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CLASSIFICATION OF KILLED ONE-DIMENSIONAL DIFFUSIONS[1]

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We show necessary and sufficient conditions for R-recurrence and R-positivity of one-dimensional diffusions killed at the origin. These conditions are stated in terms of the bottom eigenvalue function.

1. Introduction and notation. We give necessary and sufficient conditions in order that a one-dimensional diffusion X killed at 0 is R-positive. This means that the processes Y, whose law is the conditional law of X to never hit the origin, is positive recurrent. Our conditions are stated in terms of the function $\underline{\lambda}$, where $\underline{\lambda}(z)$ is the bottom of the spectrum of the eigenvalue problem associated to the diffusion killed at z.

Let us introduce precise notation. Consider the generator $\mathcal{L}u = \frac{1}{2}\,\partial_x^2 u - \alpha\,\partial_x u$. We shall assume that α is locally bounded and measurable. The results of [5], [1] and [6], although stated for $\alpha \in C^1$, can be easily generalized to our setting. We denote by X the diffusion whose infinitesimal generator is $\mathcal{L}$, or in other words the solution of the SDE

$$dX_t = dB_t - \alpha(X_t)\,dt, \qquad X_0 = x > 0,$$

where B is a standard Brownian motion. Thus, $-\alpha$ is the drift of X.

Let $T_z = \inf\{t > 0 : X_t = z\}$ be the hitting time of z. We are mainly interested in the case $z = 0$ and we denote $T = T_0$. As usual X^T corresponds to X killed at 0. The transition density of X^T on $(0, \infty)$, is given by $p(t, x, y)\,dy = \mathbb{P}_x(X_t \in dy,\ T > t)$, $x, y > 0$. Under some extra conditions on α this transition density can be computed using the Girsanov theorem by

$$\begin{aligned} p(t,x,y)\,dy &= \exp\biggl(-\int_x^y \alpha(\xi)\,d\xi\biggr) \\ &\quad \times \mathbb{E}_x\biggl(\exp\biggl(-1/2\int_0^t \alpha^2(B_s) - \alpha'(B_s)\,ds\biggr),\ B_t \in dy,\ T > t\biggr), \end{aligned} \tag{1}$$

where as customary we put $\mathbb{E}_x(f(B), A) = \mathbb{E}_x(f(B)\mathbb{1}_A)$, for an integrable function f and a measurable set A.

Most of the functions and parameters we consider in this work will depend on α. To avoid overburdening notation we shall explicit such dependence only if it is necessary. In this work we will consider the diffusion X killed at different points.

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In this sense it is useful to introduce the notation $\alpha^{(z)}$ which is the restriction of α to the region $[z, \infty)$. Since most of the time we will deal with the process X killed at 0, we shall use α synonymous to $\alpha^{(0)}$, when there is no possible confusion.

Consider $\Lambda(x) := \int_0^x e^{\gamma(\xi)}\, d\xi$, where $\gamma(\xi) := 2\int_0^\xi \alpha(\eta)\, d\eta$. We shall assume that $\alpha^{(0)}$ verifies the following hypotheses:

H. $\int_0^\infty \int_0^x e^{\gamma(\xi)}\, d\xi\, e^{-\gamma(x)}\, dx = \int_0^\infty \int_0^x e^{-\gamma(\xi)}\, d\xi\, e^{\gamma(x)}\, dx = \infty$.
H1. $\Lambda(\infty) = \infty$.

Hypothesis H is that infinity is the natural boundary of the process X^T, in particular it implies $\lim_{x\to\infty} \mathbb{P}_x(T > s) = 1$ for any $s > 0$. Hypothesis H1 is equivalent to $\mathbb{P}_x(T < \infty) = 1$ for all (or equivalently for some) $x > 0$. We observe that $\alpha^{(z)}$ also verifies H and H1.

Fix $z \in \mathbb{R}$. The eigenvalue problem $\frac{1}{2}v''(x) - \alpha(x)v'(x) = -\lambda v(x),\ v(z) = 0$, $v'(z) = 1$, has a unique solution in $[z, \infty)$ denoted by $u_{z,\lambda;\alpha}$. When there is no possible confusion about α, we shall use the simple notation $u_{z,\lambda}$. This unique solution is C^1 with an absolutely continuous derivative and it verifies, for $x \geq z$,

$$
\begin{aligned}
u'_{z,\lambda}(x) &= e^{\gamma(x)-\gamma(z)}\left(1 - 2\lambda \int_0^x u_{z,\lambda}(\xi) e^{\gamma(z)-\gamma(\xi)}\, d\xi\right),\\
u_{z,\lambda}(x) &= \int_0^x e^{\gamma(y)-\gamma(z)}\left(1 - 2\lambda \int_0^y u_{z,\lambda}(\xi) e^{\gamma(z)-\gamma(\xi)}\, d\xi\right) dy.
\end{aligned}
\tag{2}
$$

The functions $u_{z,\lambda}(x)$, $u'_{z,\lambda}(x)$ are continuous on (z, λ, x).

We denote by $\underline{\lambda}_\alpha(z)$, or simply by $\underline{\lambda}(z)$, if there is no possible confusion, the value given by

$$\underline{\lambda}(z) = \sup\{\lambda : u_{z,\lambda} \text{ is positive in } (z, \infty)\}.$$

As proved in [6], $\underline{\lambda}(z)$ is characterized by $\underline{\lambda}(z) = \sup\{\lambda : u_{z,\lambda}$ is increasing on $[z, \infty)\}$. In both cases the supremum is attained (for the former case see [5]; for the latter see [6]). From (2) once $u_{z,\lambda}$ is increasing, then necessarily it has to be strictly increasing. In particular $u_{z,\underline{\lambda}(z)}$ is strictly increasing.

In [1] it was proved that, for $x > 0$ fixed, the following limit exists and defines a diffusion Y:

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{P}_x(X_s \in A | T > t) &= e^{\underline{\lambda}(0)s}\, \mathbb{E}_x\left(\frac{u_{0,\underline{\lambda}(0)}(X_s)}{u_{0,\underline{\lambda}(0)}(x)}, X_s \in A, T > s\right)\\
&= \mathbb{P}_x(Y_s \in A).
\end{aligned}
$$

The diffusion Y satisfies the SDE

$$dY_t = dB_t - \phi(Y_t)\, dt \qquad \text{where } \phi(y) = \alpha(y) - \frac{u'_{0,\underline{\lambda}(0)}(y)}{u_{0,\underline{\lambda}(0)}(y)} \tag{3}$$

and it takes values on $(0, \infty)$. In fact, it never reaches 0 because its drift is of order $1/x$ for x near 0. The transition density for Y is

$$p^Y(t, x, y) = \frac{u_{0,\underline{\lambda}(0)}(y)}{u_{0,\underline{\lambda}(0)}(x)} e^{\underline{\lambda}(0)t}\, p(t, x, y).$$

From (1) we get, for $x > 0$, $y > 0$,

$$\begin{aligned} p^Y(t, x, y)\, dy = {} & \frac{u_{0,\underline{\lambda}(0)}(y)}{u_{0,\underline{\lambda}(0)}(x)} \exp\biggl(-\int_x^y \alpha(\xi)\, d\xi\biggr) \\ & \times \mathbb{E}_x\biggl(\exp\biggl(-\frac{1}{2}\int_0^t h_\alpha(B_s)\, ds\biggr),\ B_t \in dy,\ T > t\biggr), \end{aligned}$$

where $h_\alpha = \alpha^2 - \alpha' - 2\underline{\lambda}_\alpha(0)$. This function h_α will be used in Theorem 5 to compare the qualitative behavior of the diffusion Y for different drifts.

The following two results give some basic information about the limiting process Y. Their proofs are left to the Appendix.

THEOREM A. *Assume α satisfies* H *and* H1. *Then, $\phi(x) = \alpha(x) - u'_{0,\underline{\lambda}(0);\alpha}(x)/u_{0,\underline{\lambda}(0);\alpha}(x)$ satisfies* H *on $[z, \infty)$ for all $z > 0$. This means*

$$\int_z^\infty e^{-\gamma^Y(x)} \int_z^x e^{\gamma^Y(\xi)}\, d\xi\, dx = \int_z^\infty e^{\gamma^Y(x)} \int_z^x e^{-\gamma^Y(\xi)}\, d\xi\, dx = \infty,$$

where $\gamma^Y(y) = 2\int_c^y (\alpha(\xi) - u'_{0,\underline{\lambda}(0)}(\xi)/u_{0,\underline{\lambda}(0)}(\xi))\, d\xi$ and $c > 0$ is a fixed constant.

The second result supplies the recurrence classification of Y in terms of integrability properties of the ground state $u_{0,\underline{\lambda}(0)}$.

THEOREM B. *Assume α satisfies* H. *The process Y is*:

(i) *positive recurrent if and only if $\int_0^\infty u^2_{0,\underline{\lambda}(0)}(x) e^{-\gamma(x)}\, dx < \infty$;*
(ii) *null recurrent if and only if*

$$\int_0^\infty u^2_{0,\underline{\lambda}(0)}(x) e^{-\gamma(x)}\, dx = \infty \quad \textit{and} \quad \int_a^\infty u^{-2}_{0,\underline{\lambda}(0)}(x) e^{\gamma(x)}\, dx = \infty \qquad \textit{for } a > 0;$$

(iii) *transient if and only if $\int_a^\infty u^{-2}_{0,\underline{\lambda}(0)}(x) e^{\gamma(x)}\, dx < \infty$ for $a > 0$.*

The classification of Y induces the R-classification of the killed diffusion X^T.

DEFINITION. The process X^T, or equivalently α, is said to be R-positive (resp. R-recurrent, R-null, R-transient) if the process Y is positive recurrent (resp. recurrent, null recurrent, transient).

Under H and H1, we proved in [6] that the following equivalence is verified:

$$\underline{\lambda}(0) > 0 \iff \int_0^\infty u_{0,\underline{\lambda}(0)}(x) e^{-\gamma(x)}\, dx < \infty. \tag{4}$$

Using that $u_{0,\underline{\lambda}(0)}$ is an increasing function we deduce that $\underline{\lambda}(0) > 0$ is a necessary condition for R-positivity. Moreover, whenever X^T is R-positive it holds

$$\int_0^\infty e^{-\gamma(x)}\, dx < \infty. \tag{5}$$

The probabilistic meaning of (5) is that the process Z whose drift in $\mathbb{R}$ is $-\alpha(|x|)$, is positive recurrent. In fact, the invariant probability measure of Z has a density proportional to $e^{-\gamma(|x|)}$.

In [6] it was shown that under H and H1,

$$\underline{\lambda}(z) = \lim_{t\to\infty} -\frac{\log \mathbb{P}_x(T_z > t)}{t} \qquad \text{for any } x > z, \tag{6}$$

that is, $\underline{\lambda}(z)$ is the exponential rate at which the process X is killed at z. We observe that if H1 fails, the right-hand side of (6) vanishes, while $\underline{\lambda}(z)$ could be strictly positive.

Since $\mathbb{P}_y(T_z > t) \le \mathbb{P}_y(T_x > t)$ for $x < z < y$, the function $\underline{\lambda}$ is increasing. We point out that a simple coupling argument shows that $\underline{\lambda}_\alpha$ is increasing also in α; that is, if $\alpha \ge \beta$ on $[z, \infty)$ and both functions satisfy hypotheses H and H1 then $\underline{\lambda}_\alpha(z) \ge \underline{\lambda}_\beta(z)$ (see Corollary 1 in [6]). The study of the function $\underline{\lambda}$ is one of the main objects of this paper. In this direction we make the following definition.

DEFINITION. α has a gap at x with respect to $y < x$, if $\underline{\lambda}(x) > \underline{\lambda}(y)$.

We are mainly interested in gaps with respect to $y = 0$, in which case we just say that α has a gap at x. We notice that if α has a gap at x so it does at any $z > x$.

We shall state necessary and sufficient conditions for α to be R-positive in terms of the function $\underline{\lambda}$. In particular we will prove that if there exists some gap then the diffusion is R-positive. We point out that an analogous condition was already used in [2] to show R-positivy of Markov chains in countable spaces. The notion of R-positivity for diffusions extends the standard definition of R-positivity introduced by Vere-Jones (see [8]) for nonnegative matrices, which in terms of the Perron–Frobenius theory reduces to the fact that the inner product of the left and right positive eigenvectors is finite (see [7], Theorem 6.4). Hence, this notion turns out to be nontrivial only for processes taking values on infinite spaces. In the context of one-dimensional statistical mechanics with an infinite number of states, R-positivity of the transfer matrix associated to the Hamiltonian was shown to be a necessary and sufficient condition for the existence of a unique Gibbs state [4].

In the following section we establish the main results, whose proofs are given in Section 3. In Section 4 we give examples concerning the "last" point of increase for $\underline{\lambda}$.

Throughout the paper we shall use some basic facts about the constant drift case. If α is a nonnegative constant a, then a simple computation gives $\underline{\lambda}(x) = a^2/2$ and α is R-transient.

2. Main results. In our results we shall assume the drifts involved verify hypotheses H and H1.

THEOREM 1.

(i) *If α has a gap at some $z > 0$ then α is R-positive and α has a gap at any $x > 0$.*

(ii) *If for some $z > 0$ the function $\alpha^{(z)}$ is R-positive then $\alpha^{(y)}$ is R-positive for $0 \le y \le z$ and $\underline{\lambda}$ is strictly increasing on $[0, z]$. In particular, α has a gap at z.*

(iii) *If α does not have a gap then $\alpha^{(z)}$ is R-transient for any $z > 0$.*

We consider $\underline{\lambda}(\infty) = \lim_{x\to\infty} \underline{\lambda}(x)$ and $\bar{x} = \inf\{x \ge 0 : \underline{\lambda}(x) = \underline{\lambda}(\infty)\} \le \infty$. We notice that if $\underline{\lambda}(\infty) = \infty$ then α has necessarily a gap which implies that α is R-positive. We also point out that if α is R-transient then $\bar{x} = 0$ and $\underline{\lambda}(0) = \underline{\lambda}(\infty)$.

THEOREM 2. *The function $\underline{\lambda}$ is strictly increasing on $[0, \bar{x})$, and $\alpha^{(x)}$ is R-positive for $x \in [0, \bar{x})$. $\underline{\lambda}$ is constant on $[\bar{x}, \infty)$ and $\alpha^{(x)}$ is R-transient on $(\bar{x}, \infty)$. $\underline{\lambda}$ is continuous in $[0, \infty)$; it is C^1 on $[0, \infty)$ except perhaps at $\bar{x}$. Moreover, $\underline{\lambda}'$ satisfies, for $x \in [0, \bar{x})$,*

$$\int_x^\infty u^2_{x,\underline{\lambda}(x)}(y) \exp\left(-2\int_x^y \alpha(\xi)\, d\xi\right) dy = \frac{1}{2\underline{\lambda}'(x)}. \tag{7}$$

In particular, $\underline{\lambda}'(x) > 0$ on $[0, \bar{x})$.

Finally, when $0 < \bar{x} < \infty$ we have $\alpha^{(\bar{x})}$ is R-recurrent. It is R-null if and only if $\underline{\lambda}'(\bar{x}-) = 0$ and it is R-positive if and only if $\underline{\lambda}'(\bar{x}-) > 0$ (i.e., if $\underline{\lambda}'$ is discontinuous at $\bar{x}$).

It is worth noticing that a formula similar to (7) holds for $\underline{\lambda}(x)$

$$\int_x^\infty u_{x,\underline{\lambda}(x)}(y) \exp\left(-2\int_x^y \alpha(\xi)\, d\xi\right) dy = \frac{1}{2\underline{\lambda}(x)}.$$

This is a particular case of the relation (13) in [6], established for any $\lambda \in (0, \underline{\lambda}(x)]$.

The R-classification already obtained for $\alpha^{(x)}$, $x > 0$, can be put in terms of points of increase from the left for the function $\underline{\lambda}$. In fact, $\alpha^{(x)}$ is R-recurrent (resp. R-transient) if and only if x is a point of increase (resp. constancy) from the left for $\underline{\lambda}$. The distinction between R-null and R-positive is done by the left derivative. Thus, in order to obtain the R-classification of $\alpha^{(0)}$ we rely on extensions of $\alpha^{(0)}$ to the left of 0. Clearly the classification of $\alpha^{(0)}$ does not depend on the chosen extension. To fix notations, $\tilde{\alpha}$ is said to be an extension of $\alpha^{(0)}$ if $\tilde{\alpha}$ is defined on $[-\epsilon, \infty)$ for some $\epsilon > 0$ and $\tilde{\alpha}^{(0)} = \alpha^{(0)}$. From Theorems 1 and 2 we obtain directly the following characterization.

THEOREM 3. (i) $\alpha^{(0)}$ *is* R*-transient if and only if for some* (*any*) *extension* $\tilde{\alpha}$, 0 *is a point of constancy for* $\underline{\lambda}_{\tilde{\alpha}}$.

(ii) $\alpha^{(0)}$ *is* R*-positive if and only if for some* (*any*) *extension* $\tilde{\alpha}$ *it holds* $\underline{\lambda}'_{\tilde{\alpha}}(0-) > 0$.

As a matter of completeness:

(iii) $\alpha^{(0)}$ *is* R*-null if and only if for some* (*any*) *extension* $\tilde{\alpha}$, 0 *is a point of increase for* $\underline{\lambda}_{\tilde{\alpha}}$ *and* $\underline{\lambda}'_{\tilde{\alpha}}(0-) = 0$.

As a corollary, we get that α is R-transient whenever it is periodic and satisfies H and H1. A slight generalization is the following one. Consider a subperiodic function α in $[0, \infty)$, that is, $\alpha(x + a) \le \alpha(x)$ for some $a > 0$ and for all $x \ge 0$. We also assume α satisfies H and H1. A simple comparison argument gives $\underline{\lambda}(a) \le \underline{\lambda}(0)$; thus $\underline{\lambda}$ is a constant function. Take $\tilde{\alpha}$ any subperiodic extension of α. Again a comparison argument shows that 0 is a point of constancy for $\underline{\lambda}_{\tilde{\alpha}}$, implying that α is R-transient.

Let us fix $x > 0$ and consider the process X killed at x. The associated limiting process Y^x has a drift given by [see (3)]

$$-\phi_x(y) = \frac{u'_{x,\underline{\lambda}(x)}(y)}{u_{x,\underline{\lambda}(x)}(y)} - \alpha(y) \qquad \text{for } y > x.$$

A direct computation yields the following relation between the eigenfunctions for Y^x killed at $z > x$ and the eigenfunctions for X killed at x and z. For any $\lambda \in \mathbb{R}$ it holds

$$u_{z,\lambda-\underline{\lambda}(x);\phi_x}(y) = \frac{u_{z,\lambda;\alpha}(y)u_{x,\underline{\lambda}(x);\alpha}(z)}{u_{x,\underline{\lambda}(x);\alpha}(y)}, \tag{8}$$

where we have put $\underline{\lambda}(x) = \underline{\lambda}_\alpha(x)$. From this relation we get $\underline{\lambda}_{\phi_x}(z) = \underline{\lambda}_\alpha(z) - \underline{\lambda}_\alpha(x)$. Furthermore, the following result is verified.

PROPOSITION 4. *Let* $x \ge 0$ *and assume* $\alpha^{(x)}$ *is* R*-positive. Then, for any* $z > x$ *the function* ϕ_x *satisfies hypotheses* H *and* H1 *on* $[z, \infty)$. *Moreover,* α *has a gap at* y *with respect to* z *if and only if* ϕ_x *has a gap at* y *with respect to* z, *where* $y > z > x$. *This last condition ensures* Y^x *killed at* z *is* R*-positive*, *in particular* $\underline{\lambda}_{\phi_x}(z) > 0$.

We now establish a comparison criteria to study R-positivity.

THEOREM 5. *Assume the functions* α, β *satisfy any one of the following three conditions*:

(C1) α, β *are* C^1 *and* $h_\alpha = \alpha^2 - \alpha' - 2\underline{\lambda}_\alpha(0) \ge h_\beta = \beta^2 - \beta' - 2\underline{\lambda}_\beta(0)$ *on* $[0, \infty)$;

(C2) $\alpha \geq \beta$ *and* $\underline{\lambda}_\alpha(\infty) = \underline{\lambda}_\beta(\infty)$;
(C3) $\alpha \leq \beta$ *and* $\underline{\lambda}_\alpha(0) = \underline{\lambda}_\beta(0)$.

Then the following properties hold:

(i) *if* β *is* R*-transient then* α *is* R*-transient*;
(ii) *if* α *is* R*-positive then* β *is* R*-positive.*

We remark that among the conditions of Theorem 5, (C2) is the easiest one to verify. The other two conditions depend on $\underline{\lambda}_\alpha(0)$, $\underline{\lambda}_\beta(0)$ which in general are not simple to compute. Two special cases are studied in the following result.

COROLLARY 6. (i) *Assume that*

$$\underline{\alpha}(\infty) := \liminf_{x\to\infty} \alpha(x) > \sqrt{2\underline{\lambda}(0)} \tag{9}$$

then the process X^T *is* R*-positive. A sufficient condition for* (9) *to hold is*

$$\underline{\alpha}(\infty) \geq \left((\sup\{\alpha(x) : x \in [0, b]\})^2 + (\pi/b)^2 \right)^{1/2} \qquad \textit{for some } b > 0. \tag{10}$$

In particular the condition $\lim_{x\to\infty} \alpha(x) = \infty$ *implies* X^T *is* R*-positive.*

(ii) *Assume the following limit exists*: $\alpha(\infty) := \lim_{x\to\infty} \alpha(x) \geq 0$. *If* $\alpha(\infty) \leq \alpha(x)$ *for all* $x \geq 0$ *then* $\underline{\lambda}(0) = \alpha(\infty)^2/2$, *and the process* X^T *is* R*-transient. In particular this holds whenever* α *is a nonnegative decreasing function.*

One is tempted to believe that α is R-positive whenever it is increasing, nonconstant, and eventually positive. This is the case when α is unbounded, but in the bounded case α is not in general R-positive. In this direction the following result gives a sufficient integral condition in order that X^T is R-transient.

PROPOSITION 7. *Assume that* α *is bounded on* $[0, \infty)$ *and satisfies* $\alpha(\infty) = \lim_{y\to\infty} \alpha(y) \geq \alpha(x)$ *for all* $x \geq 0$. *Also we assume that* $\alpha(\infty) > 0$. *If*

$$\int_0^\infty \big(\alpha(\infty) - \alpha(x)\big)\big(\alpha(\infty)x + 1\big)\, dx < \frac{1}{2e},$$

then X^T *is* R*-transient.*

Let $\alpha(x) = 1 - K/(1+x)^3$. From condition (10) in Corollary 6, it follows that for large values of K, α is R-positive. In an opposite way, from Proposition 7, we find that for small values of K, α is R-transient.

We now study the eventually constant case where further explicit computations can be made. The setting is $\alpha(x) = \theta$ for all $x \geq \ell$, for some $\ell \geq 0$. When $\theta > 0$, conditions H, H1 and (5) hold.

PROPOSITION 8. *Assume α is eventually constant with $\theta > 0$. Then, there exists $\underline{\theta} = \underline{\theta}(\ell)$ such that X^T is R-positive if and only if $\theta > \underline{\theta}$, X^T is R-null if and only if $\theta = \underline{\theta}$ and X^T is R-transient if and only if $\theta < \underline{\theta}$. The value $\underline{\theta}$ is the unique solution of*

$$\frac{u'_{0,\underline{\theta}^2/2}(\ell)}{u_{0,\underline{\theta}^2/2}(\ell)} = \underline{\theta}.$$

The condition $\theta > \underline{\theta}$ is equivalent to $\underline{\lambda}(0) < \theta^2/2$. Moreover, $\underline{\lambda}(0)$ admits the following representation:

$$\underline{\lambda}(0) = \sup\left\{\lambda \le \min\big(\hat{\lambda}(\ell), \theta^2/2\big) : \frac{u'_{0,\lambda}(\ell)}{u_{0,\lambda}(\ell)} + \sqrt{\theta^2 - 2\lambda} \ge \theta\right\},$$

where $\hat{\lambda}(\ell) = \sup\{\lambda : u_{0,\lambda}$ is increasing on $[0, \ell]\}$.

In the special case $\alpha(x) = \theta_0 \mathbb{1}_{\{x<\ell\}} + \theta \mathbb{1}_{\{x \ge \ell\}}$ with $\theta > \theta_0$, the critical value $\underline{\theta}$ is given by the formula $\underline{\theta} = \theta_0 + \chi \cot(\chi\ell)$, where χ is uniquely determined by $\theta_0 = -\chi \cot(2\chi\ell)$ and $\chi \in [\pi/4\ell, \pi/2\ell)$.

In the latter special case the dependence of $\underline{\theta}$ on ℓ, θ_0, verifies the homogeneity condition

$$\underline{\theta}(\ell, \theta_0) = \frac{\underline{\theta}(1, \ell\theta_0)}{\ell}.$$

It can be proved easily that $\underline{\theta}$ is increasing on θ_0 and decreasing on ℓ, with asymptotic values

$$\lim_{\ell\to 0} \underline{\theta}(\ell, \theta_0) = \infty, \qquad \lim_{\ell\to\infty} \underline{\theta}(\ell, \theta_0) = \theta_0,$$
$$\lim_{\theta_0\to 0} \underline{\theta}(\ell, \theta_0) = \pi/4\ell, \qquad \lim_{\theta_0\to\infty} \underline{\theta}(\ell, \theta_0) = \infty.$$

Moreover, using the inequality $\pi \le 4x \cot(x)(1 - 2x \cot(2x)) \le \pi^2$ for $x \in [\pi/4, \pi/2)$, we obtain

$$\theta_0 + \frac{\pi}{4\ell(1 + 2\ell\theta_0)} \le \underline{\theta} \le \theta_0 + \frac{\pi^2}{4\ell(1 + 2\ell\theta_0)}.$$

If $\ell = 0$, that is, the drift is constant on $[0, \infty)$, the process X^T is R-transient. We observe that the above criterion gives $\underline{\theta} = \infty$.

3. Proofs of the main results. In the sequel we shall need some extra properties about the eigenfunctions $u_{z,\lambda}$. A useful tool will be supplied by the Wronskian $W[f, g]$, between two C^1 functions f and g, which is given by $W[f, g](x) = f'(x)g(x) - f(x)g'(x)$. Once f and g are fixed, we shall simply write $W(x)$ instead of $W[f, g](x)$.

LEMMA 9. *For any $a > 0$ there exists $\tilde{\lambda} > \underline{\lambda}(0)$ such that $u_{0,\lambda}$ is strictly increasing on $[0, a]$ for any $\lambda \in [\underline{\lambda}(0), \tilde{\lambda}]$.*

PROOF. The result follows from the facts that $u'_{0,\lambda}(x)$ is jointly continuous and $u_{0,\underline{\lambda}(0)}$ is strictly increasing on $[0, \infty)$. $\square$

LEMMA 10. *Assume $u_{0,\lambda}$ is increasing on $[0, a]$. Then, for all $\mu \le \lambda$ the function $u_{0,\mu}$ is also increasing on $[0, a]$. Moreover, for $x \in (0, a]$ it holds: $u_{0,\mu}(x) > u_{0,\lambda}(x)$; $u'_{0,\mu}(x) > u'_{0,\lambda}(x)$ and the ratio $u'_{0,\mu}(x)/u_{0,\mu}(x)$ is a strictly decreasing continuous function of μ on the region $(-\infty, \lambda]$. In particular, the above properties hold for $\lambda = \underline{\lambda}(0)$ on $(0, \infty)$.*

PROOF. We first notice that if $u_{0,\lambda}$ is increasing on $[0, a]$ then it is strictly increasing in the same interval. In fact, from (2) we conclude that $u'_{0,\lambda} > 0$ on $[0, a)$. Consider the Wronskian $W(x) = W[u_{0,\lambda}, u_{0,\mu}](x)$. A direct computation shows that $W(0) = 0$ and

$$W' = 2\alpha W - 2(\lambda - \mu)u_{0,\lambda}u_{0,\mu},$$

or equivalently,

$$W(x) = -2(\lambda - \mu)e^{\gamma(x)} \int_0^x e^{-\gamma(\xi)} u_{0,\lambda}(\xi)u_{0,\mu}(\xi)\, d\xi.$$

If $u_{0,\lambda}, u_{0,\mu}$ are increasing on $[0, b]$ then $W(x) < 0$ on this interval and therefore

$$\frac{u'_{0,\lambda}(x)}{u_{0,\lambda}(x)} < \frac{u'_{0,\mu}(x)}{u_{0,\mu}(x)} \qquad \text{for } x \in (0, b].$$

This implies that $u_{0,\mu}$ is increasing in $[0, a]$ (otherwise take the first $x^* < a$ where $u'_{0,\mu}(x^*) = 0$ to arrive at a contradiction). We deduce

$$\frac{u'_{0,\lambda}(x)}{u_{0,\lambda}(x)} < \frac{u'_{0,\mu}(x)}{u_{0,\mu}(x)} \qquad \text{for } x \in (0, a]. \tag{11}$$

Moreover, by integrating (11), we get for any $\varepsilon > 0$,

$$u_{0,\lambda}(x) < \frac{u_{0,\lambda}(\varepsilon)}{u_{0,\mu}(\varepsilon)} u_{0,\mu}(x).$$

Since $\lim_{\varepsilon \downarrow 0} u_{0,\lambda}(\varepsilon)/u_{0,\mu}(\varepsilon) = 1$ we obtain

$$u_{0,\lambda}(x) \le u_{0,\mu}(x) \qquad \forall\, x \in (0, a],$$

which together with (11), imply $u'_{0,\lambda}(x) < u'_{0,\mu}(x)$. Finally, the ratio $u'_{0,\mu}(x)/u_{0,\mu}(x)$ is clearly continuous on μ for any $x \in (0, a]$. $\square$

Let $z \geq x \geq 0$ be fixed. Consider the Wronskian $W = W[u_{x,\lambda}, u_{z,\mu}]$ in the region $[z, \infty)$, which is given by $W(y) = u'_{x,\lambda}(y)u_{z,\mu}(y) - u_{x,\lambda}(y)u'_{z,\mu}(y)$. One has $W(z) = -u_{x,\lambda}(z)$ and $W' = 2\alpha W + 2(\mu - \lambda)u_{x,\lambda}u_{z,\mu}$. Therefore, for $y \geq z$,

$$
\begin{aligned}
W(y) &= \exp\biggl(2\int_z^y \alpha(\xi)\,d\xi\biggr) \\
&\quad \times \biggl(W(z) + 2(\mu-\lambda)\int_z^y u_{x,\lambda}(\eta)u_{z,\mu}(\eta)\exp\biggl(-2\int_z^\eta \alpha(\xi)\,d\xi\biggr)d\eta\biggr) \\
&= \exp\biggl(2\int_z^y \alpha(\xi)\,d\xi\biggr) \\
&\quad \times \biggl(-u_{x,\lambda}(z) + 2(\mu-\lambda)\int_z^y u_{x,\lambda}(\eta)u_{z,\mu}(\eta)\exp\biggl(-2\int_z^\eta \alpha(\xi)\,d\xi\biggr)d\eta\biggr).
\end{aligned} \tag{12}
$$

LEMMA 11. *Assume that for $x < z$ fixed, $\underline{\lambda}(x) < \underline{\lambda}(z)$ is verified. Then, for $\mu \in (\underline{\lambda}(x), \underline{\lambda}(z)]$ and $y \in [z, \infty)$ we have*

$$
W\bigl[u_{x,\underline{\lambda}(x)}, u_{z,\mu}\bigr](y) < 0. \tag{13}
$$

In particular, for $y \in [z, \infty)$,

$$
\frac{u'_{x,\underline{\lambda}(x)}(y)}{u_{x,\underline{\lambda}(x)}(y)} \leq \frac{u'_{z,\underline{\lambda}(z)}(y)}{u_{z,\underline{\lambda}(z)}(y)}. \tag{14}
$$

Furthermore,

$$
\begin{aligned}
&2\bigl(\underline{\lambda}(z) - \underline{\lambda}(x)\bigr)\int_z^\infty u_{x,\underline{\lambda}(x)}(\eta)u_{z,\underline{\lambda}(z)}(\eta)\exp\biggl(-2\int_z^\eta \alpha(\xi)\,d\xi\biggr)d\eta \\
&\qquad = u_{x,\underline{\lambda}(x)}(z).
\end{aligned} \tag{15}
$$

PROOF. Let $\underline{\lambda}(x) < \mu \leq \underline{\lambda}(z)$. Assume that (13) does not hold; that is, for some finite y_0 the following strict inequality holds:

$$
2\bigl(\mu - \underline{\lambda}(x)\bigr)\int_z^{y_0} u_{x,\underline{\lambda}(x)}(\eta)u_{z,\mu}(\eta)\exp\biggl(-2\int_z^\eta \alpha(\xi)\,d\xi\biggr)d\eta > u_{x,\underline{\lambda}(x)}(z).
$$

By Lemma 9 and continuity, we get the existence of $\tilde{\lambda} \in (\underline{\lambda}(x), \mu)$ such that:

(a) $u_{x,\tilde{\lambda}}$ is increasing on $[x, y_0]$;

(b) $2(\mu - \tilde{\lambda})\int_z^{y_0} u_{x,\tilde{\lambda}}(\eta)u_{z,\mu}(\eta)\exp(-2\int_z^\eta \alpha(\xi)\,d\xi)\,d\eta > u_{x,\tilde{\lambda}}(z)$.

From (12) we have

$$
W[u_{x,\tilde{\lambda}}, u_{z,\mu}](y_0) = u'_{x,\tilde{\lambda}}(y_0)u_{z,\mu}(y_0) - u_{x,\tilde{\lambda}}(y_0)u'_{z,\mu}(y_0) > 0.
$$

Since $u_{z,\mu}$ is increasing (see Lemma 10) we get $u'_{x,\tilde{\lambda}}(y_0) > 0$ and therefore $u_{x,\tilde{\lambda}}$ is strictly increasing on a small interval $[y_0, y_0 + \delta]$. If there exists a point $y^* > y_0$ such that $u'_{x,\tilde{\lambda}}(y^*) = 0$ we arrive at a contradiction. In fact, consider y^* the smallest possible one. From (12) and relation (b) we get $W[u_{x,\tilde{\lambda}}, u_{z,\mu}](y^*) > 0$, and therefore $u'_{x,\tilde{\lambda}}(y^*) > 0$. The conclusion is that $u_{x,\tilde{\lambda}}$ is strictly increasing on $[x,\infty)$ but this is again a contradiction because $\tilde{\lambda} > \underline{\lambda}(x)$. Therefore,

$$2(\mu - \underline{\lambda}(x)) \int_z^\infty u_{x,\underline{\lambda}(x)}(\eta) u_{z,\mu}(\eta) \exp\left(-2\int_z^\eta \alpha(\xi)\, d\xi\right) d\eta \le u_{x,\underline{\lambda}(x)}(z)$$

holds, and (13) and (14) follow.

Now, let us prove (15). Take a large t_0 and find a $\tilde{\mu} > \underline{\lambda}(z)$, close enough to $\underline{\lambda}(z)$, such that $u_{z,\tilde{\mu}}$ is increasing on $[z, t_0]$. Since $\tilde{\mu} > \underline{\lambda}(z)$ there exists $t_1 > t_0$, the closest value to t_0, where $u'_{z,\tilde{\mu}}(t_1) = 0$, then $W[u_{x,\underline{\lambda}(x)}, u_{z,\tilde{\mu}}](t_1) > 0$. From (12) we get

$$2(\tilde{\mu} - \underline{\lambda}(x)) \int_z^{t_1} u_{x,\underline{\lambda}(x)}(\eta) u_{z,\tilde{\mu}}(\eta) \exp\left(-2\int_z^\eta \alpha(\xi)\, d\xi\right) d\eta > u_{x,\underline{\lambda}(x)}(z).$$

Using Lemma 10, the inequality $u_{z,\tilde{\mu}} \le u_{z,\underline{\lambda}(z)}$ holds on $[z, t_1]$. Therefore, we obtain

$$2(\tilde{\mu} - \underline{\lambda}(x)) \int_z^{\infty} u_{x,\underline{\lambda}(x)}(\eta) u_{z,\underline{\lambda}(z)}(\eta) \exp\left(-2\int_z^\eta \alpha(\xi)\, d\xi\right) d\eta > u_{x,\underline{\lambda}(x)}(z).$$

Thus, (15) is proved by passing to the limit $\tilde{\mu} \to \underline{\lambda}(z)$. □

PROOF OF THEOREM 1. (i) Let us prove the existence of a gap at some $z > 0$ is sufficient for α to be R-positive. From Lemma 11, by integrating inequality (14) (where $x = 0$) we get

$$u_{0,\underline{\lambda}(0)}(y) \le \frac{u_{0,\underline{\lambda}(0)}(y_0)}{u_{z,\underline{\lambda}(z)}(y_0)} u_{z,\underline{\lambda}(z)}(y) \qquad \text{for } 0 < z < y_0 < y.$$

From this inequality and (15) we get

$$\begin{aligned}\int_{y_0}^\infty u^2_{0,\underline{\lambda}(0)}(y) e^{-\gamma(y)}\, dy &\le \frac{u_{0,\underline{\lambda}(0)}(y_0)}{u_{z,\underline{\lambda}(z)}(y_0)} \int_{y_0}^\infty u_{0,\underline{\lambda}(0)}(y) u_{z,\underline{\lambda}(z)}(y) e^{-\gamma(y)}\, dy \\ &\le \frac{u_{0,\underline{\lambda}(0)}(y_0)}{u_{z,\underline{\lambda}(z)}(y_0)} \frac{u_{0,\underline{\lambda}(0)}(z)}{2(\underline{\lambda}(z) - \underline{\lambda}(0))} e^{-\gamma(z)} < \infty.\end{aligned}$$

This shows that α is R-positive.

Now we prove that if α has a gap at $z > 0$ then it has a gap at any $x > 0$. Without loss of generality we can assume that $x < z$. If there is not a gap at x we have $\underline{\lambda}(0) = \underline{\lambda}(x)$. For the sake of simplicity we denote $\lambda = \underline{\lambda}(0)$. Using (12), the Wronskian $W = W[u_{0,\lambda}, u_{x,\lambda}]$ is

$$\begin{aligned}W(y) &= u'_{0,\lambda}(y) u_{x,\lambda}(y) - u_{0,\lambda}(y) u'_{x,\lambda}(y) \\ &= -u_{0,\lambda}(x) \exp\left(2\int_x^y \alpha(\xi)\, d\xi\right) \qquad \text{for } x \le y.\end{aligned}$$

Therefore, we get

$$\left(\frac{u_{0,\lambda}}{u_{x,\lambda}}\right)'(y) = \frac{W(y)}{u_{x,\lambda}^2(y)} = -u_{0,\lambda}(x)\frac{\exp(2\int_x^y \alpha(\xi)\,d\xi)}{u_{x,\lambda}^2(y)}.$$

Consider $x < y_0$ and integrate the above equality on $[y_0, y]$ to obtain

$$u_{0,\lambda}(y) = u_{x,\lambda}(y)\left(\frac{u_{0,\lambda}(y_0)}{u_{x,\lambda}(y_0)} - u_{0,\lambda}(x)\int_{y_0}^{y} \frac{\exp(2\int_x^\eta \alpha(\xi)\,d\xi)}{u_{x,\lambda}^2(\eta)}\,d\eta\right). \tag{16}$$

The assumption of having a gap at $z > x$ and the assumption $\underline{\lambda}(0) = \underline{\lambda}(x)$, ensure that $\underline{\lambda}(z) > \underline{\lambda}(x)$ and $\alpha^{(x)}$ has a gap at z with respect to x. Therefore, using the part of the theorem already proved, $\alpha^{(x)}$ is R-positive. So far we have the statement

$$\alpha^{(x)} \text{ is } R\text{-positive and } \underline{\lambda}(x) = \underline{\lambda}(0). \tag{17}$$

We shall prove this leads to a contradiction. We first remark that the following integral is finite:

$$\int_x^\infty u_{x,\underline{\lambda}(x)}^2(\eta)\exp\left(-2\int_x^\eta \alpha(\xi)\,d\xi\right)d\eta < \infty,$$

which implies

$$\int_{y_0}^\infty \frac{\exp(2\int_x^\eta \alpha(\xi)\,d\xi)}{u_{x,\underline{\lambda}(x)}^2(\eta)}\,d\eta = \infty.$$

This is a contradiction with (16), because at some large y we obtain

$$u_{0,\lambda}(x)\int_{y_0}^{y} \frac{\exp(2\int_x^\eta \alpha(\xi)\,d\xi)}{u_{x,\lambda}^2(\eta)}\,d\eta > \frac{u_{0,\lambda}(y_0)}{u_{x,\lambda}(y_0)}$$

and therefore $u_{0,\underline{\lambda}(0)}(y) < 0$. Thus, we have proved that α has a gap at x.

(ii) Take $y < z$. If $\alpha^{(z)}$ is R-positive and $\underline{\lambda}(z) = \underline{\lambda}(y)$ we get a contradiction as we have done for (17). Thus, $\underline{\lambda}(z) > \underline{\lambda}(y)$, so $\alpha^{(y)}$ has a gap at z with respect to y, which implies that $\alpha^{(y)}$ is R-positive, and $\underline{\lambda}$ is strictly increasing on $[0, z]$.

(iii) We notice that α does not have a gap at any $x > 0$ and therefore $\underline{\lambda}(0) = \underline{\lambda}(x)$. From (16) we find

$$\int_{y_0}^\infty \frac{\exp(2\int_x^\eta \alpha(\xi)\,d\xi)}{u_{x,\underline{\lambda}(0)}^2(\eta)}\,d\eta < \infty.$$

Therefore, $\alpha^{(x)}$ is R-transient. $\square$

LEMMA 12. *Assume that α is R-transient. Then there exists $\epsilon > 0$ such that any solution of the problem $v'' - 2\alpha v' = -2\underline{\lambda}(0)v$ whose initial conditions satisfy $0 \le v(0) \le \epsilon$, $|v'(0) - 1| \le \epsilon$, is positive on $(0, \infty)$.*

PROOF. We begin by fixing some constants used in the proof. Let $a_1 > 1$ be the smallest solution of $\log(a_1)/a_1 = (4e)^{-1}$ and $a^* > a_1$ the smallest solution of $\log(a^*)/a^* = (2e)^{-1}$. We notice that $a^* < e$, and for any $a^* < a < e$ we have $(2e)^{-1} < \log(a)/a < e^{-1}$.

We denote by $w = u_{0,\underline{\lambda}(0)}$. We choose $\epsilon > 0$ small enough such that the following conditions are satisfied: v is positive on $(0, 1]$; $\max\{w(1)/v(1), v(1)/w(1)\} \le a_1$ and $\epsilon \int_1^\infty w^{-2}(x)e^{\gamma(x)}\,dx \le (4e)^{-1}$.

For $a \in (a^*, e)$ we shall prove that $v(x) > w(x)/a$ on $[1,\infty)$. Suppose the contrary. Since $v(1)/w(1) \ge 1/a_1 > 1/a$ we obtain that

$$1 < x(a) := \inf\{x > 1 : v(x) \le w(x)/a\} < \infty.$$

Consider the Wronskian $W = W[w, v]$. It is direct to prove that $W(x) = v(0)e^{\gamma(x)}$. Since v is positive on the interval $[1, x(a)]$ we obtain

$$w(x) = \frac{w(1)}{v(1)} v(x) \exp\biggl(\int_1^x \frac{W(y)}{w(y)v(y)}\,dy\biggr) \qquad \text{for } x \in [1, x(a)].$$

Using the relations $w(x(a)) = v(x(a))a > 0$ and $v(x) \ge w(x)/a$ on $[1, x(a)]$, we obtain

$$av(x(a)) \le \frac{w(1)}{v(1)} v(x(a)) \exp\biggl(v(0)a \int_1^{x(a)} \frac{e^{\gamma(y)}}{w^2(y)}\,dy\biggr).$$

Therefore,

$$\frac{\log(a)}{a} \le \frac{\log(w(1)/v(1))}{a} + \epsilon \int_1^\infty \frac{e^{\gamma(y)}}{w^2(y)}\,dy \le (2e)^{-1},$$

which is a contradiction. Thus, we have proved $v \ge w/a^*$ on $[1,\infty)$; in particular v is positive. $\square$

COROLLARY 13. *Assume that $\alpha^{(0)}$ is R-transient and $\tilde{\alpha}$ is an extension of $\alpha^{(0)}$. Then there is $\delta > 0$ such that $\underline{\lambda}_{\tilde{\alpha}}(x) = \underline{\lambda}_\alpha(0)$ for $x \in [-\delta, 0]$.*

PROOF. Consider $\epsilon > 0$ given by Lemma 12. If $\delta > 0$ is sufficiently small we have, for fixed $x \in [-\delta, 0)$, $v = u_{x,\underline{\lambda}_\alpha(0);\tilde{\alpha}}$ satisfies $0 \le v(0) \le \epsilon$, $|v'(0) - 1| \le \epsilon$ and v is positive on $(x, 0]$. Therefore, from the previous lemma, v is positive on (x,∞), which implies that $\underline{\lambda}_{\tilde{\alpha}}(x) \ge \underline{\lambda}_\alpha(0)$. The opposite inequality follows from the fact that $\underline{\lambda}_{\tilde{\alpha}}$ is an increasing function. $\square$

PROOF OF THEOREM 2. From Theorem 1 it follows that $\underline{\lambda}$ is strictly increasing on $[0, \bar{x})$, and in the same interval $\alpha^{(x)}$ is R-positive. Also $\alpha^{(x)}$ is R-transient in the region $(\bar{x},\infty)$.

Now let us prove that $\underline{\lambda}$ is continuous on $[0,\infty)$. We use the continuity of $u_{x,\lambda}(y)$ on x, λ, y. Consider $x \in [0, \bar{x})$. As z decreases to x, the right-hand side of (15) converges to 0 and the integral on the left-hand side stays bounded away from zero. Therefore, we deduce the right continuity of $\underline{\lambda}$ at x. For $x \in (0, \bar{x}]$ we obtain the left continuity of $\underline{\lambda}$ in the same way. The only thing left to prove is the right continuity at $\bar{x}$. If $\underline{\lambda}(\bar{x}) < \underline{\lambda}(\infty)$ we would get a contradiction with (15) by letting z decreases to $\bar{x}$, because for all $z > \bar{x}$ we have $\underline{\lambda}(\infty) = \underline{\lambda}(z)$.

An application of the dominated convergence theorem lead us to conclude from (15) that

$$\int_x^\infty u^2_{x,\underline{\lambda}(x)}(y) \exp\biggl(-2\int_x^y \alpha(\xi)\,d\xi\biggr)\,dy = \frac{1}{2\underline{\lambda}'(x)}, \tag{18}$$

and we deduce $\underline{\lambda}$ is C^1 on $[0, \bar{x})$.

Let $0 < \bar{x} < \infty$. From the definition of $\bar{x}$ we have $\underline{\lambda}(y) < \underline{\lambda}(\bar{x})$ for any $y < \bar{x}$, and according to Corollary 13, we obtain that $\alpha^{(\bar{x})}$ is R-recurrent.

From (14) if $x < z < \bar{x} < y_0 \le y$ we have

$$\frac{u^2_{x,\underline{\lambda}(x)}(y)}{u^2_{x,\underline{\lambda}(x)}(y_0)} \le \frac{u^2_{z,\underline{\lambda}(z)}(y)}{u^2_{z,\underline{\lambda}(z)}(y_0)}.$$

Using the monotone convergence theorem in (18) we can pass to the limit to $\bar{x}$ and conclude that

$$\int_{\bar{x}}^\infty u^2_{\bar{x},\underline{\lambda}(\bar{x})}(y) \exp\biggl(-2\int_{\bar{x}}^y \alpha(\xi)\,d\xi\biggr)\,dy = \lim_{x\uparrow\bar{x}} \frac{1}{2\underline{\lambda}'(x)}.$$

Therefore, $\alpha^{(\bar{x})}$ is R-positive if and only if $\lim_{x\uparrow\bar{x}} \underline{\lambda}'(x) > 0$. $\square$

PROOF OF PROPOSITION 4. From Theorem A the function ϕ_x satisfies hypothesis H in the region $[z, \infty)$, for $z > x$. Hypothesis H1 for ϕ_x in $[z, \infty)$ follows from equalities

$$\int_z^\infty \exp\biggl(2\int_z^y \phi_x(\xi)\,d\xi\biggr)\,dy = u^2_{x,\underline{\lambda}(x)}(z)\int_z^\infty \frac{\exp(2\int_z^y \alpha(\xi)\,d\xi)}{u^2_{x,\underline{\lambda}(x)}(y)}\,dy = \infty.$$

The last equality follows from the hypothesis that $\alpha^{(x)}$ is R-positive. The rest of the proof follows immediately from relation (8). $\square$

PROOF OF THEOREM 5. We first assume α and β verify condition (C1). We denote by $\lambda = \underline{\lambda}_\alpha(0)$, $\mu = \underline{\lambda}_\beta(0)$, $v = u_{0,\lambda;\alpha}$ and $w = u_{0,\mu;\beta}$. Now, consider the function $H = v'w - vw' - (\alpha - \beta)vw$. A simple computation yields

$$\begin{aligned} H' &= (\alpha+\beta)H + vw\bigl(\alpha^2 - \alpha' - 2\lambda - (\beta^2 - \beta' - 2\mu)\bigr) \\ &= (\alpha+\beta)H + vw(h_\alpha - h_\beta). \end{aligned}$$

By hypothesis, the function $h_\alpha - h_\beta$ is nonnegative, which implies

$$\begin{aligned} H(x) = {}&\exp\biggl(\int_0^x \bigl(\alpha(\xi)+\beta(\xi)\bigr)\,d\xi\biggr) \\ &\times \int_0^x v(y)w(y)\bigl(h_\alpha(y) - h_\beta(y)\bigr)\exp\biggl(-\int_0^y \bigl(\alpha(z)+\beta(z)\bigr)\,dz\biggr)\,dy \ge 0. \end{aligned}$$

Therefore, we get $v'/v - \alpha \ge w'/w - \beta$ on $(0, \infty)$. Integrating this inequality and using the relation $\lim_{\varepsilon \downarrow 0} v(\varepsilon)/w(\varepsilon) = 1$, we obtain

$$w^2(x) \exp\biggl(-2 \int_0^x \beta(z)\, dz\biggr) \le v^2(x) \exp\biggl(-2 \int_0^x \alpha(z)\, dz\biggr).$$

Then properties (i) and (ii) follow from the criteria given in Theorem B.

Now we assume (C2) holds. Let $\tilde\beta$ and $\tilde\alpha$ be any pair of extensions of β and α, respectively, defined on $[-\varepsilon, \infty)$ for some $\varepsilon > 0$ and satisfying $\tilde\beta \le \tilde\alpha$. By comparison we have the inequality $\underline{\lambda}_{\tilde\beta}(x) \le \underline{\lambda}_{\tilde\alpha}(x)$, valid for all $x \ge -\varepsilon$.

Let us prove relation (i). Since β is R-transient we have $\underline{\lambda}_{\tilde\beta}(x) = \underline{\lambda}_{\beta}(0) = \underline{\lambda}_{\beta}(\infty)$, for all $x < 0$ closed enough to 0. By hypothesis and comparison we get

$$\underline{\lambda}_{\alpha}(\infty) = \underline{\lambda}_{\beta}(\infty) = \underline{\lambda}_{\tilde\beta}(x) \le \underline{\lambda}_{\tilde\alpha}(x) \le \underline{\lambda}_{\alpha}(\infty),$$

which implies that 0 is a point of constancy for $\underline{\lambda}_{\tilde\alpha}$ proving that α is R-transient.

Now let us prove (ii). If β has a gap then it is R-positive. So for the rest of the proof, we can assume that $\underline{\lambda}_{\beta}(0) = \underline{\lambda}_{\beta}(\infty)$. By hypothesis and comparison we have $\underline{\lambda}_{\alpha}(\infty) = \underline{\lambda}_{\beta}(\infty) = \underline{\lambda}_{\beta}(0) \le \underline{\lambda}_{\alpha}(0) \le \underline{\lambda}_{\alpha}(\infty)$, so $\underline{\lambda}_{\beta}(0) = \underline{\lambda}_{\alpha}(0)$. Since $\underline{\lambda}_{\alpha}(0) = \underline{\lambda}_{\alpha}(\infty)$ and α is assumed to be R-positive, Theorem 3(ii) implies that $\underline{\lambda}'_{\tilde\alpha}(0-) > 0$. From $\underline{\lambda}_{\tilde\beta}(x) \le \underline{\lambda}_{\tilde\alpha}(x)$ we get $\underline{\lambda}'_{\tilde\beta}(0-) \ge \underline{\lambda}'_{\tilde\alpha}(0-) > 0$. By using again Theorem 3(ii) we conclude β is R-positive.

The proof that (C3) implies (i) and (ii) is similar to the previous one. $\square$

LEMMA 14. *Let $b > 0$ and consider $\hat\lambda(b) = \sup\{\lambda : u_{0,\lambda}$ is increasing on $[0, b]\}$. Then*

(19) $\quad \underline{\lambda}(0) < \hat\lambda(b) < \bigl(D^2 + (\pi/b)^2\bigr)/2 \qquad$ *where* $D = \sup\{\alpha(x) : x \in [0, b]\}$.

PROOF. The first inequality in (19) follows from Lemma 9. For proving the second inequality, consider the function $g(x) = e^{Dx} \sin(\pi x/b)$. Function g is positive on $(0, b)$; it verifies $g(0) = g(b) = 0$ and the equation $g'' - 2Dg' = -2\lambda g$, where $\lambda = (D^2 + (\pi/b)^2)/2$. Assume that $v = u_{0,\lambda}$ is increasing on $[0, b]$. Using the Wronskian $W = W[v, g]$ we deduce that $W' = 2DW + 2v'g(\alpha - D)$ and therefore

$$0 < W(b) = -g'(b)v(b) = 2e^{2Db} \int_0^b e^{-2Dx} v'(x) g(x) \bigl(\alpha(x) - D\bigr)\, dx \le 0,$$

which is a contradiction. Therefore, $u_{0,\lambda}$ cannot be increasing on $[0, b]$, proving that $\hat\lambda(b) < (D^2 + (\pi/b)^2)/2$. $\square$

PROOF OF COROLLARY 6. The proof is based on a comparison (see [6]) with the constant drift case. For proving (i), we notice that (9) implies $\underline{\lambda}(x) > \underline{\lambda}(0)$ for any large enough x. Therefore, α has a gap, which ensures that α is R-positive.

The fact that condition (10) is sufficient for (9) follows from property (19) in Lemma 14.

Now we prove (ii). For any $\epsilon > 0$ there exists x_0 large enough, such that $\underline{\lambda}(x) \le (\alpha(\infty) + \epsilon)^2/2$ for $x \ge x_0$, proving that $\underline{\lambda}(\infty) \le \alpha(\infty)^2/2$. On the other hand the condition $0 \le \alpha(\infty) \le \alpha(x)$ for all $x \ge 0$, ensures that $\underline{\lambda}(0) \ge \alpha(\infty)^2/2$, proving that $\underline{\lambda}(x) = \alpha(\infty)^2/2$ for all $x \ge 0$. The rest of the proof is based on Theorem 5. Indeed, take β the constant function $\alpha(\infty)$. The condition (C2) in Theorem 5 is satisfied and since β is R-transient we get α is also R-transient. □

PROOF OF PROPOSITION 7. Consider the nonnegative function $f(x) = \alpha(\infty) - \alpha(x)$. Let β be the constant function $\beta = \alpha(\infty)$. Denote by $\mu = \underline{\lambda}_\beta(0)$ the bottom of its spectrum, which is $\mu = \alpha(\infty)^2/2$. We shall prove $\underline{\lambda}_\alpha(0) = \mu$. Put $v = u_{0,\mu;\alpha}$ and $w = u_{0,\mu;\beta}$. We notice that $w(x) = xe^{\beta x}$. At this point we do not know if v is nonnegative.

From $w'' - 2\beta w' = -2\mu w$ and $v'' - 2(\beta - f(x))v' = -2\mu v$ we deduce that the Wronskian $W = W[w, v]$ is given by

$$W(x) = 2\exp\biggl(2\beta x - 2\int_0^x f(y)\,dy\biggr) \times \int_0^x f(z)w'(z)v(z)\exp\biggl(-2\beta z + 2\int_0^z f(y)\,dy\biggr)\,dz.$$

Since w' and f are nonnegative, if v is positive on some interval $(0, x_0]$, then W is nonnegative in that interval. This implies the inequality $v(x) \le w(x)$ for all $x \in [0, x_0]$. Hence, using the explicit form for w, we obtain the following upper bound for W:

$$W(x) \le 2e^{2\beta x}\int_0^x f(z)w'(z)w(z)e^{-2\beta z}\,dz = 2e^{2\beta x}\int_0^x f(z)z(\beta z + 1)\,dz. \tag{20}$$

On the other hand, for $x \in (0, x_0]$ we have the equality

$$w(x) = v(x)\exp\biggl(\int_0^x \frac{W(y)}{w(y)v(y)}\,dy\biggr).$$

Now consider the function $g(a) = \log(a)/(2a)$, which is nonnegative for $a \ge 1$ and attains its maximum at $a = e$, with $g(e) = 1/(2e)$. Moreover, g is strictly increasing on $[1, e)$ and strictly decreasing on $(e, \infty]$. From the hypothesis $\int_0^\infty f(z)(\beta z + 1)\,dz < 1/(2e)$, there exists a unique $\bar{a} \in [1, e)$ such that

$$\int_0^\infty f(z)(\beta z + 1)\,dz = \frac{\log(\bar{a})}{2\bar{a}}.$$

We shall prove that $v \ge w/\bar{a}$. For this purpose take any $a > \bar{a}$, sufficiently close to $\bar{a}$ in order to have $g(a) > g(\bar{a})$. Assume that $x(a) := \inf\{x > 0 : v(x) < w(x)/a\}$ is finite. Notice that $x(a) > 0$. Since v is strictly positive on $(0, x(a)]$ we have

$$av(x(a)) = w(x(a)) = v(x(a))\exp\biggl(\int_0^{x(a)} \frac{W(y)}{w(y)v(y)}\,dy\biggr).$$

Therefore, since $v(x) \geq w(x)/a$ on $[0, x(a)]$ we get from (20)

$$\begin{aligned}\log(a) &= \int_0^{x(a)} \frac{W(y)}{w(y)v(y)}\,dy \\ &\leq a\int_0^{x(a)} \frac{W(y)}{w^2(y)}\,dy \\ &\leq 2a\int_0^{x(a)} \frac{e^{2\beta y}}{w^2(y)}\int_0^y f(z)z(\beta z+1)\,dz \\ &\leq 2a\int_0^{\infty} \frac{1}{y^2}\int_0^y f(z)z(\beta z+1)\,dz \\ &= 2a\int_0^{\infty} f(z)(\beta z+1)\,dz.\end{aligned}$$

This implies that

$$g(a) = \frac{\log(a)}{2a} \leq \int_0^{\infty} f(z)(\beta z+1)\,dz = g(\bar{a}),$$

obtaining a contradiction. Thus, $x(a) = \infty$.

We have proved that $u_{0,\alpha(\infty)^2/2;\alpha} \geq w/\bar{a}$, implying that $u_{0,\alpha(\infty)^2/2;\alpha}$ is nonnegative. Hence, $\underline{\lambda}_\alpha(0) \geq \alpha(\infty)^2/2$. The opposite inequality follows from a comparison with the constant case $\alpha(\infty)$. Thus, $v = u_{0,\underline{\lambda}(0);\alpha} \geq w/\bar{a}$.

Finally, since $\alpha \leq \alpha(\infty)$ we get

$$u_{0,\underline{\lambda}(0);\alpha}(x)^{-2}\exp\biggl(2\int_0^x \alpha(\xi)\,d\xi\biggr) \leq \bar{a}^2 w(x)^{-2}e^{2\alpha(\infty)x} = (\bar{a}/x)^2,$$

and α is R-transient from Theorem B(iii). $\square$

PROOF OF PROPOSITION 8. Since for a constant drift $-\theta$ the bottom of the spectrum is $\theta^2/2$ we get $\underline{\lambda}(\ell) = \theta^2/2$ and a simple computation yields $u_{\ell,\underline{\lambda}(\ell)}(x) = (x-\ell)e^{\theta(x-\ell)}$. In particular $u_{\ell,\underline{\lambda}(\ell)}^{-2}(x)e^{2\theta(x-\ell)} = (x-\ell)^{-2}$, which is integrable near ∞. Therefore, $\alpha^{(\ell)}$ is R-transient, and the result follows when $\ell = 0$. In the sequel we shall assume that $\ell > 0$. We observe that $\underline{\lambda}(0) \leq \theta^2/2$.

We denote by $\hat{\lambda} = \hat{\lambda}(\ell) = \sup\{\lambda : u_{0,\lambda}$ is increasing on $[0, \ell]\}$. From Lemma 14 we have

$$\underline{\lambda}(0) < \hat{\lambda} < \bigl(D^2 + (\pi/\ell)^2\bigr)/2 \qquad \text{where } D = \sup\{\alpha(x) : x \in [0, \ell]\}.$$

We notice that $u'_{0,\hat{\lambda}}(\ell) = 0$; otherwise for some $\lambda > \hat{\lambda}$ we would have that $u_{0,\lambda}$ is increasing on $[0, \ell]$, contradicting the maximality of $\hat{\lambda}$.

The mapping $u'_{0,\mu^2/2}(\ell)/u_{0,\mu^2/2}(\ell) - \mu$, as a function of μ, is continuous and strictly decreasing on $[0, \sqrt{2\hat{\lambda}}\,]$, positive at 0 and negative at $\sqrt{2\hat{\lambda}}$. Therefore, there exists a unique root of this function, in $(0, \sqrt{2\hat{\lambda}}\,)$, which we denote by $\underline{\theta}$. This root verifies

$$\frac{u'_{0,\underline{\theta}^2/2}(\ell)}{u_{0,\underline{\theta}^2/2}(\ell)} = \underline{\theta} \quad \text{and} \quad \left[\theta \le \underline{\theta} \quad \Longleftrightarrow \quad \left(\frac{u'_{0,\theta^2/2}(\ell)}{u_{0,\theta^2/2}(\ell)} \ge \theta \text{ and } \theta \le \sqrt{2\hat{\lambda}}\right)\right].$$

Let us take

$$\lambda^* = \sup\left\{\lambda \le \min(\hat{\lambda}, \theta^2/2) : \frac{u'_{0,\lambda}(\ell)}{u_{0,\lambda}(\ell)} + \sqrt{\theta^2 - 2\lambda} \ge \theta\right\}.$$

As before, one can easily prove that λ^* satisfies $0 < \lambda^* < \hat{\lambda}$.

The equivalence $\theta > \underline{\theta} \Leftrightarrow \lambda^* < \theta^2/2$ plays an important role in the sequel, and it follows from

$$\text{(21)} \quad \lambda^* = \frac{\theta^2}{2} \quad \Longleftrightarrow \quad \left(\frac{u'_{0,\theta^2/2}(\ell)}{u_{0,\theta^2/2}(\ell)} \ge \theta \text{ and } \theta \le \sqrt{2\hat{\lambda}}\right) \quad \Longleftrightarrow \quad \theta \le \underline{\theta}.$$

We shall now prove that $\lambda^* = \underline{\lambda}(0)$. Take any $\lambda \le \min(\hat{\lambda}, \theta^2/2)$. The function $u_{0,\lambda}$ is increasing on $[0, \ell]$. The question is to determine the values of λ for which $u_{0,\lambda}$ is increasing in (ℓ, ∞). For this purpose consider the solution of

$$\tfrac{1}{2} f''(x) - \theta f'(x) = -\lambda f(x), \qquad x \in [\ell, \infty),$$

with boundary conditions $f(\ell) = u_{0,\lambda}(\ell)$, $f'(\ell) = u'_{0,\lambda}(\ell)$. Obviously $f = u_{0,\lambda}$ on $[\ell, \infty)$. For the analysis of this solution we consider two possible cases. When $\rho = \sqrt{\theta^2 - 2\lambda} > 0$ the solution is given by

$$f(x) = e^{\theta(x-\ell)}\big(A \sinh\big(\rho(x - \ell)\big) + B \cosh\big(\rho(x - \ell)\big)\big).$$

From the boundary conditions we obtain

$$0 < f(\ell) = u_{0,\lambda}(\ell) = B, \qquad f'(\ell) = u'_{0,\lambda}(\ell) = \theta B + \rho A.$$

The condition for having an increasing (positive solution) is equivalent to $A \ge -B$, that is, to $u'_{0,\lambda}(\ell) \ge (\theta - \sqrt{\theta^2 - 2\lambda}\,) u_{0,\lambda}(\ell)$. In other words it is equivalent to

$$\frac{u'_{0,\lambda}(\ell)}{u_{0,\lambda}(\ell)} + \sqrt{\theta^2 - 2\lambda} \ge \theta.$$

On the other hand, in the case $\lambda = \theta^2/2$ (then necessarily $\theta^2/2 \le \hat{\lambda}$), the solution is

$$f(x) = \big(C(x - \ell) + B\big)e^{\theta(x-\ell)},$$

where $B = u_{0,\theta^2/2}(\ell) > 0$ and $C = u'_{0,\theta^2/2}(\ell) - \theta u_{0,\theta^2/2}(\ell)$. The condition for having a positive solution is $C \ge 0$ which is equivalent to

$$\frac{u'_{0,\theta^2/2}(\ell)}{u_{0,\theta^2/2}(\ell)} \ge \theta.$$

In summary, we have shown that f is positive if and only if $\lambda \le \lambda^*$. In particular u_{0,λ^*} is positive on $[0,\infty)$ proving that $\lambda^* \le \underline{\lambda}(0)$. On the other hand, since $u_{0,\underline{\lambda}(0)}$ is positive and $\underline{\lambda}(0) \le \min(\hat{\lambda}, \theta^2/2)$, the argument given above allows us to conclude the equality $\lambda^* = \underline{\lambda}(0)$.

Thus, in the case $\underline{\lambda}(0) < \theta^2/2$, from (21) one gets

$$\frac{u'_{0,\underline{\lambda}(0)}(\ell)}{u_{0,\underline{\lambda}(0)}(\ell)} + \sqrt{\theta^2 - 2\underline{\lambda}(0)} = \theta,$$

which in the previous notation amounts to $A = -B$. Therefore, the solution $u_{0,\underline{\lambda}(0)}$ is, for $x > \ell$,

$$u_{0,\underline{\lambda}(0)}(x) = u_{0,\underline{\lambda}(0)}(\ell) e^{(\theta - \sqrt{\theta^2 - 2\underline{\lambda}(0)})(x-\ell)}.$$

In particular $u^2_{0,\underline{\lambda}(0)}(x)e^{-\gamma(x)} = u^2_{0,\underline{\lambda}(0)}(\ell)e^{-\gamma(\ell)}e^{-2\sqrt{\theta^2-2\underline{\lambda}(0)}(x-\ell)}$ for $x > \ell$, which is integrable and therefore X^T is R-positive.

On the other hand, if $\underline{\lambda}(0) = \theta^2/2$ one has $u_{0,\underline{\lambda}(0)} = e^{\theta(x-\ell)}(C(x-\ell)+B)$ for $x \ge \ell$, with $B > 0$ and $C \ge 0$. Then, the function

$$u^2_{0,\underline{\lambda}(0)}(x)e^{-\gamma(x)} = \bigl(C(x-\ell)+B\bigr)^2 e^{-\gamma(\ell)}$$

is not integrable near ∞.

In summary X^T is R-positive if and only if $\underline{\lambda}(0) < \theta^2/2$, which we have proved to be equivalent to $\theta > \underline{\theta}$.

Now we prove that α is R-transient if and only if $\theta < \underline{\theta}$. Remark that $\underline{\lambda}(0) = \theta^2/2$ holds under both conditions of the claimed equivalence. Since $u_{0,\theta^2/2}(x) = e^{\theta(x-\ell)}(C(x-\ell)+B)$ for $x \ge \ell$, we get that α is R-transient if and only if $C > 0$, or equivalently,

$$\theta u_{0,\theta^2/2}(\ell) < u'_{0,\theta^2/2}(\ell),$$

which holds if and only if $\theta < \underline{\theta}$.

Finally, we give an explicit formula for $\underline{\theta}$ when $\alpha(x) = \theta_0 \mathbb{1}_{\{x<\ell\}} + \theta \mathbb{1}_{\{x \ge \ell\}}$ and $\theta > \theta_0$. In this case, the solution $u_{0,\lambda}$ for $\lambda > \theta_0^2/2$, is

$$u_{0,\lambda}(x) = \frac{e^{\theta_0 x}}{\chi} \sin(\chi x),$$

where $\chi = \sqrt{2\lambda - \theta_0^2}$, and therefore $\underline{\theta} = \sqrt{2\lambda}$ is the unique solution of

$$\sqrt{2\lambda} = \frac{u'_{0,\lambda}(\ell)}{u_{0,\lambda}(\ell)} = \theta_0 + \chi \cot(\chi\ell) = \sqrt{\chi^2 + \theta_0^2}.$$

We obtain the relation $\theta_0 = -\chi \cot(2\chi\ell)$, from which we get the desired value of $\underline{\theta}$. $\square$

4. Examples.

EXAMPLE A. In the ultimately constant case, if $\bar{x} > 0$, $\alpha^{(\bar{x})}$ is always R-null (Proposition 8), then the transition from R-positive to R-transient occurs through a R-null point. We show that this is not always the case; that is, we exhibit an example where $0 < \bar{x} < \infty$ and $\alpha^{(\bar{x})}$ is R-positive. Let us construct it. Take a function g verifying the following conditions:

(i) $g > 0$ on $(0, \infty)$, $g(0) = 0$ and $g'(0) = 1$;
(ii) $\int_0^\infty g^2(x)\,dx < \infty$;
(iii) $g + g' > 0, \lim_{x\to\infty} g''(x)/(g(x) + g'(x)) = 0$ and $\int_0^\infty |g''(x)/(g(x) + g'(x))|\,dx < \infty$.

For instance $g(x) = x/(1+x)^2$ does the job.

Fix some $a > 0$. Let α be such that $\alpha(x) = 1 + g''(x-a)/(2(g(x-a) + g'(x-a)))$ for $x \ge a$. Obviously we have $\underline{\lambda}(\infty) = \alpha(\infty)^2/2 = 1/2$. Since the function $v(x) = g(x-a)e^{(x-a)}$ solves the problem $v'' - 2\alpha v' = -v$ on (a, ∞) with the boundary conditions $v'(a) = 1$, $v(a) = 0$ and it is positive, we get $\underline{\lambda}(a) = \underline{\lambda}(\infty) = 1/2$. On the other hand, from (ii) and (iii) it can be checked that $\alpha^{(a)}$ is R-positive. From Theorem 1(ii) we conclude $\bar{x} = a$.

EXAMPLE B. Let us now show that for some bounded drifts we can have $\bar{x} = \infty$. Take a sequence $0 < b_n < 1$ converging towards 1. Consider $x_0 = 0$, $x_{n+1} = x_n + \pi/\sqrt{1-b_n^2}$ and define $\alpha(x) = b_n$ for $x \in [x_n, x_{n+1})$. We have $\underline{\lambda}(\infty) = 1/2$. Let us prove that $\underline{\lambda}(x_n) < 1/2$. The solution of $v'' - 2\alpha v' = -v$ with $v(x_n) = 0$, $v'(x_n) = 1$, is given by

$$v(x) = \frac{e^{(x-x_n)}}{\sqrt{1-b_n^2}} \sin\bigl((x-x_n)\sqrt{1-b_n^2}\bigr) \qquad \text{for } x \in [x_n, x_{n+1}).$$

Since $v(x_{n+1}) = 0$ we obtain that $\underline{\lambda}(x_n) < 1/2$ and therefore $\bar{x} = \infty$.

APPENDIX

The proof of Theorem A is based on the following lemma, for which we assume neither H nor H1.

LEMMA C. *Assume α is locally bounded and measurable. Let $\lambda < 0$, then the following two conditions are equivalent*:

(i) *$u_{0,\lambda}$ is unbounded*;
(ii) $\int_0^\infty e^{\gamma(x)} \int_0^x e^{-\gamma(y)}\,dy\,dx = \infty$.

PROOF. We denote $v = u_{0,\lambda}$. From (2) and the fact that $\lambda < 0$ we get that v is strictly increasing. Moreover we have

$$v(x) = \Lambda(x) - 2\lambda \int_0^x e^{\gamma(y)} \int_0^y v(z) e^{-\gamma(z)}\, dz\, dy.$$

Hence, if $\Lambda(\infty) = \infty$, both conditions (i) and (ii) are satisfied. Therefore, for the rest of the proof we can assume $\Lambda(\infty) < \infty$.

Suppose that (ii) holds. For $x > 1$, v can be bounded from below by

$$\begin{aligned} v(x) &\geq \Lambda(x) - 2\lambda \int_1^x e^{\gamma(y)} \int_1^y v(z) e^{-\gamma(z)}\, dz\, dy \\ &\geq \Lambda(x) - 2\lambda v(1) \int_1^x e^{\gamma(y)} \int_1^y e^{-\gamma(z)}\, dz\, dy \\ &\geq \Lambda(x) - 2\lambda v(1) \int_0^x e^{\gamma(y)} \int_0^y e^{-\gamma(z)}\, dz\, dy \\ &\qquad + 2\lambda v(1) \biggl(\int_0^1 e^{\gamma(y)} \int_0^y e^{-\gamma(z)}\, dz\, dy + \Lambda(x) \int_0^1 e^{-\gamma(z)}\, dz \biggr). \end{aligned}$$

Then v is unbounded.

Now, assume $\int_0^\infty e^{\gamma(x)} \int_0^x e^{-\gamma(y)}\, dy\, dx < \infty$. We shall prove that v is bounded. Indeed, take a large x_0 such that $-2\lambda \int_{x_0}^\infty e^{\gamma(y)} \int_0^y e^{-\gamma(z)}\, dz\, dy \leq 1/2$. For $x > x_0$ we have

$$\begin{aligned} v(x) &\leq v(x_0) + \Lambda(\infty) - 2\lambda v(x) \int_{x_0}^x e^{\gamma(y)} \int_0^y e^{-\gamma(z)}\, dz\, dy \\ &\leq v(x_0) + \Lambda(\infty) + v(x)/2. \end{aligned}$$

Therefore, v is bounded by $2(v(x_0) + \Lambda(\infty))$. $\square$

PROOF OF THEOREM A. We denote $v = u_{0,\underline{\lambda}(0)}$. We also recall the notation $\gamma^Y(y) = 2\int_c^y \phi(\xi)\, d\xi = \gamma(y) - \gamma(c) - 2\log(v(y)/v(c))$, for some $c > 0$ fixed. Then

$$\begin{aligned} &\int_z^\infty e^{-\gamma^Y(y)} \int_c^y e^{\gamma^Y(\xi)}\, d\xi\, dy \\ &\qquad = \int_z^\infty \frac{v^2(y)}{v^2(c)} e^{-\gamma(y)} \int_c^y \frac{v^2(c)}{v^2(\xi)} e^{\gamma(\xi)}\, d\xi\, dy \\ &\qquad \geq \int_z^\infty e^{-\gamma(y)} \int_c^y e^{\gamma(\xi)}\, d\xi\, dy \\ &\qquad = \infty, \end{aligned}$$

where we have used the monotonicity of v and hypotheses H and H1 for α.

For the other integral involved in condition H, we consider two different situations. In the first one we assume $\underline{\lambda}(0) = 0$. In this case $v = \Lambda$ and $\phi =$

$\alpha - \Lambda'/\Lambda$. Since $d\Lambda(y) = e^{\gamma(y)}dy$, an integration by parts yields

$$\begin{aligned}\int_z^x e^{\gamma^Y(y)} \int_c^y e^{-\gamma^Y(\xi)}\, d\xi\, dy &= \int_z^x \frac{e^{\gamma(y)}}{\Lambda^2(y)} \int_c^y \Lambda^2(\xi)e^{-\gamma(\xi)}\, d\xi\, dy \\ &= \int_c^x \Lambda(y)e^{-\gamma(y)}\left(1 - \frac{\Lambda(y)}{\Lambda(x)}\right) dy.\end{aligned}$$

Since Λ increases to ∞ we can take $x_n \uparrow \infty$ such that $\Lambda(x_n) = \Lambda(n)/2$. Then

$$\begin{aligned}\int_z^\infty e^{\gamma^Y(y)} \int_c^y e^{-\gamma^Y(\xi)}\, d\xi\, dy &\geq \int_c^n \Lambda(y)e^{-\gamma(y)}\left(1 - \frac{\Lambda(y)}{\Lambda(n)}\right) dy \\ &\geq \frac{1}{2}\int_c^{x_n} \Lambda(y)e^{-\gamma(y)}\, dy,\end{aligned}$$

which converges to infinite because α satisfies H.

We are left with the case $\underline{\lambda}(0) > 0$. Consider $w = u_{z,0;\alpha}$ and $v = u_{0,\underline{\lambda}(0);\alpha}$. By (8) we have

$$u_{z,-\underline{\lambda}(0);\phi}(y) = \frac{w(y)v(z)}{v(y)}.$$

From Lemma C, the proof will be finished as soon as we prove w/v is unbounded. So let us assume $w/v \leq D$ on $[z, \infty)$. Then

$$\int_z^\infty w(y)e^{-\gamma(y)}\, dy \leq D \int_z^\infty v(y)e^{-\gamma(y)}\, dy,$$

which is finite from (4). On the other hand it is direct to check that $w(y) = e^{-\gamma(z)}(\Lambda(y) - \Lambda(z))$ and therefore

$$\int_z^\infty w(y)e^{-\gamma(y)}\, dy = e^{-\gamma(z)} \int_z^\infty \Lambda(y)e^{-\gamma(y)}\, dy - \Lambda(z) \int_z^\infty e^{-\gamma(y)}\, dy.$$

This quantity is infinite because α satisfies H and according to (5),

$$\int_z^\infty e^{-\gamma(y)}\, dy < \infty.$$

Thus, we arrive at a contradiction and w/v is unbounded. $\square$

PROOF OF THEOREM B. Let $v = u_{0,\underline{\lambda}(0);\alpha}$ and consider

$$\Lambda^Y(y) = \int_c^y e^{\gamma^Y(z)}\, dz = v^2(c) \int_c^y v^{-2}(z)e^{\gamma(z)-\gamma(c)}\, dz.$$

We first notice that $\Lambda^Y(0+) = -\infty$, because $v(x) = x + O(x^2)$ for x near 0. On the other hand if $\Lambda^Y(\infty) = \infty$ then Y is recurrent (see 5.5.22 in [3]). In the case

$\Lambda^Y(\infty) < \infty$, for any $x > 0$ it holds

$$\mathbb{P}_x\Big(\lim_{t \uparrow S} Y_t = \infty\Big) = \mathbb{P}_x\Big(\inf_{0 \le t < S} Y_t > 0\Big) = 1,$$

where S is the explosion time of Y. In this case the process Y is transient. Hence, Y is transient if and only if $\Lambda^Y(\infty) < \infty$, which is equivalent to

$$\int_c^\infty v^{-2}(z) e^{\gamma(z)}\, dz < \infty.$$

Let T_a^Y be the hitting time of $a > 0$ for the process Y. The process Y is positive recurrent when $\mathbb{E}_x(T_a^Y) < \infty$, for any $x, a \in (0, \infty)$. Using the formulas on page 353 in [3] and the fact that the speed measure for Y is given by

$$m(dx) = 2 \frac{e^{-\gamma(c)}}{v^2(c)} v^2(x) e^{-\gamma(x)}\, dx,$$

we deduce Y is positive recurrent if and only if $\int_0^\infty v^2(x) e^{-\gamma(x)}\, dx < \infty$. $\square$

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