle system. The state space, just as for the cyclic particle system, is $\{0, 1, \ldots, N-1\}^{\mathbb{Z}}$, and the N colors are arranged in a cyclic hierarchy. Time passes in discrete units. A random initial condition is specified; this is the only nondeterministic component of the process. To get the configuration at time $t+1$ from the configuration at time t, each site examines its two neighbors. For site x, if the color of either neighbor can eat the color of x, then the color of x gets eaten at time $t+1$; otherwise, the color of x remains unchanged. Once each site in $\mathbb{Z}$ determines how it will update, then all sites update synchronously at time $t+1$. The main result in [7] is that fluctuation occurs a.s. with four or fewer colors and that fixation occurs a.s. with five or more colors. Hence, for a given number of colors, the cyclic cellular automaton and the cyclic particle system exhibit the same behavior with regard to fluctuation and fixation.

The main result of this paper computes asymptotics on the mean cluster size of the three-color cyclic cellular automaton. In doing this, we demonstrate that cellular automata provide a tractable setting for proving results that are related to conjectures about certain interacting particle systems. This furthers the notion that the analysis of these deterministic systems is useful in understanding their stochastic counterparts.

It is natural to inquire about the clustering properties of the four-color cyclic cellular automaton, and this paper concludes with a discussion of this system. Although no clustering results are presented concerning this system, some empirical data will be given to suggest how this system differs from the three-color cyclic cellular automaton.

2. Definitions. Before proceeding, let us define the probability space in which we shall work, the processes with which we are concerned, and the notation that will be used to discuss them. Since the only nondeterministic component of a cyclic cellular automaton is the initial configuration, the probability space we shall define will merely serve the purpose of specifying the product measure used for the initial configuration. Instead of regarding product measure as a sequence of independent uniform random variables that define the initial configuration at each lattice site, it will be more convenient to use an implementation where a uniform random variable determines the color at the origin, and then a sequence of independent uniform random variables measure the color transitions from one site to the next. It is with this in mind that we define our probability space $(\Omega, \mathscr{F}, P)$.

Let $\Omega = \{0,1,2\} \times \{-1,0,1\}^{\mathbb{Z}}$, let $\mathscr{P}$ be the σ-algebra generated by finite cylinder sets of Ω and let P be uniform product measure on Ω. For $\omega = \omega'_0 \times (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \in \Omega$, we then have that $P(\{\omega'_0 = j\}) = P(\{\omega_i = k\}) = 1/3$, for each $i \in \mathbb{Z}$, $j \in \{0,1,2\}$ and $k \in \{-1,0,1\}$.

The *three-color cyclic cellular automaton* is denoted by $\{\eta_t\}$. An illustration of $\{\eta_t\}$ is given in Figure 1. The state space for $\{\eta_t\}$ is $S = \{0,1,2\}^{\mathbb{Z}}$, and elements of the state space are called *configurations*. For each $\eta \in S$, we write

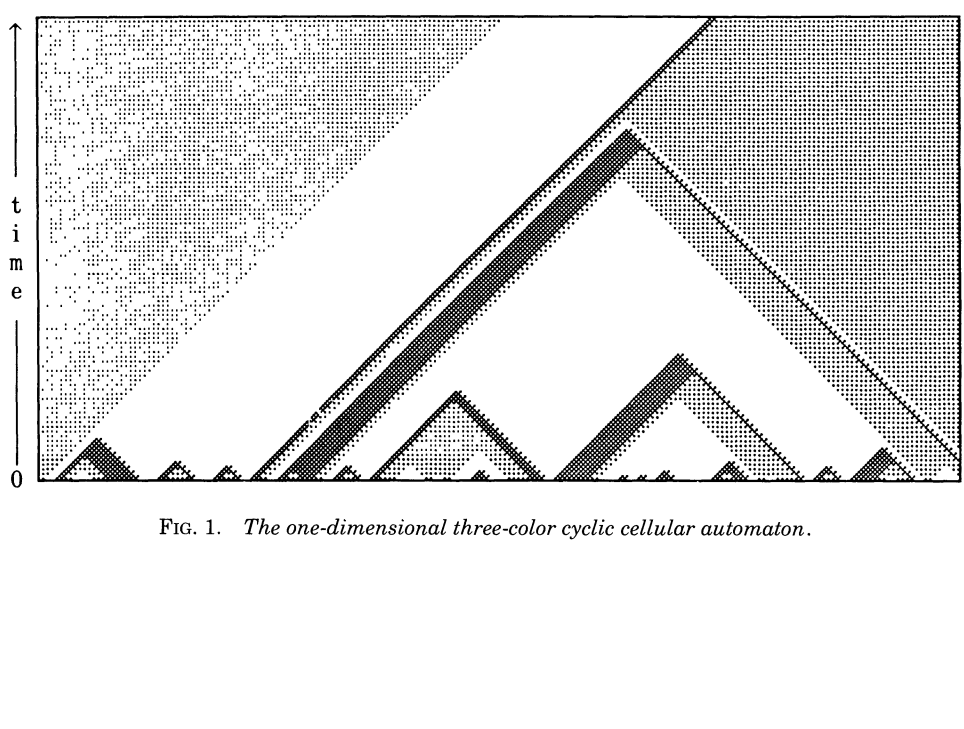


FIG. 1. *The one-dimensional three-color cyclic cellular automaton.*

$\eta(x)$ to indicate the color (either 0, 1 or 2) of η at a site $x \in \mathbb{Z}$. A sample path for $\{\eta_t\}$ is denoted by (η_t), so that for any choice of $\omega \in \Omega$, $(\eta_t) = (\eta_t)(\omega)$ consists of a sequence of configurations $\eta_0, \eta_1, \eta_2, \dots$. (As is customary, the ω dependence will be suppressed in our notation.) The parameter t can be thought of as a time parameter, so that η_t evolves from η_0.

Fix $\omega \in \Omega$. To define η_0, we first set $\eta_0(0) = \omega'_0$. For $x > 0$,

$$\eta_0(x) = \left(\eta_0(x-1) - \omega_{x-1}\right) \mod 3$$

and, for $x < 0$,

$$\eta_0(x) = \left(\eta_0(x+1) + \omega_x\right) \mod 3.$$

To define the rest of (η_t) from η_0, the following inductive rule is used: For each $x \in \mathbb{Z}$,

$$\eta_{t+1}(x) = \begin{cases} (\eta_t(x)+1) \mod 3, & \text{if } \eta_t(x+1) \equiv (\eta_t(x)+1) \mod 3 \\ & \quad \text{or } \eta_t(x-1) \equiv (\eta_t(x)+1) \mod 3, \\ \eta_t(x), & \text{otherwise.} \end{cases}$$

This completes the definition of $\{\eta_t\}$.

It is important to define another cellular automaton imbedded within $\{\eta_t\}$, which we shall call the *edge cellular automaton* $\{\zeta_t\}$. This cellular automaton will be defined so that for each time t, ζ_t will keep track of all the color transitions between consecutive sites of η_t, that is, the relation

$$\zeta_t(x) \equiv (\eta_t(x) - \eta_t(x+1)) \mod 3 \tag{2.1}$$

will hold for all $t \geq 0$. In Figure 1, $\{\zeta_t\}$ is represented by the edges between colors. In the sections to follow, our discussion will center around $\{\zeta_t\}$ rather than $\{\eta_t\}$. For this reason, the choice of $(\Omega, \mathscr{F}, P)$ was made in order to simplify the discussion of $\{\zeta_t\}$.

The state space for $\{\zeta_t\}$ is $\hat{S} = \{-1, 0, 1\}^{\mathbb{Z}}$. Notational conventions for $\{\eta_t\}$ will be used for $\{\zeta_t\}$ as well. When necessary to avoid confusion, we will call elements of S *color configurations* and elements of $\hat{S}$ *edge configurations*. To define $\{\zeta_t\}$, we show for each $\omega \in \Omega$ how to construct ζ_0, and then indicate how to define ζ_{t+1} from ζ_t for any $t \geq 0$.

Fix $\omega \in \Omega$. For each $x \in \mathbb{Z}$, define $\zeta_0(x) = \omega_x$, so that (2.1) holds when $t = 0$. We shall indicate how ζ_1 evolves from ζ_0 so that (2.1) holds in the case $t = 1$, and then an induction allows $\{\zeta_t\}$ to be defined so that (2.1) holds for all $t \geq 0$.

Let us say that ζ_0 is *vacant* at x if $\zeta_0(x) = 0$ and that ζ_0 has an *edge particle* at x if $\zeta_0(x) \neq 0$. If ζ_0 is vacant at x, then $\eta_0(x) = \eta_0(x+1)$, so that in the color configuration η_0, the edge between x and $x+1$ is within a block of one color. But if ζ_0 has an edge particle at x, then this edge is a boundary between two blocks of different colors. If $\zeta_0(x) = 1$, then $\eta_0(x) \equiv (\eta_0(x+1) + 1) \mod 3$, so that the color of the block to the left eats the color of the block to the right. By comparing η_0 and η_1, the boundary between the two blocks will have moved one site to the right by time 1. Similarly, if $\zeta_0(x) = -1$, then this boundary will have moved one site to the left by time 1. The boundaries in η_1 between blocks of different colors arise from the motion described here, although we have not yet described what happens when two such boundaries collide; this shall be addressed momentarily.

The evolution from ζ_0 to ζ_1 is motivated by the motion of these boundaries in the color configuration. We regard ζ_0 as a collection of moving particles, and ζ_1 will be the result of the motion of these particles. If $\zeta_0(x) = 1$, then we call the edge particle at x a *rightward particle*. If $\zeta_0(x) = -1$, then the edge particle at x is a *leftward particle*. To define ζ_1 from ζ_0, we move all the rightward particles one site to the right and all the leftward particles one site to the left. The only remaining detail is to indicate how collisions between particles are resolved. No collision is possible between two rightward particles or between two leftward particles because all particles move at the same speed. If a rightward particle and a leftward particle collide, they should annihilate each other, because this corresponds to a situation in the color configuration where a block of one color is getting eaten from both sides. Since only one color can be doing the eating, the colors of the blocks to either side are the same, and so when they meet after eating the block between them, there is no boundary.

This discussion motivates the construction of ζ_1 from ζ_0 which allows (2.1) to be satisfied when $t = 1$. By defining ζ_t inductively for $t = 2, 3, \dots,$ then (2.1) will be satisfied for all $t \geq 0$. Formally, given ζ_t, then for any $x \in \mathbb{Z}$,

$$\zeta_{t+1}(x) = \begin{cases} 1, & \text{if either } \zeta_t(x) = 0,\ \zeta_t(x-1) = 1,\ \zeta_t(x+1) \neq -1 \\ & \quad \text{or } \zeta_t(x) = 1,\ \zeta_t(x-1) = 1; \\ -1, & \text{if either } \zeta_t(x) = 0,\ \zeta_t(x+1) = -1,\ \zeta_t(x-1) \neq 1 \\ & \quad \text{or } \zeta_t(x) = -1,\ \zeta_t(x+1) = -1; \\ 0, & \text{otherwise.} \end{cases}$$

That the formal definition for $\{\zeta_t\}$ satisfies (2.1) for all $t \geq 0$ is left for the reader.

The idea of $\{\zeta_t\}$ being a collection of moving particles is important enough that it will be convenient to relabel the values that $\zeta_t(x)$ can take on as $\swarrow$, 0, and $\nearrow$ instead of -1, 0, and 1, respectively, so that we may redefine $\hat{S} = \{\swarrow, 0, \nearrow\}^{\mathbb{Z}}$. The initial condition ζ_0 is made up of vacant sites 0, rightward moving particles $\nearrow$, and leftward moving particles $\swarrow$. Later configurations ζ_t indicate the positions of these particles at time t, where the rules of movement allow each $\nearrow$ and $\swarrow$ to move one site in their indicated direction every time step and specify annihilation upon collision. In reference to a particular edge particle starting from a site in ζ_0, we shall feel free to speak of future events for this edge particle, such as the time at which this particle is annihilated. If an edge particle has not been annihilated by time t, we will say that this particle *lives* to time t.

3. Statement of main results. Let us define our terminology concerning clustering. In any configuration, a *cluster* is a maximal block of contiguous sites that are all the same color. To compute the mean cluster size of a configuration η, first compute the mean cluster size of η on an interval $[-n, n]$, and then let n approach infinity. Formally, the *mean cluster size* $C(\eta)$ of a configuration $\eta \in S$ is defined as

$$C(\eta) = \lim_{n \to \infty} \frac{2n}{\text{number of clusters } \eta \text{ has in } [-n, n]},$$

whenever this limit exists. Another notion we must specify is the condition necessary to conclude that a process $\{\eta_t\}$ clusters. What we want to mean by this is that when observing an interval of sites at a very large time t, then η_t is overwhelmingly likely to display only one color over this interval. So we shall say that $\{\eta_t\}$ *clusters* if and only if for any $x, y \in \mathbb{Z}$, $x < y$,

$$\lim_{t \to \infty} P(\eta_t(x) = \eta_t(x+1) = \cdots = \eta_t(y)) = 1.$$

Once we know that $\{\eta_t\}$ clusters, we can try to compute how $C(\eta_t)$, the mean cluster size of η_t, grows with t.

We can now state a result concerning the clustering of the three-color cyclic cellular automaton. In the theorem, $\sim$ means that the limit of the quotient approaches 1 as t approaches ∞.

THEOREM 1. *Let $\{\eta_t\}$ be a one-dimensional three-color cyclic cellular automaton.*

(a) *$\{\eta_t\}$ clusters.*

(b) *$C(\eta_t) \sim \sqrt{3\pi t/2}$, a.s.*

Theorem 1 gives a description of what happens in the evolution of the three-color cyclic cellular automaton. Part (b) is actually a special case of a more general theorem. To state this more general theorem, let us focus on the edge cellular automaton $\{\zeta_t\}$, which was described in terms of a system of particles moving deterministically and undergoing annihilation upon collision.

For each p in the interval $(0, 1/2]$, define a *system of deterministic annihilating particles* $\{\zeta_t^p\}$ to be a process that is defined on the probability space $(\Omega, \mathscr{F}, P^p)$, where Ω and $\mathscr{F}$ are as before, and P^p is product measure on Ω defined for each $i \in \mathbb{Z}$ as $P^p(\{\omega_i = 1\}) = P^p(\{\omega_i = -1\}) = p$ and $P^p(\{\omega_i = 0\}) = 1 - 2p$. (Events involving ω'_0 will not play a role in these systems.) The state space for $\{\zeta_t^p\}$ is $\{\swarrow, 0, \searrow\}^{\mathbb{Z}}$, and $\{\zeta_t^p\}$ is defined in much the same way as $\{\zeta_t\}$. Hence, given $\omega \in \Omega$, the initial condition ζ_0^p is defined at each site $x \in \mathbb{Z}$ via

$$\zeta_0^p(x) = \begin{cases} \swarrow, & \text{if } \omega_x = -1, \\ 0, & \text{if } \omega_x = 0, \\ \searrow, & \text{if } \omega_x = 1. \end{cases}$$

Thus, the measure determining ζ_0^p is product measure with the property that independently at each site $x \in \mathbb{Z}$,

$$(3.1)\quad P^p(\zeta_0^p(x) = \swarrow) = P^p(\zeta_0^p(x) = \nearrow) = p, \qquad P^p(\zeta_0^p(x) = 0) = 1 - 2p.$$

The evolution rule for $\{\zeta_t^p\}$ is the same as the evolution rule for the edge cellular automaton. Each $\swarrow$ represents a leftward particle, each $\nearrow$ represents a rightward particle and each 0 represents a vacant site. At each time step, every particle leaves its current site and appears one site away in the appropriate direction unless a collision occurs, in which case the colliding particles annihilate each other and no longer appear in the system. Let us observe that the edge cellular automaton is represented in this collection because it is equal in distribution to $\{\zeta_t^{1/3}\}$.

The quantity of interest for these systems is the mean interparticle distance as a function of time. In order to be precise about what we mean by this, let us define the *mean interparticle distance* $D(\zeta)$ of a configuration $\zeta \in \hat{S}$ to be

$$D(\zeta) = \lim_{n\to\infty} \frac{2n}{\text{number of particles } \zeta \text{ has in } [-n, n]}$$

whenever the limit exists. The next theorem computes asymptotics on $D(\zeta_t^p)$ as t gets large.

THEOREM 2. *For $p \in (0, 1/2]$, let $\{\zeta_t^p\}$ be a one-dimensional system of deterministic annihilating particles with initial product measure satisfying* (3.1). *Then $D(\zeta_t^p) \sim \sqrt{\pi t/2p}$, P^p-a.s.*

When Theorem 1 is interpreted in terms of the edge cellular automaton, it gives a statement about a system of deterministic annihilating particles, and since $C(\eta_t) = D(\zeta_t)$ in this context, then Theorem 1(b) is the same as Theorem 2 in the case $p = 1/3$. The computations that extend Theorem 1(b) to Theorem 2 are performed in this paper.

4. Systems of moving particles with collision rules. Theorem 2 gives a result about a class of systems of particles that move deterministically and that annihilate upon collision. A number of results concerning systems of particles undergoing some kind of motion exist in the literature. Several of these are highlighted in this section in order to show that the cellular automaton of Theorem 2 is a natural system to study outside the context of the cyclic cellular automaton from which it arose. The results mentioned in this section are compiled in Table 1 in order to identify a collection of systems that are related in a natural way, to demonstrate that a particular kind of problem has been solved for most of these systems and to emphasize the role Theorem 2 plays in filling a remaining void in this framework.

The systems to be described all have the following setup. Particles are initially placed in a one-dimensional environment (either a lattice or a continuum). Each particle moves according to some process, either random walk, Brownian motion, or deterministic motion, and the motion of any particle is independent of the motion of all other particles until it collides with another

TABLE 1
Summary of results about one-dimensional systems of particles

Collision rule, quantity of interest	**Process governing particle motion**		
	Random walk	**Brownian motion**	**Deterministic motion**
Reflection/exclusion, natural scaling divisor for tagged particle displacement	$\sqrt[4]{t}$ Arratia [3]	$\sqrt[4]{t}$ Harris [9]	$\sqrt{t}$ Harris [9]
Annihilation, growth rate of mean interparticle distance	$\sqrt{t}$ Bramson and Griffeath [4]	$\sqrt{t}$ Arratia [2][a]	$\sqrt{t}$ Theorem 2 of this paper
Coalescence, growth rate of mean interparticle distance	$\sqrt{t}$ Bramson and Griffeath [4]	$\sqrt{t}$ Arratia [1]	$\sqrt{t}$ Conjecture of this paper

[a]Result not stated in [2], but follows from the results of [2].

particle. An interaction mechanism is prescribed to decide the outcome of such collisions; the three interaction mechanisms considered are reflection/exclusion, annihilation and coalescence.

Consider first a one-dimensional lattice of particles undergoing random walk. Arratia [3] studied the exclusion process in this setting. Particles are initially laid down with density ρ and undergo independent, continuous time, simple symmetric random walks, except that any time a particle tries to jump onto an occupied site, such a jump is inhibited. Conditioning on a tagged particle starting at the origin, the displacement Y_t of this tagged particle can be examined. Arratia found that the displacement of the tagged particle in the exclusion process satisfies

$$\frac{1}{\sqrt[4]{t}} Y_t \to_d N\left(0, \frac{\sqrt{2}(1-\rho)}{\sqrt{\pi}\rho}\right).$$

(The d indicates convergence in distribution.) Note that the scaling divisor here is $t^{1/4}$ rather than $t^{1/2}$, which is the corresponding scaling divisor for a random walk on the one-dimensional lattice that moves without collisions.

Another paper examining particles undergoing random walk is by Bramson and Griffeath [4]. This paper looks at what happens when the collision mechanism causes annihilation of the particles, and also when it causes coalescence of the particles. One quantity they computed is the mean interparticle distance D_t at time t. For both the annihilating and coalescing systems, they showed that the asymptotic growth of D_t is proportional to $t^{1/2}$. More precisely, for an appropriate class of initial conditions in each case, the annihilating system satisfies

$$\frac{D_t}{\sqrt{t}} \to_p 2\sqrt{\pi}$$

and the coalescing system satisfies

$$\frac{D_t}{\sqrt{t}} \to_p \sqrt{\pi}.$$

(The p indicates convergence in probability.)

Next, consider a one-dimensional system of particles undergoing Brownian motion. Harris [9] investigated such systems where particles reflect upon collision. Initially, particles are placed on the real line according to a Poisson point process with intensity 1. A tagged particle is conditioned to start at the origin, and its displacement at time t, Y_t, is the quantity of interest. In this setting, the result is that

$$\frac{1}{\sqrt[4]{t}} Y_t \to_d N\left(0, \sqrt{\frac{2}{\pi}}\right).$$

Again, the appropriate scaling divisor is $t^{1/4}$ rather than $t^{1/2}$.

Arratia [1] looked at a system of one-dimensional Brownian motions where the particles coalesce upon collision. The initial condition places a particle at each point of the real line. Letting X_t be the configuration of the system at time t, he showed that

$$X_1 =_d \frac{1}{\sqrt{t}} X_t.$$

From this, one can conclude that the mean interparticle distance grows at a rate proportional to $t^{1/2}$. Another result due to Arratia [2] allows one to make a similar statement when annihilation is the collision rule.

Finally, consider a one-dimensional system of particles whose motions are deterministic. Harris [9] examined a system where such particles are initially placed according to a Poisson point process with intensity 1, and each particle is assigned a velocity whose value is chosen according to a normal distribution. Each particle moves at its assigned velocity until it collides with another particle, at which point reflection occurs via the mechanism that each particle takes on the velocity of the other particle in the collision. A tagged particle is conditioned to start at the origin, and its displacement at time t, Y_t, is examined. Here, the result is that

$$\frac{1}{\sqrt{t}} Y_t \to_d N\left(0, \sqrt{\frac{2}{\pi}}\right).$$

Note that the scaling divisor here is $t^{1/2}$, which differs from the scaling divisor for the tagged random walk or the tagged Brownian motion.

The omissions from this collection of results, as can be seen from Table 1, are theorems concerning the mean interparticle distance of a one-dimensional system of deterministic particles which either annihilate or coalesce upon collision. Theorem 2 fills in the gap in the case of annihilating particles, and the author conjectures that a result similar to Theorem 2 holds for coalescing particles. This conjecture is based on the realization that the mechanisms described in the proof of Theorem 2 are not severely disturbed by a change in the collision rule, and so it is unlikely that the mean interparticle distance would grow according to a different power of time, although the coefficient of the growth rate may indeed be different.

5. Some preliminary results and the proof of clustering. In this section, we introduce some tools for analyzing $\{\zeta_t^p\}$; one of these tools will be put to use to prove the clustering result of Theorem 1(a). The first lemma captures the idea that particles in $\{\zeta_t^p\}$ may not move through each other. If two particles eventually annihilate each other, then all the particles that started in the interval between these two in the initial condition must annihilate one another, that is, no particle from outside this interval may annihilate any particle from inside this interval. The proof is elementary and hence omitted. The notation $\lceil x \rceil$ represents the smallest integer greater than or equal to x; the notation $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

LEMMA 5.1. *Let $x, y \in \mathbb{Z}$ with $x < y$, and assume that the initial configuration of $\{\zeta_t^p\}$ has a rightward particle at x and a leftward particle at y; that is, $\zeta_0^p(x) = \searrow$ and $\zeta_0^p(y) = \swarrow$. Then at least one of the following three events must occur by time $\lceil (y-x)/2 \rceil$:*

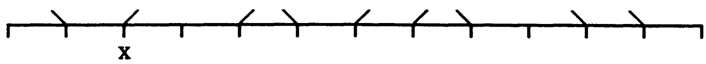
(i) *these two edge particles annihilate each other, and all the particles between them annihilate one another*;
(ii) *the $\searrow$ at x is annihilated by some $\swarrow$ in $[x+1, y-1]$*; or
(iii) *the $\swarrow$ at y is annihilated by some $\searrow$ in $[x+1, y-1]$.*

For each $x \in \mathbb{Z}$, we want to define two random walks, Z_{x+} and Z_{x-}, that will be useful in analyzing the evolution of $\{\zeta_t^p\}$. These random walks are defined on the same probability space as $\{\zeta_t^p\}$ (and, in the case $p = 1/3$, $\{\eta_t\}$). Given $\omega \in \Omega$, define

$$Z_{x+}(t) = \sum_{i=x}^{x+t-1} \omega_i,$$

$$Z_{x-}(t) = -\sum_{i=x-t+1}^{x} \omega_i.$$

The random walk Z_{x+} can be thought of as starting at 0 and increasing by 1 with every $\searrow$ encountered and decreasing by 1 with every $\swarrow$ encountered while scanning ζ_0^p to the right starting at x. Similarly, Z_{x-} scans ζ_0^p to the left starting at x, but $\swarrow$ particles increase Z_{x-} and $\searrow$ particles decrease it. Figure



$t \to$	1	2	3	4	5	6	7	8	9	10	11
$Z_{x+}(t) \to$	1	1	2	1	2	3	2	2	1	0	0

FIG. 2. *Illustration of* Z_{x+}.

2 depicts a portion of ζ_0^p and shows how Z_{x+} can be defined from it. Let us mention here that Z_{x+} and Z_{x-} are recurrent random walks under P^p (see P2.8 of Spitzer [11]). The following result gives some elementary arithmetic relationships that these random walks satisfy. The proofs follow directly from the definitions of Z_{x+} and Z_{x-}.

LEMMA 5.2. *Let $x, y \in \mathbb{Z}$ with $x < y$ and let $s, t \geq 0$.*

(a) $Z_{x+}(y - x + 1) = -Z_{y-}(y - x + 1)$.
(b) $Z_{x+}(t) = 0$ *if and only if* $Z_{(x+t-1)-}(t) = 0$.
(c) $Z_{x+}(s + t) = Z_{x+}(s) + Z_{(x+s)+}(t)$ *and* $Z_{x-}(s + t) = Z_{x-}(s) + Z_{(x-s)-}(t)$.

We shall use these random walks in order to get a condition for when an edge particle from ζ_0^p lives to time t. First, for each $x \in \mathbb{Z}$, we define the first return times τ_{x+} and τ_{x-} of the random walks Z_{x+} and Z_{x-} to zero as

$$\tau_{x+} = \min\{t > 0 \colon Z_{x+}(t) = 0\},$$
$$\tau_{x-} = \min\{t > 0 \colon Z_{x-}(t) = 0\}.$$

Note that $\tau_{x+} = \tau_{x-} = 1$ if $\zeta_0^p(x) = 0$, so that one need not move away from zero to register a return to zero under this definition of the return time. Also, $P^p(\tau_{x+} < \infty) = P^p(\tau_{x-} < \infty) = 1$ by the recurrence of Z_{x+} and Z_{x-} under P^p, so that τ_{x+} and τ_{x-} are finite P^p-a.s. Our goal is to prove the following proposition.

PROPOSITION 5.3. *Let $x \in \mathbb{Z}$ and assume $\zeta_0^p(x) = \nearrow$. Then this particle will live to time t if and only if $\tau_{x+} > 2t + 1$.*

An illustration of this proposition is given in Figure 3, where the initial condition of Figure 2 is run for a few time units.

The next lemma gives us a connection between the evolution of $\{\zeta_t^p\}$ and the first return time of Z_{x+} to zero. This will lead to the proof of Proposition 5.3.

LEMMA 5.4. (a) *For any $x \in \mathbb{Z}$ and any $t > 0$, $Z_{x+}(t) = 0$ if and only if ζ_0^p has exactly the same number of $\nearrow$ and $\swarrow$ particles between x and $x + t - 1$, inclusive, and $Z_{x-}(t) = 0$ if and only if ζ_0^p has exactly the same number of $\nearrow$ and $\swarrow$ particles between $x - t + 1$ and x, inclusive.*

(b) *Let $x \in \mathbb{Z}$ and $t > 0$. Assume that $\zeta_0^p(x) = \nearrow$ and that this particle is not annihilated by any particle starting in $[x + 1, x + t - 1]$. Then $Z_{x+}(s) > 0$ for all $s = 1, \dots, t$.*

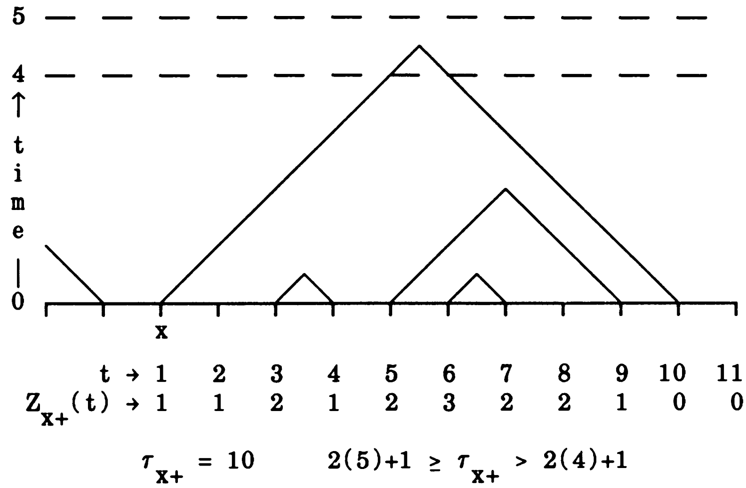


FIG. 3. *Illustration of Proposition* 5.3.

(c) *Let $x, y \in \mathbb{Z}$ with $x < y$. Assume $\zeta_0^p(x) = \nearrow$, $\zeta_0^p(y) = \swarrow$ and these two particles annihilate each other. Then $Z_{x+}(y - x + 1) = 0$ and $Z_{x+}(t) > 0$ for all $t = 1, \dots, y - x$.*

(d) *Let $x \in \mathbb{Z}$, and assume that $Z_{x+}(t) \geq 0$ for all $t = 0, \dots, \tau_{x+}$. Then either $\zeta_0^p(x) = 0$ or else $\zeta_0^p(x) = \nearrow$, $\zeta_0^p(x + \tau_{x+} - 1) = \swarrow$ and these two particles annihilate each other at time $\lfloor \tau_{x+}/2 \rfloor$.*

PROOF. Statement (a) follows directly from the definitions of Z_{x+} and Z_{x-}.

To prove (b), we induct on t. It is easy to see that (b) holds in the cases where $t = 1$ and $t = 2$. So, fix $k > 2$ and assume (b) holds whenever $t < k$. Assume the $\nearrow$ at x is not annihilated by any particle starting in $[x + 1, x + k - 1]$. From the inductive hypothesis, we only need to show $Z_{x+}(k) > 0$. If $\zeta_0^p(x + k - 1) \neq \swarrow$, then $\omega_{x+k-1} \geq 0$, so $Z_{x+}(k) = Z_{x+}(k - 1) + \omega_{x+k-1} > 0$. Hence, we can assume that $\zeta_0^p(x + k - 1) = \swarrow$. Then we are assuming that the $\nearrow$ at x is not annihilated by any particle in $[x + 1, x + k - 2]$ or by the $\swarrow$ at $x + k - 1$. By Lemma 5.1, it follows that the $\swarrow$ at $x + k - 1$ is annihilated by some $\nearrow$ at a site y in $[x + 1, x + k - 2]$. But then, all the particles in $[y, x + k - 1]$ must annihilate one another. Since every annihilation involves one $\nearrow$ and one $\swarrow$, there are equal numbers of $\nearrow$'s and $\swarrow$'s in $[y, x + k - 1]$. By part (a), $Z_{y+}(x + k - y) = 0$, and by Lemma 5.2(c) and the inductive hypothesis,

$$Z_{x+}(k) = Z_{x+}(y - x) + Z_{y+}(x + k - y) = Z_{x+}(y - x) + 0 > 0,$$

which is what we need.

To prove (c), we observe that option (i) of Lemma 5.1 occurs, so that there are equal numbers of $\nearrow$'s and $\swarrow$'s in $[x, y]$, and it follows that $Z_{x+}(y - x + 1) = 0$ by part (a). Furthermore, since the $\nearrow$ at x is not annihilated by any particle in $[x+1, y-1]$, then by part (b), $Z_{x+}(t) > 0$ for all $t = 1, \dots, y-x$.

The assumption in (d) about $Z_{x+}(t)$ being nonnegative for all t up to τ_{x+} merely rules out the possibility that $\zeta_0^p(x) = \swarrow$, because it is impossible for Z_{x+} to change from nonnegative to negative before time τ_{x+}. If $\tau_{x+} = 1$, then $\zeta_0^p(x) = 0$. So, assume $\tau_{x+} > 1$, which implies that $\zeta_0^p(x) \neq 0$. Then $\zeta_0^p(x) = \nearrow$, $Z_{x+}(\tau_{x+} - 1) > 0$ and $Z_{x+}(\tau_{x+}) = 0$ (from the definition of τ_{x+}). From this, $\omega_{x+\tau_{x+}-1} = -1$, so that $\zeta_0^p(x + \tau_{x+} - 1) = \swarrow$.

Let us determine which $\swarrow$ particle annihilates the $\nearrow$ at x. From part (c), if such an $\swarrow$ came from a site in $[x+1, x+\tau_{x+}-2]$, then $Z_{x+}(t) = 0$ for some $t < \tau_{x+}$, and from part (b), if no $\swarrow$ from a site in $[x+1, x+\tau_{x+}-1]$ annihilates the $\nearrow$ at x, then $Z_{x+}(\tau_{x+}) > 0$. The definition of τ_{x+} precludes either of these possibilities, so that the $\swarrow$ at $x + \tau_{x+} - 1$ must annihilate the $\nearrow$ at x. The time at which they annihilate each other is merely the distance each must go in order to collide. □

The proof of Proposition 5.3 now follows directly from Lemma 5.4(d). Assume that $\zeta_0^p(x) = \nearrow$. If $\tau_{x+} \le 2t + 1$, then this $\nearrow$ particle will be annihilated at time $\lfloor \tau_{x+}/2 \rfloor \le t$, while if $\tau_{x+} > 2t + 1$, then it will be annihilated at time $\lfloor \tau_{x+}/2 \rfloor > t$. So, we have a way to use results about random walk to get information about annihilation times of particles of $\{\zeta_t^p\}$.

As an application of Proposition 5.3, we shall conclude this section by proving that $\{\eta_t\}$ clusters, which is the content of Theorem 1(a). For $x, y \in \mathbb{Z}$, $x < y$ and $t \ge 0$, we have the equality of the events

$$\begin{aligned}\{\eta_t(x) &= \eta_t(x+1) = \cdots = \eta_t(y)\}\\ &= \{\zeta_t(s) = 0 \text{ for all } s = x, x+1, \dots, y-1\}\end{aligned}$$

because a color configuration interval can only display one color when there are no edge particles in the corresponding edge configuration interval. Thus, by proving the following lemma, we can apply it in the case $p = 1/3$ to conclude Theorem 1(a).

LEMMA 5.5. *Let $x, y \in \mathbb{Z}$ with $x \le y$. Then*

$$\lim_{t\to\infty} P^p\big(\zeta_t^p(s) = 0 \text{ for all } s = x, \dots, y\big) = 1. \tag{5.1}$$

PROOF. Let us observe, using the subadditivity and translation invariance of P^p, that

$$\begin{aligned}P^p&(\zeta_t^p(s) = 0 \text{ for all } s = x, \dots, y)\\ &= 1 - P^p(\zeta_t^p(s) \neq 0 \text{ for some } s = x, \dots, y)\\ &\ge 1 - (y - x + 1) P^p(\zeta_t^p(0) \neq 0)\end{aligned}$$

By symmetry between $\nearrow$ particles and $\swarrow$ particles, it suffices to show that

$$\lim_{t\to\infty} P^p(\zeta_t^p(0) = \nearrow) = 0$$

in order to conclude (5.1). But if $\zeta_t^p(0) = \nearrow$, then it must be the case that $\zeta_0^p(-t) = \nearrow$ and this $\nearrow$ particle lives to time t. Then translation invariance, Proposition 5.3, and the recurrence of the random walk Z_{0+} allow us to compute that

$$\lim_{t\to\infty} P^p(\zeta_t^p(0) = \nearrow) = \lim_{t\to\infty} P^p(\tau_{0+} > 2t + 1) = 0.$$

This proves the lemma. □

6. The computation of $D(\zeta_t^p)$. We shall prove Theorem 2 in this section, and hence be able to compute the mean cluster size of $\{\eta_t\}$. Let us recall the definition of the mean interparticle distance:

$$D(\zeta) = \lim_{n\to\infty} \frac{2n}{\text{number of particles } \zeta \text{ has in } [-n, n]}. \tag{6.1}$$

In order to analyze this quantity, we shall define indicator functions describing when particles from ζ_0^p live to time t. For each $x \in \mathbb{Z}$, let

$$\chi_{x,t}^{\nearrow} = \begin{cases} 1, & \text{if } \zeta_0^p(x) = \nearrow \text{ and this particle lives to time } t, \\ 0, & \text{otherwise;} \end{cases}$$

$$\chi_{x,t}^{\nwarrow} = \begin{cases} 1, & \text{if } \zeta_0^p(x) = \nwarrow \text{ and this particle lives to time } t, \\ 0, & \text{otherwise.} \end{cases}$$

Then the number of particles ζ_t^p has in $[-n, n]$ can be expressed as

$$\sum_{x=-n-t}^{n-t} \chi_{x,t}^{\nearrow} + \sum_{x=-n+t}^{n+t} \chi_{x,t}^{\nwarrow}. \tag{6.2}$$

Let us now observe that, for a fixed value of t, $\{\chi_{x,t}^{\nearrow}\}_{x\in\mathbb{Z}}$ and $\{\chi_{x,t}^{\nwarrow}\}_{x\in\mathbb{Z}}$ are sequences which are $2t$-independent ($\{\dots, \chi_{x-1,t}^{\nearrow}, \chi_{x,t}^{\nearrow}\}$ and $\{\chi_{y,t}^{\nearrow}, \chi_{y+1,t}^{\nearrow}, \dots\}$ are independent if $y - x > 2t$, and similarly for $\{\chi_{x,t}^{\nwarrow}\}$). This implies that each sequence is mixing and hence ergodic. These sequences are also stationary, so we may apply Birkhoff's ergodic theorem to conclude that

$$\lim_{n\to\infty} \frac{1}{2n}\left(\sum_{x=-n-t}^{n-t} \chi_{x,t}^{\nearrow} + \sum_{x=-n+t}^{n+t} \chi_{x,t}^{\nwarrow}\right) = E\left[\chi_{0,t}^{\nearrow} + \chi_{0,t}^{\nwarrow}\right], \qquad P^p\text{-a.s.} \tag{6.3}$$

The right-hand side is merely the probability that a particle is at the origin of the initial configuration and lives to time t. In light of Proposition 5.3, let us define

$$\begin{aligned} \rho_k^p &= P^p(\zeta_0^p(0) = \nearrow, \tau_{0+} > k) \\ &= P^p(Z_{0+}(1) > 0, Z_{0+}(2) > 0, \dots, Z_{0+}(k) > 0). \end{aligned}$$

Then (6.1), (6.2) and (6.3) can be combined to yield

$$D(\zeta_t^p) = \frac{1}{2\rho_{2t+1}^p}, \qquad P^p\text{-a.s.} \tag{6.4}$$

Thus, it is necessary to analyze ρ_k^p, which is done in the following lemma.

LEMMA 6.1. *For* $0 < p \le 1/2$, $\rho_k^p \sim \sqrt{p/\pi k}$.

PROOF. The proof uses a standard reflection principle argument. Let us use the notation $Z(\cdot)$ to denote $Z_{0+}(\cdot)$. We can write

$$\begin{aligned}
\rho_k^p &= \sum_{j=1}^{\infty} P^p(Z(1) > 0, \dots, Z(k-1) > 0, Z(k) = j) \\
&= p \sum_{j=1}^{\infty} P^p(Z(2) > 0, \dots, Z(k-1) > 0, Z(k) = j | Z(1) = 1) \\
&= p \sum_{j=1}^{\infty} P^p(Z(k-1) = j-1, Z(i) \neq -1 \text{ for } i = 1, \dots, k-2) \\
&= p \sum_{j=1}^{\infty} [P^p(Z(k-1) = j-1) - P^p(Z(k-1) = j-1, \\
&\qquad\qquad Z(i) = -1 \text{ for some } i = 1, \dots, k-2)].
\end{aligned}$$

By the reflection principle, we get

$$\rho_k^p = p \sum_{j=1}^{\infty} [P^p(Z(k-1) = j-1) - P^p(Z(k-1) = j+1)].$$

The sum on the right is actually a finite telescoping sum, so that

$$\rho_k^p = pP^p(Z(k-1) = 0) + pP^p(Z(k-1) = 1).$$

By symmetry,

$$\rho_k^p = pP^p(Z(k-1) = 0) + pP^p(Z(k-1) = -1).$$

We can then use the relation

$$\begin{aligned}
pP^p(Z(k-1) = 1) + (1-2p)P^p(Z(k-1) = 0) + pP^p(Z(k-1) = -1) \\
= P^p(Z(k) = 0)
\end{aligned}$$

to conclude that

$$\rho_k^p = \frac{4p-1}{2} P^p(Z(k-1) = 0) + \frac{1}{2} P^p(Z(k) = 0). \tag{6.5}$$

We now apply Proposition 7.9 of Spitzer [11] to compute asymptotics for the probability of a random walk being at the origin at a given time. In the present context, we can conclude that for $0 < p < 1/2$,

$$P^p(Z(k) = 0) \sim \frac{1}{2\sqrt{p\pi k}}$$

and in the case $p = 1/2$,

$$P^{1/2}(Z(k) = 0) = 0 \quad \text{if } k \text{ is odd;}$$

$$P^{1/2}(Z(k) = 0) \sim \sqrt{\frac{2}{\pi k}} \quad \text{if } k \text{ is even.}$$

Using these results in (6.5) gives us the conclusion of the lemma. □

Thus, applying Lemma 6.1 to (6.4), we can conclude Theorem 2, which can then be used to calculate the mean cluster size for $\{\eta_t\}$ in Theorem 1(b).

7. Numerical verification of Theorem 1(b). Theorem 1(b) was verified numerically by simulating the three-color cyclic cellular automaton with the aid of a Cellular Automaton Machine (CAM). A system of 65,536 sites with wrap-around boundary conditions was used for the simulation, and data were generated in order to estimate the asymptotic growth rate of the mean cluster size $C(\eta_t)$. Three hundred runs of the system were made, with each run lasting 15,000 time units. Every 50 time units, the number of edge particles in the imbedded edge cellular automaton was recorded. From these data, the average number of particles at each of the recording times was computed by averaging over the 300 runs. By dividing the average number of particles into the number of sites in the simulation (65,536), we computed the average of the cluster sizes $\overline{C}_t$ at each of the recording times t.

From Theorem 1(b), we know that the graph of $\ln C(\eta_t)$ versus $\ln t$ will approximate a line of slope $1/2$ at large times t. Figure 4 shows a graph of $\ln \overline{C}_t$ versus $\ln t$. We see that this graph does approximate a straight line. A least squares analysis to find the slope of the line best approximating these data gives a value of this slope as 0.502424, which is within 0.5% of theoretical value. Furthermore, we know that a graph of $C(\eta_t)$ versus $t^{1/2}$ will approximate a line of slope $(3\pi/2)^{1/2} \approx 2.171$. Figure 5 shows a graph of $\overline{C}_t$ versus $t^{1/2}$. This graph also approximates a straight line, and a least squares analysis gives that the slope of the line of best fit is 2.203980, which is within 1.5% of the theoretical value.

We would like to get empirical values of the power of time and the proportionality constant that approximate the theoretical values even better, but since Theorem 1(b) is an asymptotic result, better approximations can only be made by running the simulations for longer periods of time. Unfortunately, it is not clear that we may run a wrap-around system of 65,536 much longer than 15,000 time units without the finiteness of the simulation having a large effect on the results. To illustrate the problem, consider any site containing an $\varkappa$. Over the course of 15,000 time units, this $\varkappa$ can consider half of the ℓ particles to be moving away from it and the other half to be moving toward it, because it is impossible for this $\varkappa$ to collide with an ℓ more than 30,000 sites away to the right of it. However, over the course of 32,768 time units, this $\varkappa$ must consider all ℓ particles to be moving toward it because it and any ℓ can annihilate each other if all intervening particles annihilate one another. Al-

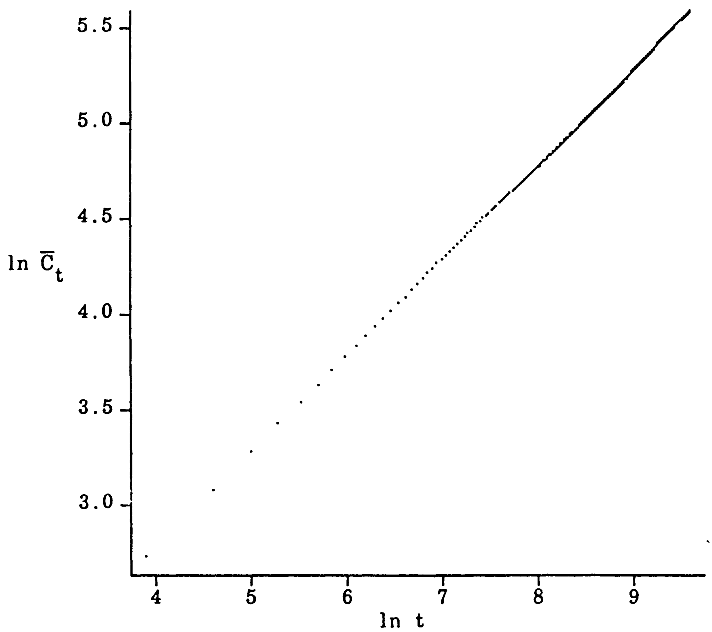


FIG. 4. *Graph of* $\ln \overline{C}_t$ *versus* $\ln t$ *for the three-color cyclic cellular automaton.*

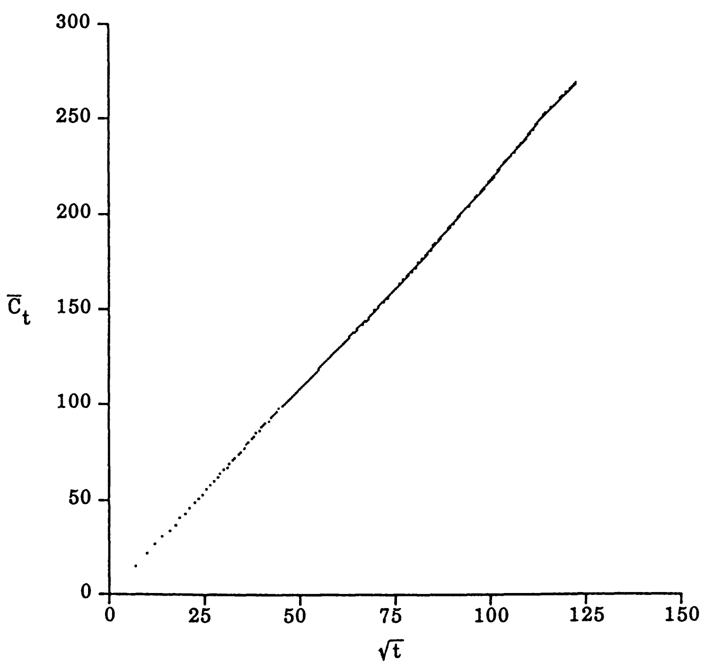


FIG. 5. *Graph of* $\overline{C}_t$ *versus* $\sqrt{t}$ *for the three-color cyclic cellular automaton.*

though it is not clear exactly how long the simulation may be run before the finiteness of the simulation has a great effect on the empirical estimates, it is clear that a simulation with many more sites is necessary in order to substantially increase the length of time over which the simulation is run in order to get better approximations to the asymptotic result. Unfortunately, running simulations on a single CAM limits our simulation size, and simulations run without the CAM are over 2000 times slower and are impractical to use for getting such statistics. Hence, for the time being, we are unable to obtain better empirical evidence of Theorem 1(b).

8. The four-color cyclic cellular automaton. The four-color cyclic cellular automaton, which we shall denote in this section by $\{\eta_t\}$, is defined in [7]. The cyclic hierarchy on the four colors specifies that color 1 can eat color 0, color 2 can eat color 1, color 3 can eat color 2, and color 0 can eat color 3. Furthermore, colors 0 and 2 form an inert pair, as do colors 1 and 3. The initial condition for $\{\eta_t\}$ is determined by product measure, where at any site $x \in \mathbb{Z}$,

$$P(\eta_0(x) = 0) = P(\eta_0(x) = 1) = P(\eta_0(x) = 2) = P(\eta_0(x) = 3) = \tfrac{1}{4}.$$

At each time t, each site looks at its two nearest neighbors. If it sees the color that can eat it, then it becomes that color at time $t + 1$; otherwise, it remains unchanged at time $t + 1$. Figure 6 illustrates $\{\eta_t\}$.

The imbedded edge cellular automaton $\{\zeta_t\}$ can be described in terms of edge particles along the same lines as in the case of three colors. But now, in addition to the rightward and leftward edge particles, r and ℓ, described in Section 2, we have a blockade particle b, which corresponds to an edge between two inert colors in the color configuration. Each site in an edge configuration is either vacant, denoted by *0*, or contains one of these three edge particles. The initial configuration for $\{\zeta_t\}$ is determined by product measure, where at any site $x \in \mathbb{Z}$,

$$P(\zeta_0(x) = 0) = P(\zeta_0(x) = r) = P(\zeta_0(x) = \ell) = P(\zeta_0(x) = b) = \tfrac{1}{4}.$$

To get the configuration at time $t + 1$ from the configuration at time t, each r particle moves one site to the right, each ℓ particle moves one site to the left and each b particle remains unmoved. If an r and an ℓ collide, they annihilate each other. But if an r or an ℓ collides with a b, the b is annihilated and the moving particle reverses direction. In other words, if an r collides with a b, the site where the collision occurs contains an ℓ, and if an ℓ collides with a b, then the site where the collision occurs contains an r.

So, the edge cellular automaton consists of particles undergoing deterministic motion among a forest of blockade particles. The moving particles bounce back and forth among the blockades, reversing direction and annihilating a blockade particle with every bounce. When two moving particles collide, they annihilate each other. These effects can be observed by looking at the color

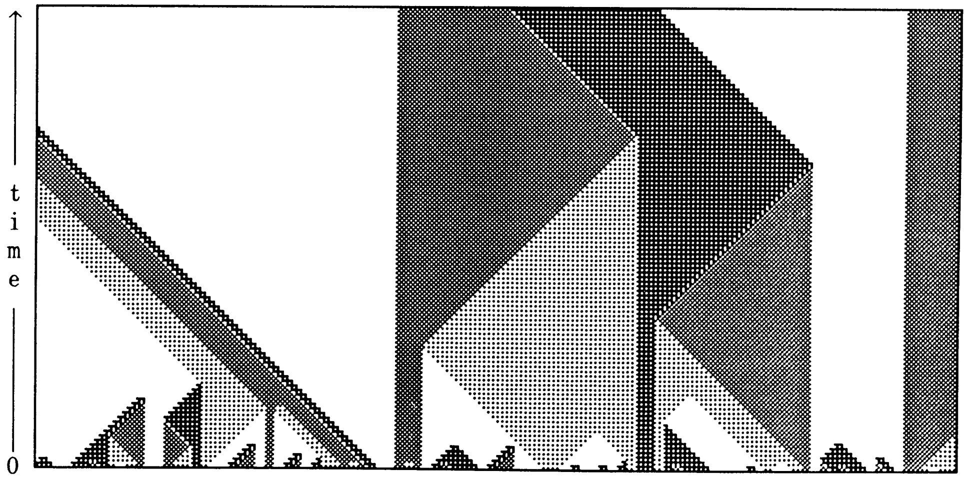


FIG. 6. *The one-dimensional four-color cyclic cellular automaton.*

interfaces of Figure 6. Any result about this edge cellular automaton will carry over to a corresponding statement about the one-dimensional four-color cyclic cellular automaton, and so it may be possible to analyze this cyclic cellular automaton by proving appropriate statements about this edge cellular automaton.

We know from [7] that $\{\eta_t\}$ fluctuates. Thus, it is natural to investigate the clustering properties of $\{\eta_t\}$. Unfortunately, the techniques used for the three-color system do not carry over to the four-color system. For the edge cellular automaton corresponding to the three-color cyclic cellular automaton, we were able to describe when a particle lives to time t in terms of the first return time of some random walk to zero. However, this sort of result is yet to be proved for the edge cellular automaton corresponding to the four-color cyclic cellular automaton. In the absence of this kind of result, some other approach is necessary to conclude that $\{\eta_t\}$ clusters.

The one-dimensional four-color cyclic cellular automaton has been simulated on a CAM. A system of 65,536 sites with wrap-around boundary conditions was used to help understand the behavior of $\{\eta_t\}$. Upon looking at these simulations, it was apparent that clustering does indeed occur for this system. Assuming the truth of this conclusion, a natural next step was to use the CAM to generate data in order to estimate the asymptotic growth rate of the mean cluster size $C(\eta_t)$. One hundred runs of the system were made, with each run lasting 90,000 time units. Every 300 time units, the number of edge particles in the imbedded edge cellular automaton, counting moving particles and blockade particles, was recorded. From these data, the average number of particles at each of the recording times was computed by averaging over the 100 runs. By dividing the average number of particles into the number of sites

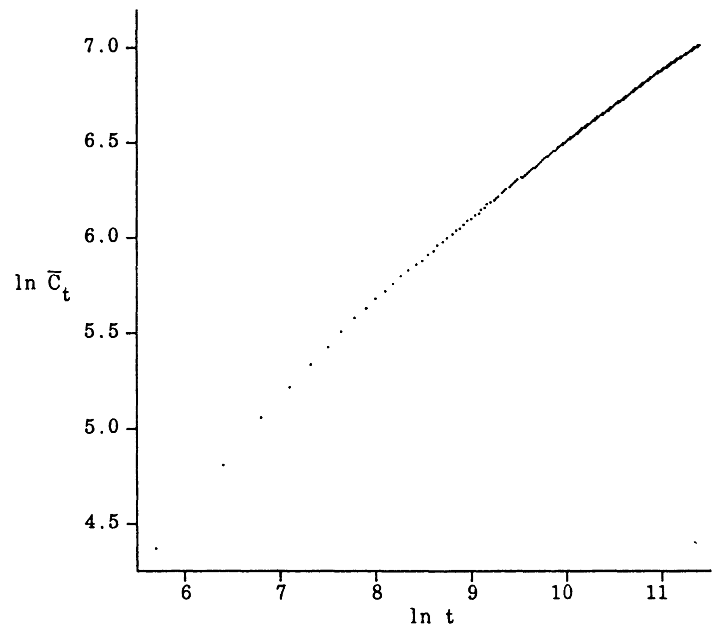


FIG. 7. *Graph of* $\ln \overline{C}_t$ *versus* $\ln t$ *for the four-color cyclic cellular automaton.*

in the simulation (65,536), we computed the average of the cluster sizes $\overline{C}_t$ at each of the recording times t.

If $C(\eta_t)$ were asymptotic to a function of the form ct^{α}, then a graph of $\ln C(\eta_t)$ versus $\ln t$ would approximate a line of slope α at large times t. Figure 7 shows a graph of $\ln \overline{C}_t$ versus $\ln t$. A least squares analysis to find the slope of the line best approximating these data gives a value of this slope as 0.407444. Repeating this least squares analysis on the latter half of the data (i.e., for times greater than 45,000), the slope is 0.346692. It seems from this analysis that the mean cluster size in the case of four colors does not grow like $t^{1/2}$, but may grow like some other power of t, perhaps $t^{1/3}$.

One may ask about the faithfulness of these simulations to the infinite system at large times such as $t = 90{,}000$ due to the fact that the simulations are run on a finite number of sites. After all, in the three-color simulations, it is impossible for r's and ℓ's to coexist after time 32,768, and so no collisions are possible at any later time. Clearly, any conclusions made about the infinite cellular automaton through observations of the three-color simulation at time 90,000 are dubious. The difference between the three-color simulations and the four-color simulations is that in the three-color simulations, some particles will have travelled great distances at large times, whereas in the four-color simulations, moving particles are bouncing back and forth and have no chance to wrap-around as long as blockade particles are still around. In all the four-color simulations observed, most of the particles remaining at time 90,000 are blockade particles. Thus, no moving particle has had a chance to be affected by the finiteness of these simulations. Furthermore, even simulations where all the moving particles are annihilated by time 90,000 are not "bad" simulations. This is because of the long period of time (compared to 90,000) required for a live particle that avoids annihilation to reach a site tens of thousands of sites away from its initial site due to the bouncing back and forth necessary to clear away all of the blockade particles in the way. Thus, Figure 7 should be useful in drawing conclusions about the infinite four-color cyclic cellular automaton. At the very least, Figure 7 is strong evidence for the conjecture that the growth rate of $C(\eta_t)$ is not proportional to $t^{1/2}$.

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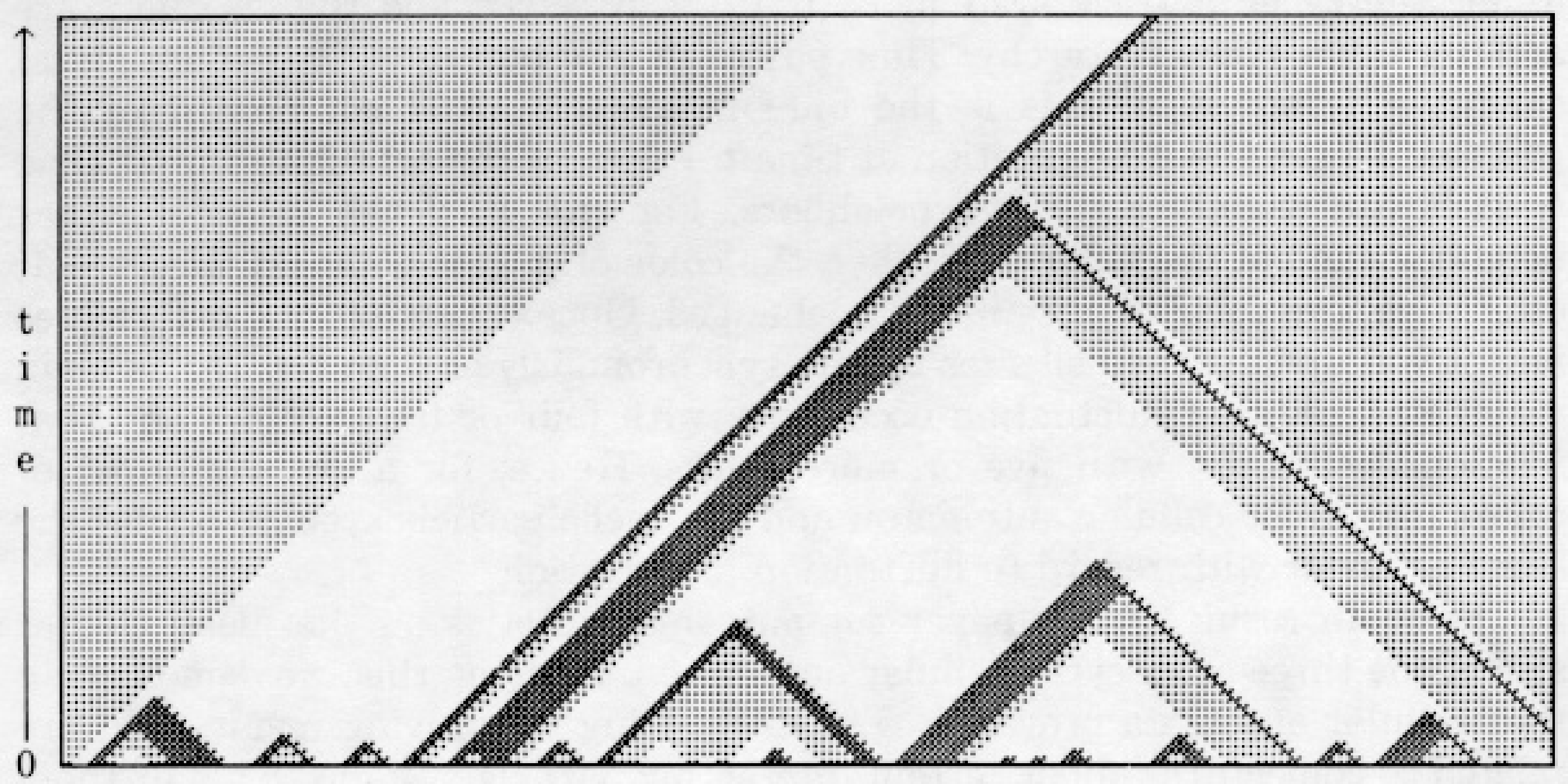
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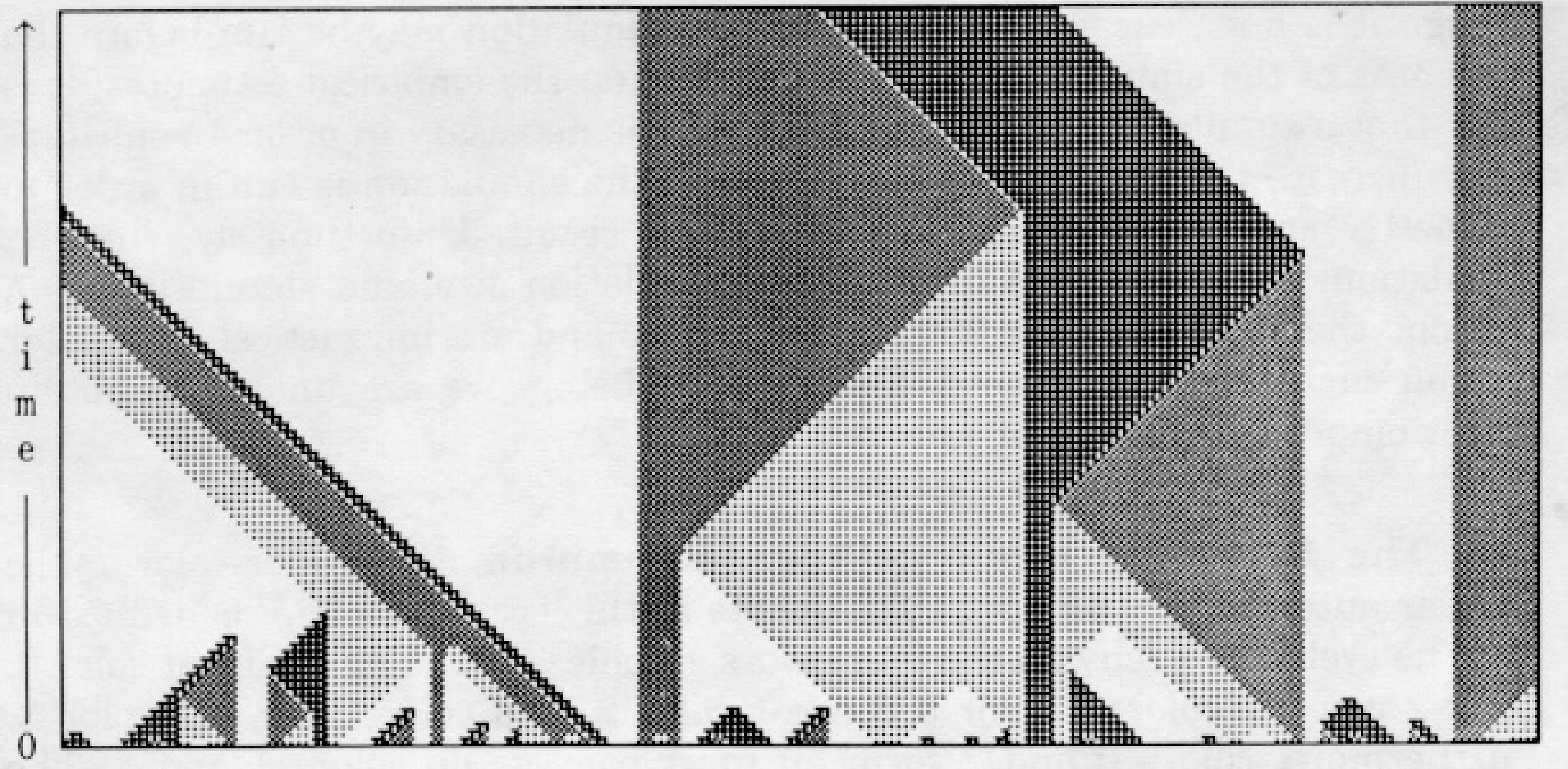
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