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SPECIAL INVITED PAPER

SAMPLE PATH PROPERTIES OF THE LOCAL TIMES OF STRONGLY SYMMETRIC MARKOV PROCESSES VIA GAUSSIAN PROCESSES[1]

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Necessary and sufficient conditions are obtained for the almost sure joint continuity of the local time of a strongly symmetric standard Markov process X. Necessary and sufficient conditions are also obtained for the almost sure global boundedness and unboundedness of the local time and for the almost sure continuity, boundedness and unboundedness of the local time in the neighborhood of a point in the state space. The conditions are given in terms of the 1-potential density of X. The proofs rely on an isomorphism theorem of Dynkin which relates the local times of Markov processes related to X to a mean zero Gaussian process with covariance equal to the 1-potential density of X. By showing the equivalence of sample path properties of Gaussian processes with the related local times, known necessary and sufficient conditions for various sample path properties of Gaussian processes are carried over to the local times. The results are used to obtain examples of local times with interesting sample path behavior.

1. Introduction. Let S be a locally compact metric space with a countable base and let $X = (\Omega, \mathscr{F}_t, X_t, P^x)$, $t \in R^+$, be a strongly symmetric standard Markov process with state space S. In saying that X is symmetric, we mean that there is a σ-finite measure $m(\cdot)$ on S such that the Markov transition function P_t satisfies

$$(P_t f, g) = (f, P_t g) \qquad \forall\, t \in R^+$$

for all measurable functions f and g in $L^2(S)$, where $(f, g) \equiv \int fg\, dm$ is the usual inner product. In saying that X is strongly symmetric, we mean that in addition to X being symmetric the measure $U^\alpha = U^\alpha(x, \cdot\,)$ given by

$$U^\alpha(x, \cdot\,) = \int_0^\infty e^{-\alpha t} P_t(x, \cdot\,)\, dt$$

is absolutely continuous with respect to m for some $\alpha > 0$ (and hence for all $\alpha > 0$). In this case, there is a canonical symmetric α-excessive density $u^\alpha =$

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$u^\alpha(x,y)$ for U^α. Moreover a strongly symmetric standard Markov process X has a symmetric transition density function $p_t(x,y)$ and

$$u^\alpha(x,y) = \int_0^\infty e^{-\alpha t} p_t(x,y)\,dt.$$

(A strongly symmetric Borel right process is standard.)

In this paper we study the sample path properties of the local time $L = \{L_t^y, (t,y) \in R^+ \times S\}$ of X. It is known that a necessary and sufficient condition for the existence of a local time for a strongly symmetric standard Markov process is that

$$u^\alpha(x,y) < \infty \qquad \forall\, x, y \in S.$$

This condition is assumed throughout this paper.

The local time is normalized by setting

$$E^x\left(\int_0^\infty e^{-\alpha t}\,dL_t^y\right) = u^\alpha(x,y). \tag{1.1}$$

[If (1.1) holds for any $\alpha > 0$, it holds for all $\alpha > 0$.] The function $u^\alpha(x,y)$ is positive definite on $S \times S$ for each $\alpha > 0$. Therefore, for each $\alpha > 0$, we can define a mean zero Gaussian process $\{G_\alpha(y),\ y \in S\}$ with covariance

$$E(G_\alpha(x)G_\alpha(y)) = u^\alpha(x,y) \qquad \forall\, x, y \in S.$$

The processes X and $\{G_\alpha(y),\ y \in S\}$, which we take to be independent, are related through the α-potential density $u^\alpha(x,y)$. We will refer to them as associated processes. To simplify the statement of our results, we will always consider X and the associated Gaussian process corresponding to $\alpha = 1$, that is, $\{G_1(y),\ y \in S\}$. In what follows we denote this process by $G = \{G(y),\ y \in S\}$.

In this paper we relate the sample path properties of the local time L of X to those of G. Our results make more precise some of the relationships between Gaussian processes and the local times of symmetric Markov processes put in evidence by an isomorphism theorem of Dynkin, which is at the foundation of all of our results. But, perhaps more significantly, since a great deal is known about Gaussian processes, we obtain concrete necessary and sufficient conditions for many important properties of the local times.

In what follows, whenever we say that the local time L_t^y of a Markov process X is continuous almost surely, we mean that we can choose the local time L_t^y in such a way that it is continuous almost surely. Also we employ the usual convention that the statement that a property of a Markov process holds almost surely means that it holds almost surely with respect to P^x for all $x \in S$.

We now present many results which show the equivalence of sample path properties of the local time L of X with those of the Gaussian process G associated with X.

THEOREM 1. *Let X be a strongly symmetric standard Markov process with finite 1-potential density u^1 and let $G = \{G(y),\ y \in S\}$ be the associated*

Gaussian process [*i.e.*, G *is a mean zero Gaussian process with covariance* $EG(x)G(y) = u^1(x,y)$]. *Let* $\{L_t^y, (t,y) \in R^+ \times S\}$ *be the local time of* X. *Then* $\{L_t^y, (t,y) \in R^+ \times S\}$ *is continuous almost surely if and only if* $\{G(y), y \in S\}$ *is continuous almost surely.*

Talagrand (1987) gives concrete necessary and sufficient conditions for the continuity almost surely of $\{G(y), y \in S\}$ in terms of the metric

$$d(x,y) = \left(E(G(x) - G(y))^2\right)^{1/2} \tag{1.2}$$
$$= \left(u^1(x,x) + u^1(y,y) - 2u^1(x,y)\right)^{1/2}.$$

Thus, we now have these conditions for the continuity almost surely of the local times of the associated Markov process. These conditions are given in Theorem 8.1.

THEOREM 2. *Let* X *be a strongly symmetric standard Markov process with continuous* 1-*potential density* u^1 *and let* $G = \{G(y), y \in S\}$ *be the associated Gaussian process. Let* $L = \{L_t^y, (t,y) \in R^+ \times S\}$ *be the local time of* X *and let* $K \subset S$ *be a compact set. Then* $\{L_t^y, (t,y) \in R^+ \times K\}$ *is continuous almost surely if and only if* $\{G(y), y \in K\}$ *is continuous almost surely.*

Clearly, Theorem 1 implies that if the Gaussian process is continuous on S, then L, the local time of the associated Markov process, has the property that $\{L_t^y, (t,y) \in R^+ \times K\}$ is continuous almost surely for all compact subsets K of S. However, we show that the relationship between G and L is a local one. Thus, for example, all we need to know is that $G(y)$ is continuous on K to determine that $\{L_t^y, (t,y) \in R^+ \times K\}$ is continuous and conversely. All the results that follow will emphasize the local nature of the relationship between the local time of a Markov process and the Gaussian process associated with the Markov process.

To our knowledge there is no general theory which allows us to assume that the local time of a Markov process has a separable version. In our proofs of the continuity of the local time we are able to construct such a version. However we are not able to do this, for example, for local times that are bounded but not necessarily continuous. For this reason, in what follows, we will often consider the local time process $\{L_t^y, (t,y) \in R^+ \times D\}$, where D is some countable subset of S. Since D is arbitrary, the results are still quite strong.

THEOREM 3. *Let* X *be a strongly symmetric standard Markov process with continuous* 1-*potential density* u^1 *and let* $G = \{G(y), y \in S\}$ *be the associated Gaussian process. Let* $\{L_t^y, (t,y) \in R^+ \times S\}$ *be the local time of* X *and let* $D \subset S$ *be countable. Then for any compact subset* K *of* S, $\{L_t^y, (t,y) \in [0,T] \times D \cap K\}$ *is bounded for all* $T < \infty$ *almost surely if and only if* $\{G(y), y \in D \cap K\}$ *is bounded almost surely.*

Let $\mathscr{T}$ be a topological space. We say that a function $f\colon \mathscr{T} \to R$ is unbounded at y_0 if

$$\limsup_{y \to y_0} |f(y)| = \infty \tag{1.3}$$

and that f has a bounded discontinuity at y_0 if

$$0 < \limsup_{y \to y_0} |f(y) - f(y_0)| < \infty. \tag{1.4}$$

In the next theorem we consider the behavior of the local time at a fixed point of S.

THEOREM 4. *Let X be a strongly symmetric standard Markov process with continuous* 1*-potential density u^1 and let $G = \{G(y),\ y \in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t, y) \in R^+ \times S\}$ be the local time of X, let $D \subset S$ be countable and $y_0 \in D$. Consider the process $L_t^y = \{L_t^y,\ (t, y) \in R^+ \times D\}$. Then, with the definitions of bounded and unbounded discontinuity given in* (1.3) *and* (1.4), *we have*:

(4.1) *L_t^y is continuous at y_0 for each $t > 0$, $P_{y_0}^y$ almost surely, if and only if $G(y)$ is continuous at y_0 almost surely*;

(4.2) *L_t^y has a bounded discontinuity at y_0 for each $t > 0$, P^{y_0} almost surely, if and only if $G(y)$ has a bounded discontinuity at y_0 almost surely*;

(4.3) *L_t^y is unbounded at y_0 for each $t > 0$, P^{y_0} almost surely, if and only if $G(y)$ is unbounded at y_0 almost surely*;

and for each $y_0 \in D$, precisely one of these three cases holds. Furthermore this theorem remains valid with the term "each t" replaced by "some t" in parts (4.1)–(4.3).

Necessary and sufficient conditions for continuity at a point and for boundedness of Gaussian processes are also given in Talagrand (1987) [or can be derived from the results of Talagrand (1987)]. These are given in Section 8.

It is well known that continuity, boundedness and unboundedness, both globally and locally, are probability 0 or 1 properties for Gaussian processes. Thus, by the above results, they are probability 0 or 1 properties for the local times of the associated Markov processes. However, a certain degree of care is necessary in expressing this phenomenon. For example, if we know that a Gaussian process is unbounded almost surely on some compact set $K \subset S$, then we know that there exists a point $y_0 \in K$ such that the process is unbounded almost surely at y_0. Roughly speaking, this implies that the local time of the associated Markov process will also be unbounded at y_0, but only if the Markov process hits y_0. Thus we can say that for L_t^y as given in Theorem 4, the events L_t^y is continuous at y_0 for each $t > 0$, L_t^y has a bounded discontinuity at y_0 for each $t > 0$ and L_t^y is unbounded at y_0 for each $t > 0$

each have P^{y_0} probability 0 or 1. Furthermore, this statement is also true with the term "each t" replaced by "some t."

To clarify some of the implications of the above results, we give the following three theorems which are immediate consequences of the above theorems.

THEOREM 5. *Let X be a strongly symmetric standard Markov process with continuous 1-potential density u^1 and local time $\{L_t^y,\ (t,y) \in R^+ \times S\}$. Let $K \subset S$ be a compact set. Then either $\{L_t^y,\ (t,y) \in R^+ \times K\}$ is continuous almost surely or else there exists an $x_0 \in K$ such that for any countable dense set $D \subset K$, with $x_0 \in D$, the event "$\{L_t^y,\ (t,y) \in R^+ \times D\}$ is continuous" has P^{x_0} measure zero.*

THEOREM 6. *Let X be a strongly symmetric standard Markov process with continuous 1-potential density u^1 and local time $\{L_t^y,\ (t,y) \in R^+ \times S\}$. Let $D \subset S$ be countable and let $K \subset S$ be a compact set. Then either $\{L_t^y,\ (t,y) \in [0,T] \times D \cap K\}$ is bounded for each $T < \infty$ almost surely or else there exists an $x_0 \in D \cap K$ such that the event "$\{L_t^y,\ (t,y) \in [0,T] \times D \cap K\}$ is bounded for some $T < \infty$" has P^{x_0} measure zero.*

THEOREM 7. *Let X be a strongly symmetric standard Markov process with continuous 1-potential density u^1 and local time $\{L_t^y,\ (t,y) \in R^+ \times S\}$. Let $D \subset S$ be countable and let $K \subset S$ be a compact set. Then:*

(7.1) *if $\{L_t^y, y \in K\}$ is continuous for some $t > 0$ almost surely, then $\{L_t^y, (t,y) \in R^+ \times K\}$ is continuous almost surely;*

(7.2) *if $\{L_t^y, y \in D \cap K\}$ is bounded for some $t > 0$ almost surely, then $\{L_t^y, (t,y) \in [0,T] \times D \cap K\}$ is bounded for each $T < \infty$ almost surely.*

The next theorem shows that continuity of the local time at each point in the state space almost surely implies that the local time is jointly continuous. (Many Markov processes, such as most Lévy processes, are continuous at each point of their parameter space almost surely, but are not continuous on any compact subset of their parameter space.)

THEOREM 8. *Let X be a strongly symmetric standard Markov process with continuous 1-potential density u^1 and local time $\{L_t^y,\ (t,y) \in R^+ \times S\}$. Let $K \subset S$ be compact and consider the local time process $L_t^y = \{L_t^y,\ (t,y) \in R^+ \times K\}$. Then L_t^y is continuous at y_0 for each $t > 0$ almost surely for all $y_0 \in K$ if and only if $\{L_t^y,\ (t,y) \in R^+ \times K\}$ is continuous almost surely. Furthermore, this theorem remains valid with the term "each t" replaced by "some t."*

Uniform and local moduli of continuity for Gaussian processes can be used to obtain similar properties for the local times of the associated Markov processes. We will state some of our results about this in Theorems 9–12, although we will defer their proofs until a later paper [Marcus and Rosen (1991)]. In these results we will assume that S is a locally compact metric space with respect to the metric d given in (1.2). [The fact that d is a metric and not a pseudometric is due to properties of $u^1(x, y)$ as we show in Lemma 3.6.] This, of course, is a natural choice, although it is not necessary in Theorems 1–8. Let $K \subset S$ be compact. Under very general conditions, whenever a Gaussian process $\{G(y),\ y \in K\}$ has continuous sample paths it also has both an exact uniform and an exact local modulus of continuity. To be more precise, we call $\omega(\delta)$ an exact uniform modulus of continuity for $\{G(y),\ y \in K\}$ if

$$\lim_{\delta \to 0} \sup_{\substack{d(x,y) \le \delta \\ x, y \in K}} \frac{|G(x) - G(y)|}{\omega(d(x,y))} = 1 \quad \text{a.s.} \tag{1.5}$$

We call $\rho(\delta)$ an exact local modulus of continuity for $\{G(y),\ y \in S\}$ at some fixed $y_0 \in S$ if

$$\lim_{\delta \to 0} \sup_{\substack{d(y,y_0) \le \delta \\ y \in S}} \frac{|G(y) - G(y_0)|}{\rho(d(y,y_0))} = 1 \quad \text{a.s.} \tag{1.6}$$

We use the expressions uniform and local moduli of continuity for functions $\omega(\delta)$ and $\rho(\delta)$ for which the equality signs in (1.5) and (1.6) are replaced by "less than or equal" signs. We will always assume, in our discussions of moduli of continuity, that $\{G(y),\ y \in K\}$ is continuous.

Our methods for studying moduli of continuity of local times enable us to only consider the Markov processes up to, but not including, their lifetime. We shall denote the lifetime of the strongly symmetric standard Markov process X by ζ.

THEOREM 9. *Let X be a strongly symmetric standard Markov process and let $G = \{G(y),\ y \in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t, y) \in R^+ \times S\}$ be the local time of X. Then if $\rho(\delta)$ is an exact local modulus of continuity for G at $y_0 \in S$,*

$$\lim_{\delta \to 0} \sup_{\substack{d(y,y_0) \le \delta \\ y \in S}} \frac{|L_t^y - L_t^{y_0}|}{\rho(d(y,y_0))} = \sqrt{2}\,(L_t^{y_0})^{1/2} \quad \textit{for almost all } t \in [0, \zeta) \ a.s. \tag{1.7}$$

We note that if $\rho(\delta)$ is simply a local modulus of continuity for $\{G(y),\ y \in S\}$, then the expression in (1.7) holds with the equality sign replaced by a "less than or equal" sign.

THEOREM 10. *Let X be a strongly symmetric standard Markov process and let $G = \{G(y),\ y \in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t, y) \in R^+ \times S\}$ be the local time of X and let $K \subset S$ be compact. Then if $\omega(\delta)$ is a* *uniform modulus of continuity for* $\{G(y),\ y \in K\}$,

$$\lim_{\delta\to 0} \sup_{\substack{d(x,y)\le\delta\\ x,y\in K}} \frac{|L_t^x - L_t^y|}{\omega(d(x,y))} \le \sqrt{2}\sup_{y\in K}\left(L_t^y\right)^{1/2} \tag{1.8}$$

for almost all $t \in [0,\zeta)$ *a.s.*

The next theorem shows that if $\omega(\delta)$ is an exact uniform modulus of continuity for G, then it is "best possible" in (1.8).

THEOREM 11. *Let X be a strongly symmetric standard Markov process and let $G = \{G(y),\ y \in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t,y) \in R^+ \times S\}$ be the local time of X and let $K \subset S$ be compact. Then if $\omega(\delta)$ is an exact uniform modulus of continuity for $\{G(y),\ y \in K\}$, there exists a $y_0 \in K$ such that*

$$\lim_{\delta\to 0} \sup_{\substack{d(x,y)\le\delta\\ x,y\in K}} \frac{|L_t^x - L_t^y|}{\omega(d(x,y))} \ge \sqrt{2}\left(L_t^{y_0}\right)^{1/2} \quad \textit{for almost all } t \in [0,\zeta) \textit{ a.s.} \tag{1.9}$$

We can improve on (1.9) when the associated Gaussian process and the state space S have sufficient regularity.

THEOREM 12. *Let X be a strongly symmetric standard Markov process and let $G = \{G(y),\ y \in S\}$ be the associated Gaussian process. Let $\{L_t^y,\ (t,y) \in R^+ \times S\}$ be the local time of X. Furthermore, let (S,d) be a locally homogeneous metric space, that is, any two points in S have isometric neighborhoods in the metric d, and let $K \subset S$ be a compact set which is the closure of its interior. Then if $\omega(\delta)$ is an exact uniform modulus of continuity for $\{G(y),\ y \in K\}$,*

$$\lim_{\delta\to 0} \sup_{\substack{d(x,y)\le\delta\\ x,y\in K}} \frac{|L_t^x - L_t^y|}{\omega(d(x,y))} = \sqrt{2}\sup_{y\in K}\left(L_y^t\right)^{1/2} \tag{1.10}$$

for almost all $t \in [0,\zeta)$ *a.s.*

The main tool in the proofs of these theorems is a version of an isomorphism theorem of Dynkin [(1983), (1984)]. This version relates the Gaussian process with covariance $u^1(x,y)$ to the local time of a Markov process which is closely related to the Markov process with 1-potential density $u^1(x,y)$. To be more specific, we develop two examples of the isomorphism in Section 4. In one we relate the Gaussian process to the local time of the associated Markov process killed at an independent exponential time. In the other, we relate the Gaussian process to certain h-transforms of the original Markov process. It seems reasonable from the form of the isomorphism that continuity of the

Gaussian process should imply continuity of the local time and we show that it does.

It is not immediately clear from the isomorphism theorem why irregularity of the Gaussian process implies irregularity of the associated local time. We show that it does by using results of Itô and Nisio (1968a) and Jain and Kallianpur (1972) for bounded discontinuities and Borell's inequality [Borell (1975)] for unboundedness. There are only certain kinds of discontinuities that a Gaussian process can have and their properties are understood very well. Several of these properties are used in our proofs. Our results on local and uniform moduli of continuity depend strongly on the fact that these are probability 0 or 1 properties for broad classes of Gaussian processes.

There is an extensive literature on the continuity of local times of Markov processes beginning with Trotter's (1958) celebrated result on the joint continuity of the local time of Brownian motion and the results of McKean (1962) and Ray (1963) which give the exact uniform modulus of continuity for the local times of Brownian motion. Boylan (1964) found a sufficient condition for the joint continuity of the local time for a wide class of Markov processes that inspired considerable efforts to obtain necessary and sufficient conditions. A variation of Boylan's result is given in Blumenthal and Getoor [(1968), V 3] following an approach of Meyer (1966). Further improvements are given in papers by Getoor and Kesten (1972) and Millar and Tran (1974). Barlow and Hawkes [Barlow (1985), Barlow and Hawkes (1985)] obtained a sufficient condition for the joint continuity of the local times of Lévy processes which Barlow showed in Barlow (1988) was also necessary. This result gives our Theorem 1 for symmetric Lévy processes. It motivated us to look for a proof of this relationship that was more directly connected to Gaussian processes. [Gaussian processes do not enter into the proofs of the results in the work of Barlow and Hawkes, although observations on the connections between Gaussian processes and the local times of Markov processes are given in Barlow (1988), Barlow and Hawkes (1985) and Hawkes (1985).] Barlow also obtains the exact uniform modulus of continuity for a wide class of Lévy processes in Barlow (1988).

We must point out that none of the references in the preceding paragraph impose the restriction that the Markov processes be symmetric. We are forced into this condition because in our work the 1-potential is taken to be the covariance function of a Gaussian process. On the other hand, the methods that we use enable us to give many necessary and sufficient conditions for properties of the local times of symmetric Markov processes and provide a new way of looking at the local times of Markov processes which we think will have other applications besides the ones given here.

This paper uses many results from the theory of Gaussian processes and Markov processes. In order to appeal to researchers in each of these specialties we have provided many details and introductory remarks. Section 2 is devoted to Gaussian processes. Statements are given of most of the main results that we use. Section 3 contains the definitions of the various Markov process terms that are used. We are particularly concerned with the consequences of our assumptions on the 1-potential density since this defines the associated Gaussian process. We show that the Markov processes that we are considering are Hunt processes. This is critical in Theorems 1–8 because it enables us to extend the continuity of the local time up to the lifetime of the process.

In Section 4 we give a proof of the isomorphism theorem of Dynkin. In Section 5 we obtain necessary conditions for the continuity and boundedness of the local times. The methods we use are from the theory of Gaussian processes. The local time enters only as a stochastic process that satisfies the conditions of the isomorphism theorem. In Section 6, in which we obtain sufficient conditions for continuity and boundedness of local times, the methods are strictly Markovian. Gaussian processes enter through the isomorphism theorem but the only property of them that we use is their assumed continuity or boundedness. At this point in the paper essentially all the results necessary to prove Theorems 1–8 have been obtained. In Section 7, which is brief, we go over the proof of each of these theorems.

In Section 8 we give Talagrand's necessary and sufficient conditions for continuity and boundedness of Gaussian processes as well as other one-sided conditions which may be easier to apply. Because of Theorems 1–8, these results are immediately applicable to local times. Lastly, in Section 9, we give examples of symmetric Markov chains with a single instantaneous state for which the local time has a bounded discontinuity. We also present a result, which we think is new for Gaussian processes, that follows from our results and Markov process considerations and make some comparisons between our work and some of the references cited above.

2. Gaussian processes. In this section we review some results about Gaussian processes which will be used in this paper. A real-valued stochastic process $\{G(z),\ z \in T\}$ is said to be a mean zero Gaussian process if $EG(z) = 0$ for all $z \in T$ and if for all $z_1, \dots, z_n \in T$ and real numbers $\alpha_1, \dots, \alpha_n$ and for all $n \geq 1$, $\sum_{i=1}^{n} \alpha_i G(z_i)$ is a real-valued normal random variable with mean zero and variance $E(\sum_{i=1}^{n} \alpha_i G(z_i))^2$. There is a large literature on Gaussian processes. Some more comprehensive treatments can be found in Adler (1991), Dudley (1973), Fernique. (1975), Jain and Marcus (1978) and Ledoux and Talagrand (1991). The last reference is recent and most comprehensive. It also contains an extensive, up-to-date bibliography. In this section we will state the results that will be used in this paper. We make no attempt to give these results in their greatest generality. To the contrary, we will try to make the statements as simple as possible consistent with the needs of this paper. The most important result from Gaussian processes used in this paper is a consequence of Borell's Brunn–Minkowski inequality in Gauss space [Borell (1975)]. It shows the symmetric nature of the extremes of a Gaussian process.

THEOREM 2.1. *Let $X = \{X(z),\ z \in T\}$ be a real-valued mean zero Gaussian process, where T is a finite set. Let a be the median of $\sup_{z \in T} X(z)$, that is,*

$$P\Big(\sup_{z\in T} X(z) \geq a\Big) = P\Big(\sup_{z\in T} X(z) \leq a\Big) = 1/2, \tag{2.1}$$

and let

$$\sigma = \sup_{z \in T} \left(EX^2(z) \right)^{1/2}.$$

Then, for $t \geq 0$, we have

$$P\left(\left\{ \sup_{z \in T} X(z) > a - \sigma t \right\} \cap \left\{ \sup_{z \in T} -X(z) > a - \sigma t \right\} \right) \geq 1 - 2\Phi(t) \tag{2.2}$$

and

$$P\left(\sup_{z \in T} X(z) < a + \sigma t \right) \geq 1 - \Phi(t), \tag{2.3}$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-u^2/2}\, du. \tag{2.4}$$

These inequalities are usually stated somewhat differently in the literature so we will give a proof of Theorem 2.1 starting from Borell's theorem [Borell (1975), Theorem 3.1] adapted to l_2^n (i.e., R^n with the Euclidean metric).

THEOREM 2.2 (Borell). *Let γ_n be the canonical Gaussian measure on R^n, that is, the measure induced by $\xi = (\xi_1, \dots, \xi_n)$, where $\{\xi_i\}_{i=1}^n$ are i.i.d. normal random variables with mean zero and variance* 1. *Let A be a measurable subset of R^n such that*

$$\gamma_n(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-\xi^T \xi/2}\, d\xi. \tag{2.5}$$

Then, for $t \geq 0$,

$$\gamma_n(A + tB) \geq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b+t} e^{-\xi^T \xi/2}\, d\xi, \tag{2.6}$$

where B is the unit ball in l_2^n.

PROOF OF THEOREM 2.1. Assume first that the covariance matrix, say R, of $\{X(z), z \in T\}$ is strictly positive definite. This implies that R is invertible and that we can write $R = P^T D P$, where P is orthogonal and D diagonal with positive entries. Let $n \equiv \operatorname{card}\{T\}$ and let ρ_n denote the measure induced by $\{X(z), z \in T\}$ on R^n. For $\mathscr{A} \subset R^n$, we have

$$\rho_n(\mathscr{A}) = \int \cdots \int_{\mathscr{A}} \left(\exp - \frac{x^T R^{-1} x}{2} \right) \frac{dx}{(2\pi)^{n/2} |R|^{1/2}}. \tag{2.7}$$

Under the change of variables $u = D^{-1/2} P x$, we get

$$\rho_n(\mathscr{A}) = \int \cdots \int_{D^{-1/2} P \mathscr{A}} \left(\exp - \frac{u^T u}{2} \right) \frac{du}{(2\pi)^{n/2}} \tag{2.8}$$

so that

$$\rho_n(\mathscr{A}) = \gamma_n(D^{-1/2}P\mathscr{A}). \tag{2.9}$$

Let $f\colon R^n \to R$ be given by $f(x) = \sup_k x_k$, where $x = (x_1, \dots, x_n)$. Let a be the number for which $\rho_n(\{f \le a\}) = 1/2$. We now set $\mathscr{A} = \{f \le a\}$. By (2.5) and (2.6) with $b = 0$, we have

$$\gamma_n(D^{-1/2}P\mathscr{A} + tB) \ge 1 - \Phi(t)$$

and, by (2.9),

$$\rho_n(\mathscr{A} + tP^TD^{1/2}B) \ge 1 - \Phi(t). \tag{2.10}$$

We will now show that

$$\mathscr{A} + tP^TD^{1/2}B \subset \{f \le a + \sigma t\}, \tag{2.11}$$

where

$$\sigma = \sup_{z \in T} \left(EX^2(z)\right)^{1/2} = \sup_k |(R)_{kk}|^{1/2}. \tag{2.12}$$

[For any matrix A, we use $(A)_{jk}$ to represent the j, kth element of A.] To see (2.11), suppose that $x = \alpha + tP^TD^{1/2}\beta$, where $\alpha \in \mathscr{A}$, $\beta \in B$; then

$$\sup_k x_k \le a + t \sup_k |(P^TD^{1/2}\beta)_k|. \tag{2.13}$$

Note that

$$(P^TD^{1/2}\beta)_k = \sum_{j=1}^n (P^TD^{1/2})_{kj}\beta_j$$

and also that

$$\begin{aligned}\sum_{j=1}^n (P^TD^{1/2})^2_{kj} &= \sum_{j=1}^n (P^TD^{1/2})_{kj}(P^TD^{1/2})_{kj} = \sum_{j=1}^n (P^TD^{1/2})_{kj}(P^TD^{1/2})^T_{jk} \\ &= \left((P^TD^{1/2})(P^TD^{1/2})^T\right)_{kk} = (P^TDP)_{kk}.\end{aligned}$$

Therefore, by the above, Schwarz's inequality and (2.12), we get

$$|(P^TD^{1/2}\beta)_k| \le \left(\sum_{j=1}^n (P^TD^{1/2})^2_{kj}\right)^{1/2} = |(R)_{kk}|^{1/2} \le \sigma. \tag{2.14}$$

We now see that (2.11) follows from (2.13) and (2.14). Thus (2.10) implies $\rho_n(\{f \le a + \sigma t\}) \ge 1 - \Phi(t)$ or, equivalently, that

$$P\left(\sup_k X_k \le a + \sigma t\right) \ge 1 - \Phi(t), \tag{2.15}$$

which gives us (2.3).

To obtain (2.2), consider the set $\mathscr{A}' = \{f \ge a\}$. Since $\rho_n(\mathscr{A}') = 1/2$, we have, as in (2.10), that

$$\rho_n(\mathscr{A}' + tP^TD^{1/2}B) \ge 1 - \Phi(t). \tag{2.16}$$

Also, we have

$$(\mathscr{A}' + tP^T D^{1/2} B) \subset \left\{ \sup_k x_k \geq a - \sigma t \right\}, \tag{2.17}$$

since if $\alpha \in \mathscr{A}'$ and $\beta \in B$ and $x = \alpha + tP^T D^{1/2}\beta$, then

$$\sup_k x_k \geq a - t \sup_k |(P^T D^{1/2}\beta)_k| \geq a - \sigma t$$

by (2.14). It follows from (2.16) and (2.17) that

$$P\left(\sup_{z \in T} X(z) > a - \sigma t \right) \geq 1 - \Phi(t). \tag{2.18}$$

Also, since $X(z)$ and $-X(z)$ are equal in distribution, we have

$$P\left(\sup_{z \in T} - X(z) > a - \sigma t \right) \geq 1 - \Phi(t). \tag{2.19}$$

The inequality in (2.2) follows from (2.18) and (2.19).

We now remove the condition that the covariance matrix of X is strictly positive definite. The Gaussian process $X = \{X(z), z \in T\}$ that we are considering can be thought of as a Gaussian vector in R^n given by $X = \{X(z_1), \dots, X(z_n)\}$. Let $\{g_i\}_{i=1}^n$ be a sequence of independent normal random variables with mean zero and variance 1 which is independent of X and let $\{b_k\}_{k=1}^\infty$ be a sequence of real numbers with $\lim_{k \to \infty} b_k = 0$. Let $X_k = \{X(z_1) + b_k g_1, \dots, X(z_n) + b_k g_n\}$. It is easy to check that the covariance matrix of X_k is strictly positive definite. Let $a_k =$ median of $\sup_{t \in T} X_k(t)$. Since the distribution of the maximum of a Gaussian vector in R^n is absolutely continuous with respect to Lebesgue measure, we see that $\lim_{k \to \infty} a_k = a$. Therefore, for $k \geq k_0$ for some k_0 sufficiently large, $a_k - \sigma t > 0$. We can now use (2.18) to get that

$$P\left(\sup_{z \in T} X_k(z) > a_k - \sigma t \right) \geq 1 - \Phi(t).$$

Since $\lim_{k \to \infty} X_k = X$ almost surely we get (2.18) for X. The same argument gives (2.19) for $-X$ and hence we get (2.2). In the same way, we can obtain (2.13) without the restriction that the covariance of X is strictly positive definite. This completes the proof of Theorem 2.1. □

REMARK 2.3. Under the same hypothesis as in Theorem 2.1, it follows from (2.15) and (2.18) that

$$P\left(\left| \sup_{z \in T} X(z) - a \right| < \sigma t \right) \geq 1 - 2\Phi(t) \quad \forall\, t \geq 0. \tag{2.20}$$

This kind of relationship is sometimes called a *concentration inequality* since it shows that $\sup_{z \in T} X(z)$ is concentrated at its median when a is large relative to σ.

It is customary to express the concentration inequality in a somewhat different form. Let $\{X(z), z \in S\}$ be a separable Gaussian process where S is an arbitrary index set. The median of $\sup_{z\in S}|X(z)|$ is well defined. It is the real number m satisfying both

$$(2.21)\qquad P\Big(\sup_{z\in S}|X(z)|\le m\Big)\ge \tfrac12 \quad\text{and}\quad P\Big(\sup_{z\in S}|X(z)|\ge m\Big)\ge\tfrac12 .$$

Another consequence of Theorem 2.2 is that

$$(2.22)\qquad P\bigg(\Big|\sup_{z\in S}|X(z)|-m\Big|<\sigma t\bigg)\ge 1-2\Phi(t)\qquad \forall\, t\ge 0$$

[see, e.g., Ledoux and Talagrand (1991), Lemma 3.1]. This form of the inequality does not give (2.18) which we need in (2.2). Nevertheless, the proofs in these different cases are similar. This was pointed out to us by M. Talagrand.

It is shown in Hoffmann-Jørgensen, Dudley and Shepp [(1979), Theorem 1.2] that the median m defined in the previous paragraph is unique. Also it follows immediately from (2.22) that

$$(2.23)\qquad \Big|E\sup_{z\in S}|X(z)|-m\Big|\le\sigma\sqrt{\frac{2}{\pi}}\,.$$

The inequality in (2.22) yields the following useful result. If $\{G(z),\ z\in T\}$, T an arbitrary index set, is a mean zero, real-valued Gaussian process such that $\sup_{z\in T}|G(z)|<\infty$ almost surely (this is actually a 0–1 event; see Theorem 2.7), then

$$(2.24)\qquad E\sup_{z\in T}|G(z)|^n<\infty\qquad\forall\, n\ge 0.$$

The following lemma will be used in Section 6.

LEMMA 2.4. *Let $G=\{G(z),\ z\in S\}$, (S,ρ) a separable metric space, be a real-valued, mean zero, Gaussian process. If G has continuous sample paths and K is a compact subset of S, then*

$$(2.25)\qquad \lim_{\delta\to0}E\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in K}}|G(y)-G(z)|^n=0\qquad\forall\, n\ge0.$$

If G is continuous at y_0, then

$$(2.26)\qquad \lim_{\delta\to0}E\sup_{\{y:\,\rho(y,y_0)\le\delta\}}|G(y)-G(y_0)|^n=0\qquad\forall\, n\ge0.$$

PROOF. The statement in (2.25) follows immediately from (2.24) and the dominated convergence theorem. To obtain (2.26) we note that if G is continuous at y_0, then it must be bounded in some neighborhood of y_0 with positive probability. Since boundedness is a 0–1 event (see Theorem 2.7), G is bounded almost surely in some neighborhood of y_0. Thus we get (2.26) as we did (2.25), by (2.24) and the dominated convergence theorem. □

A Gaussian process with continuous covariance can only have certain kinds of discontinuities. This was put in evidence by the work of Itô and Nisio (1968a) and Jain and Kallianpur (1972) on the oscillation function of a

Gaussian process. Let $G = \{G(z),\ z \in S\}$ be a mean zero Gaussian process, where $S = (S, \rho)$ is a separable metric space with metric ρ. We assume that the covariance of G is continuous on (S, ρ). Consider the function

$$W_G(z_0, \omega) = \lim_{\delta \to 0} \sup_{z, y \in B(z_0, \delta)} |G(z) - G(y)|, \tag{2.27}$$

where

$$B(z_0, \delta) = \{z \in S: \rho(z, z_0) < \delta\}. \tag{2.28}$$

We can always choose a separable version for G so that the supremum in (2.27) is taken over a countable separating set for G. Since this is the case we will always assume that G is a separable process and that limits, where indicated, are taken over countable separating sets.

THEOREM 2.5 (Jain and Kallianpur). *Let $\{G(z),\ z \in S\}$, (S, ρ) a separable metric space, be a mean zero Gaussian process with continuous covariance. Then there exists a real-valued upper semicontinuous function $\beta(z) = \beta_G(z)$, $z \in S$, which does not depend on ω, such that*

$$P(\{\omega | W_G(z, \omega) = \beta(z) \text{ for every } z \in S\}) = 1. \tag{2.29}$$

Furthermore, for every $z_0 \in S$,

$$P\left(\lim_{\delta \to 0} \sup_{z \in B(z_0, \delta)} G(z) = G(z_0) + \frac{\beta(z_0)}{2}, \right.$$
$$\left. \lim_{\delta \to 0} \inf_{z \in B(z_0, \delta)} G(z) = G(z_0) - \frac{\beta(z_0)}{2} \right) = 1. \tag{2.30}$$

We call the function $\beta(z)$ the *oscillation function* of the Gaussian process G at z. When $\beta(z_0) > 0$, (2.30) shows that the oscillations of $G(z)$ are symmetric about $G(z_0)$. This property will be used in Theorem 5.1.

Sample path properties of Gaussian processes that one may expect to satisfy 0–1 laws, generally do. This greatly simplifies the characterization of these properties. Theorem 2.5 implies many 0–1 laws for Gaussian processes. For example, if a Gaussian process satisfying the hypotheses of Theorem 2.5 is unbounded with positive probability at a point $z_0 \in S$, then $\beta(z_0) = \infty$ and hence the process is unbounded almost surely at z_0. Nevertheless, it is easier to understand and obtain 0–1 laws for Gaussian processes by considering the Karhunen–Loeve expansion for these processes. Indeed, this expansion is the starting point of the proof of Theorem 2.5. The next theorem is presented only in the degree of generality necessary for this paper. The theorem, as stated, combines the development of the orthogonal expansion in Jain and Marcus [(1974), III 3] and an important result of Itô and Nisio (1968b) [see also Jain and Marcus (1978), II 3.7].

THEOREM 2.6. *Let $G = \{G(z),\ z \in S\}$, (S, ρ) a separable metric space, be a mean zero Gaussian process with continuous covariance and let $\{\xi_j\}_{j=1}^{\infty}$ be an independent identically distributed sequence of normal random variables with mean zero and variance* 1. *Then there exists a sequence of continuous functions $\{\phi_j(z)\}_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty}\xi_j\phi_j(z)$ converges almost surely and*

$$G(z) = \sum_{j=1}^{\infty} \xi_j \phi_j(z), \qquad z \in S, \tag{2.31}$$

where equality means that the two processes G and $\sum_{j=1}^{\infty}\xi_j\phi_j(z)$ are equal in law. Moreover, when (S, ρ) is compact and G has a version with continuous sample paths, the series in (2.31) *converges uniformly almost surely and hence can be taken to be a concrete continuous version for G.*

Regardless of whether G has a continuous version, the series in (2.31) is a version for G. Since the sum of the first n terms of the series is a continuous Gaussian process, it is clear that properties of irregularity for Gaussian processes are tail events. We will list some of the 0–1 properties that we will use in this paper in the next theorem.

THEOREM 2.7. *Let $G = \{G(z),\ z \in S\}$, (S, ρ) a separable metric space, be a mean zero Gaussian process with continuous covariance. Let $z_0 \in S$ and T be a subset of S. The following events have probability* 0 *or* 1:

(2.32) *G is continuous at z_0.*

(2.33) *G has a bounded discontinuity at z_0.*

(2.34) *G is unbounded at z_0.*

(2.35) *G is continuous on T.*

(2.36) *G has a bounded discontinuity on T.*

(2.37) *G is unbounded on T.*

The following lemma will be used repeatedly in the proofs of the theorems of Section 1.

LEMMA 2.8. *Let $G = \{G(z),\ z \in K\}$, (K, ρ) a compact, separable metric space, be a mean zero Gaussian process with continuous covariance and oscillation function $\beta(z)$, $z \in K$, as defined in Theorem* 2.5. *Then*:

(2.38) *G is continuous on K almost surely if and only if $\beta(z) = 0$ for all $z \in K$,*

(2.39) *G is bounded on K almost surely if and only if $\beta(z) < \infty$ for all $z \in K$.*

PROOF. $\beta(z) = 0$ for all $z \in K$ if and only if G is continuous on K almost surely by the definition of the oscillation function. If $\beta(z_0) = \infty$ for some $z_0 \in K$, then by definition G is unbounded at z_0. Finally, suppose that G is unbounded on K with probability greater than zero. Then by (2.37), G is unbounded on K almost surely. Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a decreasing sequence of positive numbers satisfying $\lim_{j\to\infty}\varepsilon_j = 0$. Let $\{B(z_k, \varepsilon_j)\}_{k=1}^{N_j}$ be a cover of K. If

$$\sup_{z\in B(z_k,\varepsilon_j)} |G(z)| < \infty$$

on a set of positive measure, then, by Theorem 2.7, it is finite almost surely. Hence, if G is unbounded almost surely on K, there exists a $z_k \in K$ such that

$$\sup_{z\in B(z_k,\varepsilon_j)} |G(z)| = \infty \quad \text{a.s.} \tag{2.40}$$

Note that since the covariance of G is continuous, for each $z \in K$, $G(z)$ is just a mean zero random variable with finite variance. Thus (2.40) implies that

$$\sup_{y,z\in B(z_k,\varepsilon_j)} |G(y) - G(z)| = \infty \quad \text{a.s.}$$

Since K is compact there exists a subsequence $\{z_{k(i)}\}_{i=1}^{\infty}$ of $\{z_k\}_{k=1}^{\infty}$ such that $\lim_{i\to\infty} z_{k(i)} = z_0$ for some $z_0 \in K$. It is easy to see that

$$\sup_{y,z\in B(z_0,\varepsilon)} |G(y) - G(z)| = \infty \quad \text{a.s.} \qquad \forall\, \varepsilon > 0$$

and this implies that $W_G(z_0, \omega)$ defined in (2.27) is infinite almost surely and hence, by Theorem 2.5, $\beta(z_0) = \infty$. This completes the proof of Lemma 2.8. □

3. Markov processes. In this section we review some results on Markov processes which will be used in this paper. We also obtain some new properties of Markov processes which are used to relate Markov and Gaussian processes.

Let (S, ρ) be a locally compact metric space with a countable base and let $\mathscr{S}$ denote the Borel sets of S with respect to the topology induced by the metric ρ. We assume that there is a σ-finite measure $m(\cdot)$ on $(S, \mathscr{S})$. Let $b\mathscr{S}$ and $bp\mathscr{S}$ denote, respectively, the bounded and bounded positive measurable real-valued functions on $(S, \mathscr{S})$. Let $X = (\Omega, \mathscr{F}_t, X_t, P^x)$, $t \in R^+$, be a standard Markov process with state space S [Blumenthal and Getoor (1968), I 9.2]. Let θ_t denote the shift operator associated with X, ζ the lifetime of X, Δ the cemetery state for X and $\omega \in \Omega$ the elements of Ω. The transition semigroup P_t is a contraction on $bp\mathscr{S}$ which we denote by

$$P_t f(x) = P_t(x, f) = \int P_t(x, dy) f(y) = E^x(f(X_t)), \tag{3.1}$$

where E^x is the expectation operator associated with P^x. Define the semigroup $P_t^{\alpha} = e^{-\alpha t}P_t$. The resolvent or α-potential operator U^{α} is defined by

$$U^{\alpha}f(x) = U^{\alpha}(x, f) = \int_0^{\infty} e^{-\alpha t}P_t(x, f)\,dt = E^x\int_0^{\infty} e^{-\alpha t}f(X_t)\,dt. \tag{3.2}$$

Let I_A denote the indicator function of the set $A \in \mathscr{S}$. Then $U^{\alpha}(x, A) \equiv$

$U^\alpha(x, I_A)$ is a measure on $(S, \mathscr{S})$. We will sometimes refer to $U^\alpha(x, \cdot\,)$ as a measure on $(S, \mathscr{S})$.

In this paper we will assume that the Markov process X is strongly symmetric. The condition of symmetry means that the transition semigroup P_t satisfies

$$(P_t f, g) = (f, P_t g) \qquad \forall\, t \in R^+ \tag{3.3}$$

for all measurable functions f and g in $L^2(dm)$, where $(f, g) \equiv \int fg\, dm$ is the usual inner product and $\|f\|_2 \equiv (f, f)^{1/2}$. The condition of strong symmetry means that in addition to symmetry we have that for some $\alpha > 0$,

$$U^\alpha(x, \cdot\,) \text{ is absolutely continuous with respect to } m(\cdot) \qquad \forall\, x \in S. \tag{3.4}$$

THEOREM 3.1. *A strongly symmetric standard Markov process X has a unique set of transition probability densities $p_t(x,y)$ for all $x, y \in S$ and $t > 0$ such that*

$$(t, x, y) \to p_t(x,y) \quad \textit{is jointly measurable}, \tag{3.5}$$

$$p_t(x,y) = p_t(y,x), \tag{3.6}$$

$$p_{t+s}(x,y) = \int p_t(x,z) p_s(z,y)\, dm(z), \tag{3.7}$$

$$P_t f(x) = \int p_t(x,y) f(y)\, dm(y). \tag{3.8}$$

PROOF. This follows from (3.3), (3.4) [Fukushima (1980), Theorem 4.3.4] and Wittman (1986). □

It is clear from Theorem 3.1 that

$$u^\alpha(x,y) \equiv \int_0^\infty e^{-\alpha t} p_t(x,y)\, dt \tag{3.9}$$

is a symmetric density for $U^\alpha(x, \cdot\,)$ for all $\alpha > 0$.

Following convention we denote $u^0(x,y)$ by $u(x,y)$. In this paper we will be particularly concerned with $u(x,y)$ and $u^1(x,y)$. There is nothing special about $u^1(x,y)$ except that it is of the form $u^\alpha(x,y)$ for some $\alpha > 0$. Later we will identify the Markov process X with a mean zero Gaussian process. For specificity and to avoid confusion, we choose the Gaussian process with covariance function given by $u^1(x,y)$. Therefore, some of the results that follow are stated for $u^1(x,y)$ even though they are true for $u^\alpha(x,y)$ for $\alpha > 0$.

A continuous additive functional (CAF) of the Markov process X is a family of random variables satisfying:

$$\begin{aligned} &t \to A_t(\omega) \text{ is almost surely continuous and nondecreasing} \\ &\text{with } A_0(\omega) = 0 \text{ and } A_t(\omega) = A_\zeta(\omega) \text{ for all } t \geq \zeta, \end{aligned} \tag{3.10}$$

$$A_t \in \mathscr{F}_t, \tag{3.11}$$

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t \omega) \quad \text{for all } s, t \geq 0 \text{ a.s.} \tag{3.12}$$

For A_t a CAF we set

$$R_A(\omega) = \inf\{t | A_t(\omega) > 0\}.$$

A_t is called a local time at y if

$$(3.13) \qquad \begin{aligned} P^y(R_A = 0) &= 1, \\ P^x(R_A = 0) &= 0 \qquad \forall\, x \neq y. \end{aligned}$$

The reason that A_t is called a local time at y (for the Markov process X) is that the function $t \to A_t$ is the distribution function of a measure supported on the set $\{t | X_t = y\}$ [see, e.g., Blumenthal and Getoor (1968), V 3]. The next theorem gives necessary and sufficient conditions for the existence of a local time.

THEOREM 3.2. *Let X be a strongly symmetric standard Markov process. A necessary and sufficient condition for X to have a local time at y is that*

$$(3.14) \qquad u^\alpha(y,y) < \infty \quad \text{for some } \alpha \geq 0.$$

This theorem is proved using Fitzsimmons and Getoor [(1988), Proposition 4.15], Blumenthal and Getoor [(1968), V 3.13] and the fact that for strongly symmetric standard Markov processes semipolar sets are polar. [On this last point, see Blumenthal and Getoor (1968), VI 4.]

We will impose (3.14) for all $y \in S$, in all that follows.

Local times are defined up to a multiplicative constant [Blumenthal and Getoor (1968), V 3.13]. We can choose a version of the local time at y, which we will denote by L_t^y, by requiring that

$$(3.15) \qquad E^x\left(\int_0^\infty e^{-t}\, dL_t^y\right) = u^1(x,y) \qquad \forall\, x \in S$$

[see, e.g., Blumenthal and Getoor (1968), VI 4.18]. This equation uniquely determines $t \to L_t^y$, except possibly on a set of measure zero for each $y \in S$. We note that this implies that for all $0 < \alpha < \infty$,

$$(3.16) \qquad E^x\left(\int_0^\infty e^{-\alpha t}\, dL_t^y\right) = u^\alpha(x,y) \qquad \forall\, x \in S.$$

Furthermore, (3.16) holds for $\alpha = 0$ whenever (3.14) holds for $\alpha = 0$.

In the next theorem we develop the tie-in between Gaussian processes and Markov processes.

THEOREM 3.3. *Let X be a strongly symmetric standard Markov process satisfying* (3.14) *for all $y \in S$. Then its* 1*-potential density $u^1(x,y)$ is symmetric and positive definite and as a consequence satisfies*

$$(3.17) \qquad u^1(x,y) \leq \left(u^1(x,x)u^1(y,y)\right)^{1/2} \qquad \forall\, x,y \in S.$$

PROOF. The symmetry follows from (3.6) and the positive definiteness of $u^1(x,y)$ follows immediately from the fact that $p_t(x,y)$ is positive definite for all $t > 0$. To see this note that by (3.7), for all sequences of real numbers $\{a_i\}_{i=1}^n$ and all n, we have

$$\sum_{i,j=1}^{n} a_i a_j p_t(x_i, y_j) = \sum_{i,j=1}^{n} a_i a_j \int p_{t/2}(x_i, z) p_{t/2}(z, x_j)\, dm(z) \tag{3.18}$$
$$= \left\| \sum_{i=1}^{n} a_i p_{t/2}(x_i, \cdot) \right\|_2^2,$$

where $\|\cdot\|_2$ is the L^2 norm with respect to m. The statement in (3.17) is a consequence of $u^1(x,y)$ being positive definite. □

Theorem 3.3 enables us to make the following definition that associates Markov processes and their local times with Gaussian processes.

DEFINITION 3.4. Let X be a strongly symmetric standard Markov process with finite 1-potential density. Let $L = \{L_t^y, (t,y) \in R^+ \times S\}$ be the local time process of X and let $G = \{G(x), x \in S\}$ be a mean zero Gaussian process with covariance $u^1(x,y)$. These processes will be called associated processes. Whenever we say that a Markov process X is associated with a Gaussian process G, we mean, among other things, that X is a strongly symmetric standard Markov process with finite 1-potential density.

REMARK 3.5. It is clear from the above that a strongly symmetric standard Markov process with finite 1-potential density has an associated Gaussian process. However, not every Gaussian process can be associated with a strongly symmetric standard Markov process because a covariance function need not be a 1-potential density. For example, the Gaussian processes associated with real-valued symmetric Lévy processes are stationary and have spectral densities. [See, e.g., Getoor and Kesten (1972) and the material following our Theorem 8.3.] Furthermore, if a real-valued mean zero stationary Gaussian process is associated with a Markov process, then the Markov process must have independent homogeneous increments, that is, it must be a Lévy process. Thus, in particular, stationary Gaussian processes with discrete spectra cannot be associated with strongly symmetric Markov processes on the real line.

We really understand very little about what restrictions must be imposed upon a covariance for it to be the density of the potential of a symmetric Markov process. However, we do have the following results which will be useful in this paper.

LEMMA 3.6. *Let X be a strongly symmetric standard Markov process with finite 1-potential density $u^1(x,y)$. Then*

$$\int u^1(x,y)\, dm(y) \le 1 \qquad \forall\, x \in S, \tag{3.19}$$

$$u^1(x,y) \le u^1(x,x) \wedge u^1(y,y) \qquad \forall\, x,y \in S, \tag{3.20}$$

with strict inequality when $x \neq y$ and consequently

$$u^1(y,y) > 0 \qquad \forall\, y \in S. \tag{3.21}$$

PROOF. The integral relationship follows from (3.9). The inequality in (3.21) follows from (3.20) since $u^1(x,x) = 0$ implies that $u^1(x,y) = 0$ for all $y \in S$, which is impossible.

For the proof of (3.20) we define $T_{\{y\}}$ to be the first hitting time by the Markov process X of the point $y \in S$. Note that by the strong Markov property

$$\begin{aligned} u^1(x,y) &= E^x\left(\int_0^\infty e^{-s}\, dL_s^y\right) = E^x\left(\int_{T_{\{y\}}}^\infty e^{-s}\, dL_s^y\right) \\ &= E^x(e^{-T_{\{y\}}})E^y\left(\int_0^\infty e^{-s}\, dL_s^y\right) = E^x(e^{-T_{\{y\}}})u^1(y,y). \end{aligned} \tag{3.22}$$

Thus we obtain (3.20) since x and y are interchangeable. □

The next theorem shows that if a Markov process has a jointly continuous local time, then the associated 1-potential density must be continuous.

THEOREM 3.7. *Let X be a strongly symmetric standard Markov process as described in the beginning of this section and assume that the 1-potential density $u^1(x,y)$ is finite for all $x, y \in S$. Let $L_t^y = \{L_t^y, (t,y) \in R^+ \times S\}$ be the local time of X. If $\{L_t^y, y \in S\}$ is continuous for all $t \in R^+$ almost surely, then $\{u^1(x,y), (x,y) \in S \times S\}$ is continuous.*

This theorem will be proved at the end of Section 4.

A consequence of the continuity of the 1-potential density is that it implies that the Markov processes that we are considering are Hunt processes [see, e.g., Blumenthal and Getoor (1968), I 9.2 or Sharpe (1988), 47.3]. The ideas for the proof of this result, which is given next, are due to P. Fitzsimmons. The hypotheses on the Markov processes are weaker than those that we have been generally requiring. A standard Markov process X is called a Hunt process if $T_n \uparrow T$ implies that $X_{T_n} \to X_T$ almost surely on $\{T < \infty\}$, where the T_n and T are stopping times.

THEOREM 3.8. *Let X be a standard Markov process with reference measure m and with continuous α-potential density $u^\alpha(x,y)$. Then X is a Hunt process and*

$$\lim_{t \uparrow \zeta} X_t \equiv X_{\zeta -} \quad \textit{exists for all } \zeta < \infty. \tag{3.23}$$

PROOF. Our assumption that $u^\alpha(x,y)$ is continuous, together with Fatou's lemma, shows that

$$U^\alpha f(x) = \int u^\alpha(x,y) f(y)\, dm(y) \tag{3.24}$$

is lower semicontinuous for any $f \in b\mathscr{S}$. Therefore, by the resolvent equation [Blumenthal and Getoor (1968), I 8.10], we see that $U^{\beta}f$ is lower semicontinuous for any $\beta > 0$. It follows from Blumenthal and Getoor [(1970), (4.7) and (4.8)] that X is a special standard process. Therefore, by Sharpe [(1988), (47.6) (ii) and (iii)], we see that X is a Hunt process in the Ray topology. We now show that X is a Hunt process in the original topology on S.

Let ζ_i be the totally inaccessible part of ζ. It follows from Sharpe [(1988), (44.5) and (46.3)] that

$$X_{\zeta_i -} \quad \text{exists for } \zeta_i < \infty$$

and equals the corresponding limit in the Ray topology. (In fact, $X_{\zeta_i -} \in S$ since $X_\zeta = \Delta$). Now, by the above references, since X is special, it only remains to show that

$$X_{\zeta_p -} = \Delta \quad \text{for } \zeta_p < \infty, \tag{3.25}$$

since this again is the corresponding limit in the Ray topology. To see this, let us note that by (3.24),

$$h(x) = \int u^{\alpha}(x,y)\, dm(y)$$

is a strictly positive lower semicontinuous function. Therefore, it is bounded away from zero on compact subsets of S. Set

$$h(X_t)_- = \lim_{s \uparrow t} h(X_s).$$

In order to establish (3.25) we need only show

$$h\big(X_{\zeta_p}\big)_- = 0 \quad \text{for } \zeta_p < \infty. \tag{3.26}$$

Let us note that $h(x)$ is α-excessive. Therefore

$$h\big(X_{\zeta_p}\big)_- \quad \text{exists for } \zeta_p < \infty$$

and (3.26) follows from Blumenthal and Getoor [(1968), III 6.4]. This completes the proof of Theorem 3.8. □

In our proof of the joint continuity of the local time of a Markov process X, we need to consider the Markov process obtained by killing X at the first instant that it leaves a compact set and the Gaussian process associated with the killed process. We will develop the necessary machinery now. The main point that we establish is that if X can be associated with a continuous Gaussian process, then the killed process can also be associated with a continuous Gaussian process.

In what follows, we take X to be a strongly symmetric standard Markov process. Let $K \subset S$ be a compact set. Following Blumenthal and Getoor [(1968), II 1.2] we define

$$T_{K^c} = \inf\{t > 0 | X_t \notin K\}. \tag{3.27}$$

For all $f \in bp\mathscr{S}$ and $\alpha > 0$, we have Dynkin's formula

$$U^\alpha(x, f) = V^\alpha(x, f) + P^\alpha_{K^c} U^\alpha(x, f), \tag{3.28}$$

where

$$V^\alpha(x, f) = E^x \int_0^{T_{K^c}} e^{-\alpha t} f(X_t)\, dt \tag{3.29}$$

and

$$P^\alpha_{K^c}(x, f) = E^x\big(e^{-\alpha T_{K^c}} f(X_{T_{K^c}})\big). \tag{3.30}$$

By Theorem 3.1, $U^\alpha(x, \cdot\,)$ has a symmetric density $u^\alpha(x, y)$. It follows that $P^\alpha_{K^c} U^\alpha(x, \cdot\,)$ is also absolutely continuous with respect to m and that its density can be taken to be $P^\alpha_{K^c} u^\alpha(x, y)$, where the operator $P^\alpha_{K^c}$ acts on the function $u^\alpha(\cdot, y)$ as a function of its first variable.

We recall Hunt's switching formula,

$$P^\alpha_{K^c} u^\alpha(x, y) = P^\alpha_{K^c} u^\alpha(y, x), \tag{3.31}$$

which is contained in Blumenthal and Getoor [(1968), VI 1.16].

Setting

$$\tilde{v}^\alpha(x, y) = u^\alpha(x, y) - P^\alpha_{K^c} u^\alpha(x, y), \tag{3.32}$$

we see that $V^\alpha(x, \cdot\,)$ is also absolutely continuous with respect to m and has a symmetric density $\tilde{v}^\alpha(x, y)$.

The next two lemmas, which are critical in our proof of the continuity of local times, relate Markov processes and Gaussian processes. Lemma 3.9 is fundamental. Lemma 3.10 follows as a relatively simple consequence using elementary facts about Gaussian processes.

LEMMA 3.9. *Let X be a strongly symmetric standard Markov process with continuous 1-potential density $u^1(x, y)$. Let $\tilde{v}^1(x, y)$ and $P^1_{K^c} u^1(x, y)$ be as defined above. Then all three of these functions are finite, positive, symmetric, positive definite and continuous. Furthermore, $\tilde{v}^1(x, y) = 0$ if either x or y is contained in $\overline{K^c}$.*

LEMMA 3.10. *Let $u^1(x, y)$, $\tilde{v}^1(x, y)$ and $P^1_{K^c} u^1(x, y)$ be as defined above. If $u^1(x, y)$ is the covariance function of a continuous mean zero Gaussian process on S, then both $\tilde{v}^1(x, y)$ and $P^1_{K^c} u^1(x, y)$ are covariance functions of continuous mean zero Gaussian processes on S.*

We proceed to develop material needed to prove Lemma 3.9. Recall that a positive, finite real-valued function f on S is called α-excessive with respect to a standard Markov process X if

$$P^\alpha_t f(x) \le f(x) \qquad \forall\, x \in S \tag{3.33}$$

and

$$\lim_{t \to 0} P^\alpha_t f(x) \uparrow f(x) \qquad \forall\, x \in S, \tag{3.34}$$

where $P_t^\alpha = e^{-\alpha t}P_t$ for P_t the transition semigroup of X. This definition is valid for $0 \le \alpha < \infty$. A zero excessive function will simply be called an excessive function.

The α-potential density $u^\alpha(x,y)$ of a strongly symmetric standard Markov process given in (3.9) is obviously α-excessive in each variable, that is, $u^\alpha(\cdot,y)$ is an α-excessive function of the first variable for y fixed and $u^\alpha(x,\cdot)$ is an α-excessive function of the second variable for x fixed, since

$$P_t^\alpha u^\alpha(x,y) = \int_t^\infty e^{-\alpha s} p_s(x,y)\, ds.$$

By (3.30) and Fubini's theorem, we see that $P_{K^c}^\alpha u^\alpha(x,y)$ is an α-excessive function of the second variable. Furthermore, by the symmetry of $u^\alpha(x,y)$ and (3.31), $P_{K^c}^\alpha u^\alpha(x,y)$ is also an α-excessive function of the first variable. Thus both $u^\alpha(x,y)$ and $P_{K^c}^\alpha u^\alpha(x,y)$ are α-excessive in each variable. Let us note that $\tilde{v}^\alpha(x,y)$, defined in (3.32) is the difference of two functions which are α-excessive in each variable.

The next lemma will be used in the proof of Lemma 3.9.

Lemma 3.11. *Let $h_1(x,y)$ and $h_2(x,y)$ be positive finite functions on $S \times S$ which are α-excessive in each variable with respect to a strongly symmetric standard Markov process X. Let $h(x,y) = h_1(x,y) - h_2(x,y)$ also be positive and assume that*

$$(3.35) \qquad \int\int h(x,y) f(x) f(y)\, dm(x)\, dm(y) \ge 0 \qquad \forall f \in L^2(dm);$$

then $h(x,y)$ is a positive definite function on $S \times S$.

Note that, trivially, the constant 0 is an excessive function. Therefore Lemma 3.11 holds if $h(x,y)$ itself is excessive.

Proof. Assume that $h(x,y)$ satisfies (3.35). This implies that for any real-valued function $g(x)$, $x \in S$, which satisfies

$$(3.36) \qquad \int\int h(x,y)|g(x)||g(y)|\, dm(x)\, dm(y) < \infty,$$

we have

$$(3.37) \qquad \int\int h(x,y)g(x)g(y)\, dm(x)\, dm(y) \ge 0.$$

This is because we can find a sequence of functions $\{g_n(x)\}_{n=1}^\infty$ which are bounded and have compact support such that $|g_n(x)| \le |g(x)|$ for all $n \ge 1$ and $\lim_{n\to\infty} g_n(x) = g(x)$ almost surely with respect to the measure m. Since the functions $g_n(x) \in L^2(dm)$ we can use (3.35) to show that (3.37) holds with g_n replacing g for all $n \ge 1$. We can then get (3.37) by using (3.36) and the dominated convergence theorem.

Recall that, as we stated in Theorem 3.1, a strongly symmetric standard Markov process has a unique probability density $p_s(x,y)$, $\forall\ x,\ y \in S$ and $s > 0$. Consider the function

$$g(x) = \sum_{i=1}^{n} a_i e^{-\alpha s} p_s(z_i, x), \tag{3.38}$$

where $\{a_i\}_{i=1}^n$ are real numbers. Let $P^\alpha_{s,1}$ (resp., $P^\alpha_{s,2}$) denote P^α_s operating on the first variable (resp., second variable) of $h(\cdot,\cdot)$ so that

$$P^\alpha_{s,1}P^\alpha_{s,2}h(z_i,z_j) = \int\int e^{-2\alpha s}p_s(z_i,x)p_s(z_j,y)h(x,y)\,dm(x)\,dm(y) \qquad \forall\ i,j = 1,\dots,n. \tag{3.39}$$

Using this and (3.33) we have

$$\begin{aligned}
&\int\int h(x,y)|g(x)||g(y)|\,dm(x)\,dm(y)\\
&\quad\le \sum_{i,j=1}^{n} |a_i|a_j|P^\alpha_{s,1}P^\alpha_{s,2}h(z_i,z_j)|\\
&\quad\le \sum_{i,j=1}^{n} |a_i|a_j|P^\alpha_{s,1}P^\alpha_{s,2}h_1(z_i,z_j) - P^\alpha_{s,1}P^\alpha_{s,2}h_2(z_i,z_j)|\\
&\quad\le \sum_{i,j=1}^{n} |a_i|a_j|\big(P^\alpha_{s,1}P^\alpha_{s,2}h_1(z_i,z_j) + P^\alpha_{s,1}P^\alpha_{s,2}h_2(z_i,z_j)\big)\\
&\quad\le \sum_{i,j=1}^{n} |a_i|a_j|\big(h_1(z_i,z_j) + h_2(z_i,z_j)\big) < \infty.
\end{aligned} \tag{3.40}$$

Thus, (3.36) holds and using (3.39) in (3.37) we see that

$$\sum_{i,j=1}^{n} a_i a_j P^\alpha_{s,1}P^\alpha_{s,2}h(z_i,z_j) \ge 0. \tag{3.41}$$

Since $h_1(x,y)$ is α-excessive in each variable, so are $P^\alpha_{s,1}h_1(x,y)$ and $P^\alpha_{s,2}h_1(x,y)$. It follows that

$$P^\alpha_{s,1}P^\alpha_{t,2}h_1 \uparrow P^\alpha_{t,2}h_1 \quad \text{as } s \to 0$$

and

$$P^\alpha_{s,2}P^\alpha_{t,1}h_1 \uparrow P^\alpha_{t,1}h_1 \quad \text{as } s \to 0.$$

Thus for $0 < s \le t$, we have

$$P^\alpha_{s,1}P^\alpha_{s,2}h_1 \ge P^\alpha_{s,1}P^\alpha_{t,2}h_1,$$

which implies that

$$\lim_{s\to 0} P^\alpha_{s,1}P^\alpha_{s,2}h_1 \ge P^\alpha_{t,2}h_1$$

and, since this holds for all $t > 0$, we get

$$\lim_{s \to 0} P^{\alpha}_{s,1} P^{\alpha}_{s,2} h_1 \geq h_1. \tag{3.42}$$

However, since h_1 is excessive in each variable, we also have

$$P^{\alpha}_{s,1} P^{\alpha}_{s,2} h_1 \leq h_1. \tag{3.43}$$

Therefore, by (3.42) and (3.43) and the fact that the left-hand side of (3.43) is increasing as $s \to 0$, we get

$$P^{\alpha}_{s,1} P^{\alpha}_{s,2} h_1 \uparrow h_1 \quad \text{as } s \to 0. \tag{3.44}$$

Obviously (3.44) also holds for h_2. In particular, we get

$$\lim_{s \to 0} P^{\alpha}_{s,1} P^{\alpha}_{s,2} (h_1 - h_2) = h_1 - h_2. \tag{3.45}$$

Using (3.45) in (3.41) with $h = h_1 - h_2$, we get

$$\sum_{i,j=1}^{n} a_i a_j h(z_i, z_j) \geq 0$$

for all sequences $\{a_i\}_{i=1}^n$ of real numbers and $\{z_i\}_{i=1}^n$ of elements of S and for all $n \geq 1$. Hence $h(x,y)$ is positive definite. This completes the proof of Lemma 3.11. □

Proof of Lemma 3.9. That $u^1(x,y)$ is symmetric and positive definite is proved in Theorem 3.3. The positivity of $u^1(x,y)$ is obvious as can be seen in (3.9). Also, we point out in (3.31) and the paragraph containing (3.32) that $P^1_{K^c} u^1(x,y)$ and $\tilde{v}^1(x,y)$ are symmetric. It is obvious that $P^1_{K^c} u^1(x,y) \geq 0$ and since $u^1(x,y)$ is 1-excessive in each variable, we have by Blumenthal and Getoor [(1968), II 2.8] that

$$0 \leq P^1_{K^c} u^1(x,y) \leq u^1(x,y) < \infty. \tag{3.46}$$

Of course, $u^1(x,y) < \infty$ because it is assumed to be continuous.

The inequalities in (3.46) show that $P^1_{K^c} u^1(x,y)$ and $\tilde{v}^1(x,y)$ are positive and finite. It remains to show that they are positive definite.

Let

$$Q_t f(x) = E^x(f(X_t); t < T_{K^c}). \tag{3.47}$$

Using the strong Markov property it follows easily that Q_t is a semigroup of operators [Blumenthal and Getoor (1968), III 1] and V^α defined in (3.29) is its α-potential operator. Since V^α is symmetric on $L^2(dm)$, we have for $f, g \in L^2(dm)$,

$$\int_0^\infty e^{-\alpha t}(Q_t f, g)\, dt = \int_0^\infty e^{-\alpha t}(Q_t g, f)\, dt \tag{3.48}$$

for all $\alpha > 0$. Hence

$$(Q_t f, g) = (Q_t g, f) \quad \text{for almost all } t \in R^+. \tag{3.49}$$

If f and g are bounded continuous functions on S, then both $Q_t f(x)$ and

$Q_t g(x)$ are right continuous. Therefore (3.49) holds for all $t \in R^+$. Finally, using a similar argument to the one used in (3.18), we note that for all $f \in L^2(dm)$, we have

$$\begin{aligned} \int\int \tilde{v}^1(x,y) f(x) f(y) \, dm(x) \, dm(y) & \\ = (V^1 f, f) = \int_0^\infty e^{-t} (Q_t f, f) \, dt & \qquad (3.50) \\ = \int_0^\infty e^{-t} \|Q_{t/2} f\|_2^2 \, dt \geq 0. & \end{aligned}$$

It now follows from Lemma 3.11 that $\tilde{v}^1(x,y)$ is positive definite.

To show that $P^1_{K^c} u^1(x,y)$ is positive definite we note that since K^c is an open set, we have by Blumenthal and Getoor [(1968), I 11.9] and Sharpe [(1988), 12.12] that

$$P^1_{K^c} u^1(x,y) = P^1_{K^c} P^1_{K^c} u^1(x,y). \tag{3.51}$$

It follows by (3.31) that

$$\begin{aligned} P^1_{K^c} u^1(x,y) &= \int P^1_{K^c}(x,dz) P^1_{K^c} u^1(z,y) = \int P^1_{K^c}(x,dz) P^1_{K^c} u^1(y,z) \\ &= \int\int P^1_{K^c}(x,dz) P^1_{K^c}(y,dw) u^1(w,z). \end{aligned} \tag{3.52}$$

Let f be a bounded function on S with compact support and define the finite signed measure

$$\nu(\cdot) = \int f(x) P^1_{K^c}(x,\cdot) \, dm(x).$$

By (3.52),

$$\int\int P^1_{K^c} u^1(x,y) f(x) f(y) \, dm(x) \, dm(y) = \int\int u^1(w,z) \nu(dw) \nu(dz). \tag{3.53}$$

By (3.46) and the fact that $(U^1 f, f) < \infty$ for all $f \in L^2(dm)$, we see that if the left-hand side of (3.53) is greater than or equal to zero for all bounded functions f on S with compact support, then it is greater than or equal to zero for all $f \in L^2(dm)$. Therefore, by Lemma 3.11, in order to complete the proof that $P^1_{K^c} u^1(x,y)$ is positive definite we need only show that the right-hand side of (3.53) is greater than or equal to zero, and this follows from Fubini's theorem and the considerations of (3.18).

We now show that $\tilde{v}^1(x,y)$ and $P^1_{K^c} u^1(x,y)$ are continuous on $S \times S$. Since both these functions are positive definite, there exist mean zero Gaussian processes $G_v = \{G_v(x),\ x \in S\}$ and $G_P = \{G_P(x),\ x \in S\}$ with covariances $\tilde{v}^1(x,y)$ and $P^1_{K^c} u^1(x,y)$, respectively. We take G_v and G_P to be independent. It is easy to see, using (3.32), that

$$G_u(x) = G_v(x) + G_P(x), \qquad x \in S, \tag{3.54}$$

is a mean zero Gaussian process with covariance $u^1(x,y)$ and

$$E(G_u(x) - G_u(y))^2 = E(G_v(x) - G_v(y))^2 + E(G_P(x) - G_P(y))^2. \tag{3.55}$$

Since the continuity of the covariance of a Gaussian process is equivalent to the continuity of the process in L^2, it is clear, by (3.55), that the continuity of $u^1(x,y)$ implies the continuity of $\tilde{v}^1(x,y)$ and $P^1_{K^c}u^1(x,y)$.

Finally let us note that $V^1(x,A) = 0$ for $A \subset K^c$ and since $\tilde{v}^1(x,y)$ is a symmetric continuous density for $V^1(x,\cdot\,)$ it must be zero if either x or y is contained in $\overline{K^c}$. This completes the proof of Lemma 3.9. □

PROOF OF LEMMA 3.10. We showed in Lemma 3.9 that $\tilde{v}^1(x,y)$ and $P^1_{K^c}u^1(x,y)$ are positive definite on $S \times S$. Therefore, in the notation of the end of the proof of Lemma 3.9, we can find independent mean zero Gaussian processes G_v and G_P with covariances $\tilde{v}^1(x,y)$ and $P^1_{K^c}u^1(x,y)$. Let us define G_v and G_P on a single probability space and consider

$$G_1 = G_1(x) = G_v(x) + G_P(x), \qquad x \in S, \tag{3.56}$$

and

$$G_2 = G_2(x) = G_v(x) - G_P(x), \qquad x \in S.$$

It is obvious that the covariance of both G_1 and G_2 is equal to $u^1(x,y)$. Since, by hypothesis, $u^1(x,y)$ is the covariance of a mean zero continuous Gaussian process, we can find a continuous version $\overline{G}_1$ of G_1 and $\overline{G}_2$ of G_2. We now see that $\frac{1}{2}(\overline{G}_1 + \overline{G}_2)$ is a continuous mean zero Gaussian process with covariance $\tilde{v}^1(x,y)$, which is what we wanted to prove. A similar argument shows that $P^1_{K^c}u^1(x,y)$ is the covariance of a continuous Gaussian process. □

This lemma also follows immediately from Theorem 8.5 since by (3.55),

$$E(G_v(x) - G_v(y))^2 \le E(G_u(x) - G_u(y))^2$$

and similarly for G_P, where G_u is as defined in Lemma 3.9.

Let us note that whenever a local time is jointly continuous on $R^+ \times S$ it is a Radon–Nikodym derivative of an occupation measure, so that the two common (and not necessarily equivalent) ways of viewing local times are the same under this condition. The following result is essentially contained in the more general results of Blumenthal and Getoor [(1968), V 3.41] and Getoor and Kesten [(1972), Theorem 1].

THEOREM 3.12. *Let X be a strongly symmetric standard Markov process as defined in the beginning of this section. Assume that X has a jointly continuous local time $L = \{L^y_t, (t,y) \in R^+ \times S\}$. Then*

$$\int_A L^y_t \, dm(y) = \int_0^t 1_A(X_s)\, ds \tag{3.57}$$

for all $t \in R^+$ and $A \in \mathscr{S}$, almost surely with respect to P^x.

REMARK 3.13. Under the hypotheses of Theorem 3.12, m is a Radon measure. To see this note that since L is jointly continuous, we have by Theorem 3.7 that $\{u^1(x,y), (x,y) \in S \times S\}$ is continuous and by Lemma 3.6 that $u^1(y,y) > 0$ for all $y \in S$. Also

$$\int_S u^1(y,z)\, dm(z) = U^1(x,S) \leq 1.$$

Therefore, $m(B(y,\varepsilon)) < \infty$ for some $\varepsilon > 0$, where $B(y,\varepsilon)$ is the ball of radius ε centered at y in (S,ρ). Since this is true for each $y \in S$, we see that m is finite on each compact set in S, that is, that m is a Radon measure.

4. Isomorphism theorem. In this section we present a theorem due to Dynkin (1984) which provides a link between Gaussian processes and the local times of their associated strongly symmetric Markov processes (see Definition 3.4). This theorem, which we shall refer to as the isomorphism theorem, is a relationship between two independent families of random variables. One family is jointly Gaussian and the other satisfies (4.1) below.

THEOREM 4.1. *Let $l = \{l_i\}_{i=1}^\infty$ and $\mathbf{G} = \{G_i\}_{i=1}^\infty$ be R^∞-valued random variables and let G_α and G_β be real-valued random variables such that $\{\mathbf{G}, G_\alpha, G_\beta\}$ are jointly Gaussian with probability space (Ω_G, P_G) and expectation operator E_G. Let (Ω, Q) denote a probability or subprobability space of l and define $\langle G_i, G_j\rangle = E_G G_i G_j$. Assume that for any $i_1, \dots, i_n$, not necessarily distinct, we have*

$$Q\left(\prod_{j=1}^n l_{i_j}\right) = \sum_\pi \langle G_\alpha, G_{i_{\pi(1)}}\rangle \langle G_{i_{\pi(1)}}, G_{i_{\pi(2)}}\rangle \cdots \langle G_{i_{\pi(n)}}, G_\beta\rangle, \tag{4.1}$$

where π runs over all permutations of $\{1,2,\dots,n\}$. Then for all $\mathscr{C}$ measurable nonnegative functions F on R^∞, we have

$$QE_G\left(F\left(l + \frac{\mathbf{G}^2}{2}\right)\right) = E_G\left(F\left(\frac{\mathbf{G}^2}{2}\right)G_\alpha G_\beta\right), \tag{4.2}$$

where $\mathscr{C}$ denotes the σ-algebra generated by the cylinder sets of R^∞.

A proof of Theorem 4.1 is given in Dynkin (1984). We will give a more leisurely proof here which may be easier to follow because it does not make explicit mention of Feynman diagrams. However, before going on to the proof, we shall explore two examples in which (4.1) is satisfied for sequences l of values of the local times of certain Markov processes. In the first example, l may be thought of as a sequence of local times of the Markov process associated with $\mathbf{G}$ evaluated at an independent exponential time. In the second example, l is a sequence of local times of the Markov process associated with $\mathbf{G}$ considered as random variables on a probability space of a certain h-transform of the Markov process.

EXAMPLE 1. Let $X = (\Omega, \mathscr{F}_t, X_t, P^x)$, $t \in R^+$, be a standard Markov process with state space S as introduced in Section 3. Let θ_t and ζ also be as defined in Section 3. We define the measure

$$Q^x = P^x \times \mu \quad \text{on } \Omega \times R^+, \tag{4.3}$$

where the measure μ is defined by $d\mu(t) = e^{-t}\,dt$. For any $A \in \mathscr{S}$, we define the measure $Q^{x,A}(\cdot)$ by

$$Q^{x,A}(B) = Q^x\big(B \cap \{(\omega, s) | X_s(\omega) \in A\}\big)$$

for measurable sets B in $\Omega \times R^+$. For example, if $B = \{X_{t_1} \in A_1, \dots, X_{t_n} \in A_n,\ s \in I\}$, where I is some interval in R^+, then

$$Q^{x,A}(B) = \int_I e^{-s} P^x\big(\{X_{t_1} \in A_1, \dots, X_{t_n} \in A_n,\ X_s \in A\}\big)\,ds,$$

so, in particular, the total mass of $Q^{x,A}(\cdot)$ is

$$Q^{x,A}(\Omega \times R^+) = \int_0^\infty e^{-s} P^x\big(\{X_s(\omega) \in A\}\big)\,ds = U^1(x, A). \tag{4.4}$$

We note the following proposition for future reference.

PROPOSITION 4.2. *If $\{A_i\}_{i=1}^\infty$ is an increasing sequence of subsets of S with $\bigcup_{n=1}^\infty A_n = S$, then*

$$Q^{x,A_n}(B) \uparrow Q^x\big(B \cap \{(\omega, s) | s < \zeta(\omega)\}\big). \tag{4.5}$$

PROOF. This follows since

$$\begin{aligned} Q^{x,A_n}(B) \uparrow Q^{x,S}(B) &= Q^x\big(B \cap \{(\omega, s) | X_s(\omega) \in S\}\big) \\ &= Q^x\big(B \cap \{(\omega, s) | s < \zeta(\omega)\}\big). \end{aligned}$$ □

We now identify l in Theorem 4.1 with a certain sequence of local times of the Markov process X. Let $\{L_t^y, (t, y) \in R^+ \times S\}$ be the local time process of X and let $\{G_x, x \in S\}$ be a mean zero Gaussian process with covariance $u^1(x, y)$. Let $\lambda(\omega, s) = s$. We will show that for any sequence of points $y_1, y_2, \dots$ in S, the sequence of random variables $l_i = L_\lambda^{y_i}$ on $(\Omega \times R^+, Q^{x,A})$ satisfies (4.1) for $G_i = G_{y_i}$, $G_\alpha = G_x$ and $G_\beta = \int_A G_y\,dm(y)$. In order that $G_\beta = \int_A G_y\,dm(y)$ be well defined we require that

$$\int_A \int_A u^1(x, y)\,dm(x)\,dm(y) < \infty. \tag{4.6}$$

(Whenever we apply these results A will be compact and so, by Remark 3.13, (4.6) will be satisfied.)

We proceed to verify (4.1) for this example. Let 1_A denote the indicator function of the set A. We have

$$
\begin{aligned}
Q^{x,A}&\left(\prod_{j=1}^{n} L_\lambda^{y_{i_j}}\right)\\
&= \int_0^\infty e^{-s} E^x\left(\left(\prod_{j=1}^{n} L_s^{y_{i_j}}\right) 1_A(X_s)\right) ds\\
&= E^x\left(\int_0^\infty e^{-s} \prod_{j=1}^{n}\left(\int_0^s dL_r^{y_{i_j}}\right) 1_A(X_s)\, ds\right)\\
&= \sum_\pi E^x\left(\int_0^\infty e^{-s} 1_A(X_s) \int_{0<r_1<r_2\cdots<r_n<s}\cdots\int dL_{r_1}^{y_{i_{\pi(1)}}}\, dL_{r_2}^{y_{i_{\pi(2)}}}\cdots dL_{r_n}^{y_{i_{\pi(n)}}}\, ds\right)\\
&= \sum_\pi E^x\left(\int_0^\infty e^{-r_1}\, dL_{r_1}^{y_{i_{\pi(1)}}} \int_{r_1}^\infty e^{-(r_2-r_1)}\, dL_{r_2}^{y_{i_{\pi(2)}}}\cdots\right.\\
&\qquad \left.\int_{r_{n-1}}^\infty e^{-(r_n-r_{n-1})}\, dL_{r_n}^{y_{i_{\pi(n)}}} \int_{r_n}^\infty e^{-(s-r_n)} 1_A(X_s)\, ds\right).
\end{aligned} \tag{4.7}
$$

We now show by induction that

$$
\begin{aligned}
E^x&\left(\int_0^\infty e^{-r_1}\, dL_{r_1}^{y_1}\int_{r_1}^\infty e^{-(r_2-r_1)}\, dL_{r_2}^{y_2}\cdots\int_{r_n}^\infty e^{-(s-r_n)} 1_A(X_s)\, ds\right)\\
&= u^1(x,y_1)u^1(y_1,y_2)\cdots u^1(y_{n-1},y_n)\int_A u^1(y_n,z)\, dm(z)\\
&= \langle G_\alpha, G_{y_1}\rangle\langle G_{y_1}, G_{y_2}\rangle\cdots\langle G_{y_{n-1}}, G_{y_n}\rangle\langle G_{y_n}, G_\beta\rangle.
\end{aligned} \tag{4.8}
$$

For the first step in the proof by induction note that

$$
E^x\left(\int_0^\infty e^{-s} 1_A(X_s)\, ds\right) = \int_A u^1(x,z)\, dm(z). \tag{4.9}
$$

Let

$$
H = \int_0^\infty e^{-r_2}\, dL_{r_2}^{y_2}\int_{r_2}^\infty e^{-(r_3-r_2)}\, dL_{r_3}^{y_3}\cdots\int_{r_n}^\infty e^{-(s-r_n)} 1_A(X_s)\, ds \tag{4.10}
$$

and set

$$
H_{r_1} = \int_{r_1}^\infty e^{-(r_2-r_1)}\, dL_{r_2}^{y_2}\cdots\int_{r_n}^\infty e^{-(s-r_n)} 1_A(X_s)\, ds = H\circ\theta_{r_1}, \tag{4.11}
$$

where the final equality follows from (3.12). Continuing the proof by induction we assume that

$$
E^x(H) = u^1(x,y_2)u^1(y_2,y_3)\cdots\int_A u^1(y_n,z)\, dm(z). \tag{4.12}
$$

Since $E^x(H) < \infty$, $H_{r_1} \le e^{r_1}H$ and $L_{r_2}^{y_2}$ is continuous almost surely in r_2, we see that H_{r_1} is continuous almost surely. Also note that by (3.20), $E^x(H)$, considered as a function of x, is bounded. Consider

$$\begin{aligned} &E^x\left(\int_0^\infty e^{-r_1}\,dL_{r_1}^{y_1}\int_{r_1}^\infty e^{-(r_2-r_1)}\,dL_{r_2}^{y_2}\cdots\int_{r_n}^\infty e^{-(s-r_n)}1_A(X_s)\,ds\right)\\ &\qquad= E^x\left(\int_0^\infty e^{-r_1}H_{r_1}\,dL_{r_1}^{y_1}\right). \end{aligned}\tag{4.13}$$

Using the continuity of H_{r_1}, the fact that $e^{-r_1}H_{r_1}$ is decreasing and the monotone convergence theorem we see that

$$E^x\left(\int_0^\infty e^{-r_1}H_{r_1}\,dL_{r_1}^{y_1}\right) = \lim_{k\to\infty}\sum_{i=1}^\infty E^x\left(e^{-i/2^k}H\circ\theta_{i/2^k}\left(L_{i/2^k}^{y_1} - L_{(i-1)/2^k}^{y_1}\right)\right).\tag{4.14}$$

Taking the conditional expectation with respect to $\mathscr{F}_{i/2^k}$ and using the Markov property we see that the right-hand side of (4.14) equals

$$\lim_{k\to\infty}\sum_{i=1}^\infty e^{-i/2^k}E^x\left(E^{X_{i/2^k}}(H)\left(L_{i/2^k}^{y_1} - L_{(i-1)/2^k}^{y_1}\right)\right).\tag{4.15}$$

Note that by Blumenthal and Getoor [(1968), II 2.12],

$$E^{X_t}(H) = u^1(X_t,y_2)u^1(y_2,y_3)\cdots\int_A u^1(y_n,z)\,dm(z)\tag{4.16}$$

is right continuous as a function of t and, as we remarked above, it is bounded. Therefore, using the dominated convergence theorem, we see that (4.15) is equal to

$$\begin{aligned} E^x\left(\int_0^\infty e^{-r_1}E^{X_{r_1}}(H)\,dL_{r_1}^{y_1}\right) &= E^x\left(\int_0^\infty e^{-r_1}\,dL_{r_1}^{y_1}\right)E^{y_1}(H)\\ &= u^1(x,y_1)E^{y_1}(H), \end{aligned}\tag{4.17a}$$

since the measure $dL_{r_1}^{y_1}$ is supported on $\{r_1|X_{r_1}=y_1\}$. Thus by (4.14) and the proof by induction we obtain (4.8). Using (4.7) and (4.8), we obtain (4.1) for Example 1.

It follows from Theorem 4.1, which we will prove below, that

$$Q^{x,A}E_G F\left(L_\lambda^{\cdot} + \tfrac{1}{2}G_\cdot^2\right) = E_G\left(F\left(\tfrac{1}{2}G_\cdot^2\right)G_\alpha G_\beta\right).\tag{4.17b}$$

In Example 1 we are compelled, in order to satisfy (4.1), to work with $L_\lambda^{\cdot}$, where λ is an exponential random variable independent of the Markov process X. What interests us are results about $L_t^{\cdot}$ for $t\in R^+$. The next lemma is crucial in all our work because it allows us to pass from $L_\lambda^{\cdot}$ to $L_t^{\cdot}$. We continue with the notation developed above and also use ω' to denote the elements of Ω_G and τ to denote Lebesgue measure on R^+.

LEMMA 4.3. *Let $B \in \mathscr{E}$ be such that $P_G(G_\cdot^2/2 \in B) = 1$. Then for almost all $(\omega', t) \in \Omega_G \times R^+$ with respect to $P_G \times \tau$,*

$$P^x\left(L_t^{\cdot} + \frac{G_\cdot^2(\omega')}{2} \in B \text{ given that } t < \zeta\right) = 1 \tag{4.18}$$

and for almost all $\omega' \in \Omega_G$ with respect to P_G,

$$P^x\left(L_t^{\cdot} + \frac{G_\cdot^2(\omega')}{2} \in B \text{ for almost all } t \in [0, \zeta)\right) = 1. \tag{4.19}$$

Also, we can choose a countable dense set $Q \subset R^+$ such that for almost all $\omega' \in \Omega_G$ with respect to P_G,

$$P^x\left(L_t^{\cdot} + \frac{G_\cdot^2(\omega')}{2} \in B \text{ for all } t \in Q \cap [0, \zeta)\right) = 1. \tag{4.20}$$

PROOF. Let 1_{B^c} be the indicator function of the set

$$\left\{\omega' \middle| \frac{G_\cdot^2(\omega')}{2} \in B^c\right\}.$$

By our assumption, $1_{B^c} = 0$ almost surely with respect to P_G. Therefore,

$$E_G(1_{B^c} G_\alpha G_\beta) = 0$$

and consequently, by Theorem 4.1 [see also (4.17b)],

$$P_G Q^{x,A}\left(L_t^{\cdot}(\omega) + \frac{G_\cdot^2(\omega')}{2} \in B^c\right) = 0.$$

It now follows from Proposition 4.2 and (4.3) that

$$P_G P^x \mu\left(\left\{L_t^{\cdot}(\omega) + \frac{G_\cdot^2(\omega')}{2} \in B^c\right\} \cap \{t < \zeta(\omega)\}\right) = 0. \tag{4.21}$$

Therefore, for almost all $(\omega', t) \in \Omega_G \times R^+$ with respect to $P_G \times \tau$,

$$P^x\left(\left\{L_t^{\cdot}(\omega) + \frac{G_\cdot^2(\omega')}{2} \in B^c\right\} \cap \{t < \zeta(\omega)\}\right) = 0.$$

That is,

$$P^x\left(\left\{L_t^{\cdot}(\omega) + \frac{G_\cdot^2(\omega')}{2} \in B\right\} \cup \{t \geq \zeta(\omega)\}\right) = 1,$$

which is (4.18).

Using (4.21) again we see that, for almost all $\omega' \in \Omega_G$ with respect to P_G, we have

$$P^x \mu\left(\left\{L_t^{\cdot}(\omega) + \frac{G_\cdot^2(\omega')}{2} \in B^c\right\} \cap \{t < \zeta(\omega)\}\right) = 0$$

and therefore, for almost all ω with respect to P^x, we have

$$\mu\left(\left\{L_t^{\cdot}(\omega) + \frac{G_\cdot^2(\omega')}{2} \in B^c\right\} \cap \{t < \zeta(\omega)\}\right) = 0.$$

That is, for almost all $t \geq 0$, we have either that $L_t^{\cdot}(\omega) + G_\cdot^2(\omega')/2 \in B$ or $t \geq \zeta(\omega)$. This is equivalent to saying that $L_t^{\cdot}(\omega) + G_\cdot^2(\omega')/2 \in B$ for almost all $t \in [0, \zeta(\omega))$, which gives (4.19). Finally, we note that (4.20) follows immediately from (4.18). □

EXAMPLE 2. Let h be an excessive function. [See, e.g., the paragraph containing (3.33).] Although the following construction can be very general, for our purposes we will assume that h is continuous, bounded and strictly greater than zero. For $f \in b\mathscr{S}$, we define

$$P_t^{(h)} f(x) = \frac{1}{h(x)} P_t(fh)(x). \tag{4.22}$$

It is easy to see that $P_t^{(h)}$ is a semigroup. By Sharpe [(1988), Theorem 62.19] there exists a unique Markov process $(\Omega, \mathscr{F}_t, X_t, P^{x/h})$, called the h-transform of X, with transition operators $P_t^{(h)}$, for which

$$P^{x/h}\big(F(\omega)1_{\{t<\zeta(\omega)\}}\big) = \frac{1}{h(x)} P^x\big(F(\omega)h(X_t(\omega))\big) \tag{4.23}$$

for all $F \in b\mathscr{F}_t$. Note that (4.23) implies that P^x and $P^{x/h}$ have the same null sets.

In this example we assume that X has finite 0-potential density, that is, that

$$u(y,y) = u^0(y,y) < \infty \qquad \forall\, y \in S.$$

Then, by (3.16), the local times L_t^y of X satisfy

$$E^x(L_\infty^y) = u(x,y). \tag{4.24}$$

Let $\{G_y,\ y \in S\}$ be a mean zero Gaussian process with covariance $u(x,y)$. We now show that for any sequence of points $y_1, y_2 \dots$ in S, the sequence of random variables $l_i = L_\infty^{y_i}$ on $(\Omega, P^{x/h})$ with

$$h(x) = \int_A u(x,y)\, dm(y) \tag{4.25}$$

satisfies (4.1), where $G_i = G_{y_i}$, $G_\alpha = G_x$ and

$$G_\beta = \frac{1}{h(x)} \int_A G_y\, dm(y) \tag{4.26}$$

for some $A \in \mathscr{S}$. We assume that

$$\int_A \int_A u(x,y)\, dm(x)\, dm(y) < \infty, \tag{4.27}$$

so that G_β is well defined. Note that h is an excessive function with respect to X because $u(\cdot, y)$ is.

The proof of (4.1) mimics that of Example 1. Let $E^{x/h}$ denote the expectation operator of $(\Omega, \mathscr{F}_t, X_t, P^{x/h})$. Following (4.7), we have

$$
\begin{aligned}
E^{x/h}\left(\prod_{j=1}^{n} L_\infty^{y_{i_j}}\right) &= E^{x/h}\left(\prod_{j=1}^{n}\int_0^\infty dL_{r_j}^{y_{i_j}}\right)\\
&= \sum_\pi E^{x/h}\left(\int_{0\le r_1<r_2\cdots<r_n<\infty}\cdots\int dL_{r_1}^{y_{i_{\pi(1)}}}\cdots dL_{r_n}^{y_{i_{\pi(n)}}}\right) \qquad (4.28)\\
&= \sum_\pi E^{x/h}\left(\int_0^\infty dL_{r_1}^{y_{i_{\pi(1)}}}\int_{r_1}^\infty dL_{r_2}^{y_{i_{\pi(2)}}}\cdots\int_{r_{n-1}}^\infty dL_{r_n}^{y_{i_{\pi(n)}}}\right).
\end{aligned}
$$

For $n \ge 1$, set

$$
H_n = H_n(y_1,\dots,y_n) = \int_0^\infty dL_{r_1}^{y_1}\int_{r_1}^\infty dL_{r_2}^{y_2}\cdots\int_{r_{n-1}}^\infty dL_{r_n}^{y_n}
$$

and set $H_0 = 1$. We will use a proof by induction to show that

$$
E^{x/h}(H_n) = \langle G_\alpha, G_{y_1}\rangle\langle G_{y_1}, G_{y_2}\rangle\cdots\langle G_{y_{n-1}}, G_{y_n}\rangle\langle G_{y_n}, G_\beta\rangle. \qquad (4.29)
$$

Using this in (4.28) shows that (4.1) is satisfied by this example. To begin the proof by induction we see that

$$
\begin{aligned}
E^{x/h}(H_1) = E^{x/h}(L_\infty^{y_1}) &= \lim_{k\to\infty}\sum_{i=1}^\infty E^{x/h}\left(\left(L_{i/k}^{y_1} - L_{(i-1)/k}^{y_1}\right)1_{\{i/k<\zeta\}}\right)\\
&= \lim_{k\to\infty}\sum_{i=1}^\infty \frac{1}{h(x)}E^x\left(h(X_{i/k})\left(L_{i/k}^{y_1} - L_{(i-1)/k}^{y_1}\right)\right) \qquad (4.30)\\
&= \frac{1}{h(x)}E^x\left(\int_0^\infty h(X_r)\,dL_r^{y_1}\right),
\end{aligned}
$$

since by Blumenthal and Getoor [(1968), II 2.12], $h(X_r)$ is bounded and right continuous as a function of r. By the argument used in (4.17) we see that (4.30) is equal to

$$
\frac{1}{h(x)}E^x(L_\infty^{y_1})h(y_1) = \frac{1}{h(x)}u(x,y_1)h(y_1) = \langle G_\alpha, G_{y_1}\rangle\langle G_{y_1}, G_\beta\rangle. \qquad (4.31)
$$

Let us now assume that

$$
E^{x/h}(H_{n-1}) = \frac{1}{h(x)}u(x,y_1)u(y_1,y_2)\cdots u(y_{n-2},y_{n-1})h(y_{n-1})
$$

and let $H_{n-1,2} = H_n(y_2,\dots,y_n)$. Then

$$
E^{y_1/h}(H_{n-1,2}) = \frac{1}{h(y_1)}u(y_1,y_2)\cdots u(y_{n-1},y_n)h(y_n). \qquad (4.32)
$$

As in Example 1 we see that $H_{n-1,2}\circ\theta_{r_1}\uparrow H_{n-1,2}$ as $r_1\to 0$ and also that

$H_{n-1,2} \circ \theta_{r_1}$ is continuous almost surely. Therefore, by the same argument used from (4.14) onwards in Example 1, we have

$$
\begin{aligned}
E^{x/h}(H_n) &= E^{x/h}\left(\int_0^\infty H_{n-1,2} \circ \theta_{r_1}\, dL_{r_1}^{y_1}\right) \\
&= \lim_{k \to \infty} \sum_{i=1}^{\infty} \left(E^{x/h}\left(H_{n-1,2} \circ \theta_{i/2^k}\left(L_{i/2^k}^{y_1} - L_{(i-1)/2^k}^{y_1}\right)\right)\right) \\
&= \lim_{k \to \infty} \sum_{i=1}^{\infty} E^{x/h}\left(E^{(X_{i/2^k})/h}(H_{n-1,2})\left(L_{i/2^k}^{y_1} - L_{(i-1)/2^k}^{y_1}\right)\right) \\
&= E^{x/h}\left(\int_0^\infty E^{X_{r_1}/h}(H_{n-1,2})\, dL_{r_1}^{y_1}\right) = E^{x/h}(L_\infty^{y_1}) E^{y_1/h}(H_{n-1,2}).
\end{aligned} \tag{4.33}
$$

Using (4.31) and (4.32), we get (4.29). This completes the demonstration for this example.

REMARK 4.4. The approach of Example 2 also applies to the choice $h(x) = u(x, y)$ for fixed y. This case is highlighted in Dynkin (1984) and is used in Sheppard (1985) and Adler, Marcus and Zinn (1990). [In Adler, Marcus and Zinn (1990), the basic process is a Markov process killed at an independent exponential time so that $u(x, y)$, in this case, is $u^1(x, y)$ of the original process.]

PROOF OF THEOREM 4.1. We first show that

$$
E_G Q\left(\prod_{i=1}^{n}\left(l_{x_i} + \frac{G_{x_i}^2}{2}\right)\right) = E_G\left(\left(\prod_{i=1}^{n} \frac{G_{x_i}^2}{2}\right) G_\alpha G_\beta\right) \tag{4.34}
$$

for any $x_1, \dots, x_n \in \mathbb{Z}^+$, not necessarily distinct. To do this, we first recall the following well-known lemma which we prove for the convenience of the reader.

LEMMA 4.5. *Let $\{g_i\}_{i=1}^k$ be a jointly Gaussian sequence (i.e., a Gaussian process on R^k) with mean zero and let k be even. Then*

$$
E\left(\prod_{i=1}^{k} g_i\right) = \sum_{D_1 \cup \cdots \cup D_{k/2} = \{1, \dots, k\}} \prod_{i=1}^{k/2} \operatorname{cov}(D_i), \tag{4.35}
$$

where the sum is over all pairings $(D_1, \dots, D_{k/2})$ of $\{1, \dots, k\}$, that is, over all partitions of $\{1, \dots, k\}$ into disjoint sets each containing two elements and where we define

$$
\operatorname{cov}(\{i, j\}) = \operatorname{cov}(g_i, g_j) = E(g_i g_j).
$$

PROOF. We use the relationship

$$
E \exp\left(\sum_{i=1}^{k} \lambda_i g_i\right) = \exp\left(\tfrac{1}{2} E\left(\sum_{i=1}^{k} \lambda_i g_i\right)^2\right).
$$

Clearly,

$$\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_k}E\exp\left(\sum_{i=1}^{k}\lambda_i g_i\right)\Bigg|_{\lambda_1=\cdots=\lambda_k=0}=E\left(\prod_{i=1}^{k}g_i\right).$$

Also,

$$\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_k}\exp\left(\frac{1}{2}E\left(\sum_{i=1}^{k}\lambda_i g_i\right)^2\right)$$
$$=\sum_{n=0}^{\infty}\frac{1}{2^n n!}\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_k}\left(\sum_{i=1}^{k}\sum_{j=1}^{k}\lambda_i\lambda_j\operatorname{cov}(g_i,g_j)\right)^n.$$

It is easy to see that

$$\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_k}\left(\sum_{i=1}^{k}\sum_{j=1}^{k}\lambda_i\lambda_j\operatorname{cov}(g_i,g_j)\right)^n\Bigg|_{\lambda_1=\cdots=\lambda_k=0}\tag{4.36}$$

is zero when $n\neq k/2$ and when $n=k/2$ it is not zero only for those terms in $(\sum_{i=1}^{k}\sum_{j=1}^{k}\lambda_i\lambda_j\operatorname{cov}(g_i,g_j))^{k/2}$ in which each λ_i, $i=1,\dots,k$, appear only once. The terms with this property form a pairing, say $(D_1,\dots,D_{k/2})$ of $\{1,\dots,k\}$. For this pairing (4.36) is equal to $\prod_{i=1}^{k/2}\operatorname{cov}(D_i)$. Finally, it is easy to see that there are $2^{k/2}(k/2)!$ terms corresponding to each pairing of $\{1,\dots,k\}$. This establishes (4.35). □

Using Lemma 4.5 we see that

$$E_G\left(\prod_{i=1}^{n}\left(\frac{G_{u_i}G_{v_i}}{2}\right)G_\alpha G_\beta\right)=\frac{1}{2^n}\sum_{\mathscr{D}=(D_1,\dots,D_{n+1})}\operatorname{cov}(D_1)\cdots\operatorname{cov}(D_{n+1}),\tag{4.37}$$

where the sum is over all pairings $\mathscr{D}=(D_1,\dots,D_{n+1})$ of the $2n+2$ indices $\{u_i\}_{i=1}^n$, $\{v_i\}_{i=1}^n$, α and β. For example, $\operatorname{cov}(\{u_i,v_j\})=\langle G_{u_i},G_{v_j}\rangle$ and $\operatorname{cov}(\{u_i,\beta\})=\langle G_{u_i},G_\beta\rangle$. We now rewrite the right-hand side of (4.37). The reader should keep in mind that, eventually, we will set $u_i=v_i=x_i$ and obtain the left-hand side of (4.34).

Let $D^{(1)}$ be the unique element of $\mathscr{D}$ which contains α. If α is paired with either u_i or v_i, set $\pi(1)=i$ and define $\{y_{\pi(1)},z_{\pi(1)}\}$ to be $\{u_i,v_i\}$ if α is paired with u_i, but $\{v_i,u_i\}$ if α is paired with v_i. Next let $D^{(2)}$ be the unique element of $\mathscr{D}$ which contains $z_{\pi(1)}$. If $z_{\pi(1)}$ is paired with either u_j or v_j, set $\pi(2)=j$ and define $\{y_{\pi(2)},z_{\pi(2)}\}$ to be $\{u_j,v_j\}$ if $z_{\pi(1)}$ is paired with u_j, or $\{v_j,u_j\}$ if $z_{\pi(1)}$ is paired with v_j. We proceed in this manner, getting $D^{(1)},\dots,D^{(l)}$, until we get to $D^{(l+1)}$, the unique element of $\mathscr{D}$ which contains $z_{\pi(l)}$ and β. Clearly $l\le n$ but it is important to note that $l<n$ is often the case.

Let $C=C(\mathscr{D})=\{\pi(1),\dots,\pi(l)\}$ and $\mathscr{D}'=(D^{(1)},\dots,D^{(l+1)})$. Clearly $\mathscr{D}'$ is a pairing of the $2l+2$ elements $\{u_i\}_{i\in C(\mathscr{D})}$, $\{v_i\}_{i\in C(\mathscr{D})}$, α and β. Let $B=B(\mathscr{D})=\{1,\dots,n\}/C(\mathscr{D})$ and $D''=\mathscr{D}/\mathscr{D}'$. Clearly D'' is a pairing of the set of $2(n-l)$ indices consisting of $\{u_i\}_{i\in B(\mathscr{D})}$ and $\{v_i\}_{i\in B(\mathscr{D})}$. We see that we can rewrite (4.37) as

$$E_G\left(\prod_{i=1}^{n}\left(\frac{G_{u_i}G_{v_i}}{2}\right)G_\alpha G_\beta\right)$$

$$(4.38)\qquad = \frac{1}{2^n}\sum_{B\cup C=\{1,\dots,n\}}\left(\sum_{\substack{\text{pairings }(B_1,\dots,B_{|B|})\text{ of}\\ \{u_i\}_{i\in B}\cup\{v_i\}_{i\in B}}}\operatorname{cov}(B_1)\cdots\operatorname{cov}(B_{|B|})\right)$$
$$\times\left(\sum\langle G_\alpha, G_{y_{\pi(1)}}\rangle\langle G_{z_{\pi(1)}}, G_{y_{\pi(2)}}\rangle\cdots\langle G_{z_{\pi(|C|)}}, G_\beta\rangle\right),$$

where the last sum is over all permutations $(\pi(1),\dots,\pi(|C|))$ of C and over all ways of assigning $\{u_{\pi(i)}, v_{\pi(i)}\}$ to $\{y_{\pi(i)}, z_{\pi(i)}\}$. Of course, there are $2^{|C|}$ ways to make these assignments. Thus if we set $u_i = v_i = x_i$, the last sum in (4.38) is

$$(4.39)\qquad 2^{|C|}\sum_{\pi(C)}\langle G_\alpha, G_{x_{\pi(1)}}\rangle\langle G_{x_{\pi(1)}}, G_{x_{\pi(2)}}\rangle,\dots,\langle G_{x_{\pi(|C|)}}, G_\beta\rangle,$$

where now the sum is over all permutations π of C. But using (4.35) again we see that

$$(4.40)\qquad \left(\sum_{\substack{\text{pairings }(B_1,\dots,B_{|B|})\text{ of}\\ \{u_i\}_{i\in B}\cup\{v_i\}_{i\in B}}}\operatorname{cov}(B_1)\cdots\operatorname{cov}(B_{|B|})\right) = E_G\left(\prod_{i\in B}G_{u_i}G_{v_i}\right).$$

Therefore setting $u_i = v_i = x_i$ in (4.38) and using (4.39) and (4.40), we have

$$(4.41)\qquad \begin{aligned} &E_G\left(\prod_{i=1}^{n}\frac{G_{x_i}^2}{2}G_\alpha G_\beta\right)\\ &\quad= \sum_{B\cup C=\{1,\dots,n\}}E_G\left(\prod_{i\in B}\frac{G_{x_i}^2}{2}\right)\sum_{\pi(C)}\langle G_\alpha, G_{x_{\pi(1)}}\rangle,\dots,\langle G_{x_{\pi(|C|)}}, G_\beta\rangle.\end{aligned}$$

However, the left-hand side of (4.34) is

$$(4.42a)\qquad E_GQ\left(\prod_{i=1}^{n}\left(l_{x_i}+\frac{G_{x_i}^2}{2}\right)\right) = \sum_{B\cup C=\{1,\dots,n\}}E_G\left(\prod_{i\in B}\frac{G_{x_i}^2}{2}\right)Q\left(\prod_{i\in C}l_{x_i}\right)$$

and by (4.1) we see that (4.41) and (4.42a) are equivalent, thus establishing (4.34).

Let $z_1,\dots,z_n$ be fixed and let μ_1 and μ_2 be the measures on R_+^n defined by

$$(4.42b)\qquad \int F(\cdot)\,d\mu_1 = E_GQ\left(F\left(l_{z_1}+\frac{G_{z_1}^2}{2},\dots,l_{z_n}+\frac{G_{z_n}^2}{2}\right)\right)$$

and

$$(4.42c)\qquad \int F(\cdot)\,d\mu_2 = E_G\left(F\left(\frac{G_{z_1}^2}{2},\dots,\frac{G_{z_n}^2}{2}\right)G_\alpha G_\beta\right)$$

for all nonnegative measurable functions F on $\mathscr{C}$. It is convenient to use the language of probability so let us at first assume that Q is a probability measure. The measure μ_1 is determined by its characteristic function $\varphi_1(\lambda_1, \dots, \lambda_n) = E_G Q \exp(i\sum_{i=1}^n \lambda_i(l_{z_i} + G_{z_i}^2/2))$. For $\lambda_1, \dots, \lambda_n$ fixed, $\varphi_1(\lambda_1, \dots, \lambda_n)$ is determined by the distribution function of the real-valued random variable $\xi = \sum_{i=1}^n \lambda_i(l_{z_i} + G_{z_i}^2/2)$. Let μ_{2k} denote the $2k$th moment of ξ. If $\sum_{k=1}^\infty \mu_{2k} t^k/(2k)!$ converges for $t \in [0, \delta]$ for some $\delta > 0$ then the distribution function of ξ is uniquely determined by its moments. [See, e.g., Feller (1966), page 224.] Considering (4.34), we see that this sum converges if $E_G(\exp(\sum_{i=1}^n s_i G_{z_i}^2)|G_\alpha|\,|G_\beta|) < \infty$ for sufficiently small $s_i > 0$, $i = 1, \dots, n$. It is easy to see, by repeated use of the Schwarz inequality, that this is the case. Hence the measure μ_1 is uniquely determined by the moments of ξ, or equivalently, by the terms in the left-hand side of (4.34). Now if we set $\varphi_2(\lambda_1, \dots, \lambda_n) = E_G(\exp(i\sum_{i=1}^n \lambda_i G_{z_i}^2) G_\alpha G_\beta)$, we see by (4.34) and the above argument that $\varphi_1(\lambda_1, \dots, \lambda_n) = \varphi_2(\lambda_1, \dots, \lambda_n)$. Hence $\mu_1 = \mu_2$. Note that, although it is not clear to begin with that μ_2 is a positive measure, this argument shows that it is. Now let ν_1 be the measure on R^∞ determined by $QE_G(\cdot)$ and ν_2 the measure on R^∞ determined by $E_G(\cdot\, G_\alpha G_\beta)$. The above argument shows that $\nu_1 = \nu_2$ on the cylinder sets of R^∞. Since the cylinder sets of R^∞ generate $\mathscr{C}$, we see that $\nu_1 = \nu_2$ on $\mathscr{C}$. Thus we obtain Theorem 4.1 when Q is a probability measure. If Q is a subprobability measure, we need only divide both sides of (4.42b) and (4.42c) by the full measure of Q and repeat the above argument. This completes the proof of Theorem 4.1. □

The next lemma gives a useful equality that is similar to one developed in Example 1 above. We will use it in the proof of Theorem 3.7.

LEMMA 4.6. *Let X be a strongly symmetric standard Markov process with finite 1-potential density $u^1(x, y)$. Let $\{L_t^y, (t, y) \in R^+ \times S\}$ be the local time of X and let λ be an exponential random variable with mean one independent of X. Then for $\{z_i\}_{i=1}^n$ contained in S,*

$$E_\lambda E^x\left(\prod_{i=1}^n L_\lambda^{z_i}\right) = \sum_\pi u^1(x, x_{\pi_1}) u^1(z_{\pi_1}, z_{\pi_2}) u^1(z_{\pi_2}, z_{\pi_3}) \cdots u^1(z_{\pi_{n-1}}, z_{\pi_n}), \tag{4.43}$$

where E_λ denotes expectation with respect to λ and π runs over the permutations of $\{1, 2, \dots, n\}$.

PROOF. The left-hand side of (4.43) is equal to (4.7) with $1_A(X_s)$ replaced by 1. Since, with this substitution, the last integral in (4.7) is equal to 1, we see that

$$E_\lambda E^x\left(\prod_{i=1}^n L_\lambda^{z_i}\right) = \sum_\pi E^x\left(\int_0^\infty e^{-r_1}\, dL_{r_1}^{z_{i_{\pi(1)}}} \int_{r_1}^\infty e^{-(r_2 - r_1)}\, dL_{r_2}^{z_{i_{\pi(2)}}} \cdots \int_{r_{n-1}}^\infty e^{-(r_n - r_{n-1})}\, dL_{r_n}^{z_{i_{\pi(n)}}}\right).$$

As in Example 1 we continue to show by induction that

$$E^x\left(\int_0^\infty e^{-r_1}\,dL_{r_1}^{z_1}\int_{r_1}^\infty e^{-(r_2-r_1)}\,dL_{r_2}^{z_2}\cdots\int_{r_n-1}^\infty e^{-(r_n-r_{n-1})}\,dL_{r_n}^{z_n}\right) \tag{4.44}$$
$$= u^1(x,z_1)u^1(z_1,z_2)\cdots u^1(z_{n-1},z_n),$$

which is enough to prove this lemma. □

We can now prove Theorem 3.7. The ideas for the next proof were given to us by P. Fitzsimmons.

Proof of Theorem 3.7. We first show that for K compact,

$$\sup_{y\in K} u^1(y,y) < \infty. \tag{4.45}$$

Suppose that (4.45) does not hold; then we can find a sequence $\{y_n\}_{n=1}^\infty$, $y_n \in K$, and a $y \in K$ such that $\lim_{n\to\infty} y_n = y$ and

$$\lim_{n\to\infty} u^1(y_n,y_n) = \infty. \tag{4.46}$$

It follows from (3.13) that for any real number $T > 0$, $P^y(L_T^y > 0) = 1$. Let G_n be a decreasing sequence of open sets such that $y_n \in G_n$ and $\bigcap_{n=1}^\infty G_n = y$. The continuity of L_T^y implies that

$$\{L_T^y > 0\} = \bigcup_{n=1}^\infty \{L_T^z > 0, \forall\, z \in G_n\}.$$

Hence there exists an n_0 such that for all $n \ge n_0$,

$$P^y(L_T^z > 0, \forall\, z \in G_n) \ge \tfrac{1}{2}.$$

Let T_n denote the first hitting time of y_n by (X, P^y). It follows that

$$P^y(T_n \le T, \forall\, n \ge n_0) \ge \tfrac{1}{2},$$

which implies that

$$E^y(e^{-T_n}) \ge \tfrac{1}{2}e^{-T}. \tag{4.47}$$

However, by (3.22) and (3.20), we have

$$E^y(e^{-T_n}) = \frac{u^1(y,y_n)}{u^1(y_n,y_n)} \le \frac{u^1(y,y)}{u^1(y_ny_n)},$$

which implies by (4.46) that the left-hand side of (4.47) goes to zero as n approaches infinity. This contradiction establishes (4.45).

Now let λ be an exponential random variable with mean one which is independent of X. It follows from Lemma 4.6 and (3.20) and (4.45) that $\{L_\lambda^y, y \in K\}$ are uniformly bounded in $L^p(P^x \times \mu)$ for all $0 < p < \infty$, where μ is the probability measure of λ. Therefore, both $\{L_\lambda^y, y \in K\}$ and $\{L_\lambda^y L_\lambda^z, y, z \in K\}$ are uniformly integrable. It follows from our assumption about the continuity of L_t^y that both

$$y \to E^x(L_\lambda^y) = u^1(x,y) \tag{4.48}$$

and

$$(y,z) \to E^x(L_\lambda^y L_\lambda^z) = (u^1(x,y) + u^1(x,z))u^1(y,z) \tag{4.49}$$

are continuous.

To see that $\{u^1(x,y), (x,y) \in K \times K\}$ is continuous let $(y_n, z_n) \to (y_0, z_0)$ and take $x = y_0$, where all y_n, z_n, y_0 and z_0 are contained in K. We see from (4.48) and (4.49) that

$$u^1(y_0, y_n) \to u^1(y_0, y_0), \qquad u^1(y_0, z_n) \to u^1(y_0, z_0) \tag{4.50}$$

and

$$\begin{aligned} &\big(u^1(y_0, y_n) + u^1(y_0, z_n)\big)u^1(y_n, z_n) \\ &\qquad \to \big(u^1(y_0, y_0) + u^1(y_0, z_0)\big)u^1(y_0, z_0). \end{aligned} \tag{4.51}$$

Recall that by (3.21), $u^1(y_0, y_0) > 0$. Therefore, it follows from (4.50) and (4.51) that $u^1(y_n, z_n) \to u^1(y_0, z_0)$. Thus we see that $\{u^1(x,y), (x,y) \in K \times K\}$ is continuous. Since this holds for all compact sets $K \subset S$, we obtain Theorem 3.7. □

5. Discontinuity of local times. We treat two cases in this section, bounded discontinuities and infinite discontinuities. Bounded discontinuities of Gaussian processes are not uncommon. Consider

$$G(x) = \frac{B(x)}{(2x \log\log x)^{1/2}}, \qquad x \in [0,1],$$

with $G(0) = 0$, where $B(x)$ is Brownian motion. As is well known, $G(x)$ has a bounded discontinuity at $x = 0$. In fact it has oscillation function $\beta(0) = 2$ as defined in Theorem 2.5. In Section 9 we will use Theorem 5.1 to give examples of Markov processes with local times which have bounded discontinuities.

In this section we will use Theorem 4.1, the isomorphism theorem, for the processes described in Example 1 of Section 4. The notation that we use is the notation of Theorem 4.1 and Example 1. We first obtain conditions for the bounded discontinuity of local times. This result is much simpler than the one for infinite discontinuities. It actually follows almost immediately from the isomorphism theorem and Lemma 4.3.

THEOREM 5.1. *Let $\{L_t^x, (t,x) \in R^+ \times S\}$ be the local time of a strongly symmetric standard Markov process with 1-potential density $u^1(x,y)$. Assume that $u^1(x,y)$ is continuous in some neighborhood of (x_0, x_0). Let $\{G_x, x \in S\}$ be a real-valued Gaussian process with mean zero and covariance $u^1(x,y)$. Assume that G_x has oscillation function $0 \le \beta(x_0) < \infty$ at x_0 (see Theorem 2.5).*

Then for any countable dense set $C \subset S$,

$$(5.1)\qquad \frac{\beta(x_0)\sqrt{L_t^{x_0}}}{\sqrt{2}} \le \lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} L_t^x - L_t^{x_0} \le \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)\sqrt{L_t^{x_0}}}{\sqrt{2}}$$

for all $t < \zeta$ almost surely and

$$(5.2)\qquad \lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} |L_t^x - L_t^{x_0}| \le \left(\frac{\beta^2(x_0)}{4} + \frac{\beta(x_0)\sqrt{L_t^{x_0}}}{\sqrt{2}}\right)$$

for all $t < \zeta$ almost surely.

Since $0 < L_t^{x_0} < \infty$, P^{x_0} almost surely, for all $t \in (0,\zeta)$, we see from (5.1) and (5.2) that the local time of a Markov process associated with a Gaussian process that has a bounded discontinuity at x_0 (a probability 0 or 1 event by Theorem 2.7), itself has a bounded discontinuity at x_0 almost surely with respect to P^{x_0}. We suspect that the term containing $\beta^2(x_0)$ can be eliminated from (5.1) and (5.2) but do not know how to do it.

PROOF. Let C be a countable separating set for $\{G_x,\, x \in S\}$. It follows from Theorem 2.5 that

$$(5.3)\qquad \lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} G_x^2 = \left(|G_{x_0}| + \frac{\beta(x_0)}{2}\right)^2 \quad \text{a.s. } P_G.$$

Therefore, the set

$$(5.4)\qquad B = \left\{\lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} \frac{1}{2}\left(G_x^2 - G_{x_0}^2\right) = \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)|G_{x_0}|}{2}\right\}$$

has P_G measure 1. It then follows from Lemma 4.3 that for almost all $\omega' \in \Omega_G$, with respect to P_G,

$$(5.5)\qquad \begin{aligned} &\lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} L_t^x - L_t^{x_0} + \frac{G_x^2(\omega') - G_{x_0}^2(\omega')}{2} \\ &\quad = \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)}{\sqrt{2}}\sqrt{L_t^{x_0} + \frac{G_{x_0}^2(\omega')}{2}} \quad \text{for all } t \in Q \cap \{t < \zeta\} \text{ a.s.,} \end{aligned}$$

where Q is a countable dense subset of R^+. Therefore, for almost all $\omega' \in \Omega_G$, with respect to P_G,

$$\begin{aligned} &\lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} L_t^x - L_t^{x_0} + \lim_{\delta\to 0} \sup_{x\in C\cap B(x_0,\delta)} \frac{G_x^2(\omega') - G_{x_0}^2(\omega')}{2} \\ &\quad \ge \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)}{\sqrt{2}}\sqrt{L_t^{x_0} + \frac{G_{x_0}^2(\omega')}{2}} \quad \text{for all } t \in Q \cap \{t < \zeta\} \text{ a.s.} \end{aligned}$$

Since the event in (5.4) holds on a set of P_G measure one, for almost all $\omega' \in \Omega_G$, with respect to P_G, it follows that

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x - L_t^{x_0} \geq \frac{\beta(x_0)}{\sqrt{2}} \sqrt{L_t^{x_0} + \frac{G_{x_0}^2(\omega')}{2}} - \frac{\beta(x_0)|G_{x_0}(\omega')|}{2} \tag{5.6}$$

for all $t \in Q \cap \{t < \zeta\}$ a.s.

For all $\varepsilon > 0$ we can find an ω' such that (5.6) holds and $G_{x_0}^2(\omega')/2 < \varepsilon$. For this ω' we have

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x - L_t^{x_0} \geq \frac{\beta(x_0)}{\sqrt{2}} \left(\sqrt{L_t^{x_0}} - \sqrt{\varepsilon} \right) \quad \text{for } t \in Q \cap \{t < \zeta\} \text{ a.s.,} \tag{5.7}$$

and since this holds for all $\varepsilon > 0$, we get

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x - L_t^{x_0} \geq \frac{\beta(x_0)}{\sqrt{2}} \sqrt{L_t^{x_0}} \quad \text{for all } t \in Q \cap \{t < \zeta\} \text{ a.s.} \tag{5.8}$$

Let Ω', $P^{x_0}(\Omega') = 1$, be the set on which (5.8) holds. For any $t \in (0, \zeta)$, choose a sequence $t_i \in Q$ such that $t_i \uparrow t$. For $\omega \in \Omega'$ we have, by the monotonicity of the local time, that

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x(\omega) \geq L_{t_i}^{x_0}(\omega) + \frac{\beta(x_0)\sqrt{L_{t_i}^{x_0}(\omega)}}{\sqrt{2}}. \tag{5.9}$$

Since $L_t^{x_0}$ is continuous in t, we see that

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x(\omega) \geq L_t^{x_0}(\omega) + \frac{\beta(x_0)\sqrt{L_t^{x_0}(\omega)}}{\sqrt{2}}, \tag{5.10}$$

and since this is valid for all $t \in (0, \zeta)$ and all $\omega \in \Omega'$, we get the lower bound in (5.1).

To obtain the upper bound in (5.1), we use (5.5) to immediately obtain

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x - L_t^{x_0} \leq \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)}{\sqrt{2}} \sqrt{L_t^{x_0} + \frac{G_{x_0}^2(\omega')}{2}} + \frac{G_{x_0}^2(\omega')}{2} \quad \text{for all } t \in Q \cap \{t < \zeta\} \text{ a.s.} \tag{5.11}$$

for almost all $\omega' \in \Omega_G$, with respect to P_G. Thus, as above, $G_{x_0}^2(\omega')$ can be made as small as we like and we get

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} L_t^x - L_t^{x_0} \le \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)\sqrt{L_t^{x_0}}}{\sqrt{2}} \tag{5.12}$$

for $t \in Q \cap \{t < \zeta\}$ a.s.

For all $t \in (0, \zeta)$ choose $t_i \in Q$ such that $t_i \downarrow t$. Following the argument given in (5.9) and (5.10), we get the upper bound in (5.1).

To obtain (5.2) we repeat the above argument with a minor variation. Analogously to (5.4) we define the set

$$B' = \left\{ \lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} \frac{1}{2} \left| G_x^2 - G_{x_0}^2 \right| \le \frac{\beta^2(x_0)}{8} + \frac{\beta(x_0)|G_{x_0}|}{2} \right\}. \tag{5.13}$$

Since

$$|G_x^2 - G_{x_0}^2| \le |G_x - G_{x_0}|\big(2|G_{x_0}| + |G_x - G_{x_0}|\big),$$

we see by Theorem 2.5 that this set has P_G measure 1. As in (5.5), it follows by Lemma 4.3 that

$$\begin{aligned} &\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} |L_t^x - L_t^{x_0}| \\ &\quad \le \lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} \frac{\left|G_x^2(\omega') - G_{x_0}^2(\omega')\right|}{2} + \frac{\beta^2(x_0)}{8} \\ &\quad\quad + \frac{\beta(x_0)}{\sqrt{2}} \sqrt{L_t^{x_0} + \frac{G_{x_0}^2(\omega')}{2}} \quad \text{for all } t \in Q \cap \{t < \zeta\} \text{ a.s.,} \end{aligned} \tag{5.14}$$

for almost all $\omega' \in \Omega_G$, with respect to P_G. Proceeding as above, we assume that $\omega' \in B'$ and that $G_{x_0}^2(\omega')$ can be made as small as we like and get

$$\begin{aligned} &\lim_{\delta \to 0} \sup_{x \in C \cap B(x_0, \delta)} |L_t^x - L_t^{x_0}| \\ &\quad \le \frac{\beta^2(x_0)}{4} + \frac{\beta(x_0)\sqrt{L_t^{x_0}}}{\sqrt{2}} \quad \text{for all } t \in Q \cap \{t < \zeta\} \text{ a.s.} \end{aligned} \tag{5.15}$$

We can extend this result to all $t \in (0, \zeta)$ as above. This completes the proof of Theorem 5.1. □

The result for infinite discontinuities of local times is more complicated than the one for bounded discontinuities because we cannot work with sets of measure one in this case and hence cannot use Lemma 4.3. We will show that under certain conditions the local times have an infinite discontinuity at a point $z_0 \in S$. In an attempt to avoid confusion we will distinguish, initially, between this point and x, the initial point of the Markov process. Eventually, we will take x to be equal to z_0. Note that we use a slightly different notation for Gaussian processes than the notation used previously.

Let $\{G(z),\ z \in T\}$ be a real-valued Gaussian process with mean zero defined on the probability space (Ω_G, P_G), with expectation operator E, where (T, ρ) is a separable metric space. Define

$$d(u,v) = \left(E|G(u) - G(v)|^2\right)^{1/2} \qquad u,v \in T. \tag{5.16}$$

Let $z_0 \in T$ be such that $EG^2(z_0) > 0$ and consider

$$\eta(z) = G(z) - \alpha(z,z_0)G(z_0), \qquad z_0 \in T, \tag{5.17}$$

where

$$\alpha(z,z_0) = \frac{EG(z)G(z_0)}{EG^2(z_0)}. \tag{5.18}$$

It is well known and easy to see that $\{\eta(z),\ z \in T\}$ is a mean zero Gaussian process which is independent of $G(z_0)$. Let a be the median of $\sup_{z \in T} \eta(z)$ and assume that a satisfies

$$P\Big(\sup_{z\in T} \eta(z) \geq a\Big) = P\Big(\sup_{z\in T} \eta(z) \leq a\Big) = 1/2. \tag{5.19}$$

In what follows T will be a finite set. In this case it is clear that the median of $\sup_{z \in T} \eta(z)$ satisfies (5.19). Let

$$\sigma = \sup_{z\in T}\left(E\eta^2(z)\right)^{1/2}. \tag{5.20}$$

It follows from Theorem 2.1 that

$$P\Big(\Big\{\sup_{z\in T} \eta(z) > a - \sigma t\Big\} \cap \Big\{\sup_{z\in T} - \eta(z) > a - \sigma t\Big\}\Big) \geq 1 - 2\Phi(t) \tag{5.21}$$

and

$$P\Big(\sup_{z\in T} \eta(z) < a + \sigma t\Big) \geq 1 - \Phi(t), \tag{5.22}$$

where $\Phi(t)$ is given in (2.4). For future use we define

$$\phi(t) = \sqrt{2/\left(\pi EG^2(z_0)\right)}\, e^{-t^2/(2EG^2(z_0))}. \tag{5.23}$$

The next lemma contains the key idea that is used in the proof of unboundedness of local times.

LEMMA 5.2. *Let $\{G(z),\ z \in S\}$ be a real-valued Gaussian process with mean zero, where $S = (S,\rho)$ is a separable metric space on which there exists a σ-finite measure m. Let T be a finite subset of S and let $\{\eta(z),\ z \in T\}$ be as defined in* (5.17) *and* (5.18). *Define*

$$G_A = \int_A G(z)\,m(dz)$$

and

$$g(z,y) = EG(z)G(y), \qquad g(z,A) = EG(z)G_A, \qquad g(A,A) = EG_A^2.$$

We assume that

$$0 < g(z,z) < \infty \qquad \forall\, z \in S, \tag{5.24}$$

and that

$$0 < \alpha_1 \equiv \inf_{z\in T} \alpha(z,z_0) \quad \text{and} \quad \sup_{z\in T} \alpha(z,z_0) \equiv \alpha_2 < \infty. \tag{5.25}$$

Let a be the median of $\sup_{z\in T}\eta(z)$. Since T is finite, (5.19) holds. Let σ be as defined in (5.20). Then for all real numbers y and $0 \le t < a/\sigma$, we have

$$\begin{aligned} &P\biggl(\sup_{z\in T} \frac{\eta^2(z)}{2} + \alpha(z,z_0)\eta(z)y \\ &\qquad > \frac{(a-\sigma t)^2}{2} + \alpha_1(a-\sigma t)|y|\biggr) \ge 1 - 2\Phi(t) \end{aligned} \tag{5.26}$$

and

$$\begin{aligned} &P\biggl(\sup_{z\in T} \frac{\eta^2(z)}{2} + \alpha(z,z_0)\eta(z)y \\ &\qquad < \frac{(a+\sigma t)^2}{2} + \alpha_2(a+\sigma t)|y|\biggr) \ge 1 - 2\Phi(t). \end{aligned} \tag{5.27}$$

Furthermore, for the event

$$\begin{aligned} B = \biggl\{&\sup_{z\in T} \frac{\eta^2(z)}{2} + \alpha(z,z_0)\eta(z)G(z_0) \\ &\le \frac{(a-\sigma t)^2}{2} + \alpha_1(a-\sigma t)|G(z_0)|\biggr\} \end{aligned} \tag{5.28}$$

we have

$$E(1_B G(x)G_A) \le (6\Phi(t)g(x,x)g(A,A))^{1/2} \equiv H(x,A,t). \tag{5.29}$$

PROOF. We first obtain the inequality (5.26). One sees from (5.21) that on a set of measure greater than or equal to $(1-2\Phi(t))$, $\sup_{z\in T}\eta(z) > (a-\sigma t)$ and also $\sup_{z\in T} -\eta(z) > (a-\sigma t)$. Therefore, whatever the sign of y, $\sup_{z\in T}\eta(z)y > (a-\sigma t)|y|$ on a set of measure greater than or equal to $(1-2\Phi(t))$. [It is this symmetry of the large values of $\eta(z)$ that enables us to carry through our arguments. As we just saw, this is the case for Gaussian processes with bounded discontinuities.]

The inequality in (5.27) follows from (5.22). This inequality is less subtle than (5.26) because we can replace $\eta(z)y$ by $|\eta(z)y|$ in taking an upper bound. To obtain (5.29), we note that

$$E(1_B G(x)G_A) \le (E(1_B))^{1/2}(EG^2(x)G_A^2)^{1/2} \tag{5.30}$$

and by a well-known result, which we give in Lemma 4.5,

$$EG^2(x)G_A^2 = 2g^2(x,A) + g(x,x)g(A,A) \le 3g(x,x)g(A,A). \tag{5.31}$$

Now, consider $E(1_B)$. The expectation is with respect to the probability measure of the Gaussian process P_G. If we denote by P_η the probability measure of the process $\{\eta(z),\ z \in T\}$ and $P_{G(z_0)}$ the probability measure of $G(z_0)$, then P_G is equivalent to the product probability measure $P_{G(z_0)} \times P_\eta$. Let $E_{G(z_0)}$ and E_η denote the expectation operators corresponding to $P_{G(z_0)}$ and P_η, respectively. Thus $E(1_B) = E_{G(z_0)}E_\eta(1_B)$. We see from (5.26) that

$$E_\eta(1_{B^c}) \geq 1 - 2\Phi(t),$$

since, in the product space, the fixed value of $G(z_0)$ is just some real number y as in (5.26). Thus $E_\eta(1_B) \leq 2\Phi(t)$ and hence $E_{G(z_0)}E_\eta(1_B) \leq 2\Phi(t)$. Using this and (5.30) and (5.31), we get (5.29). □

The next lemma shows that, for our purposes, we can take σ as small as we like.

LEMMA 5.3. *Let $\{G(z),\ z \in S\}$ be a real-valued Gaussian process with mean zero and covariance $g(z,y)$ which is continuous in some neighborhood of (z_0, z_0). Let C be a countable dense subset of S and suppose that*

$$\lim_{\delta \to 0} \sup_{x \in C \cap B(z_0,\delta)} G(z) = \infty \quad a.s., \tag{5.32}$$

where $B(z_0,\delta) = \{z \in S: \rho(z,z_0) \leq \delta\}$. Let $\{\eta(z),\ z \in C\}$ be as defined in (5.17) *and* (5.18). *Then for any $\sigma > 0$ and $M > 0$, there exists a $\delta > 0$ such that for all $\delta' \leq \delta$, there exists a finite set $D_{M,\sigma,\delta'} \subset C \cap B(z_0,\delta')$ such that*

$$P\Big(\sup_{z \in D_{M,\sigma,\delta'}} \eta(z) \geq M\Big) \geq \tfrac{1}{2} \tag{5.33}$$

and

$$\sup_{z \in D_{M,\sigma,\delta'}} E\eta^2(z) \leq \sigma^2. \tag{5.34}$$

PROOF. Since $g(z,y)$ is assumed to be continuous in some neighborhood of (z_0, z_0) in (S,ρ) it follows that for some $\delta > 0$ sufficiently small, $\sup_{z \in B(z_0,\delta')} E\eta^2(z) \leq \sigma^2$, $\forall\ \delta' \leq \delta$. On the other hand, (5.32) clearly holds with $G(z)$ replaced by $\eta(z)$. The fact that we can take $D_{M,\sigma,\delta'}$ finite in (5.33) follows from the basic continuity property of probability measures. □

We can now show that the local time of a Markov process associated with a Gaussian process that is unbounded at a point is itself almost surely unbounded at that point.

THEOREM 5.4. *Let $\{L_t^z,\ (t,z) \in R^+ \times S\}$ be the local time process of a strongly symmetric standard Markov process with 1-potential density $u^1(z,y)$. Assume that $u^1(z,y)$ is continuous in some neighborhood of (z_0,z_0). Let $\{G(z),\ z \in S\}$ be a real-valued Gaussian process with mean zero and covariance*

$g(z, y) = u^1(z, y)$. Suppose that there exists a countable dense subset $C \subset S$ for which

$$\lim_{\delta \to 0} \sup_{z \in C \cap B(z_0, \delta)} G(z) = \infty \quad a.s.\ P_G. \tag{5.35}$$

Then

$$\lim_{\delta \to 0} \sup_{z \in C \cap B(z_0, \delta)} L_t^z = \infty \qquad \forall\, t > 0 \quad a.s.\ P^{z_0}. \tag{5.36}$$

PROOF. Let $\{\eta(z),\ z \in C\}$ be as defined in (5.17) and (5.18). For any large number M and small number $\sigma > 0$, let $D_{M,\sigma,\delta'} \subset C$ be a finite set for which (5.33) and (5.34) are satisfied, where $\delta' \le \delta$ for some $\delta > 0$. Let α_1 and α_2 be as defined in (5.25). Since $g(z_0, z_0) > 0$, by (5.24) we can always take δ small enough so that $\alpha_1 \ge 1/2$ and $\alpha_2 \le 2$. For convenience set $T = T_{M,\sigma,\delta'} = D_{M,\sigma,\delta'}$. Since T is finite, $\sup_{z \in T} \eta(z)$ has a unique median $a \ge M$ which satisfies (5.19). Let $l_z = L_\lambda^z$ be as given in Example 1 of Section 4. It follows from the isomorphism theorem applied to

$$\begin{aligned} F(\tau) = \text{Indicator function of } \Big\{ &\sup_{z \in T} \tau(z) - \alpha^2(z, z_0)\tau(z_0) \\ &\le \frac{(a - \sigma t)^2}{2} + \alpha_1(a - \sigma t)\sqrt{2\tau(z_0)} \Big\} \end{aligned} \tag{5.37}$$

that

$$\begin{aligned} P_G Q^{x,A}\Big(&\sup_{z \in T} l_z - \alpha^2(z, z_0) l_{z_0} + \frac{G^2(z)}{2} - \alpha^2(z, z_0)\frac{G^2(z_0)}{2} \\ &\le \frac{(a - \sigma t)^2}{2} + \sqrt{2}\,\alpha_1(a - \sigma t)\sqrt{l_{z_0} + \frac{G^2(z_0)}{2}} \Big) = E_G(1_B G(x) G_A). \end{aligned} \tag{5.38}$$

This is just a particular case of (4.2) for F as given in (5.37) and for the processes of Example 1. We take A to be a compact set in S and

$$\begin{aligned} B = \Big\{ &\sup_{z \in T} \frac{G^2(z)}{2} - \alpha^2(z, z_0)\frac{G^2(z_0)}{2} \\ &\le \frac{(a - \sigma t)^2}{2} + \alpha_1(a - \sigma t)|G(z_0)| \Big\}. \end{aligned} \tag{5.39}$$

This set B is exactly the same as the set B in (5.28). Therefore, the left-hand side of (5.38) is bounded by

$$H(x, A, t), \tag{5.40a}$$

where $H(x, A, t)$ is given in (5.29). (Let us assume here that $\sigma t < 1$ and $a > 2$. This will be the case when we choose M and σ below.)

By the discussion following (5.31) and employing the relations given in (5.17) and (5.18), we can write the inequality given by (5.38) and (5.40a) as

$$Q^{x,A}P_{G(z_0)}P_\eta\Bigg(\sup_{z\in T} l_z - \alpha^2(z,z_0)l_{z_0} + \frac{\eta^2(z)}{2} + \alpha(z,z_0)\eta(z)G(z_0) \le \frac{(a-\sigma t)^2}{2} + \sqrt{2}\,\alpha_1(a-\sigma t)\sqrt{l_{z_0}+\frac{G^2(z_0)}{2}}\Bigg) \le H(x,A,t). \tag{5.40b}$$

We now apply (5.27), with $y = G(z_0)$, to see that

$$Q^{x,A}P_{G(z_0)}\Bigg(\sup_{z\in T} l_z - \alpha^2(z,z_0)l_{z_0} \le \frac{(a-\sigma t)^2}{2} + \sqrt{2}\,\alpha_1(a-\sigma t)\sqrt{l_{z_0}+\frac{G^2(z_0)}{2}} - \frac{(a+\sigma t)^2}{2} - \alpha_2(a+\sigma t)|G(z_0)|\Bigg) \le H(x,A,t) + 2\Phi(t), \tag{5.40c}$$

where α_2 is given in (5.25). It follows from this that

$$Q^{x,A}P_{G(z_0)}\Big(\sup_{z\in T} l_z - \alpha^2(z,z_0)l_{z_0} \le \sqrt{2}\,\alpha_1(a-\sigma t)\sqrt{l_{z_0}} - \alpha_2(a+\sigma t)|G(z_0)| - 2\sigma t a\Big) \le H(x,A,T) + 2\Phi(t), \tag{5.41a}$$

and, since $\alpha_1 \ge 1/2$ and $\alpha_2 \le 2$, that

$$Q^{x,A}P_{G(z_0)}\Big(\sup_{z\in T} l_z - \alpha^2(z,z_0)l_{z_0} \le \frac{1}{\sqrt{2}}(a-\sigma t)\sqrt{l_{z_0}} - 2(a+\sigma t)|G(z_0)| - 2\sigma t a\Big) \le H(x,A,t) + 2\Phi(t). \tag{5.41b}$$

Take $x = z_0$. We will now be more explicit about our choices of M and σ. Let $0 < \bar{\varepsilon}_1^{1/2} < Q^{z_0,A}(\Omega)$ satisfy

$$\bar{\varepsilon}_1 < (96)^{-4}. \tag{5.42}$$

For $\varepsilon_1 \le \bar{\varepsilon}_1$, define $b(\varepsilon_1)$ such that

$$\int_0^{b(\varepsilon_1)} \phi(t)\,dt = \varepsilon_1^{1/2}, \tag{5.43}$$

where $\phi(t)$ is given in (5.23). Note that

$$b(\varepsilon_1) \le 3\varepsilon_1^{1/2} \quad \text{for all } \varepsilon_1 \text{ such that } 0 < \varepsilon_1 \le \bar{\varepsilon}_1. \tag{5.44}$$

We choose $t = t(\varepsilon_1)$ such that

$$H(z_0, A, t) + 2\Phi(t) = \varepsilon_1 \tag{5.45}$$

and pick σ so that

$$\sigma \le \frac{\varepsilon_1^{1/4}}{16t}. \tag{5.46}$$

For some real number N, set

$$M = \frac{4N}{\varepsilon_1^{1/4}}. \tag{5.47}$$

Recall that a $\ge M$ and since $M \ge 2$, we have for our choices of σ and t that

$$(a - \sigma t) \ge a/2 \quad \text{and} \quad (a + \sigma t) \le 2a. \tag{5.48}$$

As we stated above, there exists a $\delta > 0$ so that (5.41b) holds for $T = T_{M,\sigma,\delta'}$ for all $\delta' \le \delta$. Using (5.45) and (5.48) in (5.41b), we have

$$\begin{aligned} P_{G(z_0)}Q^{z_0, A}\Big(\sup_{z \in T} l_z - \alpha^2(z, z_0) l_{z_0} &\le a\sqrt{l_{z_0}/8} \\ &\quad - 4a|G(z_0)| - 2\sigma t a \Big) \le \varepsilon_1. \end{aligned} \tag{5.49}$$

Note that $P_{G(z_0)}Q^{z_0, A}$ is a product measure on $(R^+ \times \Omega)$. Let us denote the elements of this space by (y, ω). We see from (5.49) that the set

$$\begin{aligned} D(y, \omega) = \Big(\sup_{z \in T} l_z(\omega) &- \alpha^2(z, z_0) l_{z_0}(\omega) \\ &\ge a\sqrt{l_{z_0}/8} - 4a|y| - 2\sigma t a \Big) \end{aligned} \tag{5.50}$$

has $P_{G(z_0)}Q^{z_0, A}$ measure greater than or equal to $Q^{z_0, A}(\Omega) - \varepsilon_1$. Let $b = b(\varepsilon_1)$ be as defined in (5.43). We claim that there exists a $y \le b$ such that

$$\int_\Omega I_D(y, \omega) Q^{z_0, A}(d\omega) \ge Q^{z_0, A}(\Omega) - \varepsilon_1^{1/2}, \tag{5.51}$$

since, if not, by (5.43) and the remark following (5.50), we have

$$\begin{aligned} &\int_0^\infty \int_\Omega I_D(y, \omega) Q^{z_0, A}(d\omega) \phi(y)\, dy \\ &\quad = \int_0^b \int_\Omega I_D(y, \omega) Q^{z_0, A}(d\omega) \phi(y)\, dy + \int_b^\infty \int_\Omega I_D(y, \omega) Q^{z_0, A}(d\omega) \phi(y)\, dy \\ &\quad < \varepsilon_1^{1/2}\big(Q^{z_0, A}(\Omega) - \epsilon_1^{1/2}\big) + \big(1 - \varepsilon_1^{1/2}\big) Q^{z_0, A}(\Omega) < Q^{z_0, A}(\Omega) - \varepsilon_1, \end{aligned}$$

which contradicts the fact that the set D has $P_{G(z_0)}Q^{z_0, A}$ measure greater than or equal to $Q^{z_0,A}(\Omega) - \varepsilon_1$. Therefore, for some $0 \le y \le b(\varepsilon_1)$,

$$(5.52)\quad Q^{z_0,A}\Big(\sup_{z\in T} l_z - \alpha^2(z,z_0)l_{z_0} < a\sqrt{l_{z_0}/8} - 4ay - 2\sigma ta\Big) < \varepsilon_1^{1/2}.$$

It follows from (5.52) and (5.44) that

$$(5.53)\qquad Q^{z_0,A}\Big(\sup_{z\in T} l_z < a\Big(\sqrt{l_{z_0}/8} - 12\varepsilon_1^{1/2} - 2\sigma t\Big)\Big) < \varepsilon_1^{1/2}.$$

Let

$$(5.54)\qquad Q^{z_0,A}\big(l_{z_0} \le 2\varepsilon_1^{1/2}\big) = \varepsilon_2.$$

We note that $\varepsilon_2 \to 0$ as $\varepsilon_1 \to 0$ because $L^t_{z_0} > 0$ for all $t > 0$ almost surely with respect to P^{z_0}. It follows from (5.53) and (5.54) that

$$(5.55)\qquad Q^{z_0,A}\Big(\sup_{z\in T} l_z < a\big(\tfrac{1}{2}\varepsilon_1^{1/4} - 12\varepsilon_1^{1/2} - 2\sigma t\big)\Big) < \epsilon_1^{1/2} + \epsilon_2.$$

Finally, by (5.42) and (5.46),

$$Q^{z_0,A}\Big(\sup_{z\in T} l_z < a\varepsilon_1^{1/4}/4\Big) < \varepsilon_1^{1/2} + \varepsilon_2,$$

and by (5.47) and the fact that $a \ge M$, we get

$$(5.56)\qquad Q^{z_0,A}\Big(\sup_{z\in T} l_z < N\Big) < \varepsilon_1^{1/2} + \varepsilon_2.$$

Since (5.56) is valid for $T = T_{M,\sigma,\delta'}$ for all $\delta' \le \delta$, we have

$$Q^{z_0,A}\Big(\lim_{\delta\to 0}\ \sup_{z\in C\cap B(z_0,\delta)} l_z < N\Big) < \varepsilon_1^{1/2} + \varepsilon_2$$

and since this holds for all $\varepsilon_1 > 0$ sufficiently small and $\varepsilon_2 \to 0$ as $\varepsilon_1 \to 0$, we get

$$(5.57)\qquad Q^{z_0,A}\Big(\lim_{\delta\to 0}\ \sup_{z\in C\cap B(z_0,\delta)} l_z < N\Big) = 0.$$

Finally, since this holds for all N, we have

$$(5.58)\qquad Q^{z_0,A}\Big(\lim_{\delta\to 0}\ \sup_{z\in C\cap B(z_0,\delta)} l_z < \infty\Big) = 0.$$

Let $\{A_k\}_{k=1}^{\infty}$ be an increasing sequence of compact sets such that $A_k \uparrow S$. Then, by (5.58) with $A = A_k$ and Proposition 4.2, we have

$$(5.59)\qquad Q^{z_0}\Big(\lim_{\delta\to 0}\ \sup_{z\in C\cap B(z_0,\delta)} l_z < \infty \cap \{(\omega,t)|t < \zeta(\omega)\}\Big) = 0,$$

which is equivalent to

$$(5.60)\qquad \int_0^\infty P^{z_0}\Big(\lim_{\delta\to 0}\ \sup_{z\in C\cap B(z_0,\delta)} L^z_t < \infty \cap \{t < \zeta\}\Big)e^{-t}\,dt = 0.$$

It now follows from Fubini's theorem that

$$(5.61)\quad P^{z_0}\Big(\lim_{\delta\to 0}\sup_{z\in C\cap B(z_0,\delta)} L_t^z < \infty \cap \{t<\zeta\}\Big) = 0 \quad \text{for almost all } t\in R^+.$$

Let $Q\subset R^+$ be a countable dense set for which (5.61) holds. Then

$$(5.62)\qquad \lim_{\delta\to 0}\sup_{z\in C\cap B(z_0,\delta)} L_t^z = \infty \qquad \forall\, t\in Q\cap[0,\zeta)\text{ a.s. } P^{z_0}.$$

Therefore, by the monotonicity of the local time, we get (5.36). □

REMARK 5.5. It may be useful to point out that in some ways the proof of Theorem 5.4 is similar to the proof of Theorem 5.1. In Theorem 5.1, a key role is played by the fact that the set B in (5.4) has measure one. This is used twice in obtaining (5.8). We cannot obtain an analogue of this when the Gaussian process is unbounded because in this case the oscillation function β is infinite. Nevertheless we can use different versions of Borell's inequality to bound $\sup_{t\in T}\frac{1}{2}(G^2(z)-\alpha^2(z,z_0)G^2(z_0))$ above and below, when T is a finite index set. [Note that this term is essentially the same as $\sup_{t\in T}\frac{1}{2}(G^2(z)-G^2(z_0))$, since $\alpha^2(z,z_0)\to 1$ as $z\to z_0$ and is easier to work with.] This is what we do in Lemma 5.2 since

$$\frac{\eta^2(z)}{2}+\alpha(z,z_0)\eta(z)G(z_0)=\frac{1}{2}\big(G^2(z)-\alpha^2(z,z_0)G^2(z_0)\big).$$

[In Lemma 5.2, stressing the independence of η and $G(z_0)$, we consider $G(z_0)$ fixed and equal to a real number y.] The key point is that we can put $|y|$ in (5.26). This inequality is used to estimate the set B in (5.28) which is used with the isomorphism theorem to give (5.40b). The inequality in (5.27) gives (5.40c). Then as in the proof of Theorem 5.1, we consider l_z and $G(z_0)$ on a product probability space and take $|G(z_0)|$ as small as we wish. Theorem 5.4 is complicated by the fact that we must consider the relationship between a, σ, t and T.

6. Continuity and boundedness of local times. Let X be a strongly symmetric standard Markov process with lifetime ζ and local time $L=\{L_t^y,\,(t,y)\in R^+\times S\}$. In Example 1 of Section 4, we considered the stochastic process $\{L_\lambda^y,\,y\in S\}$ obtained by replacing t in L_t^y by an independent exponentially distributed random variable λ, because this process and the Gaussian process associated with X are what appear in the isomorphism theorem. It follows easily from the isomorphism theorem that if the associated Gaussian process has a version with continuous sample paths almost surely, then so does $\{L_\lambda^y,\,y\in S\}$. Therefore, the problem that confronts us is to show that if $\{L_\lambda^y,\,y\in S\}$ has a version with continuous paths almost surely, then so does $\{L_t^y,\,(t,y)\in R^+\times S\}$. This, unfortunately, turns out to be rather complicated for us. In Theorem 6.2 we show the continuity almost surely of $\{L_t^y,\,(t,y)\in[0,\zeta)\times S\}$ and in Theorem 6.3, using the fact that X is a Hunt process, we extend the joint continuity of L_t^y to $R^+\times S$. In Theorem 6.1, we consider the continuity of the local times of a restricted class of strongly symmetric standard Markov processes. However, Theorem 6.1 is actually the main result of this section. We apply it, in theorem 6.2, to the more general processes that we are concerned with by considering versions of these processes that are killed when they exit a compact set.

When we say that a stochastic process $\hat{L} = \{\hat{L}_t^y, (t, y) \in R^+ \times S\}$ is a version of the local time of a Markov process X, we mean more than the traditional statement that one stochastic process is a version of the other. Besides this, we also require that the version is itself a local time for X, that is, that for each $y \in S$, $\hat{L}^y_\cdot$ is a local time for X at y, as defined in Section 3. To be more specific, suppose that $L = \{L_t^y, (t, y) \in R^+ \times S\}$ is a local time for X. When we say that we can find a version of the local time which is jointly continuous on $T \times S$, where $T \subset R^+$, we mean that we can find a stochastic process $\hat{L} = \{\hat{L}_t^y, (t, y) \in R^+ \times S\}$ which is continuous on $T \times S$ almost surely with respect to P^x for all $x \in S$ and which satisfies, for each x, $y \in S$,

$$\hat{L}_t^y = L_t^y \qquad \forall\, t \in R^+ \text{ a.s. } P^x. \tag{6.1}$$

Following convention, we often say that a Markov process has a continuous local time, when we mean that we can find a continuous version for the local time.

THEOREM 6.1. *Let X be a strongly symmetric standard Markov process as defined in Section 3 but with the following additional condition: That $u(x, y) = u^0(x, y)$, given by (3.9), is bounded and uniformly continuous on $(S \times S)$. Then, if $u(x, y)$ is the covariance of a mean zero continuous Gaussian process $\{G(y), y \in S\}$, we can find a version of the local time of X which is jointly continuous on $[0, \zeta) \times S$.*

PROOF. Let $f \in bp\mathscr{S}$ be strictly positive and in $L^1(dm)$. Define

$$h(x) = Uf(x) = \int u(x, y) f(y)\, dm(y).$$

By our assumptions on u and f, $h(x)$ is bounded and uniformly continuous on S. Furthermore, h is an excessive function with respect to X [because $u(x, \cdot)$ is] and $h(x) > 0$ for all $x \in S$. [To see this last point, let $1(\cdot)$ be the indicator function of S and note that $Uf(x) = 0$ implies $U1(x) = 0$, which implies that $U^\alpha 1(x) = 0$ for all $\alpha \geq 0$. But, for a standard Markov process, $\lim_{\alpha \to \infty} \alpha U^\alpha 1(x) = 1$.]

For $A \in \mathscr{S}$ define the finite measure

$$m_1(A) = \int_A f(y)\, dm(y),$$

so that, obviously,

$$h(x) = \int u(x, y)\, dm_1(y).$$

For this function h we consider the h-transform of X discussed in Example 2 of Section 4 but with the measure m replaced by m_1 so that (4.25), (4.26) and (4.27) are satisfied. This is a Markov process on the probability space $(\Omega, P^{x/h})$ with expectation operator $E^{x/h}$. It follows, by Theorem 3.2, that X has a local time $L = \{L_t^y, (t,y) \in R^+ \times S\}$. In Example 2 of Section 4 we show that $\{L_\infty^{y_i}\}_{i=1}^\infty$, $y_i \in S$, considered as a sequence of random variables on $(\Omega, P^{x/h})$, satisfies (4.1) of Theorem 4.1. Therefore, by Theorem 4.1, (4.2) holds for this sequence and the mean zero Gaussian process with covariance $u(x,y)$. In particular, it follows from (4.30) and (4.31) that

$$E^{x/h}(L_\infty^y) = \frac{u(x,y)h(y)}{h(x)}. \tag{6.2}$$

This shows that $L_\infty^y \in L^1(\Omega, P^{x/h})$ for all $y \in S$.

We show first that L is jointly continuous on $R^+ \times S$ almost surely with respect to $P^{x/h}$. Consider the martingale

$$A_t^y = E^{x/h}(L_\infty^y | \mathscr{F}_t) \tag{6.3}$$

and note that

$$L_\infty^y = L_t^y + L_\infty^y \circ \theta_t = L_t^y + 1_{\{t<\zeta\}} L_\infty^y \circ \theta_t.$$

Therefore,

$$\begin{aligned} A_t^y &= L_t^y + E^{x/h}\big(1_{\{t<\zeta\}} L_\infty^y \circ \theta_t | \mathscr{F}_t\big) \\ &= L_t^y + 1_{\{t<\zeta\}} E^{x/h}\big(L_\infty^y \circ \theta_t | \mathscr{F}_t\big) = L_t^y + 1_{\{t<\zeta\}} E^{X_t/h}\big(L_\infty^y\big), \end{aligned} \tag{6.4}$$

where we use the Markov property for the h-transform process. It follows from (6.4) and (6.2), along with the convention that $1/h(\Delta) = 0$, that

$$A_t^y = L_t^y + \frac{u(X_t, y)h(y)}{h(X_t)}. \tag{6.5}$$

At this point, let us note that if $f(x)$ is an excessive function for the semigroup P_t, then $f(x)/h(x)$ is an excessive function for the semigroup $P_t^{(h)}$ defined in (4.22). Also, by Sharpe [(1988), 62.19], X_t is a right process for $P^{x/h}$. Consequently, $f(X_t)/h(X_t)$ is right continuous almost surely with respect to $P^{x/h}$. [See Sharpe (1988), 7.1.] Therefore, if we take $f(x) = u(x,y)$, we see that A_t^y is right continuous. Let D be a countable dense subset of S and F a finite subset of D. Since

$$\sup_{\substack{\rho(y,z)\le\delta \\ y,z\in F}} A_t^y - A_t^z = \sup_{\substack{\rho(y,z)\le\delta \\ y,z\in F}} |A_t^y - A_t^z|$$

is a right continuous, nonnegative submartingale, we have, for any $\varepsilon > 0$,

$$\begin{aligned} P^{x/h}\Bigg(\sup_{t\ge 0} \sup_{\substack{\rho(y,z)\le\delta \\ y,z\in F}} A_t^y - A_t^z \ge \varepsilon\Bigg) &\le \frac{1}{\varepsilon} E^{x/h}\Bigg(\sup_{\substack{\rho(y,z)\le\delta \\ y,z\in F}} L_\infty^y - L_\infty^z\Bigg) \\ &\le \frac{1}{\varepsilon} E^{x/h}\Bigg(\sup_{\substack{\rho(y,z)\le\delta \\ y,z\in D}} L_\infty^y - L_\infty^z\Bigg). \end{aligned} \tag{6.6}$$

Let E_G be the expectation operator of the Gaussian process. It follows from Example 2 of the isomorphism theorem that

$$E^{x/h}\left(\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in D}} L_\infty^y - L_\infty^z\right)$$

$$\le E_G\left(\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in D}} \frac{G^2(y)}{2} - \frac{G^2(z)}{2}\right) \tag{6.7}$$

$$+\sqrt{3}\left(E_G\left(\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in D}} \left|\frac{G^2(y)}{2} - \frac{G^2(z)}{2}\right|^2\right) E_G(G^2(x))E_G\left(G_\beta^2\right)\right)^{1/2},$$

where G_β is given in (4.26). [Also, on the right-hand side of (4.2) we use the Cauchy–Schwarz inequality twice and the fact that for a mean zero normal random variable, say η, $E\eta^4 = 3(E\eta^2)^2$.] It now follows from Lemma 2.4 that for any $\bar{\varepsilon} > 0$, we can choose a $\delta > 0$ such that the first term in (6.6) is less than $\bar{\varepsilon}$. Combining (6.5) and (6.6) we now see that

$$P^{x/h}\left(\sup_{t\ge 0}\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in F}} L_t^y - L_t^z \ge 2\varepsilon\right)$$

$$\le \bar{\varepsilon} + P^{x/h}\left(\sup_{t\ge 0}\frac{1}{h(X_t)}\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in D}} (u(X_t,y)h(y) - u(X_t,z)h(z)) \ge \varepsilon\right) \tag{6.8}$$

$$\le \bar{\varepsilon} + P^{x/h}\left(\sup_{t\ge 0}\frac{1}{h(X_t)} \ge \frac{\varepsilon}{\gamma(\delta)}\right),$$

where

$$\gamma(\delta) = \sup_{x\in S}\sup_{\substack{\rho(y,z)\le\delta\\ y,z\in D}} (u(x,y)h(y) - u(x,z)h(z)).$$

By the remark immediately following (6.5), with $f(x) = 1$, and by the remarks immediately following Sharpe [(1988), 7.1], we see that $1/h(X_t)$ is a right continuous nonnegative supermartingale with respect to $P^{x/h}$. Therefore,

$$P^{x/h}\left(\sup_{t\ge 0}\frac{1}{h(X_t)} \ge \frac{\varepsilon}{\gamma(\delta)}\right) \le \frac{\gamma(\delta)}{\varepsilon}E^{x/h}\left(\frac{1}{h(X_0)}\right) = \frac{\gamma(\delta)}{\varepsilon h(x)}. \tag{6.9}$$

Since both h and u are bounded and uniformly continuous on S, by choosing $\delta > 0$ sufficiently small we can make the right-hand side of (6.9) less than $\bar{\varepsilon}$. By this observation, (6.8) and (6.9), and taking the limit over a sequence of finite sets increasing to D, we see that for any ε and $\bar{\varepsilon} > 0$ we can find a $\delta > 0$

such that

$$P^{x/h}\left(\sup_{t\geq 0}\sup_{\substack{\rho(y,z)\leq\delta\\ y,z\in D}} L_t^y - L_t^z \geq 2\varepsilon\right) \leq 2\bar{\varepsilon}. \tag{6.10}$$

It follows by the Borel–Cantelli lemma that we can find a sequence $\{\delta_i\}_{i=1}^{\infty}$, $\delta_i > 0$, such that $\lim_{i\to\infty}\delta_i = 0$ and

$$\sup_{t\geq 0}\sup_{\substack{\rho(y,z)\leq\delta_i\\ y,z\in D}} L_t^y - L_t^z \leq \frac{1}{2^i} \tag{6.11}$$

for all $i \geq I(\omega)$, almost surely with respect to $P^{x/h}$.

Fix $T < \infty$ and a compact set $K \subset S$. We will now show that L_t^y is uniformly continuous on $[0, T] \times (K \cap D)$ almost surely with respect to $P^{x/h}$. That is, for each ω in a set of full measure with respect to $P^{x/h}$, we can find an $I(\omega)$ such that for $i \geq I(\omega)$,

$$\sup_{\substack{|s-t|\leq\delta'\\ s,t\in[0,T]}}\sup_{\substack{\rho(y,z)\leq\delta_i'\\ y,z\in D\cap K}} |L_s^y - L_t^z| \leq \frac{1}{2^i}, \tag{6.12}$$

where $\{\delta_i'\}_{i=1}^{\infty}$ is a sequence of real numbers such that $\delta_i' > 0$ and $\lim_{i\to\infty}\delta_i' = 0$.

To prove (6.12), fix ω and assume that $i \geq I(\omega)$, so that (6.11) holds. Let $Y = \{y_1, \dots, y_n\}$ be a finite subset of $K \cap D$ such that

$$K \subset \bigcup_{j=1}^{n} B(y_j, \delta_{i+2}),$$

where $B(y, \delta)$ is a ball of radius δ and center y. Since each $L_t^{y_j}(\omega)$, $j = 1, \dots, n$, is uniformly continuous on $[0, T]$, we can find a finite increasing sequence $t_1 = 0, t_2, \dots, t_{k-1} < T$, $t_k \geq T$, such that $t_m - t_{m-1} = \delta''_{i+2}$ for all $m = 1, \dots, k$, where δ''_{i+2} is chosen so that

$$|L_{t_{m+1}}^{y_j}(\omega) - L_{t_{m-1}}^{y_j}(\omega)| \leq \frac{1}{2^{i+2}} \qquad \forall j = 1, \dots, n, \quad \forall m = 1, \dots, k-1. \tag{6.13}$$

Let $s_1, s_2 \in [0, T]$ and assume that $s_1 \leq s_2$ and that $s_2 - s_1 \leq \delta''_{i+2}$. There exists an $1 \leq m \leq k-1$ such that

$$t_{m-1} \leq s_1 \leq s_2 \leq t_{m+1}.$$

Assume also that $y, z \in K \cap D$ satisfy $\rho(y, z) \leq \delta_{i+2}$. We can find a $y_j \in Y$ such that $y \in B(y_j, \delta_{i+2})$. If $L_{s_2}^y(\omega) \geq L_{s_1}^z(\omega)$, we have

$$\begin{aligned} |L_{s_2}^y(\omega) - L_{s_1}^z(\omega)| &\leq |L_{t_{m+1}}^y(\omega) - L_{t_{m-1}}^z(\omega)| \\ &\leq |L_{t_{m+1}}^y(\omega) - L_{t_{m+1}}^{y_j}(\omega)| + |L_{t_{m+1}}^{y_j}(\omega) - L_{t_{m-1}}^{y_j}(\omega)| \\ &\quad + |L_{t_{m-1}}^{y_j}(\omega) - L_{t_{m-1}}^{y}(\omega)| + |L_{t_{m-1}}^{y}(\omega) - L_{t_{m-1}}^{z}(\omega)|, \end{aligned} \tag{6.14}$$

where we use the fact that L_t^y is nondecreasing in t. The second term to the right of the last inequality in (6.14) is less than or equal to $2^{-(i+2)}$ by (6.13). The other three terms are also less than or equal to $2^{-(i+2)}$ by (6.11), since $\rho(y, y_j) \le \delta_{i+2}$ and $\rho(y, z) \le \delta_{i+2}$. Taking $\delta_i' = \delta_{i+2}'' \wedge \delta_{i+2}$, we get (6.12) on the larger set $[0, T'] \times (K \cap D)$ for some $T' \ge T$. Obviously, this implies (6.12) as stated in the case when $L_{s_2}^y(\omega) \ge L_{s_1}^z(\omega)$. A similar argument gives (6.12) when $L_{s_2}^y(\omega) \le L_{s_1}^z(\omega)$. Thus (6.12) is established.

In what follows we say that a function is locally uniformly continuous on a measurable set in a locally compact metric space if it is uniformly continuous on all compact subsets of the set. Let K_n be a sequence of compact subsets of S such that $S = \bigcup_{n=1}^{\infty} K_n$. Let

$$\hat{\Omega} = \{\omega | L_t^y(\omega) \text{ is locally uniformly continuous on } [0, \zeta) \times (S \cap D)\}. \tag{6.15}$$

Let $\mathscr{R}$ denote the rational numbers. Then

$$\hat{\Omega}^c = \bigcup_{\substack{s \in \mathscr{R} \\ 1 \le n \le \infty}} \{\omega | L_t^y(\omega) \text{ is not uniformly continuous on } [0, s] \times (K_n \cap D);\ s < \zeta\}.$$

It follows from (6.12) and (4.23), using the fact that $h > 0$, that $P^x(\hat{\Omega}^c) = 0$ for all $x \in S$, or equivalently, that

$$P^x(\hat{\Omega}) = 1 \qquad \forall\, x \in S. \tag{6.16}$$

We now construct a stochastic process $\hat{L} = \{\hat{L}_t^y, (t, y) \in R^+ \times S\}$ which is continuous on $[0, \zeta) \times S$ and which is a version of L. For $\omega \in \hat{\Omega}$, let $\{\tilde{L}_t^y(\omega), (t,y) \in [0, \zeta) \times S\}$ be the continuous extension of $\{L_t^y(\omega), (t, y) \in [0, \zeta) \times (S \cap D)\}$ to $[0, \zeta) \times S$. Set

$$\begin{aligned} \hat{L}_t^y(\omega) &= \tilde{L}_t^y(\omega), && \text{if } t < \zeta(\omega), \\ \hat{L}_t^y(\omega) &= \liminf_{\substack{s \uparrow \zeta(\omega) \\ s \in \mathscr{R}}} \tilde{L}_t^y(\omega), && \text{if } t \ge \zeta(\omega), \end{aligned} \tag{6.17}$$

and for $\omega \in \hat{\Omega}^c$, set

$$\hat{L}_t^y \equiv 0 \qquad \forall\, t, y \in R^+ \times S. \tag{6.18}$$

$\{\hat{L}_t^y, (t, y) \in R^+ \times S\}$ is a well-defined stochastic process which, clearly, is jointly continuous on $[0, \zeta) \times S$. We now show that $\hat{L}$ satisfies (6.1). Recall that for each $z \in D$, $\{L_t^z, t \in R^+\}$ is increasing almost surely with respect to P^x. Hence, the same is true for $\{\tilde{L}_t^y, t \in R^+\}$ and so the limit inferior in (6.17) is actually a limit, almost surely with respect to P^x. Thus $\{\hat{L}_t^y, t \in R^+\}$ is continuous and constant for $t \ge \zeta$, almost surely with respect to P^x. Similarly, $L_y'^t$, the local time for X at y is, by definition, continuous in t and constant for $t \ge \zeta$, almost surely with respect to P^x. Now let us note that we could just as well have obtained (6.12) with D replaced by $D \cup \{y\}$ and hence obtained (6.16) with D replaced by $D \cup \{y\}$ in the definition of $\hat{\Omega}$. Therefore if we take a sequence $\{y_i\}_{i=1}^{\infty}$ with $y_i \in D$ such that $\lim_{i\to\infty} y_i = y$ we have that

$$\lim_{i\to\infty} L_t^{y_i} = L_t^y \quad \text{locally uniformly on } [0,\zeta) \text{ a.s. } P^x.$$

By the definition of $\hat{L}$, we also have

$$\lim_{i\to\infty} L_t^{y_i} = \hat{L}_t^y \quad \text{locally uniformly on } [0,\zeta) \text{ a.s. } P^x.$$

This shows that

$$\hat{L}_t^y = L_t^y \qquad \forall\, t < \zeta \text{ a.s. } P^x. \tag{6.19}$$

Now, since $\hat{L}_t^y$ and L_t^y are continuous in t and constant for $t \ge \zeta$, we get (6.1). This completes the proof of Theorem 6.1. □

THEOREM 6.2. *Let X be a strongly symmetric standard Markov process with finite 1-potential density $u^1(x,y)$ and let $\{G(y),\, y \in S\}$ be the associated Gaussian process. If $\{G(y),\, y\in S\}$ has continuous sample paths, we can find a version of the local time of X which is jointly continuous on $[0,\zeta)\times S$.*

PROOF. Let K be a compact subset of S such that K is the closure of its interior (i.e., $K = \overline{K^0}$). Let $Z = \{Z_t, \tilde{P}^x\}$ be the Markov process obtained by killing X at T_{K^c}, where T_{K^c} is defined in (3.27), so that ζ_z, the lifetime of Z, is equal to T_{K^c}. The transition probability and potential operators of Z are given by Q_t and V^α, defined in (3.29) and (3.47), restricted to K^0. We have

$$\tilde{P}^x(F 1_{t<\zeta_z}) = P^x(F 1_{t<T_{K^c}}) \qquad \forall\, F \in \mathscr{F}_t. \tag{6.20}$$

Let $\alpha = 1$. Under the assumptions of this theorem we showed in (3.32) and Lemma 3.10 that V^1 has a 1-potential density $\tilde{v}^1(x,y)$ which is the covariance of a continuous Gaussian process and which, in particular, is continuous on $S\times S$ and is equal to zero if either x or y is contained in $\overline{K^c}$. Let $v^1(x,y)$ denote the restriction of $\tilde{v}^1(x,y)$ to $K^0\times K^0$. By the above, $v^1(x,y)$ is bounded and uniformly continuous. Note that $v^1(x,y)$ is a finely continuous density for $V^1(x,dy)$ and hence is the canonical density for Z defined by (3.9).

Define

$$\tilde{L}_t^y = L_{t\wedge T_{K^c}}^y, \qquad y \in K^0. \tag{6.21}$$

This is clearly a CAF for Z. We now show that $\tilde{L}_t^y$ is a local time for Z. By (6.16) we have, for $x,y\in K^0$,

$$\begin{aligned}
\tilde{E}^x\left(\int_0^\infty e^{-t}\,d\tilde{L}_t^y\right) &= \lim_{n\to\infty}\sum_{i=1}^{\infty}\tilde{E}^x\left(\left(\int_{(i-1)/2^n}^{i/2^n} e^{-t}\,d\tilde{L}_t^y\right)1_{\{i/2^n<\zeta_z\}}\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{\infty}E^x\left(\left(\int_{(i-1)/2^n}^{i/2^n} e^{-t}\,dL_t^y\right)1_{\{i/2^n<T_{K^c}\}}\right)\\
&= E^x\left(\int_0^{T_{K^c}} e^{-t}\,dL_t^y\right) = E^x\left(\int_0^\infty e^{-t}\,dL_t^y\right) - E^x\left(\int_{T_{K^c}}^\infty e^{-t}\,dL_t^y\right)\\
&= u^1(x,y) - P_{K^c}^1 u^1(x,y) = v^1(x,y).
\end{aligned}$$

It follows from Blumenthal and Getoor [(1968), IV 2.13] that $\tilde{L}_t^y$ is a local time for Z.

Let Y be the Markov process obtained by killing Z at an independent exponential time, that is, Z is killed at time λ, where $P(\lambda \geq u) = \exp(-u)$. The 0-potential density for Y is the 1-potential density for Z. Thus we have a Markov process Y with state space K^0 and bounded, uniformly continuous 0-potential density $v^1(x, y)$ on $K^0 \times K^0$. This process satisfies the hypotheses of Theorem 6.1 and it is easy to see that $\tilde{L}_{t \wedge \lambda}^y$ is a local time for Y. It now follows from (6.21) and the proof of Theorem 6.1, in particular from (6.15) and (6.16), that L_t^y is locally uniformly continuous on $[0, T_{K^c} \wedge \lambda) \times D \cap K^0$ almost surely with respect to $P^x \times \mu$, where, as in Section 4, μ is the probability measure of λ and D is a countable dense subset of S. Next, we see by Fubini's theorem that L_t^y is locally uniformly continuous on $[0, T_{K^c} \wedge q_i) \times D \cap K^0$ for all $q_i \in Q$ almost surely with respect to P^x, where Q is a countable dense subset of R^+. Therefore, we have that L_t^y is locally uniformly continuous on $[0, T_{K^c}) \times D \cap K^0$ almost surely with respect to P^x.

Let $K_n = \overline{K_n^0} \subset K_{n+1}^0$ be an increasing sequence of compact sets such that $\bigcup_{n=1}^{\infty} K_n = S$. By Blumenthal and Getoor [(1968), I 9.3], $\lim_{n \to \infty} T_{K_n^c} = \zeta$. Therefore, by the above argument with $K = K_n$ for each n, we see that L_t^y is locally uniformly continuous on $[0, \zeta) \times D$ almost surely with respect to P^x. We now complete the proof exactly as in Theorem 6.1, beginning with the paragraph immediately following (6.16). □

In Theorem 6.2, we obtained conditions for the joint continuity of the local time on $[0, \zeta) \times S$. Clearly, the local time is constant for $t \geq \zeta$. Theorem 6.3, which is a consequence of the fact that the Markov processes that we are considering are Hunt processes, enables us to show that the local time remains continuous as $t \to \zeta$. The ideas for the next proof were given to us by P. Fitzsimmons.

THEOREM 6.3. *Let X be a strongly symmetric standard Markov process with finite 1-potential density $u^1(x, y)$ and let $G = \{G(y), y \in S\}$ be the associated Gaussian process. If $\{G(y), y \in S\}$ has continuous sample paths, we can find a version of $\{L_t^y, (t, y) \in R^+ \times S\}$, the local time of X, which is jointly continuous on $R^+ \times S$.*

PROOF. Since G is continuous, its covariance $u^1(x, y)$ is continuous. Therefore, by Theorem 3.8, X is a Hunt process. [Note that the assumption that $u^1(x, y)$ exists implies that the measure m on the state space of X is a reference measure.] We first assume that $u(x, y) = u^0(x, y)$ is uniformly continuous and bounded on S. Let ζ_i be the totally inaccessable part of ζ, the lifetime of X, and let

$$C_t = \left(1_{[\zeta_i, \infty)}\right)^p(t), \qquad t \in R^+, \tag{6.22}$$

be the dual predictable projection of $1_{(\zeta_t,\infty]}(t)$. By Sharpe [(1988), Section 35 and page 392], C_t is a CAF.

Let ν_c denote the Revuz measure of $\{C_t,\ t \in R^+\}$. Thus ν_c is a σ-finite measure on S with the property that

$$(6.23)\qquad E^x\left(\int_0^\infty g(X_s)\,dC_s\right) = \int u(x,y)g(y)\,d\nu_c(y)$$

for all $g \in b\mathscr{S}$ [Revuz (1970)]. Let $f \in b\mathscr{S}$ be strictly positive and in $L^1(m + \nu_c)$ and set

$$(6.24)\qquad h(x) = \int u(x,y)f(y)\,d(m+\nu_c)(y).$$

We will show below that

$$(6.25)\qquad P^x(\cdot,\zeta_i < \infty) \ll P^{x/h}(\cdot), \qquad \forall\, x \in S,$$

where $P^{x/h}$ is the probability measure of the h-transform of X as defined between (4.22) and (4.23) for h as given in (6.24).

We now show that this theorem follows from (6.25). Note that h is bounded, strictly positive and the potential of the finite measure $f d(m+\nu_c)$. Therefore, following the proof of Theorem 6.1, we can obtain (6.12) for the local time of X almost surely with respect to $P^{x/h}$. By (6.25) and the fact that (6.12) holds for all $T > 0$, we see that

$$(6.26)\qquad \{L_t^y(\omega),\ (t,y) \in R^+ \times D\} \text{ is locally uniformly continuous}$$

whenever $\omega \in \{\zeta_i < \infty\}$, except possibly on a set of P^x measure zero. To remove the restriction that $\omega \in \{\zeta_i < \infty\}$, we note that by Sharpe [(1988), (44.5) and (47.10)(iii)],

$$(6.27)\qquad \zeta_i(\omega) = \begin{cases} \zeta(\omega), & \text{if } X_{\zeta-} \in S,\\ \infty, & \text{otherwise,}\end{cases}$$

so that if $\zeta_i(\omega) = \infty$, then either $\zeta(\omega) = \infty$ or $X_{\zeta-} = \Delta$. In either case, the fact that the proof of Theorem 6.1 shows that almost surely with respect to P^x,

$$\{L_t^y,\ (t,y) \in [0,\zeta) \times D\}$$

is locally uniformly continuous, implies that (6.26) holds even when $\zeta_i = \infty$. (Here we use the fact that L_t^y only increases for those t such that $X_t = y$.) Having established that (6.26) holds almost surely with respect to P^x, we use this and follow the proof of Theorem 6.2 to show that (6.26) also holds under the hypotheses on u^1 given in this theorem and we can obtain a jointly continuous extension of (6.26) to $R^+ \times S$. Finally, we proceed as in Theorem 6.1 to show that the extension is a local time for X.

It remains to establish (6.25). We first define the CAF,

$$A_t = \begin{cases} t + C_t, & t \le \zeta,\\ \zeta + C_\zeta, & t \ge \zeta.\end{cases}$$

By the absolute continuity theorem [Blumenthal and Getoor (1968), V 2.6], there is a function $a(\cdot) \in b\mathscr{S}$ such that

$$C_t = \int_0^t a(X_s)\, dA_s. \tag{6.28}$$

Let k_s, $s \in R^+$, denote the usual killing operators [Sharpe (1988), 11.3]. Let F be a random variable on (Ω, P^x). We have

$$\begin{aligned} E^x\big(Ff(X_{\zeta-}), \zeta_i < \infty\big) &= E^x\big(F \circ k_{\zeta_i} f(X_{\zeta_i -}), \zeta_i < \infty\big) \\ &= E^x\left(\int_0^\infty F \circ k_s f(X_{s-})\, d1_{[\zeta_i,\infty)}(s)\right) \\ &= E^x\left(\int_0^\infty F \circ k_s f(X_{s-})\, dC_s\right), \end{aligned} \tag{6.29}$$

since $F \circ k_s f(X_{s-})$ is predictable [Sharpe (1988), Section 11] and C_s is the dual predictable projection of $1_{[\zeta_i,\infty)}(s)$. By (6.28) the last line in (6.29) equals

$$\begin{aligned} &E^x\left(\int_0^\infty F \circ k_s f(X_{s-}) a(X_s)\, dA_s\right) \\ &\quad = E^x\left(\int_0^\infty F \circ k_s a(X_{s-}) f(X_s)\, dA_s\right) \\ &\quad = E^x\left(\int_0^\infty F \circ k_s a(X_{\zeta-}) \circ k_s f(X_s)\, dA_s\right), \end{aligned} \tag{6.30}$$

since A_s is continuous and $\{s | X_s \neq X_{s-}\}$ is countable. Note that by (6.23),

$$\begin{aligned} E^x\left(\int_0^\infty f(X_s)\, dA_s\right) &= E^x\left(\int_0^\infty f(X_s)\, ds + \int_0^\infty f(X_s)\, dC_s\right) \\ &= \int u(x,y) f(y) d(m(y) + \nu_c(y)) = h(x) \end{aligned} \tag{6.31}$$

so that, in the notation of Sharpe [(1988), page 299], $h = u_B$, where

$$B_t = \int_0^t f(X_s)\, dA_s.$$

It now follows from Sharpe [(1988), 62.24] that the last term in (6.30) equals

$$E^{x/h}\big(Fa(X_{\zeta-})\big)h(x). \tag{6.32}$$

[Note that there is an obvious typographical error in Sharpe (1988), 62.24.] Since $f(X_{\zeta-}) > 0$ on $\zeta_i < \infty$, we see by the equality of the first line of (6.29) with (6.32) that (6.25) holds. This completes the proof of Theorem 6.3. □

The following variation of Theorem 6.1 will be used in the proof of Theorem 2.

THEOREM 6.4. *Let X be a strongly symmetric standard Markov process as defined in Section 3 but with the following additional condition: that*

$u(x,y) = u^0(x,y)$, given by (3.9), is bounded and uniformly continuous on $(S \times S)$. Let K be a compact subset of S. Then if $u(x,y)$ is the covariance of a mean zero continuous Gaussian process $\{G(y),\, y \in K\}$, we can find a version of the local time of X which is jointly continuous on $[0,\zeta) \times K$.

PROOF. The proof follows the proof of Theorem 6.1 except that following (6.5) we take D to be a countable dense subset of K. In going from (6.6) to (6.8), the key step, we only need the hypothesis that the Gaussian process is continuous on K. □

The next theorem is a modification of Theorem 6.1 which allows us to treat boundedness of the local time.

THEOREM 6.5. *Let X be a strongly symmetric standard Markov process as defined in Section 3 but with the following additional condition: that $u(x,y) = u^0(x,y)$, given by (3.9), is bounded and uniformly continuous on $(S \times S)$. Let K be a compact subset of S and D a countable dense subset of K. Then, if $u(x,y)$ is the covariance of a mean zero bounded Gaussian process $\{G(y),\, y \in D \cap K\}$, the local time of X is locally bounded on $[0,\zeta) \times D \cap K$ almost surely.*

PROOF. This proof is a rather obvious modification of the proof of Theorem 6.1. (Unless otherwise noted the following notation is the same as in Theorem 6.1.) Following (6.5) we take D to be a countable dense subset of K. In place of (6.6), we use

$$P^{x/h}\Big(\sup_{t\geq 0}\sup_{y\in F} A_t^y \geq M\Big) \leq \frac{1}{M}E^{x/h}\Big(\sup_{y\in F} L_\infty^y\Big) \leq \frac{1}{M}E^{x/h}\Big(\sup_{y\in D\cap K} L_\infty^y\Big), \tag{6.33}$$

where $M > 0$. Using the isomorphism theorem as in (6.7), but applied to the supremum of $G^2(y)/2$ over $D \cap K$, we see that analogously to (6.8) and (6.9) we have

$$P^{x/h}\Big(\sup_{t\geq 0}\sup_{y\in F} L_t^y \geq 2M\Big) \leq \frac{C}{M} + P^{x/h}\Big(\sup_{t\geq 0}\frac{1}{h(X_t)} \geq \frac{M}{\bar{\gamma}}\Big) \leq \frac{C}{M} + \frac{\bar{\gamma}}{Mh(x)}, \tag{6.34}$$

where C is finite by (2.4) since $\{G(y),\, y \in D \cap K\}$ is assumed to be bounded. Furthermore,

$$\bar{\gamma} \equiv \sup_{x\in S}\sup_{y\in D\cap K} u(x,y)h(y) < \infty. \tag{6.35}$$

It follows by (6.34) and (6.35) that

$$\sup_{t\geq 0}\ \sup_{y\in D\cap K} L_t^y < \infty \quad \text{a.s. } P^{x/h}.$$

An argument analogous to the one employed in (6.15) and (6.16) shows that

$$\sup_{0\leq t\leq T}\ \sup_{y\in D\cap K} L_\infty^y < \infty \quad \text{for all } T<\zeta \text{ a.s. } P^x.$$

This completes the proof of Theorem 6.5. □

The next theorem is a modification of Theorem 6.1 which allows us to treat continuity of the local time at a point in the state space of the Markov process.

THEOREM 6.6. *Let X be a strongly symmetric standard Markov process as defined in Section 3 but with the following additional condition: that $u(x,y)=u^0(x,y)$, given by (3.9), is bounded and uniformly continuous on $(S\times S)$. Let $\{L_t^y,\ (t,y)\in R^+\times S\}$ be the local time of X. If $u(x,y)$ is the covariance of a mean zero Gaussian process $\{G(y),\ y\in S\}$, which is continuous at a point $y_0\in S$, then for any countable subset D of S with $y_0\in D$, $\{L_t^y,\ (t,y)\in R^+\times D\}$ is continuous at y_0 for all $t\in[0,\zeta)$ almost surely.*

PROOF. The proof follows the proof of Theorem 6.1 except that following (6.5) we take the supremum over $\{y,\ z\in F|\rho(y,y_0)\leq\delta,\ \rho(z,y_0)\leq\delta\}$. In (6.7), the key relationship, we use

$$\lim_{\delta\to 0} E_G\left(\sup_{\substack{\rho(y,y_0)\leq\delta,\,\rho(z,y_0)\leq\delta\\ y,z\in D}}\left|\frac{G^2(y)}{2}-\frac{G^2(z)}{2}\right|^2\right)=0,$$

which follows from (2.26). Following the proof of Theorem 6.1 and making the obvious changes we get the proof of this theorem. □

7. Proofs of Theorems 1–8. All the ingredients for the proofs of these theorems have been assembled in Sections 5 and 6. Nevertheless, it seems useful to go over the proof of each of these theorems and make clear which of the above results are used in each case.

PROOF OF THEOREM 1. If $\{G(y),\ y\in S\}$ is continuous almost surely, then it follows from Theorem 6.3 that $\{L_t^y,\ (t,y)\in R^+\times S\}$ is continuous almost surely. Conversely, suppose that $L=\{L_t^y,\ (t,y)\in R^+\times S\}$ is continuous almost surely. It follows by Theorem 3.7 that $u^1(x,y)$ is continuous on $S\times S$. Let K be a compact subset of S. Then $\beta(y)=0$ for all $y\in K$, where β is the oscillation function of G defined in Theorem 2.5, since if $\beta(y_0)>0$ for some $y_0\in K$, then either Theorem 5.1 or Theorem 5.4 shows that L has either a bounded discontinuity at y_0 or is unbounded at y_0 almost surely with respect to P^{y_0}. By (2.38) of Lemma 2.8, we see that G is continuous almost surely on

K. Since this is true for all compact subsets of S, we see that the continuity of L implies the continuity of G. □

PROOF OF THEOREM 2. Assume that $\{G(y),\ y \in K\}$ is continuous. To show that L_t^y is jointly continuous on $R^+ \times K$, we repeat the proofs of Theorems 6.1–6.3, noting several rather obvious changes. The modification of Theorem 6.1 that we will use is given as Theorem 6.4. Consider the proof of Theorem 6.2. In the first four paragraphs of this proof the compact set K is used in two contexts. To determine the set K^c on which X is killed and to determine the domain of $L_t^{\cdot}$, let $K_1 \subset S$ be a compact set such that $K \subset K_1^0$, where K is the set in the hypothesis of this theorem (Theorem 2), and reconsider the first four paragraphs of the proof of Theorem 6.2 with X killed when it hits K_1^c but with K still the domain of $L_t^{\cdot}$. Also, in these paragraphs, replace Theorem 6.1 by Theorem 6.4 and understand the statement "proof of Theorem 6.4" to mean the minor alteration of the proof of Theorem 6.1 which proves Theorem 6.4. Making these changes we obtain, in place of the final sentence in the fourth paragraph of Theorem 6.2, the statement, "Therefore, we have that L_t^y is locally uniformly continuous on $[0, T_{K_1^c}) \times D \cap K$." We conclude our analogy with the proof of Theorem 6.2 as in the final paragraph of that proof but whereas we take $K_n \uparrow S$ in defining $T_{K_n^c}$, we continue to restrict the domain of $L_t^{\cdot}$ to K. Thus we get that under the hypothesis of this theorem, L_t^y is jointly continuous on $[0, \zeta) \times K$ almost surely.

As in the extension of Theorem 6.1 to Theorem 6.3, the continuity of $u^1(x, y)$ along with Theorem 3.8 shows that X is a Hunt process. An obvious analogue of Theorem 6.3 now shows that L_t^y is jointly continuous on $R^+ \times K$ almost surely.

For the converse, suppose that $L = \{L_t^y,\ (t, y) \in R^+ \times K\}$ is continuous almost surely. The argument given in the second half of the proof of Theorem 1 shows that G is continuous on K. □

PROOF OF THEOREM 3. Assume that $\{G(y),\ y \in K\}$ is bounded. Thus Theorem 6.5 holds. We adapt the argument of Theorem 6.2 to Theorem 6.5 similarly to the way we adapted it to Theorem 6.4 in the proof of Theorem 2. Again we distinguish between the sets K_n which define $T_{K_n^c}$, the time at which X is killed, and the fixed set K such that $D \cap K$ is the domain of $L_t^{\cdot}$. Following the argument of Theorem 6.2, first considering the Markov process Z and then Y and using it in Theorem 6.5 with some fixed K_n (in the notation of Theorem 6.2), we get that L_t^y is locally bounded on $[0, T_{K_n^c} \wedge \lambda) \times (D \cap K^0)$ almost surely with respect to $P^x \times \mu$, where λ and μ are as in Theorem 6.2. [We note that the bound in (6.34) does depend on the set K_n since we have $x \in K_n^0$.] Passing to the limit as $K_n \uparrow S$ and noting that $\lim_{n \to \infty} T_{K_n^c} = \zeta$, we obtain that L_t^y is locally bounded on $[0, \zeta \wedge \lambda) \times (D \cap K^0)$ almost surely with respect to $P^x \times \mu$, from which we get, by Fubini's theorem, that L_t^y is bounded on $[0, T] \times D \cap K$ for all $T < \zeta$ almost surely with respect to P^x. By the continuity of $u^1(x, y)$ and Theorem 3.8, we have that X is a Hunt process and the argument of Theorem 6.3 enables us to remove the condition $T < \zeta$.

This completes the assertion that the boundedness of G implies the boundedness of the local time.

Suppose that $\{G(y),\ y \in D \cap K\}$ is unbounded on a set of positive probability. It then follows from (2.39) and Theorem 5.4 that $\{L_t^y,\ (t,y) \in [0,T] \times D \cap K\}$ is unbounded for all $T > 0$. This contradiction establishes the converse. $\square$

Proof of Theorem 4. Consider $\beta(y_0)$, the oscillation function of G at y_0. Since $\beta(y_0)$ is either zero, greater than zero but finite, or infinite, it follows from Theorem 2.5 that one and only one of the following three possibilities holds:

$$G \text{ is continuous at } y_0 \text{ almost surely.} \tag{7.1}$$

$$G \text{ has a bounded discontinuity at } y_0 \text{ almost surely.} \tag{7.2}$$

$$G \text{ is unbounded at } y_0 \text{ almost surely.} \tag{7.3}$$

Therefore, to prove this theorem it is sufficient to show that each of the possibilities (7.1)–(7.3) implies the corresponding behavior of the local time stated in this theorem. The case governed by (7.3) immediately follows from Theorem 5.4. Suppose that (7.1) holds. Then we have Theorem 6.6. We extend this result by following the proof of Theorem 6.2 as we did in the proof of Theorem 2, taking $K_n \uparrow S$ in defining $T_{K_n^c}$ while considering the continuity of L_t^y at y_0. An analogue of Theorem 6.3 shows that L_t^y is continuous at y_0 almost surely for each $t > 0$.

Before considering the case governed by (7.2), let us note that if $\beta(y_0) < \infty$, then since it is upper semicontinuous, it is bounded in some compact neighborhood K of y_0. Therefore, by Lemma 2.8, G is bounded on K almost surely and by Theorem 3, L_t^y is bounded at y_0. Now suppose that (7.2) holds. Since $\beta(y_0) < \infty$ in this case, we see from the above remarks that L_t^y is bounded at y_0. On the other hand, since $\beta(y_0) > 0$, it follows from the left-hand side of (5.1) that L_t^y has a bounded discontinuity at y_0 for $t \in [0, \zeta)$, almost surely with respect to P^{y_0}. However, it is easy to see by taking the limit as $t \uparrow \zeta$ that the left-hand side of (5.1) remains valid for $t = \zeta$. Since L_t^y is constant for $t > \zeta$, we see that under (7.2), L_t^y has a bounded discontinuity at y_0 for all $t > 0$ almost surely with respect to P^{y_0}. $\square$

Proof of Theorem 5. By Theorem 2, if $G = \{G(y),\ y \in K\}$ is continuous almost surely we can find a version of the local time which is continuous almost surely. If $\{G(y),\ y \in K\}$ is not continuous, then by Lemma 2.8 we have that the oscillation function of G at x_0, $\beta(x_0)$, > 0 for some $x_0 \in K$. This implies, by Theorems 5.1 and 5.4 along with the fact that G is separable on any countable dense subset of K, that the event "$\{L_t^y,\ (t,y) \in R^+ \times D\}$ is continuous" has P^{x_0} measure zero. $\square$

PROOF OF THEOREM 6. This theorem follows from Theorem 4, Lemma 2.8 and Theorem 5.4 by essentially the same argument used in the proof of Theorem 5. □

PROOF OF THEOREM 7. If L_t^y is not continuous on $R^+ \times K$ almost surely, then by Theorem 2 the associated Gaussian process $\{G(y),\ y \in S\}$ is not continuous on K. Thus by Lemma 2.8, $\beta(y) > 0$ for some $y \in K$. Theorem 7, part (7.1), now follows readily from Theorems 5.1 and 5.4. Theorem 7, part (7.2), follows similarly, beginning with Theorem 3. □

PROOF OF THEOREM 8. This proof is identical to the proof of Theorem 7, part 7.1. □

8. Necessary and sufficient conditions for continuity and boundedness of Gaussian processes. In this paper we show how sample path properties of the local time of strongly symmetric standard Markov processes can be obtained from corresponding properties of Gaussian processes. The main point of all this is that, in most cases, necessary and sufficient conditions are known from these properties of the Gaussian processes and therefore we now have them for the local times. In full generality these conditions are complex. However, after a general discussion, we will introduce some simplifying assumptions and obtain results that are tractable and often easy to apply.

The question of necessary and sufficient conditions for continuity and boundedness of Gaussian processes has recently been completely settled by Talagrand [see, e.g., Talagrand (1987) and Ledoux and Talagrand (1991)]. For a real-valued Gaussian process $G = \{G(y),\ y \in S\}$, where S is some index set, there is a natural pseudometric

$$d = d(x,y) = \left(E(G(x) - G(y))^2\right)^{1/2} \tag{8.1}$$

on S. [In all that follows we will assume that $EG(y) = 0$ for all $y \in S$.] The fundamental result on the continuity of sample paths of G is obtained when (S,d) is a compact metric (or pseudometric) space. However, for Gaussian processes which are associated with local times, we see from Lemma 3.6 that $d(x,y) > 0$ whenever $x \neq y$ so (S,d) is a compact metric space. The following theorem restates some of the results in Ledoux and Talagrand [(1991), Theorem 11.18] and Talagrand [(1987, Theorem 1].

THEOREM 8.1. *Let $G = \{G(y),\ y \in S\}$ be a mean zero Gaussian process. If there exists a probability measure m on (S,d) such that*

$$\sup_{y \in S} \int_0^{\operatorname{diam} S} \left(\log \frac{1}{m(B(y,\varepsilon))}\right)^{1/2} d\varepsilon < \infty \tag{8.2}$$

[*where* diam S *denotes the diameter of* S *and* $B(y,\varepsilon)$ *is defined analogously to*

(2.28)], *then* $\sup_{y \in S} G(y) < \infty$ *almost surely. If, in addition,*

$$\lim_{\delta \to 0} \sup_{y \in S} \int_0^{\delta} \left(\log \frac{1}{m(B(y,\varepsilon))} \right)^{1/2} d\varepsilon = 0, \tag{8.3}$$

then G has a version with continuous sample paths. Furthermore, both (8.2) *and* (8.3) *are also necessary conditions in the sense that if there does not exist a probability measure m on (S, d) for which* (8.2) [*resp.*, (8.3)] *holds, then* $\sup_{y \in S} G(y) = \infty$ *almost surely* (*resp., there is no continuous version of G*).

Note that if there is no continuous version of G, then the oscillation function of G cannot be identically zero on S. Therefore G has an almost surely bounded or unbounded discontinuity at some point $y_0 \in S$.

As is pointed out in Ledoux and Talagrand (1991), following the proof of Theorem 11.18, a Gaussian process is continuous on a compact metric space (S, ρ) if and only if it is continuous on (S, d) for d as given in (8.1) and d is continuous on (S, ρ). This is important for us because we do not, in general, specify the metric on the locally compact metric space which is the state space of X. [By Theorem 3.7 we need only consider the case when d is continuous on (S, ρ). Under this condition if (S, ρ) is compact, so is (S, d).]

If (S, ρ) is only locally compact, then, of course, a Gaussian process is continuous on (S, ρ) if and only if it is continuous on all compact subsets of S.

We next consider conditions for continuity and boundedness of a Gaussian process at a point $y_0 \in S$. These conditions are not explicitly stated in Talagrand (1987) or Ledoux and Talagrand (1991), although they follow directly from material presented in Talagrand (1991). We include a proof for completeness. We are grateful to M. Talagrand for a helpful discussion relating to the next proof.

THEOREM 8.2. *Let $G = \{G(y), y \in S\}$, (S, d) a compact metric space, be a mean zero Gaussian process. A necessary and sufficient condition for G to be continuous at $y_0 \in S$ is that there exists a probability measure m on (S, d) and a sequence of closed balls $D_n \subset S$ with $\bigcap_{n=1}^{\infty} D_n = \{y_0\}$ such that*

$$\lim_{n \to \infty} \sup_{y \in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{1}{m(B_n(y,\varepsilon))} \right)^{1/2} d\varepsilon = 0, \tag{8.4}$$

where

$$B_n(y, \varepsilon) = \{x \in D_n : d(x,y) \le \varepsilon\}.$$

A necessary and sufficient condition for G to be bounded at $y_0 \in S$ is that G is bounded on D_n for some n, that is, that (8.2) *holds on* (D_n, d) *for some n.*

PROOF. Assume that G is continuous at y_0. Then there exists a closed ball D_1 around y_0 such that

$$P\Big(\sup_{y \in D_1} G(y) < \infty \Big) > 0. \tag{8.5}$$

This implies by Theorem 2.7 and (2.24) that

$$E \sup_{y \in D_1} G(y) < \infty. \tag{8.6}$$

Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of closed balls satisfying the conditions of the hypothesis. It follows from Talagrand [(1987), Theorem 1] that there exist probability measures m_n on (D_n, d) such that

$$\sup_{y \in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{1}{m_n(B_n(y, \varepsilon))} \right)^{1/2} d\varepsilon \le KE \sup_{y \in D_n} G(y), \tag{8.7}$$

where K is a universal constant. We note that since

$$\sup_{y \in D_n} G(y) \le G(y_0) + \sup_{y \in D_n} |G(y) - G(y_0)|,$$

it follows from the continuity of $G(y)$ at y_0 and (2.24) that

$$\lim_{n \to \infty} E \sup_{y \in D_n} G(y) = 0. \tag{8.8}$$

Without loss of generality we can choose m_n and D_n such that $\operatorname{diam} D_n \le 2^{-n}$ and

$$\sup_{y \in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{1}{m_n(B_n(y, \varepsilon))} \right)^{1/2} d\varepsilon \le 2^{-n}. \tag{8.9}$$

Let $\tilde{m}_n$ be the probability measure on S given by

$$\tilde{m}_n(A) = m_n(A \cap D_n) \qquad \forall A \in \mathscr{S}.$$

We define the probability measure m on S by

$$m(\cdot) = \sum_{n=1}^{\infty} \frac{\tilde{m}_n(\cdot)}{2^n}$$

and note that by (8.9),

$$\begin{aligned} &\sup_{y \in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{1}{m(B_n(y, \varepsilon))} \right)^{1/2} d\varepsilon \\ (8.10)\quad &\le \sup_{y \in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{2^n}{\tilde{m}_n(B_n(y, \varepsilon))} \right)^{1/2} d\varepsilon \\ &\le (\operatorname{diam} D_n)(n \log 2)^{1/2} + \sup_{y \in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{1}{\tilde{m}_n(B_n(y, \varepsilon))} \right)^{1/2} d\varepsilon. \end{aligned}$$

Hence by (8.8) and (8.7), we get (8.4) when G is continuous at y_0. On the other hand, assume that (8.4) is satisfied for some measure m. We note that by

Ledoux and Talagrand [(1991), Theorem 11.18],

$$(8.11)\qquad E\sup_{y\in D_n} G(y) \le K \sup_{y\in D_n} \int_0^{\operatorname{diam} D_n} \left(\log \frac{1}{m(B_n(y,\varepsilon))}\right)^{1/2} d\varepsilon,$$

since m is a subprobability measure on D_n. Therefore, if (8.4) holds, we have

$$(8.12)\qquad \lim_{n\to\infty} E \sup_{y\in D_n} G(y) = 0.$$

Writing $G(y) = G(y_0) + G(y) - G(y_0)$ and noting that $EG(y_0) = 0$, we see that

$$(8.13)\qquad E\sup_{y\in D_n} G(y) = E \sup_{y\in D_n} G(y) - G(y_0) \ge \tfrac{1}{2} E \sup_{y\in D_n} |G(y) - G(y_0)|,$$

where, at the last step, we use the symmetry of $G(y) - G(y_0)$. We see from (8.12) and (8.13) that if (8.4) holds, then G is continuous at y_0 almost surely.

Suppose that $G(y)$ is bounded at y_0. Then by the same argument used at the beginning of the proof, we see that (8.5) holds for some closed ball which in this instance we can call D_n. The rest of the proof in this case now follows from (2.24). □

The integrals in Theorems 8.1 and 8.2 are difficult to evaluate in general but there are many weaker or more restrictive conditions for continuity or boundedness of Gaussian processes that are easier to verify. Of course, these results are implied by Theorems 8.1 and 8.2, although most were obtained earlier [see, e.g., Dudley (1973), Fernique (1975) and Jain and Marcus (1978)].

Necessary and sufficient conditions for continuity of stationary Gaussian processes were obtained by Dudley (1973) and Fernique (1975). These results are often given in terms of metric entropy but we will present them in an alternate form [see Jain and Marcus (1974) and Marcus and Pisier (1981), Chapter 2, Lemma 3.6], involving the nondecreasing rearrangement of $d(x,y)$.

THEOREM 8.3. *Let S be a locally compact Abelian group and $G = \{G(y),\ y\in S\}$ a mean zero stationary Gaussian process on S, that is, $EG^2(y)$ = Const. for all $y\in S$ and $d(x,y) = \sigma(x-y)$ for some function σ on S. Let K be a compact subset of S and define*

$$\mu_\sigma(\varepsilon) = \lambda\{x\in K\oplus K | \sigma(x) < \varepsilon\}$$

and

$$(8.14)\qquad \overline{\sigma(u)} = \sup\{y | \mu_\sigma(y) > u\},$$

where λ is Haar measure on S. Then G has continuous sample paths if and only if

$$(8.15\text{a})\qquad I(\bar\sigma) \equiv \int_0^{1/2} \frac{\overline{\sigma(u)}}{u(\log 1/u)^{1/2}}\, du < \infty.$$

Note also that it follows from Jain and Kallianpur [(1972), Theorem 1, (1.7)] that under the hypotheses of Theorem 8.3, G is bounded on K if and only if it is continuous on K.

Condition (8.15a) is the one given by Barlow and Hawkes [Barlow (1975), Barlow and Hawkes (1985)] for the joint continuity of the local time of a Lévy process on the real line. Let $\{X(t),\, t \in \mathbb{R}^+\}$ be a symmetric Lévy process, that is,

$$E \exp(i\lambda X(t)) = \exp(-t\psi(\lambda)), \tag{8.15b}$$

where

$$\psi(\lambda) = 2\int_0^\infty (1 - \cos \lambda u)\nu(du) \tag{8.16}$$

for ν a Lévy measure. We see from Theorem 3.2 that X has a local time if and only if $(1 + \psi(\lambda))^{-1} \in L^1(R^+)$. [See also Bretagnolle (1971) and Kesten (1969).] It is easy to see, by considering the characteristic function of X, that when $(1 + \psi(\lambda))^{-1} \in L^1(R^+)$,

$$\begin{aligned} &u^1(x,y) + u^1(y,y) - 2u^1(x,y) \\ &\qquad = \frac{2}{\pi}\int_0^\infty \frac{1 - \cos \lambda(x-y)}{1 + \psi(\lambda)}\, d\lambda \equiv \sigma^2(|x-y|). \end{aligned} \tag{8.17}$$

Thus by (1.2) and Theorem 8.3 the stationary Gaussian process with $d(x,y) = \sigma(|x-y|)$ has continuous sample paths if and only if (8.15a) holds. This is the condition in Barlow (1988), although, as we remarked in Section 1, in Barlow (1988) it is not required that X be symmetric. For the readers' interest we note that the necessary and sufficient condition for the joint continuity of the local time of a Lévy process on R^+ which is given in Barlow (1988) is still (8.15a) but with

$$\sigma^2(|x-y|) = \frac{2}{\pi}\int_0^\infty (1 - \cos \lambda(x-y))\operatorname{Re}\left(\frac{1}{1+\psi(\lambda)}\right) d\lambda,$$

where $\psi(\lambda)$ now is the exponent of the characteristic function of a real-valued, not necessarily symmetric, infinitely divisible random variable.

The next theorem gives a simple sufficient condition for continuity of Gaussian processes on a compact subset of R^n.

THEOREM 8.4. *Let $G = \{G(y),\, y \in S\}$, S a compact subset of R^n, be a mean zero Gaussian process. If there exists a nonnegative, nondecreasing function $\hat{\sigma}$ such that*

$$d(x,y) \le \hat{\sigma}(|x-y|) \tag{8.18}$$

and $I(\hat{\sigma}) < \infty$, then G has continuous sample paths.

Necessary conditions for continuity or boundedness can be approached by using comparison results such as those given in Jain and Marcus [(1978), II, Lemma 4.4].

THEOREM 8.5. *Let* $G_1 = \{G_1(y),\ y \in S\}$ *and* $G_2 = \{G_2(y),\ y \in S\}$ *be mean zero Gaussian processes with corresponding pseudometrics* $d_1(x,y)$ *and* $d_2(x,y)$ *as defined in* (8.1). *Suppose*

$$d_1(x,y) \le d_2(x,y) \qquad \forall\, x,y \in S.$$

Then if G_2 *is continuous (resp., bounded),* G_1 *is continuous (resp., bounded).*

In specific problems one can use Theorem 8.5 in conjunction with the necessary part of Theorem 8.3 to show that a Gaussian process is not continuous or unbounded.

Lastly we mention some metric entropy conditions which, in some cases, are not difficult to apply. The upper bound in the next theorem is due to Dudley (1973) and the lower bound to Sudakov (1969).

THEOREM 8.6. *Let* $G = \{G(y),\ y \in S\}$, S *some index set, be a mean zero Gaussian process. Let* $N(S,\varepsilon)$ *be the minimum number of closed balls in the pseudometric* d *that covers* S. *Then there exists a universal constant* K *such that*

$$\begin{aligned} K^{-1} \sup_{\varepsilon>0} \varepsilon(\log N(S,\varepsilon))^{1/2} &\le E \sup_{y\in S} G(y) \\ &\le K\int_0^\infty (\log N(S,\varepsilon))^{1/2}\, d\varepsilon. \end{aligned} \tag{8.19}$$

Furthermore, G *is continuous on* (S,d) *if the integral in* (8.19) *is finite.*

In (8.19) the criterion for boundedness of G is given as an inequality. Similar inequalities exist for all the other conditions that we have given—they can be found in the above references.

9. Examples and comments. In this paper we devote considerable attention to the study of local times with bounded discontinuities which come from symmetric Markov processes with continuous 1-potential densities. In part (a) of this section we will consider a class of symmetric Markov chains with an instantaneous state which will provide examples of processes with such local times. In part (b) we present the one result which we have found that uses Markov process theory to give conditions for sample path continuity of Gaussian processes which we do not think can be obtained from current results in the theory of Gaussian processes. Lastly, in part (c) we explain how our results imply some earlier ones on the joint continuity of the local times of more general Markov processes than Lévy processes.

(a) *Local times for certain Markov chains.* We will first consider a class of strongly symmetric standard Markov processes, with continuous 1-potential density and with local times that have a bounded discontinuity. These examples were suggested to us by an example of M. Barlow. They are variations of an example by Kolmogorov (1951) of a Markov chain with a single instantaneous state.

We define a Markov chain through its 1-potential. The state space of the chain is the sequence $S = \{1/2, 1/3, \dots, 1/n, \dots, 0 = 1/\infty\}$ with the topology inherited from the real line. Clearly S is a compact metric space with one limit point. Let $\{q_n\}_{n=2}^{\infty}$ and $\{r_n\}_{n=2}^{\infty}$ be strictly positive real numbers such that

$$\sum_{n=2}^{\infty} \frac{q_n}{r_n} < \infty \quad \text{and} \quad \lim_{n\to\infty} q_n = \infty. \tag{9.1}$$

We define a finite measure m on S by $m(1/n) = q_n/r_n$ and $m(0) = 1$ and a resolvent $\{U^\alpha, \alpha > 0\}$ on $C(S)$ in terms of its density $u^\alpha(x,y)$, $x, y \in S$, with respect to m. That is, for all bounded positive functions f on S,

$$U^\alpha f(x) = \int_S u^\alpha(x,y) f(y) m(dy), \tag{9.2}$$

where

$$u^\alpha(0,0) = \frac{1}{\alpha + \sum_{j=2}^{\infty}\bigl(\alpha q_j/(\alpha + r_j)\bigr)}, \tag{9.3}$$

$$u^\alpha(0,1/i) = u^\alpha(1/i,0) = u^\alpha(0,0)\frac{r_i}{\alpha + r_i},$$

$$u^\alpha(1/i,1/j) = \delta_{ij}\frac{r_j}{q_j(\alpha + r_j)} + u^\alpha(0,0)\frac{r_i}{\alpha + r_i}\frac{r_j}{\alpha + r_j}.$$

One can check that $u^\alpha(x,y)$ is symmetric and continuous on S and that U^α satisfies

$$U^\alpha: C(S) \to C(S), \tag{9.4}$$

$$\|\alpha U^\alpha\| \le 1, \qquad \alpha U^\alpha 1 = 1, \tag{9.5}$$

$$U^\alpha - U^\beta + (\alpha - \beta) U^\alpha U^\beta = 0, \tag{9.6}$$

$$\lim_{\alpha\to\infty} \alpha U^\alpha f(x) = f(x) \qquad \forall\, x \in S. \tag{9.7}$$

[For (9.7) we use the dominated convergence theorem.] It follows from Sharpe [(1988), 9.26] that $\{U^\alpha, \alpha > 0\}$ is the resolvent of a strongly symmetric standard Markov process X. By Theorem 3.2, X has a local time which we will denote by $L = \{L_t^{1/n}, (t, 1/n) \in R^+ \times S\}$. We will now examine the behavior of the local time in the neighborhood of L_t^0.

THEOREM 9.1. *Let X, S and L be as described above. Let*

$$\beta = \beta(\{q_n\}) = \limsup_{n\to\infty} \left(\frac{2 \log n}{q_n^*}\right)^{1/2}, \tag{9.8}$$

where $\{q_n^*\}_{n=2}^\infty$ *is a nondecreasing rearrangement of* $\{q_n\}_{n=2}^\infty$. *Then*:

(i)

$$(9.9)\qquad (2L_t^0)^{1/2}\beta \le \limsup_{n\to\infty} L_t^{1/n} - L_t^0 \le \frac{\beta^2}{2} + (2L_t^0)^{1/2}\beta$$

for all $t \ge 0$ *almost surely.*

(ii) *If* $\beta = 0$, $\{L_t^{1/n}\}_{n=2}^\infty$ *is continuous at* 0 *and*

$$(9.10)\qquad \limsup_{n\to\infty} \frac{L_t^{1/n} - L_t^0}{(\log n/q_n)^{1/2}} = 2(L_t^0)^{1/2} \quad \textit{for almost all } t \ge 0 \ a.s.$$

(iii) *Assume that* $\{q_n\}_{n=2}^\infty$ *is nondecreasing and* $\beta = \infty$. *Then*

$$(9.11)\qquad \limsup_{n\to\infty} \sup_{1\le k\le n} \frac{L_t^{1/k}}{(\log n/q_n)^{1/2}} \ge 2(L_t^0)^{1/2}$$

for all $t \ge 0$ *almost surely.*

In the following lemma we will collect the results on Gaussian processes that will be used in the proof of Theorem 9.1.

LEMMA 9.2. *Let* $\{\xi_n\}_{n=1}^\infty$ *be a sequence of mean zero normal random variables. Let* $\{a_n\}_{n=1}^\infty$ *and* $\{v_n\}_{n=1}^\infty$ *be strictly positive real numbers such that*

$$(9.12)\qquad \sum_{n=1}^\infty \frac{v_n}{a_n} < \infty \quad \textit{and} \quad \lim_{n\to\infty} v_n = \infty.$$

Suppose that

$$E\xi_n^2 = \frac{1}{v_n} \quad \textit{and} \quad E\xi_n\xi_m = \frac{1}{a_n a_m}, \qquad n \ne m.$$

Then, with $\{v_n^*\}_{n=1}^\infty$ *denoting a nondecreasing rearrangement of* $\{v_n\}_{n=1}^\infty$, *we have*

$$(9.13)\qquad \limsup_{n\to\infty} \xi_n = \limsup_{n\to\infty} \left(\frac{2\log n}{v_n^*}\right)^{1/2} \quad a.s.$$

Also

$$(9.14)\qquad \limsup_{n\to\infty} \frac{|\xi_n|}{(2\log n/v_n)^{1/2}} = 1 \quad a.s.$$

Furthermore, if v_n *is nondecreasing and*

$$(9.15)\qquad \limsup_{n\to\infty} \left(\frac{2\log n}{v_n}\right)^{1/2} = \infty,$$

then for any integer m *and* $0 < \varepsilon < 10^{-2}$, *there exists an* $n_0 = n_0(m, \varepsilon)$ *such* *that for all* $n \geq n_0$,

$$\text{median}\Big(\sup_{m \leq k \leq n} \xi_k\Big) \geq (1-\varepsilon)\left(\frac{2\log n}{v_n}\right)^{1/2}. \tag{9.16}$$

PROOF. Let n^* be a rearrangement of n such that $\{v_{n^*}\}$ is nondecreasing. Clearly, (9.12) remains unchanged if we sum on n^* rather than on n. Hence we will assume from now on that $\{v_n\}_{n=1}^\infty$ is nondecreasing. Let $\{\eta_n\}_{n=0}^\infty$ be independent normal random variables with mean zero and variance 1. By (9.12), we can choose N sufficiently large such that $(1/v_n - 1/a_n^2) > 0$ for all $n \geq N$. For $n \geq N$, define

$$\tilde{\eta}_n = \left(\frac{1}{v_n} - \frac{1}{a_n^2}\right)^{1/2}\eta_n + \frac{1}{a_n}\eta_0. \tag{9.17}$$

We see by (9.12) that $\{\tilde{\eta}_n\}_{n=N}^\infty$ and $\{\xi_n\}_{n=N}^\infty$ are equivalent Gaussian sequences. Therefore, with $b_n = (1/v_n - 1/a_n^2)^{1/2}$, we have

$$\limsup_{n\to\infty} \xi_n =_{\mathscr{D}} \limsup_{n\to\infty} \tilde{\eta}_n = \limsup_{n\to\infty} b_n\eta_n, \tag{9.18}$$

where $=_{\mathscr{D}}$ means "equal in distribution." The last equality follows since (9.12) implies that $\lim_{n\to\infty} 1/a_n = 0$. The last expression in (9.18) is easy to obtain because $\{\eta_n\}_{n=1}^\infty$ are independent. Using

$$P\Big(\sup_{N \leq n \leq N_1} b_n\eta_n > \lambda\Big) = 1 - \prod_{n=N}^{N_1}(1 - P(b_n\eta_n > \lambda)) \tag{9.19}$$

and the standard tail estimate for normal random variables, along with the fact that, by (9.12), $\lim_{n\to\infty} b_n = 0$, we see that

$$\limsup_{n\to\infty} \xi_n = \lambda^*, \tag{9.20}$$

where

$$\lambda^* = \inf\left\{\lambda\colon \sum_{n=1}^\infty e^{-\lambda^2/(2b_n^2)} < \infty\right\}. \tag{9.21}$$

Furthermore, we see by (9.12) that λ^* does not change if we replace $1/b_n^2$ in (9.21) by v_n for $n \geq 1$. Since we have taken $\{v_n\}_{n=1}^\infty$ to be nondecreasing we see that in order to complete the proof we need only show that

$$\limsup_{n\to\infty}\left(\frac{2\log n}{v_n}\right)^{1/2} = \inf\left\{\lambda\colon \sum_{n=1}^\infty e^{-\lambda^2 v_n/2} < \infty\right\}. \tag{9.22}$$

Let

$$\delta = \limsup_{n\to\infty} \frac{2\log n}{v_n}.$$

Then, given $\varepsilon > 0$, there exists an $N = N(\varepsilon)$ such that for all $n \ge N$,

$$\frac{v_n}{(2\log n)}\delta \ge \left(1 - \frac{\varepsilon}{2}\right) \tag{9.23}$$

and, if $\delta < \infty$, there exists an infinite subsequence n_j of n such that

$$\frac{v_{n_j}}{2\log n_j}\delta \le \left(1 + \frac{\varepsilon}{2}\right). \tag{9.24}$$

It follows from (9.23) that $\lambda^* \le ((1+\varepsilon)\delta)^{1/2}$. To obtain a lower bound for λ^*, we recall the well-known fact that if $\{c_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of real numbers such that $\sum_{n=1}^{\infty} c_n < \infty$, then $c_n = o(1/n)$. Assume $\delta < \infty$. If $\lambda = ((1-\varepsilon)\delta)^{1/2}$, it follows from (9.24) that

$$e^{-\lambda^2 v_{n_j}/2} \ge \frac{1}{n_j^{1-\varepsilon/2}},$$

which implies that $\lambda^* \ge ((1-\varepsilon)\delta)^{1/2}$. Since ε can be made arbitrarily small we get (9.22) and hence (9.13) when it is finite. If $\delta = \infty$, for all $\lambda > 0$ there exists an infinite subsequence n_j such that

$$\frac{v_{n_j}}{2\log n_j} \le \frac{1}{\lambda}.$$

Thus

$$e^{-\lambda^2 v_{n_j}/2} \ge \frac{1}{n_j},$$

which implies that $\lambda^* \ge \lambda$ and since this is true for all λ, we get $\lambda^* = \infty$. Thus we have established (9.13) in this case also.

The statement in (9.14) is easy and well known. The upper bound follows immediately from the Borel–Cantelli lemma and does not depend on $E\xi_n\xi_m$ for $n \ne m$ but only on $E\xi_n^2$. The lower bound in (9.14) is proved using Slepian's lemma. Let $\{\eta_k\}_{k=1}^{\infty}$ and ρ be independent normal random variables with mean zero and variance one. For some $0 < \varepsilon < 1/2$, set

$$\mu_k = \sqrt{v_k}\,\xi_k, \qquad \nu_k = \eta_k(1-\varepsilon)^{1/2} + \varepsilon^{1/2}\rho.$$

Let $N_0 = N_0(\varepsilon)$ be such that $E\sqrt{v_j}\,\xi_j\sqrt{v_k}\,\xi_k \le \varepsilon$ for all $j, k \ge N_0$. Note that for all $j, k \ge N_0$, we have

$$E\mu_k^2 = E\nu_k^2 \quad\text{and}\quad E\mu_j\mu_k \le E\nu_j\nu_k, \qquad j \ne k.$$

Therefore, by Slepian's lemma [see, e.g., Jain and Marcus (1978), II, Lemma 4.3], for all $N \ge N_0$,

$$P\left(\sup_{N\le k\le\infty}\frac{\sqrt{v_k}\,\xi_k}{(2\log k)^{1/2}} > (1-2\varepsilon)^{1/2}\right)$$
$$\ge P\left(\sup_{N\le k\le\infty}\frac{\eta_k(1-\varepsilon)^{1/2}+\varepsilon^{1/2}\rho}{(2\log k)^{1/2}} > (1-2\varepsilon)^{1/2}\right) = 1,$$

which is easy to obtain since the $\{\eta_k\}$ are independent. Since this is valid for all $\varepsilon > 0$, we get (9.14).

To show that (9.15) implies (9.16), we note that by (9.12) we can choose an $m_1 \ge m$ such that

$$\sup_{k \ge m_1} \frac{v_k}{a_k^2} < \frac{\varepsilon}{2}.$$

Then by (9.17) and the remark immediately following it we see that

$$\begin{aligned} &P\left(\sup_{m_1 \le k \le n} \xi_k \ge (1-\varepsilon)\left(\frac{2\log n}{v_n}\right)^{1/2}\right) \\ (9.25)\qquad &\ge P\left(\sup_{m_1 \le k \le n} \eta_k \ge (1-\varepsilon')(2\log n)^{1/2} + \varepsilon'|\eta_0|\right) \end{aligned}$$

for some $\varepsilon' > 0$. Since $\{\eta_k\}$ are independent normal random variables with mean zero and variance one, it is easy to check [see (9.19)] that the last term in (9.25) goes to 1 as n goes to infinity. The statement in (9.16) follows from this observation and (9.25). □

Proof of Theorem 9.1. By Theorem 5.1 we obtain (9.9) once we show that 2β is the oscillation function of a mean zero Gaussian process $\{G(x),\ x \in S\}$ with covariance $u^1(x,y)$ given in (9.1) and (9.3). Consider

$$(9.26)\qquad \xi_n = G(1/n) - G(0), \qquad n = 2, 3, \dots .$$

Then for $n \ge 2$, we have

$$\begin{aligned} E\xi_n^2 &= u^1\left(\frac{1}{n}, \frac{1}{n}\right) - 2u^1\left(\frac{1}{n}, 0\right) + u^1(0,0) \\ (9.27)\qquad &= \frac{r_n}{q_n(1+r_n)} + u^1(0,0)\left(1 - \frac{r_n}{1+r_n}\right)^2 = \frac{1+\alpha(n)}{q_n}, \end{aligned}$$

where $\lim_{n\to\infty} \alpha(n) = 0$. Also, for $n \ne m$, $n, m \ge 2$,

$$(9.28)\qquad E\xi_n\xi_m = u^1(0,0)\left(1 - \frac{r_n}{1+r_n}\right)\left(1 - \frac{r_m}{1+r_m}\right) = \frac{u^1(0,0)}{(1+r_n)(1+r_m)}.$$

Setting, for $n \ge 2$,

$$(9.29)\qquad v_n = \frac{q_n}{1+\alpha(n)} \quad \text{and} \quad a_n = \frac{1+r_n}{(u^1(0,0))^{1/2}},$$

we see that (9.12) is satisfied. Thus by Lemma 9.2, we see that the oscillation function of $G(x)$ at $x = 0$ is twice the right-hand side of (9.2), which is equal to 2β. Thus we have established (9.9). [Note that we also get an analogue of (5.2).]

To obtain (9.10), we note that it follows from (9.14) that

$$\limsup_{n\to\infty} \frac{|G(1/n) - G(0)|}{(2(1+\alpha(n))\log n/q_n)^{1/2}} = 1 \quad \text{a.s.} \tag{9.30}$$

This is equivalent to

$$\limsup_{\delta\to 0} \sup_{\substack{d(1/n,0)\le\delta \\ n\ge 2}} \frac{|G(1/n) - G(0)|}{(2d(1/n,0)\log n)^{1/2}} = 1 \quad \text{a.s.} \tag{9.31}$$

It now follows from Theorem 9 that

$$\lim_{\delta\to 0} \sup_{\substack{d(1/n,0)\le\delta \\ n\ge 2}} \frac{|L_t^{1/n} - L_t^0|}{(2d(1/n,0)\log n)^{1/2}} = \sqrt{2}\,(L_t^0)^{1/2} \tag{9.32}$$

for almost all $t \ge 0$ a.s.,

which is equivalent to (9.10).

The proof of (9.11) is actually contained in the proof of Theorem 5.4. Consider the Gaussian process $G = \{G(1/m), G(1/(m+1)), \dots, G(1/n), G(0)\}$, with covariance u^1 as given in (9.3). As in (5.17)–(5.20), we define

$$\begin{aligned} \eta\left(\frac{1}{k}\right) &= G\left(\frac{1}{k}\right) - \alpha\left(\frac{1}{k},0\right)G(0), \\ \alpha\left(\frac{1}{k},0\right) &= \frac{EG(1/k)G(0)}{EG^2(0)} = \frac{r_k}{1+r_k}, \\ \sigma^2 &= \sup_{m\le k\le n} E\eta^2\left(\frac{1}{k}\right) = \sup_{m\le k\le n} \frac{r_k}{q_k(1+r_k)}, \\ a = a_{m,n} &= \operatorname{median}\left(\sup_{m\le k\le n} \eta\left(\frac{1}{k}\right)\right). \end{aligned} \tag{9.33}$$

Note that (9.16) holds with $\xi_k = G(1/k)$. However, since the right-hand side of (9.16) goes to infinity as n goes to infinity, we see that it also holds with ξ_k replaced by $\eta(1/k)$. Therefore, for any integer m and $0 < \varepsilon < 10^{-2}$, there exists an $n_0 = n_0(m,\varepsilon)$ such that for all $n \ge n_0$,

$$a_{m,n} \ge (1-\varepsilon)\left(\frac{2\log n}{q_n}\right)^{1/2}. \tag{9.34}$$

Following the proof of Theorem 5.4 for G, with $z_0 = 0$, we get (5.14b) where l is related to the local time of the associated Markov process as in Example 1 in Section 4, exactly as it is in the proof of Theorem 5.4. We choose $\bar{\varepsilon}_1$, ε_1 and t as in (5.42), (5.43) and (5.45). We can choose σ to satisfy (5.46) by taking m sufficiently large. We do not bother with N and M but note that by (9.34) we can choose a as large as we like by taking n large enough. We continue through the proof of Theorem 5.4 until (5.53). Using this and the upper bound on $2\sigma t$, we obtain

$$Q^{0,A}\left(\sup_{z\in T}\frac{l_z}{a}<\sqrt{\frac{l_0}{8}}-12\varepsilon_1^{1/2}-\frac{\varepsilon_1^{1/4}}{8}\right)<\varepsilon_1^{1/2}, \tag{9.35}$$

where $T=\{1/m, 1/(m+1), \dots, 1/n, 0\}$. It follows immediately from (9.35) that

$$Q^{0,A}\left(\sup_{m\le k\le n}\frac{l_{1/k}}{a_{m,n}}\ge\sqrt{\frac{l_0}{8}}-\varepsilon_1^{1/4}\right)\ge Q^{0,A}(\Omega)-\varepsilon_1^{1/2},$$

and using (9.34) we see that

$$\begin{aligned}Q^{0,A}&\left(\limsup_{n\to\infty}\sup_{m\le k\le n}\frac{l_{1/k}}{(2\log n/q_n)^{1/2}}\ge(1-\varepsilon)\left(\sqrt{\frac{l_0}{8}}-\varepsilon_1^{1/4}\right)\right)\\&\ge Q^{0,A}(\Omega)-\epsilon_1^{1/2}.\end{aligned}$$

Since this is true for all ε and ε_1 greater than zero and since $l_{1/k}$ is finite almost surely for each $k\ge 2$, we get

$$\limsup_{n\to\infty}\sup_{2\le k\le n}\frac{l_{1/k}}{(2\log n/q_n)^{1/2}}\ge\sqrt{\frac{l_0}{8}} \tag{9.36}$$

on a set of full measure with respect to $Q^{0,A}$. The constant $1/\sqrt{8}$ is the best we can get following the proof of Theorem 5.4 as it is written. But the proof can be strengthened. Instead of taking $\alpha_1\ge 1/2$ and $a-\sigma t>a/2$ in the proof of Theorem 5.4, we could have taken $\alpha_1\ge(1-\gamma)$ and $a-\sigma t\ge(1-\gamma)a$ for any $\gamma>0$ and proceeded with the proof with only minor modifications. In this way we can increase the constant on the right-hand side of (9.36) by a factor of 4. Using this improvement we see, as in the proof of Theorem 5.4, that (9.36) implies that

$$\int_0^\infty P^0\left(\limsup_{n\to\infty}\sup_{2\le k\le n}\frac{L_t^{1/k}}{(2\log n/q_n)^{1/2}}<\sqrt{2}\left(L_t^0\right)^{1/2}\cap\{t<\zeta\}\right)e^{-t}\,dt=0.$$

It now follows, as in (5.61) and (5.62), that for $Q\subset R^+$ a countable dense set,

$$\limsup_{n\to\infty}\sup_{2\le k\le n}\frac{L_t^{1/k}}{(2\log n/q_n)^{1/2}}\ge\sqrt{2}\left(L_t^0\right)^{1/2}\qquad\forall\, t\in Q\cap[0,\zeta)\text{ a.s. } P^0.$$

Finally, by the monotonicity of the local time we get (9.11). □

Following the proof of Theorem 5.1, it is easy to see that if we square the denominator on the left-hand side of (9.11), the resulting limit is bounded above by a constant. For more on this and related points see Marcus (1990).

The processes considered in Theorem 9.1 are symmetrized versions of an extension by Reuter (1969) of Kolmogorov's example. Kolmogorov's example is treated extensively in Chung [(1967), pages 278–283], where one can find a lucid description of the sample paths of the Markov chains that we are considering. These processes have Q matrix

$$\begin{array}{c} 0 \\ 1/2 \\ 1/3 \\ 1/4 \\ \cdot \end{array}\begin{pmatrix} -\infty & q_2 & q_3 & q_4 & \cdot \\ r_2 & -r_2 & 0 & 0 & \\ r_3 & 0 & -r_3 & 0 & \\ r_4 & 0 & 0 & -r_4 & \\ \cdot & & & & \ddots \end{pmatrix}.$$

Walsh (1978) gives an example of a diffusion with a discontinuous local time but in his example u^1 is discontinuous. (Recall our Theorem 3.7.) In fact, Walsh's process is not symmetric and if one changes the measure to obtain a symmetric process, then both u^1 and the local time become continuous.

(b) *Gaussian processes.* We next present a result about Gaussian processes which follows from Theorem 1 that we cannot obtain by Gaussian process methods alone.

THEOREM 9.3. *Let X be a strongly symmetric standard Markov process with α-potential density u^α. If u^{α_0} is the covariance of a continuous Gaussian process, then u^α is the covariance of a continuous Gaussian process for each $\alpha > 0$ and so is u^0 if it is finite. If u^0 is the covariance of a continuous Gaussian process, then u^α is the covariance of a continuous Gaussian process for each $\alpha \geq 0$.*

PROOF. Our initial choice of the 1-potential density to express our results was completely arbitrary. We could have used the α-potential density for any $\alpha > 0$. Thus if u^{α_0} is the covariance of a continuous Gaussian process, it follows from Theorem 1, proved with respect to u^{α_0}, that X has a jointly continuous local time. Since $u^\alpha < \infty$ for some $\alpha > 0$ implies $u^\alpha < \infty$ for all $\alpha > 0$, it follows from Theorem 1, proved with respect to u^α, that u^α is the covariance of a continuous Gaussian process. If $u^0 < \infty$, it follows from Theorem 1, proved with respect to u^0, that it, too, is the covariance of a continuous Gaussian process. If u^0 is the covariance of a continuous Gaussian process, then of course, it is finite and hence $u^\alpha < \infty$ for all $\alpha \geq 0$. By the same reasoning as above, u^α is the covariance of a continuous Gaussian process for each $\alpha \geq 0$. $\square$

We can get part of Theorem 9.3 by standard Gaussian techniques. Denote

$$d_\alpha(x,y) = \left(u^\alpha(x,x) + u^\alpha(y,y) - 2u^\alpha(x,y)\right)^{1/2}. \tag{9.37}$$

By Theorem 3.3, the probability transition density function $p_t(x,y)$ of X is positive definite. Thus $(p_t(x,x) + p_t(y,y) - 2p_t(x,y)) \geq 0$. This implies that

$$d_{\alpha_0}(x,y) > d_{\alpha_1}(x,y) \quad \text{if } \alpha_0 < \alpha_1.$$

Thus, by Theorem 8.5, if u^{α_0} is the covariance of a continuous Gaussian process, so is u^α for each $\alpha > \alpha_0$. However, this argument says nothing about u^α for $0 < \alpha < \alpha_0$.

(c) *Sufficient conditions for the joint continuity of local times.* The most general sufficient condition in the literature, for the joint continuity of local times, that we know of is due to Getoor and Kesten (1972). Let X be a strongly symmetric standard Markov process with state space (S, ρ) as described in Section 3. Define

$$\psi^1(x,y) = E^x(e^{-T_{\{y\}}}), \qquad h(x,y) = \left(1 - \psi^1(x,y)\psi^1(y,x)\right)^{1/2}, \tag{9.38}$$

where $T_{\{y\}}$ is the first hitting time of y. The sufficient condition in Getoor and Kesten (1972) for the existence of a jointly continuous version of the local time of X is given in terms of the function h. We will show that if $h(x,y)$ is continuous on S, then h is equivalent to the metric d defined in (1.2). This means that all the results that we obtained for continuity of the local time, expressed in terms of d, are equally valid with d replaced by h. In particular, Theorem 8.1 is valid with $B(y, \varepsilon) = \{x \in S\colon h(x,y) < \varepsilon\}$. [Actually S is taken to be a closed interval of the real line in Getoor and Kesten (1972). On the other hand, symmetry is not required in Getoor and Kesten (1972).] The following lemma shows that h is equivalent to the metric d.

LEMMA 9.4. *Let h and d be given as in* (9.38) *and* (1.2), *respectively, and let K be a compact subset S. Then*

$$\inf_{z\in K}\left(\frac{1}{2u^1(z,z)}\right)^{1/2} d(x,y) \le h(x,y) \le \sup_{z\in K}\left(\frac{1}{u^1(z,z)}\right)^{1/2} d(x,y). \tag{9.39}$$

Furthermore, if $h(x,y) \to 0$ as $x \to y$ for all $x, y \in K$, then there exist constants $0 < C_0 \le C_1 < \infty$ depending on K such that

$$C_0\, d(x,y) \le h(x,y) \le C_1\, d(x,y) \qquad \forall\, x, y \in K. \tag{9.40}$$

PROOF. One sees from Getoor and Kesten (1972) that

$$\psi^1(x,y) = \frac{u^1(x,y)}{u^1(y,y)}, \qquad x, y \in S.$$

Therefore,

$$\begin{aligned} u^1(x,x) - u^1(x,y) &= u^1(x,x)(1 - \psi^1(y,x)) \\ &\le u^1(x,x)(1 - \psi^1(x,y)\psi^1(y,x)), \end{aligned} \tag{9.41}$$

since both $\psi^1(x,y)$ and $\psi^1(y,x)$ are less than or equal to 1. (See Lemma 3.6.) Note that (9.41) also holds with x and y replaced throughout. Combining the two statements, we get the left-hand side of (9.39). By Lemma 3.6 again, we note that

$$
\begin{aligned}
1 &- \psi^1(x,y)\psi^1(y,x) \\
&\le 1 - \psi^1(x,y) + 1 - \psi^1(y,x) \\
&\le \frac{1}{u^1(x,x)}\left(u^1(x,x) - u^1(x,y)\right) + \frac{1}{u^1(y,y)}\left(u^1(y,y) - u^1(x,y)\right),
\end{aligned}
$$

which gives the right-hand side of (9.39). To get (9.40) we need only note that the conditions on h given before (9.40) imply that $u^1(x,x)$ is continuous on K. This completes the proof of Lemma 9.4. □

As we just remarked, all our results on continuity and boundedness can be expressed in terms of h. In particular if K is a closed interval of the real line and

$$
p(v) = \sup_{\substack{|x-y|\le v \\ x,y\in K,}} h(x,y),
$$

then it follows from Theorem 1 and 8.4 that if

$$
\int_0^{1/2} \frac{p(v)}{v(\log 1/v)^{1/2}}\,dv < \infty,
$$

then X has a jointly continuous local time. This condition is a little weaker than (2.17) in Theorem 2 in Getoor and Kesten (1972) and is the best possible of its type. Note that by Theorem 8.4, this result is also valid on R^n.

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REFERENCES

ADLER, R. J. (1991). *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*. IMS, Hayward, Calif.

ADLER, R. J. and EPSTEIN, R. (1986). Central limit theorem for Markov paths and some properties of Gaussian random fields. *Stochastic Process. Appl.* **24** 157–202.

ADLER, R. J., MARCUS, M. B. and ZINN, J. (1990). Central limit theorems for the local times of certain Markov processes and the squares of Gaussian processes. *Ann. Probab.* **18** 1126–1140.

BARLOW, M. T. (1985). Continuity of local times for Lévy processes. *Z. Wahrsch. Verw. Gebiete.* **69** 23–35.

BARLOW, M. T. (1988). Necessary and sufficient conditions for the continuity of local time of Lévy processes. *Ann. Probab.* **16** 1389–1427.

BARLOW, M. T. and HAWKES, J. (1985). Applications de l'entropie métrique à la continuité des temps locaux des processus de Lévy. *C. R. Acad. Sci. Paris Ser A–B* **301** 237–239.

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. Academic, New York.

BLUMENTHAL, R. M. and GETOOR, R. K. (1970). Dual processes and potential theory. *Proc. Twelfth Biennial Seminar of Canadian Math. Congress*, 137–156.

BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss space. *Invent. Math.* **30** 207–216.

BOYLAN, E. S. (1964). Local times for a class of Markov processes. *Illinois J. Math.* **8** 19–39.

BRETAGNOLLE, J. (1971). Resultats de Kesten sur les processus à accroisements indépendent. *Sem. de Probabilité V. Lecture Notes in Math.* **191** 21–36. Springer, New York.

CHUNG, K. L. (1967). *Markov chains with Stationary Transition Probabilities*, 2nd ed. Springer, New York.

DUDLEY, R. M. (1973). Sample functions of the Gaussian process. *Ann. Probab.* **1** 66–103.

DYNKIN, E. B. (1983). Local times and quantum fields. In *Seminar on Stochastic Process. Progress in Probability* **7** 64–84. Birkhäuser, Boston.

DYNKIN, E. B. (1984). Gaussian and non-Gaussian random fields associated with Markov processes. *J. Funct. Anal.* **55** 344–376.

FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

FERNIQUE, X. (1975). Regularité des trajectoires des fonctions aléatoire Gaussiennes. *Ecole d'Eté de Probabilités de Saint-Flour IV. Lecture Notes in Math.* **480** 1–96. Springer, New York.

FITZSIMMONS, P. J. and GETOOR, R. K. (1988). On the potential theory of symmetric Markov processes. *Math. Ann.* **281** 495–512.

FUKUSHIMA, M. (1980). *Dirichlet Forms and Markov Processes*. North-Holland, Amsterdam.

GEMAN, D. and HOROWITZ, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67.

GETOOR, R. K. and KESTEN, H. (1972). Continuity of local times of Markov processes. *Compositio Math.* **24** 277–303.

HAWKES, J. (1985). Local times as stationary processes. In *From Local Times to Global Geometry, Control and Physics. Pitman Research Notes in Math.* **150**. Longman, Chicago.

HOFFMANN-JØRGENSEN, J., SHEPP, L. A. and DUDLEY, R. M., (1979). On the lower tails of Gaussian seminorms. *Ann. Probab.* **7** 319–342.

ITÔ, K. and NISIO, M. (1968a). On the oscillation function of a Gaussian process. *Math. Scand.* **22** 209–233.

ITÔ, K. and NISIO, M. (1968b). On the convergence of sums of independent Banach space valued random variables. *Osaka J. Math.* **5** 35–48.

JAIN, N. C. and KALLIANPUR, G. (1972). Oscillation function of a multiparameter Gaussian process. *Nagoya Math. J.* **47** 15–28.

JAIN, N. C. and MARCUS, M. B. (1974). Sufficient conditions for the continuity of stationary Gaussian processes and applications to random series of functions. *Ann. Inst. Fourier (Grenoble)* **24** 117–141.

JAIN, N. C. and MARCUS, M. B. (1978). Continuity of subgaussian processes. In *Probability on Banach Spaces, Advances in Probability* **4** 81–196. Dekker, New York.

KESTEN, H. (1969). *Hitting Probabilities of Single Points for Processes with Stationary Independent Increments*. Memoir 93, Amer. Math. Soc., Providence, R.I.

KOLMOGOROV, A. N. (1951). On the differentiability of the transition probabilities in homogeneous Markov processes with a denumerable number of states. *Učen. Zap. MGY* **148** 53–59.

LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Space*. Springer, New York.

MARCUS, M. B. (1990). Rate of growth of local times of strongly symmetric Markov processes. In *Seminar on Stochastic Processes. Progress in Probability* **5** 253–250. Birkhäuser, Boston.

MARCUS, M. B. and PISIER, G. (1981). *Random Fourier Series with Applications to Harmonic Analysis. Ann. Math. Studies* **101**. Princeton Univ. Press.

MARCUS, M. B. and ROSEN, J. (1991). Moduli of continuity of local times of strongly symmetric Markov processes via Gaussian processes. *J. Theoret. Probab.* To appear.

MCKEAN, JR. H. P. (1962). A Hölder condition for Brownian local time. *J. Math. Kyoto* **1–2** 195–201.

MEYER, P. A. (1966). Sur les lois de certaines fonctionelles additives: Applications aux temps locaux. Publ. Inst. Statist. Univ. Paris, **15** 295–310.

MILLAR, P. W. and TRAN, L. T. (1974). Unbounded local times. *Z. Wahrsch. Verw. Gebiete* **30** 87–92.

RAY, D. B. (1963). Sojourn times of a diffusion process. *Illinois. J. Math.* **7** 615–630.

REUTER, G. (1969). Remarks on a Markov chain example of Kolmogorov. *Z. Wahrsch. Verw. Gebiete* **13** 315–320.

REVUZ, D. (1970). Mesures associées aux fonctionnelles additives de Markov I. *Trans. Amer. Math. Soc.* **148** 501–531.

SHARPE, M. (1988). *General Theory of Markov Processes*. Academic, New York.

SHEPPARD, P. (1985). On the Ray–Knight Markov property of local times. *J. London Math. Soc.* **2** 377–384.

SUDAKOV, V. N. (1969). Gaussian measures, Cauchy measures and ε-entropy. *Soviet Math. Dokl.* **10** 310.

TALAGRAND, M. (1987). Regularity of Gaussian processes. *Acta Math.* **159** 99–149.

TROTTER, H. F. (1958). A property of Brownian motion paths. *Illinois J. Math.* **2** 425–433.

WALSH, J. (1978). A diffusion with a discontinuous local time. *Temps locaux. Asterisque* **52–53** 37–46.

WITTMAN, R. (1986). Natural densities for Markov transition probabilities. *Probab. Theory Related Fields* **73** 1–10.

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