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A PROOF OF STEUTEL'S CONJECTURE

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A proof is given of a conjecture, due to F. W. Steutel: the set of positive α-values, for which any positive mixture of $\Gamma(\alpha)$-distributions is infinitely divisible, is the interval $(0, 2]$.

Introduction. Steutel (1970, 1980) discusses mixtures of gamma distributions and shows that the set of positive numbers α for which all positive mixtures of $\Gamma(\alpha)$-distributions on $[0, \infty)$ are infinitely divisible is an interval $(0, \alpha_0]$. He shows that $1 \le \alpha_0 \le 2$ and conjectures that $\alpha_0 = 2$. Here a $\Gamma(\alpha)$-distribution is a distribution with density-function $g_{\lambda,\alpha}(x) = (\lambda^\alpha/\Gamma(\alpha))x^{\alpha-1}e^{-\lambda x}$, $\lambda > 0$, and its Laplace-transform is $\check{F}_{\lambda,\alpha}(s) = (\lambda/(\lambda+s))^\alpha$. The mixing takes place with respect to λ (a contribution from $\lambda = \infty$ is permitted). Setting $\lambda = 1/t$, Steutel (1980) formulates the following:

Conjecture 1. *If G is an arbitrary distribution function on $[0, \infty)$, then*

$$\int_0^\infty \left(\frac{1}{1+st}\right)^2 dG(t) \tag{1}$$

is the Laplace-transform of an infinitely divisible distribution.

For the details of the argument that follows, see Steutel [(1970), Chapter 1.]

Clearly, we can form a sequence (G_n) of distribution-functions which are stepfunctions with 0 as a point of continuity, so that $G_n(t) \to G(t)$ at all points t of continuity of the mixing function G.

According to the closure theorem for infinitely divisible distributions, it suffices to show infinite divisibility for the mixture of $\Gamma(2)$-distributions obtained using such a stepfunction G_n as mixing function, that is, we can consider, instead of (1), a distribution with Laplace-transform $\check{F}(s) = \sum_{j=1}^n p_j(\lambda_j/(\lambda_j+s))^2$ with positive probabilities p_j, $1 \le j \le n$, and finite positive values λ_j, $1 \le j \le n$. The function $\check{F}(s)$ has $n-1$ pairs $(z_j, \tilde{z}_j)$, $j = 1, 2, \dots, n-1$, of nonreal zeros, if these zeros are counted with multiplicity; we shall always let $\Im z_j > 0$. We have the representation

$$\log \check{F}(s) = \int_0^\infty (e^{-sx} - 1)\frac{1}{x}k(x)\,dx, \tag{2}$$

where $k(x) = 2\sum_{j=1}^n e^{-\lambda_j x} - 2\sum_{j=1}^{n-1}\Re(e^{-z_j x})$.

To show that $\check{F}(s)$ is the Laplace-transform of an infinitely divisible distribution it suffices to show that $k(x) \ge 0$ for $x > 0$ (in fact, (2) then becomes a

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canonical representation). Obviously, it is enough to show that

$$\sum_{j=1}^{n-1} e^{-\lambda_j x} \geq \sum_{j=1}^{n-1} e^{-x\Re z_j} \tag{3}$$

for $x > 0$.

Karamata's inequality states that if f is a real, convex and nondecreasing function, and if $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are real numbers, satisfying $x_1 \geq x_2 \geq \cdots \geq x_n$, $y_1 \geq y_2 \geq \cdots \geq y_n$, $\Sigma_{k=1}^m x_k \geq \Sigma_{k=1}^m y_k$, $m = 1, 2, \ldots, n$, then $\Sigma_{k=1}^n f(x_k) \geq \Sigma_{k=1}^n f(y_k)$. Applying Karamata's inequality with $f(x) = e^x$, $n-1$ *for* n, $-\lambda_k x$ for x_k and $-x\Re z_k$ for y_k, $k = 1, 2, \ldots, n-1$, we see that to prove Conjecture I we need only prove

CONJECTURE II [Steutel (1980)]. *For* $n \in \mathbb{N} \setminus \{1\}$, $A_j > 0$, $j = 1, 2, \ldots, n$, *and* $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, *define*

$$A(z) = \sum_{j=1}^{n} \frac{A_j}{(z-\lambda_j)^2}. \tag{4}$$

Let the $2n-2$ *zeros* z_k *and* $\tilde{z}_k$, $k = 1, 2, \ldots, n-1$, *of* $A(z)$ *be ordered such that* $\Re z_1 \leq \Re z_2 \leq \cdots \leq \Re z_{n-1}$. *Then*

$$\sum_{j=1}^{p} \lambda_j < \sum_{k=1}^{p} \Re z_k \tag{5}$$

for $p = 1, 2, \ldots, n-1$.

Reformulation of the problem. Conjecture II will be proved by induction with respect to p. Following the argument of Steutel (1970), subsection 2.7, we note that the imaginary part of $A(z)$ is $2\Im z \Sigma_{j=1}^n A_j(\lambda_j - \Re z)/|\lambda_j - z|^4$, so that for any zero z_k of $A(z)$, the n different numbers $\lambda_j - \Re z_k$, $j = 1, 2, \ldots, n$, cannot all have the same sign. In fact, we can conclude that $\lambda_1 < \Re z_1 \leq \Re z_2 \leq \cdots \leq \Re z_{n-1} < \lambda_n$. In particular, this shows that the inequality (5) is satisfied for $p = 1$ and any $n \geq 2$.

Assume next that Conjecture II is false, and that we have a counter-example. Then there is a certain minimal p (≥ 2), and for this value of p a minimal n ($\geq p+1$), so that the conditions of the conjecture are satisfied, but the inequality (5) is not.

Since the problem is invariant with respect to common affine transformations $\lambda_j \mapsto a\lambda_j + b$, $z_k \mapsto az_k + b$, $a > 0$, b real, we can assume that $\lambda_1 = 0$ and $\lambda_n = 1$.

For the above mentioned values of p and n, we consider the quantity $G = \Sigma_{k=1}^p \Re z_k - \Sigma_{j=1}^p \lambda_j$, where the numbers z_k, $1 \leq k \leq p$, are roots with positive imaginary part of an equation $A(z) = 0$ of type (4); the set of roots of this equation is ordered so that $j < k$ implies $\Re z_j \leq \Re z_k$, and so that a root of multiplicity m appears m consecutive times in the ordering. Now G will be a function of the variables $A_1, \ldots, A_n$ and $\lambda_2, \ldots, \lambda_{n-1}$, notationally considered coordinates of the n-tuple $\mathbf{A}$ and the $(n-2)$-tuple $\boldsymbol{\lambda}$.

The function $G = G(\mathbf{A}, \boldsymbol{\lambda})$ is defined in the open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{n-2}$ whose elements $(\mathbf{A}, \boldsymbol{\lambda})$ satisfy $A_j > 0$ for $j = 1, 2, \dots, n$, and $0 < \lambda_2 < \lambda_3 < \cdots < \lambda_{n-1} < 1$. As G is not changed when the numbers A_j, $j = 1, \dots, n$, are multiplied by a common positive factor, we shall sometimes include the extra condition $\sum_{j=1}^n A_j = 1$.

By assumption, G is not everywhere positive. Clearly, G is a continuous function (use, for instance, Rouché's theorem to prove this), and G has an infimum $\gamma > -p$. There will now be two cases to consider: G does not or does attain its infimum.

CASE 1 (G does not attain its infimum). Then γ must be negative, and we have a sequence $(\mathbf{A}^{(m)}, \boldsymbol{\lambda}^{(m)})$, $m \in \mathbb{N}$, of points in Ω so that $G(\mathbf{A}^{(m)}, \boldsymbol{\lambda}^{(m)}) \to \gamma$ for $m \to \infty$. Stipulating $\sum_{j=1}^n A_j^{(m)} = 1$ for all m, we can choose a subsequence, also denoted $(\mathbf{A}^{(m)}, \boldsymbol{\lambda}^{(m)})$, converging towards a point $(\mathbf{A}^{(0)}, \boldsymbol{\lambda}^{(0)})$ of $\partial\Omega$. Then the coordinates of $\mathbf{A}^{(0)}$ and $\lambda^{(0)}$ do not satisfy all of the inequalities $A_j^{(0)} > 0$, $j = 1, 2, \dots, n$, $0 < \boldsymbol{\lambda}_2^{(0)} < \cdots < \lambda_{n-1}^{(0)} < 1$. As we shall see, this permits us to construct a function of type (4) with fewer than n variables A_j and still violating (5) (possibly even for a smaller value of p), contradicting the minimality assumptions.

To describe the situation we divide the set $\{1, 2, \dots, n\}$ of suffixes into classes, so that two suffixes j and k belong to the same class if and only if $\lambda_j^{(0)} = \lambda_k^{(0)}$ (remember that $\lambda_1^{(0)} = 0$ and $\lambda_n^{(0)} = 1$). For each class C we calculate the number $A_C = \sum_{j \in C} A_j^{(0)}$. Those classes C for which $A_C > 0$ are numbered $C_1, C_2, \dots, C_q$, so that, for $i = 1, \dots, q - 1$, each integer in C_i is less than each integer in C_{i+1}. Then

$$\sum_{j=1}^{n} \frac{A_j^{(0)}}{\left(z - \lambda_j^{(0)}\right)^2} = \sum_{k=1}^{q} \frac{B_k}{\left(z - \mu_k\right)^2} = B(z),$$

where, for $C = C_k$, we have put $B_k = A_C$ and μ_k equal to the common value λ_C of $\lambda_j^{(0)}$ for $j \in C_k$. The function $B(z)$ has $q - 1$ zeros w_k with $\Im w_k > 0$. Each zero w_k is a limit of a sequence $(w_k^{(m)})$ of zeros of the functions $A_{(m)}(z) = \sum_{j=1}^n A_j^{(m)}/(z - \lambda_j^{(m)})^2$. However, each function $A_{(m)}(z)$ has $n - 1$ zeros $z_k^{(m)}$, $1 \le k \le n - 1$ with positive imaginary part, if we count with multiplicity. These are also roots of the polynomial equation $P_m(z) = 0$, where

$$P_m(z) = \sum_{j=1}^{n} A_j^{(m)} \prod_{i \neq j} \left(z - \lambda_i^{(m)}\right)^2 = \prod_{k=1}^{n-1} \left(z - z_k^{(m)}\right)\left(z - \bar{z}_k^{(m)}\right)$$

(we have here used the normalization $\sum_{j=1}^n A_j^{(m)} = 1$). For $m \to \infty$, $P_m \to P_0$, where P_0 is some real monic polynomial of degree $2n - 2$. Now, the polynomial P_0 has the pairs of zeros $(w_k, \bar{w}_k)$, $k = 1, 2, \dots, q - 1$, but also, for each class C, a $(2m_C)$-fold zero λ_C, where m_C equals $\operatorname{card} C$ for $A_C = 0$ and $(\operatorname{card} C) - 1$ for $A_C > 0$ [this follows directly from the expression $P_0(z) = \sum_{j=1}^n A_j^{(0)} \prod_{i \neq j} (z - \lambda_i^{(0)})^2$]. It is now easy to see that $n - q$ (the number of superfluous $\lambda_j^{(0)}$-values) equals $\sum_C m_C$, so that $n - 1 = q - 1 + \sum_C m_C$, and we have, in fact, accounted for all the zeros of P_0. At least one m_C is nonzero, so that $q < n$.

Among the pairs of zeros $(z_k, \tilde{z}_k)$, $1 \le k \le n-1$, of P_0 we consider those for which $z_k \ne \tilde{z}_k$, that is, those pairs which are also among the $(w_k, \tilde{w}_k)$, $1 \le k \le q-1$. A subset of these have original index at most equal to p. Let the cardinality of this subset be $r \le p$. Then we have, because of the minimality of n and p, that $\Sigma_{k=1}^{r} \Re w_k \ge \Sigma_{j=1}^{r} \mu_j$ (with equality only if $r = 0$).

The set of the remaining (degenerate) pairs of zeros $(z_k^{(0)}, \tilde{z}_k^{(0)})$ with $k \le p$ equals the set of the $p - r$ pairs $(\lambda_j^{(0)}, \lambda_j^{(0)})$ of lowest suffix for which $\lambda_j^{(0)}$ was discarded (i.e., not among the μ_k). Thus $\Sigma_{k=1}^{p} \Re z_k^{(0)}$ is at least as great as the sum of p particular values $\lambda_j^{(0)}$, and so also at least as great as the sum of the smallest p values $\lambda_j^{(0)}$, in obvious contradiction to the surmise that γ should be negative.

CASE 2 (G assumes its infimum). We are left with the possibility that the infimum γ of G is a minimum, that is, there is a point $(\mathbf{A}, \boldsymbol{\lambda}) \in \Omega$, so that $G(\mathbf{A}, \boldsymbol{\lambda}) = \gamma \le 0$.

We shall utilize the fact that no change of the parameters $\mathbf{A}$ and $\boldsymbol{\lambda}$ can give a decrease in the value of $G(\mathbf{A}, \boldsymbol{\lambda})$. Let $(\mathbf{B}, \boldsymbol{\mu}) \in \mathbb{R}^n \times \mathbb{R}^{n-2}$. For small positive values of ε, $(\mathbf{A} + \varepsilon\mathbf{B},\ \boldsymbol{\lambda} + \varepsilon\boldsymbol{\mu}) \in \Omega$. To estimate $G(\mathbf{A} + \varepsilon\mathbf{B},\ \boldsymbol{\lambda} + \varepsilon\boldsymbol{\mu})$ we need the following lemma, which will be proved at the end of the paper.

LEMMA. *Let two functions f and g be holomorphic in an open region Ω [notation: $\{f, g\} \subset H(\Omega)$]. Let $z_0 \in \Omega$ be a zero of f of exact multiplicity $m \ge 1$. Assume that $g(z_0) \ne 0$, and that for each small $\varepsilon > 0$ the function $h_\varepsilon \in H(\Omega)$ is defined and satisfies $h_\varepsilon = f + \varepsilon g + O(\varepsilon^2)$ uniformly for z in a fixed neighbourhood of z_0.*

Then any given neighbourhood N of z_0, containing only this zero of f, contains m distinct zeros $z_{\varepsilon,k}$, $k = 1, 2, \dots, m$, of h_ε, if only ε is sufficiently small. For each k we have

$$z_{\varepsilon,k} - z_0 = \varepsilon_1 \xi_k + o(\varepsilon_1),$$

where ε_1 is the positive solution of the equation $\varepsilon_1^m = \varepsilon$, and ξ_k, $1 \le k \le m$, are the m roots of the equation $\xi^m = -a_0$, where $a_0 = g_1(z_0)$. Here g_1 (holomorphic in a neighbourhood of z_0) is defined as g/f_1, where $f_1(z) = f(z)/(z - z_0)^m$. Furthermore,

$$\sum_{k=1}^{m} (z_{\varepsilon,k} - z_0) = -\varepsilon a_{m-1} + o(\varepsilon),$$

where $a_{m-1} = g_1^{(m-1)}(z_0)/(m-1)!$

Going on with the estimate of $G(\mathbf{A} + \varepsilon\mathbf{B},\ \boldsymbol{\lambda} + \varepsilon\boldsymbol{\mu})$, we must first account for the splitting-up of each multiple zero of $A(z)$. In our notation we have $z_k = z_{k+1} = \cdots = z_{k+m_k-1}$, if m_k is the multiplicity, and for sufficiently small ε we have the corresponding new zeros $z_{\varepsilon, k+q}$, $0 \le q \le m_k - 1$. These satisfy the equation

$$\sum_{j=1}^{n} \frac{A_j + \varepsilon B_j}{(z - \lambda_j - \varepsilon\mu_j)^2} = 0, \tag{6}$$

where $\mu_1 = \mu_n = 0$, whereas we need not keep the normalization $\sum_{j=1}^{n} A_j = 1$, so that the parameters B_j, $j = 1, 2, \dots, n$, are independent.

In a neighbourhood of $z = z_k$, (6) is rewritten as

$$\sum_{j=1}^{n} \frac{A_j}{(z - \lambda_j)^2} + \varepsilon \sum_{j=1}^{n} \left(\frac{B_j}{(z - \lambda_j)^2} + \frac{2A_j\mu_j}{(z - \lambda_j)^3} \right) = O(\varepsilon^2). \tag{7}$$

We define the polynomial P by the equation

$$\frac{P(z)}{\prod_{q=1}^{n}(z - \lambda_q)^2} = \sum_{j=1}^{n} \frac{A_j}{(z - \lambda_j)^2}. \tag{8}$$

Then P has z_k as an m_k-fold zero, and we can define the polynomial

$$P_k(z) = \frac{P(z)}{(z - z_k)^{m_k}}, \tag{9}$$

and the rational function

$$g_k(z) = \frac{\prod_{q=1}^{n}(z - \lambda_q)^2}{P_k(z)}, \tag{10}$$

holomorphic in a neighborhood of z_k. Using the definitions (8), (9) and (10), we rewrite (7) as

$$(z - z_k)^{m_k} = -\varepsilon \sum_{j=1}^{n} \left(\frac{B_j g_k(z)}{(z - \lambda_j)^2} + \frac{2A_j\mu_j g_k(z)}{(z - \lambda_j)^3} \right) + O(\varepsilon^2). \tag{11}$$

In this part of the proof we have $p \geq 2$, that is, $n \geq p + 1 \geq 3$, so that we can put $\mu_2 = \pm 1$, $\mu_j = 0$ for $j \neq 2$, $B_j = 0$ for all j. Then, according to the first part of the lemma, the zero $z_k = \cdots = z_{k+m_k-1}$ does split up, and for $0 \leq q \leq m_k - 1$ we can order the zeros, so that

$$z_{\varepsilon,k+q} - z_{k+q} = \varepsilon^{1/m}|a_0|^{1/m} \exp\left(\frac{i}{m} \arg(-a_0) + i\frac{2\pi q}{m} \right) + o(\varepsilon^{1/m}),$$

where $a_0 = 2\mu_2 A_2 g_k(z_k)/(z_k - \lambda_j)^3$. If, for some k, we have $k \leq p < k + m_k - 1$, we see that we can choose $p + 1 - k$ zeros $z_{\varepsilon,k+q}$ and the sign of μ_2, so that the sum of the real parts of the chosen zeros $z_{\varepsilon,k+q}$ is less than $(p + 1 - k)\Re z_k$ by a quantity of order $\varepsilon^{1/m}$. On the other hand, if $k + m_k - 1 \leq p$, we must use the second part of the lemma, and the sum of the real parts of the $z_{\varepsilon,k+q}$, $0 \leq q \leq m_k - 1$, differs from $m_k \Re z_k$ only by $O(\varepsilon)$. We conclude that in the situation $k \leq p < k + m_k - 1$ we do not have a minimum of $G(\mathbf{A}, \boldsymbol{\lambda})$, so there is a contradiction. We may thus now assume $p = k + m_k - 1$ for some k.

But then it is time to change the notation: Let $z_1, z_2, \ldots, z_s$ be the different zeros (with positive imaginary parts) of $A(z)$, and let m_k be the multiplicity of the zero z_k, $1 \le k \le s$. In this notation the last assumption becomes $\sum_{k=1}^{r} m_k = p$ for some r. The function G is now redefined as

$$G(\mathbf{A}, \boldsymbol{\lambda}) = \sum_{k=1}^{r} m_k \Re z_k - \sum_{j=1}^{p} \lambda_j. \tag{12}$$

According to the second part of the lemma, we have

$$\begin{aligned} G(\mathbf{A} + \varepsilon \mathbf{B}, \boldsymbol{\lambda} + \varepsilon \boldsymbol{\mu}) = G(\mathbf{A}, \boldsymbol{\lambda}) &- \varepsilon \sum_{j=2}^{p} \mu_j \\ &- \varepsilon \Re \sum_{k=1}^{r} \left(\sum_{j=1}^{n} B_j b_{k, m_k - 1}(\lambda_j) + \sum_{j=2}^{n-1} 2\mu_j A_j c_{k, m_k - 1}(\lambda_j) \right) \\ &+ o(\varepsilon), \end{aligned} \tag{13}$$

where we have expanded

$$\frac{g_k(z)}{(z - \lambda_j)^2} = \sum_{q=0}^{\infty} b_{k,q}(\lambda_j)(z - z_k)^q \tag{14}$$

and

$$\frac{g_k(z)}{(z - \lambda_j)^3} = \sum_{q=0}^{\infty} c_{k,q}(\lambda_j)(z - z_k)^q. \tag{15}$$

Equations (14) and (15) are to be considered as identities in the complex variable λ_j. Starting from the expansions

$$g_k(z) = \sum_{q=0}^{\infty} d_{k,q}(z - z_k)^q, \tag{16}$$

$$(z - \lambda_j)^{-2} = \sum_{q=0}^{\infty} (-1)^q (q+1) \frac{(z - z_k)^q}{(z_k - \lambda_j)^{q+2}}, \tag{17}$$

we find

$$b_{k,q}(\lambda_j) = \sum_{t=2}^{q+2} \frac{1}{(z_k - \lambda_j)^t} (-1)^t (t-1) d_{k, q+2-t}, \tag{18}$$

and in particular

$$b_{k, m_k - 1}(\lambda_j) = \sum_{q=2}^{m_k + 1} \frac{a_{k,q}}{(z_k - \lambda_j)^q}, \tag{19}$$

with certain coefficients $a_{k,q}$ independent of λ_j. Differentiating (14) with respect to λ_j, we get for $q \geq 0$,

$$2c_{k,q}(\lambda_j) = b'_{k,q}(\lambda_j), \tag{20}$$

and in particular,

$$2c_{k,m_k-1}(\lambda_j) = \sum_{q=2}^{m_k+1} \frac{q a_{k,q}}{(z_k - \lambda_j)^{q+1}}. \tag{21}$$

Since $G(\mathbf{A}, \boldsymbol{\lambda})$ is minimal, the terms of first order in ε in (13) must be ≥ 0 for any choice of parameters $\boldsymbol{\mu}$ and $\mathbf{B}$.

Letting $B_j = \pm 1$ for some j, all other $B_k = 0$, and all $\mu_k = 0$, we find

$$\sum_{k=1}^{r} \sum_{q=2}^{m_k+1} \Re \frac{a_{k,q}}{(z_k - \lambda_j)^q} = 0, \qquad 1 \leq j \leq n. \tag{22}$$

Letting $\mu_j = \pm 1$ for some j, all other $\mu_k = 0$, and all $B_k = 0$ results in the two equations

$$1 + 2A_j \sum_{k=1}^{r} \Re c_{k,m_k-1}(\lambda_j) = 0, \qquad 2 \leq j \leq p, \tag{23a}$$

$$\sum_{k=1}^{r} \Re c_{k,m_k-1}(\lambda_j) = 0, \qquad p+1 \leq j \leq n-1. \tag{23b}$$

Assume first that we had $n \geq 2p+1$. Then (22) could be considered as a set of n linear homogeneous equations in the $2p$ quantities $a_{k,q}$ and $\tilde{a}_{k,q}$, $1 \leq k \leq r$, and for each k, $2 \leq q \leq m_k + 1$. Remember that $\sum_{k=1}^{r} m_k = p$. According to (21) and (23a), not all of the $a_{k,q}$ and $\tilde{a}_{k,q}$ vanish; thus the first $2p$ rows of the coefficient matrix of (22) are linearly dependent: there are complex numbers C_j, not all zero, so that

$$\sum_{j=1}^{2p} \frac{C_j}{(z_k - \lambda_j)^q} = \sum_{j=1}^{2p} \frac{C_j}{(\tilde{z}_k - \lambda_j)^q} = 0 \tag{24}$$

for every pair (k, q) in the range given above. Replacing C_j by $\tilde{C}_j$ for all j gives the same equations, which means that C_j could be replaced by the real quantities $C_j + \tilde{C}_j$ or $i(\tilde{C}_j - C_j)$, which are not all zero. Thus we can assume that C_j is real for all j, and even that some C_j is positive. Define $c = \min\{A_j/C_j | C_j > 0\}$; then, if we put

$$D_j = \begin{cases} A_j - cC_j, & \text{for } 1 \leq j \leq 2p, \\ A_j, & \text{for } 2p+1 \leq j \leq n, \end{cases}$$

the D_j are nonnegative; some, but not all of them, are zero. Each root z_k with $k \leq r$ is also a root of the equations $\sum_{j=1}^{n} D_j/(z - \lambda_j)^q = 0$, $2 \leq q \leq m_k + 1$, that is, a zero of $\sum_{j=1}^{n} D_j/(z-\lambda_j)^2$ with at least multiplicity m_k. Writing

$$\sum_{j=1}^{n} \frac{D_j}{(z-\lambda_j)^2} = \sum_{j=1}^{n'} \frac{D'_j}{(z-\lambda'_j)^2},$$

where all D'_j are positive, we have $n' < n$, and $G(\mathbf{D}', \boldsymbol{\lambda}')$ is the difference between a minimal sum of real parts of p zeros z'_k [as usual with $\Im z'_k > 0$, and each counted only as many times as its multiplicity as a zero for $D(z)$] and the sum of the p smallest values of λ'_j. The first sum is not greater than the sum $\sum_{k=1}^{r} m_k \Re z_k$, and the last sum is not less than $\sum_{j=1}^{p} \lambda_j$ (some of these λ_j may not be among the λ'_j).

But n was assumed to be minimal with the property that the function G could have nonpositive values for the given value of p. We have a contradiction and must conclude that $n \le 2p$.

Next, we shall need an identity, obtained by expansion in partial fractions:

$$\frac{\prod_{q=1}^{n}(z-\lambda_q)^2}{(z-\lambda_j)^2 P(z)} = 1 + \sum_{k=1}^{s} \frac{1}{(z-z_k)^{m_k}} \sum_{q=0}^{m_k-1} b_{k,q}(\lambda_j)(z-z_k)^q + \sum_{k=1}^{s} \frac{1}{(z-\bar{z}_k)^{m_k}} \sum_{q=0}^{m_k-1} \bar{b}_{k,q}(\lambda_j)(z-\bar{z}_k)^q, \tag{25}$$

where we have used (14) to expand

$$\frac{\prod_{q=1}^{n}(z-\lambda_q)^2}{(z-\lambda_j)^2 P_k(z)} = \frac{g_k(z)}{(z-\lambda_j)^2}$$

[see (9) and (10) for the definitions of P_k and g_k; to simplify notation we have again assumed $\sum_{j=1}^{n} A_j = 1$].

In (25) the coefficient of $1/z$ in an expansion at $z = \infty$ is

$$-2\sum_{q=1}^{n} \lambda_q + 2\lambda_j + 2\sum_{k=1}^{s} m_k \Re z_k = 2\sum_{k=1}^{s} \Re b_{k,m_k-1}(\lambda_j) = 2\sum_{k=1}^{s}\sum_{q=2}^{m_k+1} \Re \frac{a_{k,q}}{(z_k-\lambda_j)^q}. \tag{26}$$

Defining the complex number

$$c = \sum_{q=1}^{n} \lambda_q - \sum_{k=1}^{s} m_k \Re z_k \tag{27}$$

we see that the n distinct real numbers λ_j satisfy the equation

$$\lambda - c - \frac{1}{2}\sum_{k=1}^{s}\sum_{q=2}^{m_k+1}\left(\frac{a_{k,q}}{(z_k-\lambda)^q} + \frac{\bar{a}_{k,q}}{(\bar{z}_k-\lambda)^q}\right) = 0. \tag{28}$$

Using (22), we see that the λ_j, $1 \leq j \leq n$, satisfy also the simpler equation

$$\lambda - c - \frac{1}{2} \sum_{k=r+1}^{s} \sum_{q=2}^{m_k+1} \left(\frac{a_{k,q}}{(z_k - \lambda)^q} + \frac{\tilde{a}_{k,q}}{(\tilde{z}_k - \lambda)^q} \right) = 0. \tag{29}$$

The λ_j, $1 \leq j \leq n$, are then also zeros of the polynomial

$$\begin{aligned} Q(\lambda) = &\left(\prod_{k=r+1}^{s} ((z_k - \lambda)(\tilde{z}_k - \lambda))^{m_k+1} \right) \\ &\times \left(\lambda - c - \frac{1}{2} \sum_{k=r+1}^{s} \sum_{q=2}^{m_k+1} \left(\frac{a_{k,q}}{(z_k - \lambda)^q} + \frac{\tilde{a}_{k,q}}{(\tilde{z}_k - \lambda)^q} \right) \right) \end{aligned} \tag{30}$$

By differentiation we get, for $1 \leq j \leq n$, using (19) and (20),

$$Q'(\lambda_j) = \left(\prod_{k=r+1}^{s} |z_k - \lambda_j|^{2(m_k+1)} \right) \left(1 - 2 \sum_{k=r+1}^{s} \Re c_{k, m_k - 1}(\lambda_j) \right). \tag{31}$$

For $2 \leq j \leq p$ we use Eq. (23a) and find

$$Q'(\lambda_j) = \left(\prod_{k=r+1}^{s} |z_k - \lambda_j|^{2(m_k+1)} \right) \left(1 - \frac{1}{A_j} - 2 \sum_{k=1}^{s} \Re c_{k, m_k - 1}(\lambda_j) \right), \tag{32a}$$

while, for $p + 1 \leq j \leq n - 1$, we find from (23b),

$$Q'(\lambda_j) = \left(\prod_{k=r+1}^{s} |z_k - \lambda_j|^{2(m_k+1)} \right) \left(1 - 2 \sum_{k=1}^{s} \Re c_{k, m_k - 1}(\lambda_j) \right). \tag{32b}$$

Again we shall need an identity, obtained by partial fraction expansion and valid for $1 \leq j \leq n$:

$$\begin{aligned} \frac{\prod_{q=1}^{n}(z - \lambda_q)^2}{(z - \lambda_j)^3 P(z)} = &\frac{1}{z - \lambda_j} \frac{\prod_{q \neq j}(\lambda_j - \lambda_q)^2}{P(\lambda_j)} \\ &+ \sum_{k=1}^{s} \frac{1}{(z - z_k)^{m_k}} \sum_{q=0}^{m_k - 1} c_{k,q}(\lambda_j)(z - z_k)^q \\ &+ \sum_{k=1}^{s} \frac{1}{(z - \tilde{z}_k)^{m_k}} \sum_{q=0}^{m_k - 1} \tilde{c}_{k,q}(\lambda_j)(z - \tilde{z}_k)^q, \end{aligned} \tag{33}$$

where we have used (15). From (8) we also get

$$A_j = \frac{P(\lambda_j)}{\prod_{q \neq j}(\lambda_j - \lambda_q)^2}, \tag{34}$$

so that the coefficient of $1/z$ in an expansion of the two sides of (33) at $z = \infty$ becomes

$$1 = \frac{1}{A_j} + 2 \sum_{k=1}^{s} \Re c_{k, m_k - 1}(\lambda_j). \tag{35}$$

We see that (32a) and (32b) simplify to

$$Q'(\lambda_j) = 0, \qquad 2 \le j \le p, \tag{36a}$$

$$Q'(\lambda_j) = \left(\prod_{k=r+1}^{s} |z_k - \lambda_j|^{2(m_k+1)} \right) \Big/ A_j > 0, \qquad p+1 \le j \le n-1. \tag{36b}$$

Thus the numbers λ_j, $2 \le j \le p$, are double zeros of Q, and we can define the new polynomial

$$Q_1(\lambda) = \frac{Q(\lambda)}{\lambda(\lambda - 1)\left(\prod_{q=2}^{p}(\lambda - \lambda_q)^2\right)\prod_{q=p+1}^{n-1}(\lambda - \lambda_q)}. \tag{37}$$

For $p + 1 \le j \le n - 1$ we find

$$Q_1(\lambda_j) = \frac{Q'(\lambda_j)}{\lambda_j(\lambda_j - 1)\left(\prod_{q=2}^{p}(\lambda_j - \lambda_q)^2\right)\prod_{q=p+1, q \ne j}^{n-1}(\lambda_j - \lambda_q)}. \tag{38}$$

In particular, the quantities

$$\kappa_j = \frac{Q_1(\lambda_j)}{\prod_{q=p+1, q \ne j}^{n-1}(\lambda_j - \lambda_q)}, \qquad p+1 \le j \le n-1,$$

are negative.

The degree of the polynomial Q_1 is found from (30) and (37) to be

$$\begin{aligned} &1 + \textstyle\sum_{k=r+1}^{s} 2(m_k + 1) - (n + p - 1) \\ &\quad = 1 + 2(n - 1 - p) + 2(s - r) - n - p + 1 \\ &\quad = n - 3p + 2(s - r) \le 3n - 5p - 2 \le n - p - 2, \end{aligned}$$

where we have used the two inequalities $n \le 2p$ and $s - r \le n - 1 - p$. The polynomial Q_1 is then completely determined by its values at the $n - 1 - p$ points λ_j, $p + 1 \le j \le n - 1$. In fact, we have

$$\begin{aligned} Q_1(\lambda) &= \sum_{j=p+1}^{n-1} Q_1(\lambda_j) \prod_{\substack{q=p+1 \\ q \ne j}}^{n-1} \frac{\lambda - \lambda_q}{\lambda_j - \lambda_q} \\ &= \sum_{j=p+1}^{n-1} \kappa_j \prod_{\substack{q=p+1 \\ q \ne j}}^{n-1} (\lambda - \lambda_q). \end{aligned}$$

We see that the leading coefficient in this polynomial is negative. However, it is obvious from the definition of Q_1 [see (37)] that this coefficient should be equal to 1.

This contradiction shows that our assumption is false, and Conjecture II is proved.

PROOF OF THE LEMMA. Expanding

$$g_1(z) = \sum_{j=0}^{m-1} a_j(z - z_0)^j + (z - z_0)^m g_2(z),$$

where q_2 is holomorphic in a neighbourhood of z_0, the equation $h_\varepsilon(z) = 0$ can be written as

$$\begin{aligned}(z - z_0)^m &= -\varepsilon \frac{\sum_{j=0}^{m-1} a_j(z - z_0)^j}{1 + \varepsilon g_2(z)} + O(\varepsilon^2) \\ &= -\varepsilon \sum_{j=0}^{m-1} a_j(z - z_0)^j + O(\varepsilon^2).\end{aligned}$$

Introduce the new variable $w = (z - z_0)/\varepsilon_1$. Then

$$w^m = -a_0 - \sum_{j=1}^{m-1} a_j \varepsilon_1^j w^j + O(\varepsilon_1^m). \tag{39}$$

As a consequence of Rouché's theorem, this equation has, for $a_0 \neq 0$ and for sufficiently small ε_1, exactly one root in each of m disjoint regions, each containing one number $(-a_0)^{1/m}$. This proves the first part of the lemma.

Still assuming $a_0 \neq 0$ we consider the polynomial $w^m + \sum_{j=0}^{m-1} a_j \varepsilon_1^j w^j$, whose roots are denoted by ζ_k, $1 \leq k \leq m$. As we have seen, each ζ_k approaches a value of $(-a_0)^{1/m}$ as $\varepsilon_1 \to 0$. Label the roots $z_{\varepsilon,k}$ of h_ε similarly. Define $\eta_k = (z_{\varepsilon,k} - z_0)/\varepsilon_1$, $1 \leq k \leq m$. Then $\eta_k - \zeta_k \to 0$ for $\varepsilon_1 \to 0$. The numbers η_k satisfy (39), and this can be written $\prod_{j=1}^m (\eta_k - \zeta_j) = O(\varepsilon)$. Now, $|\eta_k - \zeta_j|$ is bounded below for $j \neq k$ and ε sufficiently small, implying that $\eta_k - \zeta_k = O(\varepsilon)$ for $k = 1, 2, \dots, m$. Thus also $\sum_{k=1}^m \eta_k = \sum_{k=1}^m \zeta_k + O(\varepsilon) = -\varepsilon_1^{m-1} a_{m-1} + O(\varepsilon_1^m)$. Finally $\sum_{k=1}^m (z_{\varepsilon,k} - z_0) = -\varepsilon_1^m a_{m-1} + O(\varepsilon_1^{m+1}) = -\varepsilon a_{m-1} + o(\varepsilon)$, as stated. This concludes the proof of the lemma. □

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