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GRADIENT ESTIMATES OF DIRICHLET HEAT SEMIGROUPS AND APPLICATION TO ISOPERIMETRIC INEQUALITIES[1]

BY FENG-YU WANG

Beijing Normal University

By using probabilistic approaches, some uniform gradient estimates are obtained for Dirichlet heat semigroups on a Riemannian manifold with boundary. As an application, lower bound estimates of isoperimetric constants are presented in terms of functional inequalities.

1. Introduction. Let M be a connected complete Riemannian manifold of dimension d. Consider $L := \Delta + Z$, where Z is a C^1-vector field. Assume that there is $K \geq 0$ such that

$$(\mathrm{Ric} - \langle \nabla_{\cdot} Z, \cdot \rangle)(X, X) \geq -K|X|^2, \qquad X \in TM. \tag{1.1}$$

If M has no boundary, then the semigroup P_t of the L-diffusion process satisfies the gradient estimate

$$\|\nabla P_t f\|_\infty \leq \frac{C}{\sqrt{t}} \|f\|_\infty, \qquad t \in (0, 1],\ f \in \mathscr{B}_b^+(M), \tag{1.2}$$

where $C > 0$ is a constant depending only on K and $\mathscr{B}_b^+(M)$ stands for the set of all nonnegative bounded measurable functions. When M has a convex boundary ∂M (i.e., the second fundamental form of ∂M is nonnegative), (1.2) remains true for the Neumann heat semigroup (i.e., the semigroup of the reflecting L-diffusion process). Indeed, if ∂M is either empty or convex, then (1.1) implies $|\nabla P_t f| \leq e^{Kt} P_t |\nabla f|$ for all $t \geq 0$ and all $f \in C_b^1(M)$. This gradient estimate appeared first in Donnelly and Li (1982) for $L = \Delta$ and was established in Elworthy (1992), Bakry and Ledoux (1996), Qian (1997), Wang (1997) and some other references by different means. Then, as shown by the proof of Lemma 4.2 in Bakry and Ledoux (1996), one has

$$\|\nabla P_t f\|_\infty^2 \leq \frac{K\|f\|_\infty^2}{1 - e^{-2Kt}} \leq \frac{K\|f\|_\infty^2}{(1 - e^{-2Kt})(t \wedge 1)}, \qquad t > 0. \tag{1.3}$$

In particular, if $K = 0$, then $\|\nabla P_t f\|_\infty \leq \|f\|_\infty / \sqrt{2t}$.

A remarkable application of the uniform gradient estimate was made by Ledoux (1994) to obtain isoperimetric inequalities using Poincaré and log-Sobolev ones. It is well known that a Poincaré–Sobolev type of inequality follows from the

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corresponding isoperimetric inequality [see, e.g., Chavel (1984)]. When $Z = 0$, Buser (1992) proved a converse result; that is, he obtained a lower bound estimate of Cheeger's isoperimetric constant by using the Poincaré inequality. His proof was considerably simplified by Ledoux through a gradient estimate of type (1.2). Ledoux's argument has been used in Wang (2000) and Hu (2002) to obtain lower bounds of various isoperimetric constants from general Poincaré–Sobolev inequalities.

The purpose of this paper is to establish (1.2) for the Dirichlet heat semigroup and then apply the gradient estimate to isoperimetric inequalities. From now on, we assume that $\partial M \neq \varnothing$ and let P_t be the Dirichlet semigroup generated by L, that is,

$$P_t f(x) := \mathbb{E} f(x_t^x) \mathbb{1}_{\{t<\tau^x\}}, \qquad f \in \mathscr{B}_b^+(M), t \geq 0, \tag{1.4}$$

where $(x_t^x)_{t\geq 0}$ is the L-diffusion process starting from x and τ^x is the hitting time of the process to the boundary.

The gradient estimate of the Dirichlet semigroup has been studied in Thalmaier and Wang (1998) and Wang (1997, 1998). But in these references a uniform estimate is available only on a domain with a positive distance to the boundary. Therefore, as far as we know, (1.2) is still to be established in the present setting. In fact, for the Dirichlet semigroup, the curvature condition (1.1) is no longer sufficient to imply (1.2): as shown by Examples 3.1 and 3.2, to derive (1.2), one has to make additional assumptions on the boundary as well as the vector field Z.

Let N be the inward normal unit vector field of ∂M. Define $\mathbf{b}: T_p \partial M \times T_p \partial M \to \mathbb{R}$ by

$$\mathbf{b}(\xi, \eta) := -\langle \nabla_\xi N, \eta \rangle,$$

which is symmetric, and $B(\xi, \eta) := b(\xi, \eta)N$ is known as the second fundamental form of ∂M. We assume that there is $\sigma \geq 0$ such that

$$\operatorname{Tr} \mathbf{b} := \sum_{n=1}^{d-1} \mathbf{b}(\xi_n, \xi_n) \geq -\sigma(d-1), \qquad \{\xi_n\}_{n=1}^{d-1} \in O(\partial M); \tag{1.5}$$

that is, the mean curvature of ∂M is bounded below by $-\sigma(d-1)$, where $O(\partial M)$ is the orthonormal frame bundle of ∂M. Moreover, we assume that there exists $\delta \in \mathbb{R}$ such that

$$|Z| \leq \delta. \tag{1.6}$$

Finally, let $k \geq 0$ be such that

$$\operatorname{Ric}(X, X) \geq -k(d-1)|X|^2, \qquad X \in TM. \tag{1.7}$$

Our main result is the following theorem.

THEOREM 1.1. *Assume that* (1.5), (1.6) *and* (1.7) *hold*. *Let* $c := (d-1)[\sigma \vee \sqrt{k}\,] + \delta$. *Let* P_t *be defined by* (1.4). *For any* $f \in \mathscr{B}_b^+(M)$ *with* $\|f\|_\infty > 0$, *we have*

$$\begin{aligned}\frac{\|\nabla P_t f\|_\infty}{\|f\|_\infty} &\le \left(4+\frac{3}{4}\right)c + \frac{\sqrt{2c}(1+4^{2/3})^{1/4}(1+5/2^{1/3})}{(\pi t)^{1/4}} \\ &\quad + \frac{\sqrt{1+2^{1/3}}(1+4^{2/3})}{2\sqrt{\pi t}} \\ &=: C(t), \qquad t>0. \end{aligned} \tag{1.8}$$

Consequently,

$$\|\nabla P_t f\|_\infty \le \frac{C(1)\|f\|_\infty}{\sqrt{t \wedge 1}}, \qquad t>0,\ f \in \mathscr{B}_b^+(M). \tag{1.9}$$

If, *in particular*, $k=\sigma=\delta=0$, *then*

$$\|\nabla P_t f\|_\infty \le \frac{\|f\|_\infty\sqrt{1+2^{1/3}}(1+4^{2/3})}{2\sqrt{\pi t}}, \qquad t>0,\ f \in \mathscr{B}_b^+(M). \tag{1.10}$$

As an application of Theorem 1.1, we have the following result on isoperimetric constants.

THEOREM 1.2. *Let* $Z = \nabla V$ *for some* $V \in C^2(M)$ *and let* $\mu(dx) := e^{V(x)}\,dx$, *where* dx *stands for the volume element*. *Assume* (1.5), (1.6) *and* (1.7).

(i) *Let* $\hat{M} := M \setminus \partial M$. *If*

$$\lambda_1 := \inf\{\mu(|\nabla f|^2) : f \in C_0^\infty(\hat{M}), \mu(f^2)=1\} > 0, \tag{1.11}$$

then

$$\kappa := \inf_A \frac{\mu_\partial(\partial A)}{\mu(A)} \ge \sup_{t>0} \frac{1-e^{-\lambda_1 t}}{C(1)(t \vee \sqrt{t}\,)} \ge \frac{1-e^{-1}}{C(1)}\big(\lambda_1 \wedge \sqrt{\lambda_1}\big), \tag{1.12}$$

where here and in the rest of the paper A *runs over all bounded smooth domains in* $\hat{M}$ *and* $\mu_\partial(\partial A)$ *is the area of the boundary of* A *induced by* μ. *If*, *in particular*, $k=\sigma=\delta=0$, *then*

$$\kappa \ge \frac{2\sqrt{\pi\lambda_1}}{\sqrt{1+2^{1/3}}(1+4^{2/3})}\sup_{r>0}\frac{1-e^{-r}}{\sqrt{r}}.$$

(ii) *Assume that* $k=\sigma=\delta=0$. *If there exists* $p>1$ *such that the following Nash inequality holds*:

$$\mu(f^2)^{1+2/p} \le C\mu(|f|)^{4/p}\mu(|\nabla f|^2), \qquad f \in C_0^\infty(\hat{M}), \tag{1.13}$$

where $C>0$ *is a constant*, *then*

$$\kappa_p := \inf_A \frac{\mu_\partial(\partial A)}{\mu(A)^{(p-1)/p}} \ge \frac{2p\sqrt{2\pi}}{\sqrt{(1+2^{1/3})Cp}(1+4^{2/3})(p+1)^{(p+1)/p}}. \tag{1.14}$$

(iii) *In general*, (1.13) *implies*

$$\kappa_p' := \inf_A \frac{\mu_\partial(\partial A)}{\{\mu(A)^{(p-1)/p}\} \wedge \{\mu(A)^{(p-2)/p}\}} \\ \geq \frac{p}{2\sqrt{2}C(1)(1+p/2)^{(p+2)/p}[(Cp) \vee \sqrt{Cp}\,]}.$$

REMARK. By Cheeger's inequality, one has $\lambda_1 \geq \kappa^2/4$. Next, it is well known that [see, e.g., Chavel (1984)]

$$\mu\big(|f|^{p/(p-1)}\big)^{(p-1)/p} \leq \frac{1}{\kappa_p}\mu(|\nabla f|), \qquad f \in C_0^\infty(\hat{M}).$$

Then, letting $\mu(|f|) = 1$ and using Hölder's inequalities, we obtain

$$\mu(f^2) = \mu\big(|f|^{2/(p+1)}|f|^{2p/(p+1)}\big) \leq \mu\big((f^2)^{p/(p-1)}\big)^{(p-1)/(p+1)} \\ \leq \left(\frac{\mu(|\nabla f^2|)}{\kappa_p}\right)^{p/(p+1)} \leq \left\{\frac{2}{\kappa_p}\sqrt{\mu(|\nabla f|^2)\mu(f^2)}\right\}^{p/(p+1)}.$$

Therefore, (1.13) holds for

$$C = \frac{4}{\kappa_p^2}.$$

Theorem 1.2 contains certain converses of the above classical results.

Theorem 1.1 is proved in the next section by using the coupling method developed by Kendall (1986) and Cranston (1991). To obtain a uniform gradient estimate, the key step is to estimate the joint distribution of the coupling time and the hitting time to the boundary. In Section 3, we present two examples to show that any of conditions (1.5) and (1.6) cannot be dropped from Theorem 1.1. Finally, we prove Theorem 1.2 in Section 4 following Ledoux's argument. An extension of Theorem 1.2 is also presented (see Proposition 4.1).

2. Proof of Theorem 1.1. The main idea of the proof comes from Cranston (1991) where Kendall's coupling was refined and applied to the gradient estimate of bounded harmonic functions. To derive uniform estimates of type (1.2), we need to estimate the joint distribution of the coupling time and the hitting times to the boundary.

Let $(x_t^x, y_t^y)_{t\geq 0}$ be a coupling of the L-diffusion processes starting from x and y, respectively, with absorbing boundary ∂M. Let τ_1^x and τ_2^y denote, respectively, the hitting times to ∂M of x_t^x and y_t^y. For $R > \rho(x, y)$, the Riemannian distance between x and y, we put

$$S_R^{x,y} := \inf\{t \geq 0 : \rho(x_t^x, y_t^y) \geq R\}, \qquad T^{x,y} := \inf\{t \geq 0 : x_t^x = y_t^y\}.$$

As usual, we let x_t^x and y_t^y move together since the coupling time $T^{x,y}$.

For simplicity, from now on we remove the superscripts from the notation of processes and stopping times, but it is important to keep in mind the dependence on the starting points x and y.

For any $t > 0$ and any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}\{T \wedge \tau_1 \wedge \tau_2 \le \varepsilon t\} &\subset \{T \le \tau_1 \wedge \tau_2 \wedge t\} \cup \{\tau_1 \wedge \tau_2 \le \varepsilon t, T \ge \tau_1 \wedge \tau_2\} \\ &\subset \{T \le \tau_1 \wedge \tau_2 \wedge t\} \cup \{\tau_1 \le (\varepsilon t) \wedge T, \tau_2 > t\} \\ &\quad \cup \{\tau_2 \le (\varepsilon t) \wedge T, \tau_1 > t\} \cup \{\tau_1 \vee \tau_2 \le t\}.\end{aligned}$$

It is clear that when $T \le \tau_1 \wedge \tau_2 \wedge t$ one has $\tau_1 = \tau_2$ and $x_t = y_t$, and when $\tau_1 \vee \tau_2 \le t$ one has $\mathbb{1}_{\{t<\tau_1\}} = \mathbb{1}_{\{t<\tau_2\}} = 0$. We obtain, for $f \in \mathscr{B}_b^+(M)$,

$$\begin{aligned}&|P_t f(x) - P_t f(y)| \\ &\quad \le \mathbb{E}\big|f(x_t)\mathbb{1}_{\{t<\tau_1\}} - f(y_t)\mathbb{1}_{\{t<\tau_2\}}\big| \\ &\quad \le \|f\|_\infty \big\{\mathbb{P}(T \wedge \tau_1 \wedge \tau_2 > \varepsilon t) \\ &\qquad\qquad + \mathbb{P}(\tau_1 \le \varepsilon t \wedge T, \tau_2 > t) + \mathbb{P}(\tau_2 \le \varepsilon t \wedge T, \tau_1 > t)\big\} \\ &\quad \le \|f\|_\infty \big\{\mathbb{P}(T \wedge \tau_1 \wedge \tau_2 > \varepsilon t) + \mathbb{P}(\tau_1 \le \varepsilon t \wedge S_R, \tau_2 > t) \\ &\qquad\qquad + \mathbb{P}(\tau_2 \le \varepsilon t \wedge S_R, \tau_1 > t) + \mathbb{P}(T \wedge \tau_1 \wedge \tau_2 \ge S_R)\big\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R > \rho(x, y).\end{aligned}$$

Therefore, for $\|f\|_\infty > 0$,

$$\begin{aligned}&\frac{|\nabla P_t f|(x)}{\|f\|_\infty} \\ &\quad \le \limsup_{y \to x} \frac{1}{\rho(x, y)} \big\{\mathbb{P}(T \wedge \tau_1 \wedge \tau_2 > \varepsilon t) + \mathbb{P}(\tau_1 \le \varepsilon t \wedge S_R, \tau_2 > t) \\ &\qquad\qquad + \mathbb{P}(\tau_2 \le \varepsilon t \wedge S_R, \tau_1 > t) + \mathbb{P}(T \wedge \tau_1 \wedge \tau_2 \ge S_R)\big\}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad R > 0.\end{aligned} \tag{2.1}$$

Thus, to derive upper bounds of $\|\nabla P_t f\|_\infty / \|f\|_\infty$, we need to estimate those probabilities involved in (2.1). To this end, we present the following three lemmas.

LEMMA 2.1. *Let $(r_t)_{t \ge 0}$ be the one-dimensional diffusion process generated by $a\frac{d^2}{dr^2} + b(r)\frac{d}{dr}$, where $a > 0$ is a constant and $b \in C^1(\mathbb{R})$. Let $r_0 > 0$ and $\tau_0 := \inf\{t \ge 0 : r_t = 0\}$. Let*

$$\xi(r) := \int_0^r \exp\left[-\frac{1}{a}\int_0^s b(t)\,dt\right] ds, \qquad r \in \mathbb{R},$$

$$c(u) := \frac{1}{a} \sup_{t \in [0,u]} \int_0^t b(s)\,ds, \qquad u > 0.$$

We have

$$\mathbb{P}(\tau_0 > t) \le \xi(r_0) \inf_{s>r_0} \left\{ \frac{1}{\xi(s)} + \frac{e^{c(s)}}{\sqrt{a\pi t}} \right\}, \qquad t > 0.$$

PROOF. It is easy to see that $a\xi'' + b\xi' = 0$. Then, by Itô's formula,

$$d\xi(r_t) = \sqrt{2a}\xi'(r_t)\,dB_t = \sqrt{2a}\xi' \circ \xi^{-1}(\xi(r_t))\,dB_t,$$

where $(B_t)_{t\ge 0}$ is the one-dimensional Brownian motion. Hence, $(\xi(r_t))_{t\ge 0}$ is the one-dimensional diffusion process on $(0, \xi(\infty))$ generated by $a(\xi' \circ \xi^{-1})^2(r)\frac{d^2}{dr^2}$. Next, let

$$\begin{aligned} T(t) &:= \frac{1}{2a}\int_0^t \frac{ds}{(\xi' \circ \xi^{-1}(B_s))^2} \\ &= \frac{1}{2a}\int_0^t \exp\left[\frac{2}{a}\int_0^{\xi^{-1}(B_s)} b(r)\,dr\right] ds, \qquad t \ge 0. \end{aligned}$$

Then the time-changed Brownian motion $B_{T^{-1}(t)}$ is also generated by $a(\xi' \circ \xi^{-1})^2(r)\frac{d^2}{dr^2}$. Therefore, letting $B_0 = \xi(r_0)$ and

$$\tau' := \inf\{t \ge 0 : B_t = 0\}, \qquad \sigma_u := \inf\{t \ge 0 : B_t \ge \xi(u)\}, \qquad u > r_0,$$

we obtain [see, e.g., Karatzas and Shreve (1998) for the distribution of τ']

$$\begin{aligned} \mathbb{P}(\tau_0 > t) &\le \mathbb{P}\big(T(\tau') > t, \tau' \le \sigma_u\big) + \mathbb{P}(\tau' > \sigma_u) \\ &\le \mathbb{P}\big(\tau' > 2ate^{-2c(u)}\big) + \mathbb{P}(\tau' > \sigma_u) \\ &= \frac{2}{\sqrt{2\pi}}\int_0^{\xi(r_0)e^{c(u)}/\sqrt{2at}} e^{-s^2/2}\,ds + \frac{\xi(r_0)}{\xi(u)} \\ &\le \xi(r_0)\left(\frac{e^{c(u)}}{\sqrt{a\pi t}} + \frac{1}{\xi(u)}\right). \end{aligned}$$

□

To study the hitting time of the L-diffusion process to ∂M, we need to estimate $L\rho_{\partial M}$, where $\rho_{\partial M}$ is the Riemannian distance function to the boundary. Let $\mathrm{cut}(\partial M)$ denote the set of focal cut points of ∂M [see, e.g., Chavel (1984)]. We will use the following Laplacian comparison theorem due to Kasue (1982); see the Appendix for a complete proof.

THEOREM 2.2 [Kasue (1982)]. *Let $x \notin \partial M \cup \mathrm{cut}(\partial M)$ and let $l_\cdot : [0, \rho_{\partial M}(x)] \to M$ be the minimal geodesic linking ∂M and x. Assume* (1.5) *holds for some $\sigma \in \mathbb{R}$. Let $R \in C[0, \rho_{\partial M}(x)]$ be such that*

$$\mathrm{Ric}(l', l')(s) \ge -(d-1)R(s), \qquad s \in [0, \rho_{\partial M}(x)]. \tag{2.2}$$

If $h \in C^2[0, \rho_{\partial M}(x)]$ is a strictly positive function satisfying

$$h'' \geq Rh, \qquad h(0) = 1, \qquad h'(0) \geq \sigma, \tag{2.3}$$

then

$$\Delta \rho_{\partial M}(x) \leq \frac{(d-1)h'(\rho_{\partial M}(x))}{h(\rho_{\partial M}(x))}.$$

LEMMA 2.3. *Under conditions* (1.5)–(1.7), *we have*

$$L\rho_{\partial M}(x) \leq \delta + (d-1)\big[\sigma \vee \sqrt{k}\,\big] =: c, \qquad x \notin \partial M \cup \mathrm{cut}(\partial M). \tag{2.4}$$

PROOF. For $x \notin \partial M \cup \mathrm{cut}(\partial M)$, let l_s be in Theorem 2.2. Below we simply denote $\rho = \rho_{\partial M}(x)$. Let

$$h(t) := \cosh \sqrt{k}t + \frac{\sigma}{\sqrt{k}} \sinh \sqrt{k}t, \qquad t \in [0, \rho].$$

One has

$$h''(t) - kh(t) = 0, \qquad h(0) = 1, \qquad h'(0) = \sigma,$$

where $1/\sqrt{k} \sinh \sqrt{k}t := t$ for $k = 0$. By Theorem 2.2,

$$\Delta \rho \leq \frac{(d-1)[\sqrt{k} \sinh \sqrt{k}\rho + \sigma \cosh \sqrt{k}\rho]}{\cosh \sqrt{k}\rho + (\sigma/\sqrt{k}) \sinh \sqrt{k}\rho} \leq (d-1)\big[\sigma \vee \sqrt{k}\,\big].$$

Then the proof is complete since $\langle \nabla \rho, Z \rangle \leq |Z| \leq \delta$. $\square$

To apply (2.1), we let (x_t, y_t) with $(x_0, y_0) = (x, y)$ be the coupling by reflection of the L-diffusion processes constructed by Kendall (1986) and Cranston (1991), which is a diffusion process on $\hat{M} \times \hat{M}$ up to the time $\tau_1 \wedge \tau_2$ [see Proposition 1 in Cranston (1991)]; if one of the processes first hits ∂M before the coupling time, then let it stay at the hitting point and let the other move independently. For this coupling, we have [see, e.g., Chen and Wang (1994) and Wang (1994)]

$$\begin{aligned} d\rho(x_t, y_t) &\leq 2\sqrt{2}\, dB_t + 2\big[(d-1)\sqrt{k} + \delta\big]\, dt \\ &\leq 2\sqrt{2}\, dB_t + 2c\, dt, \qquad t \leq \tau_1 \wedge \tau_2. \end{aligned} \tag{2.5}$$

Let

$$\begin{aligned} H_1(u) &:= \inf_{s>u} \left\{ \frac{c}{2(1 - e^{-cs/2})} + \frac{e^{cs/2}}{2\sqrt{\varepsilon \pi t}} \right\}, \\ H_2(u) &:= \inf_{s>u} \left\{ \frac{c}{1 - e^{-cs}} + \frac{e^{cs}}{\sqrt{(1-\varepsilon)\pi t}} \right\}, \qquad u > 0. \end{aligned}$$

LEMMA 2.4. *For the above coupling, we have*

$$\mathbb{P}(T \wedge \tau_1 \wedge \tau_2 > \varepsilon t) \le H_1\big(\rho(x,y)\big)\rho(x,y), \tag{2.6}$$

$$\begin{aligned}&\mathbb{P}\big(\tau_1 \le (\varepsilon t) \wedge S_R, \tau_2 > t\big) + \mathbb{P}\big(\tau_2 \le (\varepsilon t) \wedge S_R, \tau_1 > t\big)\\ &\quad \le 2H_2(R)\rho(x,y),\end{aligned} \tag{2.7}$$

$$\mathbb{P}(T \wedge \tau_1 \wedge \tau_2 > S_R) \le \frac{c\rho(x,y)}{2(1-e^{-cR/2})} \tag{2.8}$$

for any $x, y \in \hat{M}$, any $t > 0$ and any $R > \rho(x,y)$.

PROOF. (a) Let $(r_t)_{t\ge 0}$ solve the stochastic differential equation

$$dr_t = 2\sqrt{2}\, dB_t + 2c\, dt, \qquad t \ge 0, r_0 = \rho(x,y).$$

Let $\tau_0 := \inf\{t \ge 0 : r_t = 0\}$, where B_t is in (2.5). By (2.5) we have $r_t \ge \rho(x_t, y_t)$ up to the time $\tau_1 \wedge \tau_2$. Then

$$\mathbb{P}(T \wedge \tau_1 \wedge \tau_2 \ge \varepsilon t) \le P(\tau_0 \ge \varepsilon t).$$

Since $(r_t)_{t\ge 0}$ is generated by $4\frac{d^2}{dr^2} + 2c\frac{d}{dr}$, (2.6) follows from Lemma 2.1.

(b) Kendall (1987) established Itô's formula for the distance of the Brownian motion to a fixed point in M. It is easy to see that his argument also works for the distance of the L-diffusion process to ∂M. Then we have

$$d\rho_{\partial M}(y_t) = \sqrt{2}\, dB_t + 1\!\!1_{\{y_t \notin \mathrm{cut}(\partial M)\}} L\rho_{\partial M}(y_t)\, dt - dL_t, \qquad t \le \tau_2,$$

where B_t is a Brownian motion on $\mathbb{R}$ and $(L_t)_{t\ge 0}$ is an increasing process with support contained by $\{t \ge 0 : y_t \in \mathrm{cut}(\partial M)\}$. Since $\mathrm{cut}(\partial M)$ is a zero-volume set so that the Lebesgue measure of $\{t : y_t \in \mathrm{cut}(\partial M)\}$ is 0, it follows from Lemma 2.3 that

$$d\rho_{\partial M}(y_t) \le \sqrt{2}\, dB_t + c\, dt, \qquad t \le \tau_2.$$

Letting $a = 1$ and $b(r) = c$, it follows from Lemma 2.1 and a comparison theorem that, for any $y \in \hat{M}$,

$$\mathbb{P}\big(\tau_2 > (1-\varepsilon)t\big) \le \frac{1}{c} H_2\big(\rho_{\partial M}(y)\big)\big(1 - e^{-c\rho_{\partial M}(y)}\big). \tag{2.9}$$

Since when $\tau_1 \le (\varepsilon t) \wedge \tau_2 \wedge S_R$ one has

$$\rho_{\partial M}(y_{\tau_1}) \le \rho(x_{\tau_1}, y_{\tau_1}) = \rho\big(x_{\tau_1\wedge\tau_2\wedge(\varepsilon t)}, y_{\tau_1\wedge\tau_2\wedge(\varepsilon t)}\big) \le R,$$

letting $\mathbb{P}^z$ stand for the distribution of the L-diffusion process starting from z for any $z \in M$, we obtain, by (2.9) with y replaced by y_{τ_1},

$$\begin{aligned}
&\mathbb{P}\big(\tau_1 \le (\varepsilon t) \wedge S_R, \tau_2 > t\big) \\
&\quad \le \mathbb{E}\mathbb{1}_{\{\tau_1 \le (\varepsilon t)\wedge\tau_2\wedge S_R\}}\mathbb{P}^{y_{\tau_1}}\big(\tau_2 > (1-\varepsilon)t\big) \\
&\quad \le \frac{H_2(R)}{c}\mathbb{E}\big(1-\exp\big[-c\rho\big(x_{\tau_1\wedge\tau_2\wedge(\varepsilon t)}, y_{\tau_1\wedge\tau_2\wedge(\varepsilon t)}\big)\big]\big).
\end{aligned}$$

Next, it follows from (2.5) and Itô's formula that

$$\begin{aligned}
d\big(1-e^{-c\rho(x_t,y_t)}\big) &\le ce^{-c\rho(x_t,y_t)}\big\{2\sqrt{2}\,dB_t + 2c\,dt - 4c\,dt\big\} \\
&\le ce^{-c\rho(x_t,y_t)}2\sqrt{2}\,dB_t.
\end{aligned}$$

Then

$$\mathbb{E}\big(1-\exp\big[-c\rho\big(x_{\tau_1\wedge\tau_2\wedge(\varepsilon t)}, y_{\tau_1\wedge\tau_2\wedge(\varepsilon t)}\big)\big]\big) \le 1-e^{-c\rho(x,y)} \le c\rho(x,y).$$

Thus, $\mathbb{P}(\tau_1 \le (\varepsilon t) \wedge S_R, \tau_2 > t) \le H_2(R)\rho(x,y)$. Similarly, the same estimate holds by exchanging τ_1 and τ_2. Therefore, (2.7) holds.

(c) Let $(r_t)_{t\ge 0}$ solve

$$dr_t = 2\sqrt{2}\,dB_t + 2c\,dt, \qquad r_0 = \rho(x,y).$$

Let $\tau'_r := \inf\{t \ge 0 : r_t = r\}$, $r \ge 0$. We have $r_t \ge \rho(x_t, y_t)$ up to time $\tau_1 \wedge \tau_2$. Then

$$\begin{aligned}
\mathbb{P}(T\wedge\tau_1\wedge\tau_2 > S_R) &\le \mathbb{P}(\tau'_0 > \tau'_R) \\
&= \frac{\int_0^{\rho(x,y)}\exp[-cr/2]\,dr}{\int_0^R \exp[-cr/2]\,dr} \le \frac{c\rho(x,y)}{2(1-e^{-cR/2})}. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 1.1. Combining (2.1) with (2.6)–(2.8), we obtain

$$\frac{\|\nabla P_t f\|_\infty}{\|f\|_\infty} \le H_1(0) + 2H_2(R) + \frac{c}{2(1-e^{-cR/2})}, \qquad R > 0. \tag{2.10}$$

It is easy to see that, for $r, t > 0$, the minimum of the function

$$h(s) := \frac{r}{1-e^{-rs}} + \frac{1}{t}e^{2rs}, \qquad s > 0,$$

is reached at s with $e^{rs} = 1 + \sqrt{rt/2}$. Then

$$\begin{aligned}
H_1(0) &= \frac{\sqrt{2c} + c(\varepsilon\pi t)^{1/4}}{2(\varepsilon\pi t)^{1/4}} + \frac{(\sqrt{2}+\sqrt{c}(\varepsilon\pi t)^{1/4})^2}{4\sqrt{\varepsilon\pi t}} \\
&= \frac{\sqrt{2c}}{(\varepsilon\pi t)^{1/4}} + \frac{3c}{4} + \frac{1}{2\sqrt{\varepsilon\pi t}}.
\end{aligned}$$

Similarly, letting $R = \frac{1}{c}\log[1+\sqrt{c/2}((1-\varepsilon)\pi t)^{1/4}]$, one obtains

$$H_2(R) = H_2(0) = \frac{2\sqrt{2c}}{((1-\varepsilon)\pi t)^{1/4}} + \frac{3c}{2} + \frac{1}{\sqrt{(1-\varepsilon)\pi t}}.$$

Moreover, letting $\alpha := 1+\sqrt{c/2}((1-\varepsilon)\pi t)^{1/4}$, we have

$$\begin{aligned}\frac{c}{2(1-e^{-cR/2})} &= \frac{c}{2(1-\alpha^{-1/2})} = \frac{c(1+\alpha^{-1/2})\alpha}{2(\alpha-1)}\\ &\le \frac{c\alpha}{\alpha-1} = c + \frac{\sqrt{2c}}{((1-\varepsilon)\pi t)^{1/4}}.\end{aligned}$$

Therefore, it follows from (2.10) that

$$\begin{aligned}\frac{\|\nabla P_t f\|_\infty}{\|f\|_\infty} &\le \left(4+\frac{3}{4}\right)c + \frac{\sqrt{2c}}{(\varepsilon\pi t)^{1/4}} + \frac{5\sqrt{2c}}{((1-\varepsilon)\pi t)^{1/4}}\\ &\quad + \frac{1}{2\sqrt{\varepsilon\pi t}} + \frac{2}{\sqrt{(1-\varepsilon)\pi t}}.\end{aligned}$$

Then (1.8) follows by taking $\varepsilon = (1+4^{2/3})^{-1}$. This choice of ε is optimal for the summation of the last two terms and hence is optimal for small time. $\square$

3. Examples. In this section we present two examples to show that conditions (1.5) and (1.6) are somehow essential for the uniform gradient estimate (1.2).

EXAMPLE 3.1. Let $M = \{(x,y)\in\mathbb{R}^2 : x\ge 0\}$. Consider $L = \Delta + ay\frac{\partial}{\partial x} - by\frac{\partial}{\partial y}$, where $a,b>0$ are two constants. Then $k=\sigma=0$ and (1.1) holds for some $K\ge 0$, but $\|\nabla P_t 1\|_\infty = \infty$ for all $t>0$.

PROOF. It suffices to prove that $\|\nabla P_t 1\|_\infty = \infty$. The proof consists of two steps.

(a) Let (x_t, y_t) be the L-diffusion process starting from $(x,y)\in\mathbb{R}^2$. For $r>0$, let $\tau_r := \{t\ge 0 : y_t \le r\}$. We intend to prove

$$\lim_{y\to\infty}\inf_x \mathbb{P}(\tau_r > t) = 1, \qquad t>0. \tag{3.1}$$

Note that one may let (x_t, y_t) solve the stochastic differential equation

$$\begin{aligned} dx_t &= \sqrt{2}\,dB_t^1 + ay_t\,dt, &\qquad x_0 &= x,\\ dy_t &= \sqrt{2}\,dB_t^2 - by_t\,dt, &\qquad y_0 &= y,\end{aligned}$$

where B_t^1 and B_t^2 are two independent Brownian motions on $\mathbb{R}$. Thus, the motion of y_t does not depend on that of x_t, and $(y_t)_{t\ge 0}$ is a diffusion process generated by

$$L_2 := \frac{d^2}{dy^2} - by\frac{d}{dy}.$$

Then $\mathbb{P}(\tau_r > t)$ is independent of x.

Obviously, $\mathbb{P}(\tau_r > t)$ is increasing in y. So, if (3.1) does not hold, there exists $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}(\tau_r > t) \le \lim_{y\to\infty} \mathbb{P}(\tau_r > t) \le \varepsilon, \qquad y \in \mathbb{R}.$$

Thus, by the Markovian property,

$$\mathbb{P}(\tau_r > 2t) = \mathbb{E}1_{\{\tau_r > t\}}\mathbb{E}^{y_t}1_{\{\tau_r > t\}} \le \varepsilon\mathbb{P}(\tau_r > t) \le \varepsilon^2, \qquad y \in \mathbb{R},$$

where, for a point $z \in \mathbb{R}$, $\mathbb{E}^z$ stands for the expectation taken w.r.t. the distribution of the L_2-diffusion process starting from z. Similarly, one has

$$\mathbb{P}(\tau_r > nt) \le \varepsilon^n, \qquad n \ge 1,\ y \in \mathbb{R}.$$

Then

$$\mathbb{E}\tau_r = \int_0^\infty \mathbb{P}(\tau_r > s)\,ds \le t + \sum_{n=1}^\infty \mathbb{P}(\tau_r > nt) \le t + \sum_{n=1}^\infty \varepsilon^n < \infty. \tag{3.2}$$

On the other hand, letting

$$G(s) := \int_r^s \exp[u^2/2]\,du \int_u^\infty \exp[-t^2/2]\,dt, \qquad s \in \mathbb{R},$$

one has $L_2 G = -1$. Thus, for $N > y > r$,

$$G(N)\mathbb{P}(\tau_r > \tau_N) = \mathbb{E}G(y_{\tau_r\wedge\tau_N}) = G(y) - \mathbb{E}(\tau_r \wedge \tau_N). \tag{3.3}$$

Moreover, let $F(s) := \int_0^s \exp[t^2/2]\,dt,\ s \in \mathbb{R}$. We have $L_2 F = 0$ and hence

$$F(y) = \mathbb{E}F(y_{\tau_r\wedge\tau_N}) = [F(N) - F(r)]\mathbb{P}(\tau_r > \tau_N) + F(r).$$

Thus,

$$\mathbb{P}(\tau_r > \tau_N) = \frac{F(y) - F(r)}{F(N) - F(r)}, \qquad N > y > r.$$

Combining this with (3.3), we arrive at

$$\mathbb{E}(\tau_r \wedge \tau_N) = G(y) - \frac{G(N)[F(y) - F(r)]}{F(N) - F(r)}.$$

By letting first $N \uparrow \infty$ then $y \uparrow \infty$ and noting that $G(N)/F(N) \to 0$ as $N \to \infty$, we obtain

$$\sup_y \mathbb{E}\tau_r = G(\infty) = \infty,$$

which is contradictory to (3.2).

(b) Let $\tau := \inf\{t \ge 0 : x_t = 0\}$. For any $r > 1$, by (3.1) we may choose $y > r$ such that $\mathbb{P}(\tau_r > t) \ge \frac{1}{2}$. We have, up to the time τ_r,

$$dx_t = \sqrt{2}\,dB_t^1 + ay_t\,dt \ge \sqrt{2}\,dB_t^1 + ra\,dt.$$

Let $(x'_t)_{t\ge 0}$ solve the equation

$$dx'_t = \sqrt{2}\,dB^1_t + ar\,dt, \qquad r_0 = x,$$

and let $\tau' := \inf\{t \ge 0 : x'_t = 0\}$. We have $x'_t \le x_t$ for $t \le \tau_r$ and hence

$$\begin{aligned} P_t 1(x, y) &:= \mathbb{P}(\tau > t) \ge \mathbb{P}(\tau \wedge \tau_r > t) \ge \mathbb{P}(\tau' \wedge \tau_r > t) \\ &= \mathbb{P}(\tau' > t)\mathbb{P}(\tau_r > t) \ge \tfrac{1}{2}\mathbb{P}(\tau' > t), \end{aligned}$$

where we have used the fact that $(x'_t)_{t\ge 0}$ and $(y_t)_{t\ge 0}$ are independent. Since

$$d(1 - \exp[-arx'_t]) = ar\exp[-arx'_t]\sqrt{2}\,dB^1_t$$

is a martingale up to the time τ', one has (note that $x'_0 = x$)

$$1 - \exp[-arx] = E\big(1 - \exp\big[-arx'_{t\wedge\tau'}\big]\big) \le \mathbb{P}(\tau' > t).$$

Then

$$\|\nabla P_t 1\|_\infty \ge \limsup_{x\downarrow 0} \frac{P_t 1(x, y)}{x} = \limsup_{x\downarrow 0} \frac{\mathbb{P}(\tau > t)}{x} \ge \frac{1}{2}ar.$$

Since $r > 1$ is arbitrary, we have $\|\nabla P_t 1\|_\infty = \infty$. $\square$

EXAMPLE 3.2. Let $M = \{x \in \mathbb{R}^d : |x| \ge \varepsilon\}$, where $d \ge 2$. For $L = \Delta$ we have $k = \delta = 0$ but $\lim_{\varepsilon\to 0}\|\nabla P_t 1\|_\infty = \infty$.

PROOF. Let $(x_t)_{t\ge 0}$ be the diffusion process generated by Δ with $x_0 = x$, $|x| > \varepsilon$. We have

$$d|x_t| = \sqrt{2}\,dB_t + \frac{d-1}{|x_t|}\,dt$$

up to the time $\tau := \inf\{t \ge 0 : |x_t| = \varepsilon\}$. For $d > 2$, it is easy to check that $(|x_t|^{2-d})_{t\ge 0}$ is a martingale up to τ. We have

$$|x|^{2-d} = \mathbb{E}|x_{\tau\wedge t}|^{2-d} \ge \varepsilon^{2-d}\mathbb{P}(\tau \le t).$$

Then

$$\mathbb{P}(\tau > t) = 1 - \mathbb{P}(\tau \le t) \ge 1 - \frac{\varepsilon^{d-2}}{|x|^{d-2}}.$$

Thus,

$$\|\nabla P_t 1\|_\infty \ge \limsup_{|x|\downarrow\varepsilon} \frac{\mathbb{P}(\tau > t)}{|x| - \varepsilon} \ge \lim_{r\downarrow\varepsilon} \frac{r^{d-2} - \varepsilon^{d-2}}{r^{d-2}(r - \varepsilon)} = \frac{d-2}{\varepsilon},$$

which goes to ∞ as $\varepsilon \to 0$. For $d = 2$, $(-\log|x_t|)_{t\ge 0}$ is a martingale up to τ and hence the above argument leads to

$$\|\nabla P_t 1\|_\infty \ge \lim_{r\to\varepsilon+} \frac{\log\varepsilon - \log r}{(-\log r)(r - \varepsilon)} = \frac{1}{\varepsilon(-\log\varepsilon)} \to \infty \qquad \text{as } \varepsilon \to 0. \qquad \square$$

4. Isoperimetric inequalities: proof of Theorem 1.2.

PROOF OF THEOREM 1.2. By (1.9) we have [see formula (6) in Ledoux (1994)]

$$\mu(|f - P_t f|) \le 2C(1)\big(\sqrt{t} \vee t\big)\mu(|\nabla f|), \qquad t > 0.$$

For bounded smooth domain $A \subset \hat{M}$, let $f = \mathbb{1}_A$. Since P_t is symmetric, we obtain

$$\begin{aligned} 2C(1)\big(\sqrt{t} \vee t\big)\mu_\partial(\partial A) &\ge 2\mu(A) - 2\mu(\mathbb{1}_A P_t \mathbb{1}_A) \\ &= 2\mu(A) - 2\mu\big((P_{t/2}\mathbb{1}_A)^2\big) \end{aligned} \tag{4.1}$$

for all $t > 0$. If $\lambda_1 > 0$, we have $\mu((P_{t/2}\mathbb{1}_A)^2) \le e^{-\lambda_1 t}\mu(A)$. Then

$$\kappa \ge \sup_{t>0} \frac{1 - e^{-\lambda_1 t}}{C(1)(\sqrt{t} \vee t)} \ge \frac{1 - e^{-1}}{C(1)}\big(\sqrt{\lambda_1} \wedge \lambda_1\big).$$

If, in particular, $k = \delta = \sigma = 0$, then (1.10) holds. According to Ledoux (1994), we have

$$\mu(|f - P_t f|) \le \sqrt{(1 + 2^{1/3})/\pi}\,(1 + 4^{2/3})\sqrt{t}\mu(|\nabla f|), \qquad t > 0.$$

Thus,

$$\begin{aligned} &\sqrt{(1 + 2^{1/3})/\pi}(1 + 4^{2/3})\sqrt{t}\mu_\partial(\partial A) \\ &\qquad \ge 2\mu(A) - 2\mu\big((P_{t/2}\mathbb{1}_A)^2\big), \qquad t > 0. \end{aligned} \tag{4.2}$$

Therefore,

$$\kappa \ge \frac{2\sqrt{\pi\lambda_1}}{\sqrt{1 + 2^{1/3}}(1 + 4^{2/3})} \sup_{r>0} \frac{1 - e^{-r}}{\sqrt{r}}.$$

Then the proof of Theorem 1.2(i) is complete.

To prove (ii), we note that (1.13) implies

$$\|P_t\|_{L^1(\mu)\to L^2(\mu)} \le \left(\frac{Cp}{4t}\right)^{p/4}, \qquad t > 0. \tag{4.3}$$

Indeed, for $f \in L^2(\mu)$ with $\mu(|f|) = 1$, one obtains from (1.13) that

$$\frac{d}{dt}\mu\big((P_t f)^2\big) = -2\mu(|\nabla P_t f|^2) \le -\frac{2}{C}\mu\big((P_t f)^2\big)^{1+2/p}, \qquad t > 0.$$

Thus,

$$\mu\big((P_t f)^2\big)^{-2/p} - \mu(f^2)^{-2/p} \ge \frac{4t}{Cp},$$

which implies (4.3). If $k = \delta = \sigma = 0$, by (4.2) and (4.3) we obtain

$$\sqrt{(1 + 2^{1/3})/\pi}\,(1 + 4^{2/3})\sqrt{t}\mu_\partial(\partial A) \ge 2\mu(A) - 2\left(\frac{Cp}{2t}\right)^{p/2}\mu(A)^2.$$

For $\varepsilon \in (0,1)$, putting $t = (Cp/2)(\mu(A)/\varepsilon)^{2/p}$, we arrive at

$$\frac{\mu_\partial(\partial A)}{\mu(A)^{(p-1)/p}} \geq \frac{2\sqrt{2\pi}(1-\varepsilon)\varepsilon^{1/p}}{\sqrt{1+2^{1/3}}(1+4^{2/3})\sqrt{Cp}}.$$

Therefore, (1.14) follows by minimizing the right-hand side in ε.

Finally, by (4.1) and (4.3) we have

$$2C(1)\big(t \vee \sqrt{t}\big)\mu_{\partial M}(\partial A) \geq \mu(A) - \left(\frac{Cp}{2t}\right)^{p/2}\mu(A)^2.$$

Taking $t = (Cp/2)(\mu(A)/\varepsilon)^{2/p}$ for $\varepsilon \in (0,1)$, we obtain

$$\begin{aligned}\kappa_p' &\geq \sup_{\varepsilon\in(0,1)} \frac{(1-\varepsilon)\varepsilon^{2/p}}{\sqrt{2}C(1)[(Cp)\vee\sqrt{Cp}\,]} \\ &= \frac{p}{4C(1)(1+p/2)^{(p+2)/p}[(Cp)\vee\sqrt{Cp}\,]}.\end{aligned}$$ □

Note that the Nash inequality is a special case of the following general functional inequality introduced in Wang (2000):

(4.4) $$\mu(f^2) \leq r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \qquad r>0,\ f \in C_0^\infty(\hat{M}),$$

where $\beta:(0,\infty)\to(0,\infty)$ is a decreasing function with $\beta(0+)=\infty$ and $\beta(\infty)=0$. The following is an extension of Theorem 1.2, which follows from (1.10) and the proof of Theorem 3.4(2) in Wang (2000).

PROPOSITION 4.1. *Assume that* $K \geq 0$ *and* $k \geq \delta$. *Then* (4.4) *implies*

$$\inf_A \frac{\mu_\partial(\partial A)\sqrt{\beta^{-1}(1/[4\mu(A)])}}{\mu(A)} \geq \frac{2\sqrt{\pi}}{\sqrt{(1+2^{1/3})\log 3}(1+4^{2/4})}.$$

APPENDIX

Proof of Theorem 2.2. Theorem 2.2 appeared as Lemma (2.8) in Kasue (1982) without proof, since he believed that the proof is similar to that of his Lemma (2.5). As far as I understand, his proof of Lemma (2.5) is presented for the case where the reference submanifold is a point. It is nontrivial to modify his proof for the case where the reference submanifold is a hypersurface, because in this case one has to check the following initial condition of reference vector fields:

(A.1) $$S_{l_0'}X - \nabla_{l_0'}X \in (T_{l_0}\partial M)^\perp,$$

where $l.$ is the minimal geodesic linking ∂M and a point $x \in \hat{M}$, and $S_{l_0'}X$ is the projection of $-\nabla_X N$ to $T_{l_0}\partial M$ [recall that N is the inward unit normal vector field of ∂M, so one has $l_0' = N(l_0)$]. Since we do not have any explicit proof of

Theorem 2.2, it might be helpful to include a complete proof below for the reader's reference.

PROOF OF THEOREM 2.2. In the situation of Theorem 2.2, simply denote $\rho = \rho_{\partial M}(x)$. Let $\{E_i\}_{i=1}^{d-1}$ be parallel vector fields along $l.$ such that $\{E_i(0)\}_{i=0}^{d-1} \in O_{l_0}(\partial M)$ are eigenvectors of $S_{l_0'}$. Write

$$S_{l_0'} E_i = \sigma_i E_i, \qquad i \le d-1.$$

Let $g \in C^2(\mathbb{R})$ such that $0 \le g \le 1$, $g(s) = 1$ for $s \le 1$ and $g(s) = 0$ for $s \ge 2$. For any $\varepsilon \in (0, \rho/2)$, let

$$h_{i,\varepsilon}(s) := (1 + \sigma_i s) g(s/\varepsilon) + h(s)\bigl(1 - g(s/\varepsilon)\bigr), \qquad i \le d-1, s \in [0, \rho].$$

Then it is easy to see that $h_{i,\varepsilon} E_i / h(\rho)$ satisfies (A.1). Now, let J_i be the ∂M-Jacobi field [i.e., a Jacobi field along $l.$ satisfying (A.1)] with $J_i(\rho) = E_i(\rho)$. By the second variational formula and the index inequality [see, e.g., (1.1) in Kasue (1982)], we obtain

$$\begin{aligned}
\Delta\rho &= \sum_{i=1}^{d-1} I(J_i, J_i)(\rho) \\
&\le \sum_{i=1}^{d-1} I\bigl(h_{i,\varepsilon} E_i / h(\rho), h_{i,\varepsilon} E_i / h(\rho)\bigr)(\rho) \\
&= \frac{1}{h(\rho)^2} \sum_{i=1}^{d-1} \biggl\{ \langle S_{l_0'} E_i(0), E_i(0) \rangle + \int_0^{2\varepsilon} [h_{i,\varepsilon}'(s)^2 + R_i(s) h_{i,\varepsilon}(s)^2]\, ds \biggr\} \\
&\quad + \frac{1}{h(\rho)^2} \int_{2\varepsilon}^{\rho} [(d-1)h'(s)^2 - h(s)^2 \operatorname{Ric}(l_s', l_s')]\, ds,
\end{aligned} \tag{A.2}$$

where $-R_i(s)$ is the sectional curvature of the plane containing l_s' and $E_i(s)$. It is easy to see that

$$\sum_{i=1}^{d-1} \langle S_{l_0'} E_i(0), E_i(0) \rangle = -\sum_{i=1}^{d-1} \mathbf{b}\bigl(E_i(0), E_i(0)\bigr) \le (d-1)\sigma. \tag{A.3}$$

Next, by (2.2) and (2.3) we have

$$\begin{aligned}
\int_{2\varepsilon}^{\rho} \biggl(h'(s)^2 - \frac{1}{d-1} \operatorname{Ric}(l_s', l_s') h(s)^2 \biggr) ds &\le \int_{2\varepsilon}^{\rho} (hh')'(s)\, ds \\
&= h(\rho)h'(\rho) - h(2\varepsilon)h'(2\varepsilon).
\end{aligned} \tag{A.4}$$

Finally, since $R_i h_{i,\varepsilon}$ is bounded on $[0, \rho]$ and since for $s \in [0, 2\varepsilon]$, one has [recall that $h(0) = 1$]

$$|h'_{i,\varepsilon}(s)| = \left|\sigma_i g(s/\varepsilon) + \frac{g'(s/\varepsilon)}{\varepsilon}[1 + \sigma_i s - h(s)] + h'(s)\big(1 - g(s/\varepsilon)\big)\right|$$
$$\le |\sigma_i| + \frac{\sup_{[0,\rho]}|g'|}{\varepsilon}\Big(2|\sigma_i|\varepsilon + 2\varepsilon \sup_{[0,\rho]}|h'|\Big) + \sup_{[0,\rho]}|h'| \le D$$

for some constant $D > 0$ and all $i \le d - 1$, it follows that

$$\lim_{\varepsilon\to 0}\sum_{i=1}^{d-1}\int_0^{2\varepsilon}\big[h'_{i,\varepsilon}(s)^2 + R_i(s)h_{i,\varepsilon}(s)^2\big]\,ds = 0. \tag{A.5}$$

Therefore, substituting (A.3)–(A.5) into (A.2) and letting $\varepsilon \to 0$ [recall that $h(0) = 1$ and $h'(0) \ge \sigma$], we complete the proof. □

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DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING 100875
PEOPLE'S REPUBLIC OF CHINA
E-MAIL: wangfy@bnu.edu.cn