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ON POISSON EQUATION AND DIFFUSION APPROXIMATION 2[1]

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Three different results are established which turn out to be closely connected so that the first one implies the second one which in turn implies the third one. The first one states the smoothness of an invariant diffusion density with respect to a parameter. The second establishes a similar smoothness of the solution of the Poisson equation in $\mathbb{R}^d$. The third one states a diffusion approximation result, or in other words an averaging of singularly perturbed diffusion for "fully coupled SDE systems" or "SDE systems with complete dependence."

1. Introduction. Our goal is to establish a general result of diffusion approximation, more precisely to study the limit in law of $\{Y_t^\varepsilon; t \geq 0\}$ as $\varepsilon \to 0$, where

$$dX_t^\varepsilon = \varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \varepsilon^{-1}\sigma(X_t^\varepsilon, Y_t^\varepsilon)\,dB_t,$$
$$dY_t^\varepsilon = F(X_t^\varepsilon, Y_t^\varepsilon)\,dt + \varepsilon^{-1} G(X_t^\varepsilon, Y_t^\varepsilon)\,dt + H(X_t^\varepsilon, Y_t^\varepsilon)\,dB_t,$$

where X_t^ε takes values in $\mathbb{R}^d$, Y_t^ε in $\mathbb{R}^\ell$, and $\{B_t; t \geq 0\}$ is a d-dimensional standard Brownian motion. The novelty, compared to our previous work [15] and other contributions to this field, see in particular [1, 2, 6, 7, 13] and the references therein, is the dependence of the coefficients of the "fast" component X^ε upon the process Y^ε. Another essential feature of our setting is noncompactness of the state space. In order to tackle this problem, we need to study the solution of a Poisson equation associated to the process $\{X_t^y; t \geq 0\}$, where

$$dX_t^y = b(X_t^y, y)\,dt + \sigma(X_t^y, y)\,dB_t$$

and its regularity with respect to the variables x and y. At the same time, we obtain regularity results for the density $p_\infty(x, y)$ of the invariant measure μ^y of the process $\{X_t^y; t \geq 0\}$. Both those problems are also important as such; an example which shows this can be found in [7]; cf. the assumptions of Theorem 7.9.1 and the footnote concerning the invariant density regularity, while discrete time version of our work should be of interest for analyzing certain stochastic algorithms; see [3]. Our results seem to be useful for studying some climate model (see [10]) where different aspects of ordinary differential equations, SDEs and dynamical systems

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with averaging are considered. To compare our setting with (3.1) + (3.3) of [10] one should change the time scale to εt. A short version of our work in discrete time concerning mainly an invariant density has been appeared in [14]. We mention also a relevant paper with a close idea how investigate Poisson equations, [9].

Our Poisson equation takes the form

$$L(x,y)u(x,y) = -f(x,y), \qquad x \in \mathbb{R}^d, \tag{1}$$

where $y \in \mathbb{R}^\ell$ is a parameter, and

$$L(x,y) = \sum_{i,j=1}^{d} a_{ij}(x,y)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{d} b_i(x,y)\frac{\partial}{\partial x_i},$$

with $a = \sigma\sigma^*/2$, under the condition that for each $y \in \mathbb{R}^\ell$,

$$\int_{\mathbb{R}^d} f(x,y)\mu^y(dx) = 0. \tag{2}$$

Recall that there is no boundary condition; we are looking for solutions in the class of functions which grows at most polynomially in $|x|$, as $|x| \to \infty$. The condition which guarantees uniqueness of the solution is then (cf. [15])

$$\int u(x,y)\mu^y(dx) = 0. \tag{3}$$

Both coefficients a and b are assumed to be bounded; a is uniformly continuous with respect to x variable. The existence of the invariant probability measure μ^y is insured by the following recurrence condition:

$$\lim_{|x|\to\infty} \sup_y b(x,y)x = -\infty; \tag{H_b}$$

cf. [16]. We also assume nondegeneracy of the diffusion coefficient uniformly with respect to y, that is, we assume that there exist two constants $0 < \lambda < \Lambda < \infty$ such that

$$\lambda I \le a(x,y) \le \Lambda I, \tag{H_a}$$

from which uniqueness of the invariant measure follows.

We shall specify the required regularity assumptions when we shall need them. $\mathbb{N}$ denotes $\{0,1,\ldots\}$, while $\mathbb{N}^* = \{1,\ldots\}$. The notation $a \in C_{\mathrm{b}}^{i+\alpha,j}$ with $0 < \alpha < 1$ below means that the function has j bounded derivatives in y variable and i derivatives in x variable, and all derivatives $\partial_x^{i'}\partial_y^{j'}a$, $0 \le i' \le i$, $0 \le j' \le j$, are Hölder continuous with respect to x variable with exponent α uniformly in y. We denote by ($\mathrm{H}^{r+\alpha,j}$), with some $j \in \mathbb{N}^*$, $r \in \mathbb{N}$, $0 < \alpha < 1$, the condition

$$a, b \in C_{\mathrm{b}}^{r+\alpha,j}, \qquad r \in \mathbb{N}^*. \tag{$H^{r+\alpha,j}$}$$

We need to study the regularity of u in the variables x, y, and find expressions for the derivatives of order one and two with respect to y.

In our previous paper [15], we extensively used the representation

$$u(x, y) = \int_0^\infty E_{x,y} f(X_s^y, y)\,ds$$
$$= \int_0^\infty dt \int_{\mathbb{R}^d} dx' f(x', y) p_t(x, x'; y).$$

Here we shall rewrite the same formula as

$$u(x, y) = \int_0^1 dt \int_{\mathbb{R}^d} dx' f(x', y) p_t(x, x'; y) + \int_1^\infty dt\, p_t(x, f; y), \tag{4}$$

where, by definition,

$$p_t(x, f, y) := \int f(x', y) p_t(x, x'; y)\,dx'$$
$$\equiv \int f(x', y)[p_t(x, x'; y) - p_\infty(x', y)]\,dx'.$$

Hence, the derivative $\partial_y u$ should have a representation of the form

$$\partial_y u(x, y) = \int_0^1 dt D_y p_t(x, f; y) + \int_1^\infty dt D_y p_t(x, f; y),$$

where $D_y p_t(x, f; y)$ is a full derivative of the function p_t with respect to y. To explore this elementary idea in order to get this and similar results for derivatives in x, y of arbitrary order, we have to show the differentiability of the transition density, or at least of the function $p_t(x, f; y)$ and study its behavior both near $t = 0$ and as $t \to \infty$. For $p_t(x, f; y)$ it turns out to be possible under wider assumptions than for $p_t(x, x'; y)$.

We shall essentially use arguments from PDE theory, rather than probabilistic ones as we did in [15]. Our estimations will be based on two types of bounds. The first type is a set of estimates for the fundamental solution of a nondegenerate second order parabolic PDE, due independently to Eidelman [4, 5] and Friedman [8]. The second is a polynomial inequality for the convergence rate of the fundamental solution as $t \to \infty$; see [16, 17].

We now give some indication concerning a notation which will be used repeatedly in the paper. If u is a function of the d-dimensional variable x with values in $\mathbb{R}$, we shall write $\partial_x u(x)$ to denote the d-dimensional vector whose ith coordinate is $\partial_{x_i} u(x)$. Similarly $\partial_x^2 u(x)$ will denote a $d \times d$ matrix, and $\partial_x^j u(x)$ denotes for $j > 2$ a tensor with j indices. These notation are classical and quite obvious, but since we shall use them so to speak as if the corresponding quantity were a scalar, we prefer to be explicit at least once about the dimensions.

The organization of the paper is as follows. The three main theorems, Theorem 1 on the transition and invariant densities, Theorem 2 on smoothness and bounds for semigroups and Theorem 3 on the solution of the Poisson equation, are stated in Section 2. The proof of Theorems 1 and 2 is the object of Sections

4 and 5 correspondingly, while the proof of Theorem 3 is given in Section 2 as a consequence of Theorem 1. Section 3 is devoted to the application of Theorem 2 to the diffusion approximation result which was introduced above. Section 4 gives estimates on the transition and invariant densities, together with their derivatives with respect to x. Section 4.3 establishes the differentiability of the transition density with respect to y and Section 4.4 the same differentiability of the invariant density. Section 4.5 analyzes the x-differentiability of the y-derivatives from Sections 4.3 and 4.4. The result concerning higher order y-derivatives is given in Section 4.5. Finally, differentiability of solutions of PDEs is discussed in Section 5.

2. Invariant density and Poisson equation. We first recall the existence and uniqueness result and some estimates for solutions of equation (1) from [15], adjusted to our present setting which is a bit less general than in [15].

PROPOSITION 1. *Under conditions* $(H_a) + (H_b) +$ (*uniform continuity of the matrix* a) + [*growth of* $f(x, y)$ *in* x *not faster than polynomially for any* y], *there exists a solution of* (1) *in the class of functions from the Sobolev space* $\bigcap_{p>1} W^2_{p,\mathrm{loc}}$ *which are locally bounded and grow at most polynomially in* $|x|$, *as* $|x| \to \infty$, *unique up to an additive constant which can be chosen so that for any* y *the centering equality* (3) *holds*.

Moreover, for this solution, (4) *holds true along with the following bounds*:

If for some $\beta \ge 0$,

$$|f(x, y)| \le C(y)(1 + |x|^\beta),$$

then for any $\beta' > \beta + 2$,

$$|u(x, y)| + |\nabla_x u(x, y)| \le C_1(y)(1 + |x|)^{\beta'} \tag{5}$$

with some $C_1(y)$.

If for some $\beta < 0$,

$$|f(x, y)| \le C(y)(1 + |x|)^{\beta-2},$$

then u *and* $\nabla_x u$ *are bounded*,

$$|u(x, y)| + |\nabla_x u(x, y)| \le C_1(y) \tag{6}$$

with some $C_1(y)$.

If for some $\beta > 4$,

$$|f(x, y)| \le C(y)(1 + |x|^{\beta-2}),$$

then for some constant $C_1(y)$,

$$|u(x, y)| + |\nabla_x u(x, y)| \le C_1(y)(1 + |x|^\beta). \tag{7}$$

We can now state our main results.

THEOREM 1. *Let* (H_a), (H_b) *and* $(H^{2+\alpha,j})$ *with an* $0<\alpha<1$ *hold true. Then the tensor of partial derivatives* $\partial_y^j p_t(x,x',y) =: p_t^{(j)}(x,x',y)$ *exists, together with the limit*

$$\lim_{t\to\infty} p_t^{(j)}(x,x',y) =: p_\infty^{(j)}(x',y)$$

and

$$p_\infty^{(j)}(x',y) = \partial_y^j p_\infty(x',y).$$

Moreover, for $i=0,1$ *and also for* $i=2$ *if* $j=0$,

$$|\partial_{x'}^i p_t^{(j)}(x,x';y)| \le Ct^{-(d+i)/2}\exp(-c|x-x'|^2/t), \qquad 0<t\le 1, \tag{8}$$

and for any

$$|\partial_{x'}^i p_t^{(j)}(x,x';y) - \partial_{x'}^i p_\infty^{(j)}(x',y)| \le C\frac{1+|x|^m}{(1+|x'|^{m'})(1+t)^k}, \qquad t>1, \tag{9}$$

$$|\partial_{x'}^i p_\infty^{(j)}(x',y)| \le \frac{C}{(1+|x'|^{m'})}. \tag{10}$$

We also have that, for $i=0,1$ *and also for* $i=2$ *if* $j=0$,

$$|\partial_x^i p_t^{(j)}(x,x';y)| \le Ct^{-(d+i)/2}\exp(-c|x-x'|^2/t), \qquad 0<t\le 1, \tag{11}$$

and for $i=1$ *and also for* $i=2$ *if* $j=0$, *for any* m', k *there exist such* C, m *that*

$$|\partial_x^i p_t^{(j)}(x,x';y)| \le \frac{C(1+|x|^m)}{(1+|x'|^{m'})(1+t)^k}. \tag{12}$$

The last two inequalities are helpful in estimating derivatives of higher orders of the function u.

PROOF OF THEOREM 1. The proof follows from the results of Section 4.

Case $j=0$. Existence of the limit p_∞ follows from [17]. Inequalities (8) and (11) with $j=0$ are classical, due to Eidelman and Friedman; see Proposition 2 based on [8]. Inequalities (9), (10) and (12) are proved in Proposition 3.

Case $j=1$. Inequality (8) follows from Theorem 5 ($i=0$) and Proposition 4 ($i=1$). Inequalities (9) and (10) are proved in Theorem 7. Inequality (11) follows from Proposition 4. Finally, (12) is proved in Theorem 8.

Case $j=2$. All inequalities are proved in Theorem 9. □

It is plausible that inequalities (8)–(10) hold true with $i=2$, and (11)–(12) with $2\le i\le 4$, too. Generalizations under assumption $(H^{r+\alpha,j})$ with $r>2$ would be also natural. However, we will not use them: we are interested in partial derivatives of second order needed to apply the Itô formula. So neither of these extensions are discussed here.

Derivatives $\partial_y^j p_t(x,f;y)$ will be denoted by $p_t^{(j)}(x,f;y)$.

THEOREM 2. *Let $j \in \mathbb{N}$ and $(H_a) + (H_b)$ hold true. If either $((H^{\alpha,1}) + (f \in C^{1+\alpha,j}))$ or $((H^{2+\alpha,1}) + (f \in C^{\alpha,j}))$ is valid with some $\alpha > 0$, then the function $p_t^{(j)}(x, f, y)$ has the following properties: there exists a limit*

$$\lim_{t\to\infty} p_t^{(j)}(x, f; y) = \partial_y^j p_\infty(f, y) =: p_\infty^{(j)}(f, y), \tag{13}$$

for any $k > 0$ there exist $C, m > 0$ such that for all $t \geq 1$,

$$\left|p_t^{(j)}(x, f, y) - p_\infty^{(j)}(f, y)\right| \leq C\frac{(1 + |x|^m)}{(1 + t)^k}, \tag{14}$$

and for $i = 1$ and also for $i = 2$ if $j = 0$,

$$\left|\partial_x^i p_t^{(j)}(x, f, y)\right| \leq C\frac{(1 + |x|^m)}{(1 + t)^k}. \tag{15}$$

The proof is given in Section 5.

The next result concerns the regularity of solutions of the Poisson equation. We restrict ourselves to the derivatives up to the order 2 needed in Theorems 3 and 4 below. However we note that higher order derivatives in all variables can be also studied similarly using Theorem 1.

THEOREM 3. *Let conditions $(H_a) + (H_b)$ and either $((H^{\alpha,1}) + (f \in C^{1+\alpha,2}))$ or $((H^{2+\alpha,1}) + (f \in C^{\alpha,2}))$, with some $\alpha > 0$, be satisfied, and the centered [see (2)] function f be such that*

$$|f(x, y)| + |\partial_y f(x, y)| + |\partial_y^2 f(x, y)| \leq C(y)(1 + |x|^m).$$

Then the solution of (1) and (3) satisfies $u(x, \cdot) \in C^2$ for any x, and the following bounds hold true with some m', m'', m''' and some constant $C_1(y)$:

$$|\partial_y u(x, y)| \leq C_1(y)\bigl(1 + |x|^{m'}\bigr), \tag{16}$$

$$|\partial_y^2 u(x, y)| \leq C_1(y)\bigl(1 + |x|^{m''}\bigr) \tag{17}$$

and

$$|\partial_y \partial_x u(x, y)| \leq C_1(y)\bigl(1 + |x|^{m'''}\bigr). \tag{18}$$

PROOF. The existence of a solution u follows from the Proposition 1. Let us denote

$$q_t := p_t^{(1)}, \qquad r_t := p_t^{(2)}.$$

Clearly, q is an ℓ-dimensional vector, and r an $\ell \times \ell$ matrix.

The bound for u_y follows from the representation

$$u_y(x, y) = \int_0^1 q_s(x, f, y)\, ds + \int_1^\infty q_s(x, f, y)\, ds,$$

standard estimates, including derivative with respect to y, for the integral $\int_t^1 p_s(x, f, y)\, ds$ which solves a Cauchy problem for a parabolic equation in the region $[0, 1] \times \mathbb{R}^d$ (with an initial value at $t = 1$), and the estimates in Theorem 2 applied to the second integral here. The estimates for u_{yy} and u_{xy} follow similarly from the bounds of the same Theorem 2. $\square$

3. Diffusion approximation. The aim of this section is to apply Theorem 3 to the study of the asymptotic behavior, as $\varepsilon \to 0$, of the $\mathbb{R}^\ell$-valued process $\{Y_t^\varepsilon; t \geq 0\}$, where

$$\begin{aligned} dX_t^\varepsilon &= \varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon)\, dt + \varepsilon^{-1} \sigma(X_t^\varepsilon, Y_t^\varepsilon)\, dB_t, \\ dY_t^\varepsilon &= F(X_t^\varepsilon, Y_t^\varepsilon)\, dt + \varepsilon^{-1} G(X_t^\varepsilon, Y_t^\varepsilon)\, dt + H(X_t^\varepsilon, Y_t^\varepsilon)\, dB_t, \\ X_0^\varepsilon &= x, \qquad Y_0^\varepsilon = y. \end{aligned}$$

We shall make the following assumptions. For each $K > 0$, there exists a constant C_K such that for all $y, y' \in \mathbb{R}^\ell$, $|x| \leq K$,

$$(H_L) \qquad \begin{aligned} &|F(x, y) - F(x, y')| + |G(x, y) - G(x, y')| + |H(x, y) - H(x, y')| \\ &\quad \leq C_K |y - y'|. \end{aligned}$$

Moreover, we assume the following growth condition on these coefficients, together with a regularity condition and a centering condition on G. There exist positive K, α, m_1, m_2 and m_3 such that for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^\ell$:

$$(H_P) \qquad \begin{aligned} |F(x, y)| &\leq K(1 + |y|)(1 + |x|^{m_1}), \\ |H(x, y)| &\leq K(1 + |y|^{1/2})(1 + |x|^{m_2}), \end{aligned}$$

(H_G) the following smoothness, growth and centering conditions on function G are satisfied:

$$G(x, \cdot) \in C^2(\mathbb{R}^\ell), \qquad |G(x, y) - G(x', y)| \leq K|x - x'|^\alpha,$$
$$|G(x, y)| + |\partial_y G(x, y)| + |\partial_y^2 G(x, y)| \leq K(1 + |x|^{m_3}),$$
$$\int_{\mathbb{R}^d} G(x, y) \mu^y(dx) = 0.$$

Let $\bar{X}_t^\varepsilon := X_{\varepsilon^2 t}^\varepsilon$ and $B_t^\varepsilon := \varepsilon^{-1} B_{\varepsilon^2 t}$. Then

$$\bar{X}_t^\varepsilon = x + \int_0^t b(\bar{X}_s^\varepsilon, Y_{\varepsilon^2 u}^\varepsilon)\, du + \int_0^t \sigma(\bar{X}_s^\varepsilon, Y_{\varepsilon^2 u}^\varepsilon)\, dB_u^\varepsilon,$$

so that $\bar{X}_\cdot^\varepsilon$ is asymptotically identical in law to $X_\cdot^y$.

Our strategy for proving that $Y_\cdot^\varepsilon$ converges in law to a diffusion, to be precise, is the same as the one used in [15], which follows ideas from previous works; see, for example, [13]. Let $f : \mathbb{R}^d \to \mathbb{R}$ be an arbitrary smooth function. For each $y \in \mathbb{R}^\ell$, let $\{u(x, y), x \in \mathbb{R}^d\}$ denote the solution of the Poisson equation

$$Lu(x, y) + \langle \nabla_y f(y), G(x, y)\rangle = 0.$$

Clearly $u(x, y) = \langle \nabla_y f(y), PG(x, y)\rangle$, where

$$PG(x, y) := \int_0^\infty dt \int_{\mathbb{R}^d} p_t(x, x'; y) G(x', y)\, dx'.$$

The basic identity, which is used both for checking the tightness of the sequence Y^ε and identifying the limit, is obtained by applying Itô's formula in order to express

$$f_\varepsilon(X_t^\varepsilon, Y_t^\varepsilon) - f_\varepsilon(X_0^\varepsilon, Y_0^\varepsilon),$$

where $f_\varepsilon(x, y) = f(x) + \varepsilon u(x, y)$. Needless to say, the results of the above sections are essential in order to establish the required smoothness of u, and express its derivatives.

We shall use the following notation:

$$(PG)(x, y) = \int_0^\infty p_t(x, G; y)\, dt,$$

$$(Q_i G)(x, y) = \int_0^\infty \partial_{y^i} p_t(x, G; y)\, dt,$$

$$(R_{ij} G)(x, y) = \int_0^\infty \partial_{y^i y^j} p_t(x, G; y)\, dt.$$

Clearly

$$u(x, y) = \langle \nabla f(y), PG(x, y)\rangle.$$

Now for $1 \le i, j \le \ell$,

$$\begin{aligned}
\partial_{y_i} u(x, y) &= \langle \partial_{y_i} \nabla f(y), PG(x, y)\rangle + \langle \nabla f(y), Q_i G(x, y)\rangle \\
&\quad + \langle \nabla f(y), P[\partial_{y_i} G](x, y)\rangle, \\
\partial_{y_i} \partial_{y_j} u(x, y) &= \langle \partial^2_{y_i y_j} \nabla f(y), PG(x, y)\rangle \\
&\quad + \langle \partial_{y_i} \nabla f(y), Q_j G(x, y)\rangle + \langle \partial_{y_j} \nabla f(y), Q_i G(x, y)\rangle \\
&\quad + \langle \partial_{y_i} \nabla f(y), P[\partial_{y_j} G](x, y)\rangle + \langle \partial_{y_j} \nabla f(y), P[\partial_{y_i} G](x, y)\rangle \\
&\quad + \langle \nabla_f(y), R_{ij} G(x, y)\rangle + \langle \nabla f(y), P[\partial^2_{y_i y_j} G](x, y)\rangle.
\end{aligned}$$

THEOREM 4. *Let conditions* $(H^{2+\alpha,1})$, (H_a), (H_b), (H_L), (H_P) *and* (H_G) *be satisfied. Then for any* $T > 0$, *the family of processes* $\{Y_t^\varepsilon, 0 \le t \le T\}_{0<\varepsilon\le 1}$ *is* *weakly relatively compact in* $C([0,T];\mathbb{R}^\ell)$. *Any accumulation point* Y *is a solution of the martingale problem associated to the operator*

$$\mathcal{L} = \tfrac{1}{2}\tilde{a}_{ij}(y)\partial_{y_i}\partial_{y_j} + \tilde{b}_i(y)\partial_{y_i},$$

where

$$\begin{aligned}\tilde{b}(y) = \bar{F}(y) &+ \sum_i \int G_i(x,y)\partial_{y_i}(PG)(x,y)\mu^y(dx) \\ &+ \sum_{i,k}\int (H\sigma^*)_{ik}(x,y)\partial_{x_k}\partial_{y_i}(PG)(x,y)\mu^y(dx)\end{aligned}$$

and

$$\tilde{a}(y) = (\bar{\mathcal{H}} + \bar{\mathcal{G}} + \bar{\mathcal{K}})(y),$$

with

$$\begin{aligned}\bar{F}(y) &= \int F(x,y)\mu^y(dx), \\ \bar{\mathcal{H}}(y) &= \int HH^*(x,y)\mu^y(dx), \\ \bar{\mathcal{G}}(y) &= \int \big[G(x,y)PG^*(x,y) + PG(x,y)G^*(x,y)\big]\mu^y(dx), \\ \bar{\mathcal{K}}_{ij}(y) &= \sum_{k=1}^d \int \big[(H\sigma^*)_{ik}(x,y)\partial_{x_k}(PG_j)(x,y) \\ &\qquad + (H\sigma^*)_{jk}(x,y)\partial_{x_k}(PG_i)(x,y)\big]\mu^y(dx).\end{aligned}$$

If, moreover, the martingale problem associated to $\mathcal{L}$ *is well posed* (*it is easy to state sufficient conditions for that*), *then* $Y^\varepsilon \Rightarrow Y$, *where* Y *is the unique* (*in law*) *diffusion process with generator* $\mathcal{L}$.

Notice that all integrals in the definition of $\mathcal{L}$ are well defined.

The proof is exactly the same as that in [15], except for one minor modification, which we now explain, but we shall not repeat the proof. Namely, we had to reinforce the boundedness assumption on G, since allowing that G grows linearly in y, with bounded (with respect to y) derivatives in y would not prevent QG and RG, hence the partial derivatives in y of PG to grow linearly in y, which would destroy the compactness argument in [15], unless we assume that F and H are bounded in y.

Note that the condition $|G(x,y) - G(x',y)| \le C|x-x'|^\alpha$, $C, \alpha > 0$ is sufficient for $\partial_x^2 \bar{G} \in C(\mathbb{R}^d \times \mathbb{R}^\ell)$ due to Eidelman or Friedman results. This allows to use the original Itô formula while in the first part of the paper we used the Itô–Krylov version.

4. Properties of a fundamental solution.

4.1. *Auxiliary bounds for the transition density.* Let us recall certain results concerning upper bounds for fundamental solutions of the Cauchy problem. We use [8] adjusted to our (homogeneous) case; cf. Theorem 9.2 and Theorem 9.7; see also Theorems 3.5 and 3.6 in [5] and [4]. Concerning the case $|m| = 2$ in the first part of Proposition 2, see remark around inequality (9.4.19) in [8].

In the next statement, we drop the index y. Note however that all the constants are independent of $y \in \mathbb{R}^d$. This implies inequalities (8) and (11) for $j = 0$ from Theorem 1.

PROPOSITION 2. *Assume that condition (H_a) holds and that b is bounded.*

1. *Let $a(x, y), b(x, y)$ be Hölder continuous (with exponent $\alpha > 0$) in x uniformly in y with some uniform constants. Then the transition density $p_t(x, x'; y)$ exists and satisfies the following bounds: for any $T > 0$ there exist some constants $C, c > 0$ such that for any $0 \le |m| \le 2$ and $0 \le t \le T$,*

$$\left|\partial_x^m p_t(x, x'; y)\right| \le Ct^{-(d+|m|)/2} \exp(-c|x' - x|^2/t).$$

2. *Let $a(\cdot, y)$, $b(\cdot, y) \in C_{\mathrm{b}}^{n+\alpha}$ for some $n \in \mathbb{N}$, $\alpha > 0$ and with bounds uniform in y. Then $\partial_x^{m+i}\partial_{x'}^{j} p_t(x, x')$ exists and is a continuous function of (t, x, x') in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for all $0 \le |i| + |j| \le n$, $0 \le |m| < 2$ and for $0 < t \le T$,*

$$\left|\partial_x^{m+i}\partial_{x'}^{j} p_t(x, x'; y)\right| \le Ct^{-(|m|+|i|+|j|+d)/2} \exp(-c|x' - x|^2/t),$$

$$\left|\partial_x^{m}\partial_{x'}^{j} p_t(x' - x, x'; y)\right| \le Ct^{-(|m|+d)/2} \exp(-c|x|^2/t),$$

with some constants $C, c > 0$.

The next type of bounds we will use are based on the ergodic estimates of the process $\{X_t^y; t \ge 0\}$. We first recall a result from [17].

LEMMA 1. *Assume that conditions (H_a) and (H_b) hold and there exists a density $p_t(x, x'; y)$. Then for each $m > 0$, there exists a constant C_m such that for all $x \in \mathbb{R}^d$, $t \ge 0$ and $y \in \mathbb{R}^k$,*

$$\int_{\mathbb{R}^d} |x'|^m p_t(x, x'; y)\, dx' \le C_m(1 + |x|^{m+2}).$$

Notice that the power $m + 2$ in the right-hand side can be replaced by m in the case of constant nondegenerate diffusion; see [16].

We next establish the following lemma.

LEMMA 2. *For each $m, c > 0$, there exists a constant $C > 0$ such that for all $0 < t \le 1$,*

$$t^{-d/2} \int_{\mathbb{R}^d} (1 + |x'|^m)^{-1} \exp\left(-c\frac{|x - x'|^2}{2t}\right) dx' \le \frac{C}{1 + |x|^m}$$

and

$$t^{-d/2}\int_{\mathbb{R}^d}(1+|x'|^m)\exp\biggl(-c\frac{|x-x'|^2}{2t}\biggr)\,dx' \le C(1+|x|^m).$$

Though of course a nonprobabilistic proof of this lemma is available, we give probabilistic arguments, which are probably as simple as any other possible proof.

PROOF OF LEMMA 2. It suffices to prove the first assertion for $|x|\ge 1$. Up to a factor which depends only on c and d, the above left-hand side equals ($\{B_t; t\ge 0\}$ denotes again a d-dimensional standard Brownian motion)

$$\begin{aligned}\mathbb{E}\biggl(\frac{1}{1+|x+B_t/\sqrt{c}|^m}\biggr) &= \mathbb{E}\biggl(\frac{1}{1+|x+B_t/\sqrt{c}|^m}; |B_t|\le\frac{\sqrt{c}|x|}{2}\biggr)\\ &\quad+\mathbb{E}\biggl(\frac{1}{1+|x+B_t/\sqrt{c}|^m}; |B_t|>\frac{\sqrt{c}|x|}{2}\biggr)\\ &\le \frac{1}{1+|x/2|^m}+\mathbb{P}(|B_t|>\sqrt{c}|x|/2),\end{aligned}$$

from which the result follows with $C=2^m(1+2c^{m/2}\mathbb{E}|B_1|^m)$, using Chebyshev's inequality, and $|x|^{-1}\le 2(1+|x|)^{-1}$ which follows from $|x|\ge 1$.

The second statement follows from the inequality

$$\sup_{0\le t\le 1}\mathbb{E}(1+|x+B_t/\sqrt{c}|^m)\le C(1+|x|^m). \qquad \square$$

We are now ready to prove the following proposition.

PROPOSITION 3. *Let $n\in\mathbb{N}$. Assume that conditions* (H_a) *and* (H_b) *hold, and moreover, that the coefficients a and b are Hölder continuous in x, uniformly with respect to y, and that for each $y\in\mathbb{R}^\ell$, $a(\cdot,y), b(\cdot,y)\in C_{\mathrm{b}}^n(\mathbb{R}^d)$, where the bounds for the functions and their derivatives are independent of y.*

Then for any $k, j\in\mathbb{R}_+$, there exist $C, m>0$ such that for all $x, x'\in\mathbb{R}^d$, $y\in\mathbb{R}^\ell$, $t\ge 1$,

$$|\partial_{x'}^n p_\infty(x',y)|\le\frac{C}{1+|x'|^j}, \tag{19}$$

$$|\partial_{x'}^n p_t(x,x';y)|\le C\frac{1+|x|^m}{(1+|x'|^j)} \tag{20}$$

and

$$|\partial_{x'}^n p_t(x,x';y)-\partial_{x'}^n p_\infty(x',y)|\le C\frac{1+|x|^m}{(1+t)^k(1+|x'|^j)}. \tag{21}$$

Also, provided $n \geq 1$ *and for each* $y \in \mathbb{R}^\ell$, $a(\cdot, y)$, $b(\cdot, y) \in C_{\mathrm{b}}^{(n-2)^+}(\mathbb{R}^d)$, *where the bounds for the functions and their derivatives are independent of* y,

$$\left|\partial_x^n p_t(x, x'; y)\right| \leq C \frac{1 + |x|^m}{(1+t)^k (1 + |x'|^j)}. \tag{22}$$

The last inequality also holds for mixed derivatives $\partial_{x'}^n \partial_x^{n'} p_t$, *provided again* $n \geq 1$ *and the required regularity becomes the fact that* $a(\cdot, y)$ *and* $b(\cdot, y)$ *belong to* $C_{\mathrm{b}}^{(n'-1)^+ + n}$.

PROOF. Remind that a and b are Hölder continuous with respect to x. Along with the assumption $a, b \in C^n$ it is essential for the estimates of $\partial_{x'}^n p_t$ and $\partial_x^n p_t$. We first note that a simple consequence of Proposition 2 is that $|\partial_{x'}^n p_1(x, x'; y)| \leq c_n$. Combining this with the Chapman–Kolmogorov relation implies that $|\partial_{x'}^n p_t(x, x'; y)| \leq c$ for all $t \geq 1$. This implies (20) and (19) in case $|x'| \leq 1$. We now prove (20) in case $|x'| > 1$:

$$\begin{aligned}
\left|\partial_{x'}^n p_t(x, x'; y)\right| &= \left| \int_{\mathbb{R}^d} p_{t-1}(x, x''; y) \partial_{x'}^n p_1(x'', x'; y)\, dx'' \right| \\
&\leq C \int_{\mathbb{R}^d} p_{t-1}(x, x''; y) e^{-c|x'-x''|^2}\, dx'' \\
&\leq C \int_{|x''| > |x'|/2} p_{t-1}(x, x''; y)\, dx'' + C \exp[-c|x'|^2/4] \\
&\leq C \left(\frac{2}{|x'|}\right)^j \mathbb{E}_x\left(|X_{t-1}|^j\right) + C \exp[-c|x'|^2/4] \\
&\leq C \frac{1 + |x|^{j+2}}{1 + |x'|^j},
\end{aligned}$$

where we have used Lemma 1 in the final step. Inequality (20) is proved and (19) is proved exactly in the same way. We now want to prove (21):

$$\begin{aligned}
&\left|\partial_{x'}^n p_t(x, x'; y) - \partial_{x'}^n p_\infty(x', y)\right| \\
&\quad \leq \int_{\mathbb{R}^d} \left|p_{t-1}(x, x''; y) - p_\infty(x'', y)\right| \times \left|\partial_{x'}^n p_1(x'', x'; y)\right| dx'' \\
&\quad \leq C \|\mu_{t-1}^x - \mu_\infty\| \\
&\quad \leq C \frac{1 + |x|^m}{1 + (t-1)^k},
\end{aligned}$$

where the last inequality can be found in [16]. Note that k is arbitrary, and m depends on k. Inequality (21) follows from this and the next inequality, which is a direct consequence of (20) + (19):

$$\left|\partial_{x'}^n p_t(x, x'; y) - \partial_{x'}^n p_\infty(x', y)\right| \leq c \frac{1 + |x|^{j+2}}{1 + |x'|^j}.$$

We finally prove (22). First note that for $n \geq 1$,

$$\int_{\mathbb{R}^d} \partial_x^n p_1(x, x''; y) p_\infty(x', y)\, dx'' = \partial_x^n p_\infty(x', y) \\ = 0.$$

Hence, from (21),

$$\begin{aligned} \left|\partial_x^n p_t(x, x'; y)\right| &= \left| \int_{\mathbb{R}^d} \partial_x^n p_1(x, x''; y) \left[p_{t-1}(x'', x'; y) - p_\infty(x', y)\right] dx'' \right| \\ &\leq C \int_{\mathbb{R}^d} e^{-c|x-x''|^2} \frac{1 + |x''|^m}{(1+t)^k (1 + |x'|^j)}\, dx'' \\ &\leq C \frac{1 + |x|^m}{(1+t)^k (1 + |x'|^j)}, \end{aligned}$$

where the last step follows from Lemma 2.

We omit the proof of the last assertion for $\partial_{x'}^n \partial_x^{n'} p_t$. $\square$

4.2. *First derivative* $q_t = \partial_y p_t$. We now establish the differentiability of p_t with respect to y. The result of this section is the following.

THEOREM 5. *Assume that conditions* (H_b), (H_a) *and* $(H^{1+\alpha,1})$ *hold. Then for each* $t > 0$, $x, x' \in \mathbb{R}^d$, $y \in \mathbb{R}^\ell$, $p_t(x, x'; y)$ *is differentiable in the variable* y, *and the gradient* $\partial_y p_t(x, x'; y)$ *is given by the formula*

$$\begin{aligned} \partial_y p_t(x, x', y) &= q_t(x, x'; y) \\ &:= \int_0^t ds \int_{\mathbb{R}^d} p_s(x, x''; y) \frac{\partial L}{\partial y}(x'', y) p_{t-s}(x'', x'; y)\, dx. \end{aligned} \tag{23}$$

The function q_t *is bounded and continuous in* y *for any* $t > 0, x, x'$. *Moreover, there exist* $C, c > 0$ *such that for all* $0 < t \leq 1$, x, x', y,

$$|q_t(x, x'; y)| \leq \frac{C}{t^{d/2}} \exp\left(-\frac{c|x - x'|^2}{t}\right). \tag{24}$$

PROOF. We first check the integrability, that is, that formula (23) makes sense. First the integrability over dx'' for fixed $0 < s < t$ is no problem, given the estimates of Proposition 2. Here we use the assumption $a, b \in C_{\mathrm{b}}^{0,1}$, so that the expression $\partial L/\partial y$ is well defined. Now the second factor in the integrand is a linear combination of partial derivatives of $p_{t-s}(x, x''; y)$ with respect to x of order one and two, with bounded coefficients. Since we can always integrate by parts one partial derivative in the dx'' integral, we see that the integrand inside the ds integral expressing q_t is a sum of terms of the type

$$\int_{\mathbb{R}^d} \alpha_{ij}(x'', y) \frac{\partial^i p_s}{\partial^i x_n''} p(x, x''; y) \frac{\partial^j p_{t-s}}{\partial^j x_m''}(x'', x'; y)\, dx'',$$

where $i, j = 0, 1, 1 \leq n, m \leq d$. It now follows from Proposition 2 that the integral of the absolute value of the integrand in (23) is dominated by a constant times

$$\begin{aligned}
&\int_0^t ds \int_{\mathbb{R}^d} s^{(d+1)/2}(t-s)^{(d+1)/2} \exp\left[-c\frac{|x-x''|^2}{s}\right] \exp\left[-c\frac{|x''-x'|^2}{t-s}\right] dx'' \\
&\quad \leq \frac{c}{t^{d/2}} \exp\left[-c\frac{|x-x'|^2}{t}\right] \int_0^t \frac{ds}{\sqrt{s(t-s)}} \\
&\quad \leq \frac{C}{t^{d/2}} \exp\left(-c\frac{|x-x'|^2}{t}\right)
\end{aligned}$$

which proves (24). Here we used the assumption $a, b \in C_{\mathrm{b}}^{1,0}$ (see Proposition 2).

The same estimate holds when p_s and p_{t-s} are evaluated at different points y and y'.

We now prove that p_t is differentiable and that its derivative is given by (23). For $1 \leq i \leq \ell$, let e_i denote the unit vector in the ith direction of $\mathbb{R}^\ell$, and let $h \neq 0$. We define

$$q_t^{i,h}(x, x'; y) := \frac{p_t(x, x'; y + he_i) - p_t(x, x'; y)}{h}.$$

In what follows, we delete most of the indices for notational simplicity, including the index i, and write $y + h$ instead of $y + he_i$. The idea is to notice that

$$\frac{\partial q_t^h(y)}{\partial t} = -L(y)q_t^h(y) + \frac{L(y+h) - L(y)}{h} p_t(y+h), \qquad q_0^h = 0, \tag{25}$$

so that q_t^h should be given by the formula

$$\begin{aligned}
q_t^h(x, x'; y) = \int_0^t ds \int_{\mathbb{R}^d} & p_s(x, x''; y) \\
& \times \frac{L(y+h) - L(y)}{h} p_{t-s}(x'', x'; y+h)\, dx''
\end{aligned} \tag{26}$$

where we can pass to the limit as $h \to 0$ to get the desired assertion.

These last statements need some justifications since p_t and hence q_t^h have singularities at $t = 0$, so that $q_0^h = 0$ should be understood in a weak sense. A simple way to justify our claims is to start with $p_t(x, \phi; y) = \int \phi(x') p_t(x, x'; y)\, dx'$ and $q_t^h(x, \phi; y) := \int \phi(x') q_t^h(x, x'; y)\, dx' \, \mathbb{1}(t > 0)$, $\phi \in C_0^\infty$.

Due to [12], Theorem 4.5.2, the function $p_t(x, \phi; y)$ is a classical solution of the problem

$$\partial_t p_t(x, \phi; y) = L(y) p_t(x, \phi; y), \qquad p_0(x, \phi; y) = \phi(x).$$

So $q_t^h(x, \phi; y)$ satisfies a smoothed version of equation (25), and is given by the formula (26), but with $p_{t-s}(x'', x', y)$ replaced by $p_{t-s}(x'', \phi, y)$. Now we let $\phi_k \to \delta_{x'}$, and use the estimate leading to the proof of (24) and Lebesgue's dominated convergence theorem. This shows (26).

We can now invoke the same argument in order to take the limit as $h \to 0$ in the right-hand side of (26), provided $p_{t-s}(x'', x'; y)$ and $\partial_{x''} p_{t-s}(x'', x'; y)$ are continuous in y (we get rid of the second derivative by integration by parts). The first continuity (and even uniform Lipschitz continuity) follows from (26) and implies the continuity of $y \to \int \partial_{x''} p_t(x'', x'; y)\phi(x')\,dx'$, provided $\phi \in C^1(\mathbb{R}^d)$ and has compact support. It remains to exploit the boundedness of $\partial_x p_t$ in order to extract out of $\partial_x p_t(x, \cdot; y_n)$ a sequence which converges on a countable dense set of (x')'s, and continuity in x', uniformly with respect to y in order to conclude that the limit, indeed, equals to $\partial_x p_t(x, x', y)$, as $y_n \to y$. So, we finally get (23).

Continuity of q in y follows from the convergence of the integrand in (23) uniformly with respect to y. $\square$

4.3. *Behavior of the first derivative q_t as $t \to \infty$.* Assumption $(H^{1+\alpha,1})$ is used in this section. It turns out that similar calculus can be applied to study the behavior of q_t as well as next derivatives $\partial_y^k p_t$, by induction, provided certain preliminary properties are established. Hence, we organized this and next section as follows: we introduce new notation, $\bar{p}$ and $\bar{L}$ for this induction. Whilst the first reading of this and next subsections, $\bar{p}_t(x, x', y) = p_t(x, x', y)$, and $\bar{L} = L$. Then we will establish additional bounds in order to be able to do the next induction step, and at the next steps, $\bar{p}$ and $\bar{L}$ will mean some new functions. Notice that in the next steps we do not assume $\bar{p}_t$ to be a density or to be positive.

Define

$$f_t^1(x, x'; y) := \frac{\partial \bar{L}(x, y)}{\partial y} \bar{p}_t(x, x'; y)$$

or, a bit more generally,

$$f_t^1(x, x'; y) := \sum_i \frac{\partial \bar{L}^i(x, y)}{\partial y} \bar{p}_t^i(x, x'; y),$$

with any $\bar{L}^i(x, y) = \sum \bar{a}_{kj}^i(x, y)\partial_{x_k}\partial_{x_j} + \sum \bar{b}_k^i(x, y)\partial_{x_k}$ such that all $(\bar{a}^i, \bar{b}^i)$ satisfy the same nondegeneracy, boundedness and smoothness conditions as (a, b). Just for simplicity, we do all the calculus with only one operator $\bar{L}$, however, it remains absolutely similar for any finite sum.

Consider the functions

$$\begin{aligned}\bar{q}_t(x, x'; y) &:= \int_0^t ds \int dx''\, p_s(x, x''; y) f_{t-s}^1(x'', x'; y)\\ &= \int_0^t ds \int dx''\, p_{t-s}(x, x''; y) f_s^1(x'', x'; y)\end{aligned} \tag{27}$$

and

$$\bar{q}_\infty(x', y) := \int_0^\infty ds \int_{\mathbb{R}^d} dx'' p_\infty(x'', y) f_s^1(x'', x', y). \tag{28}$$

We need to show that

$$\bar{q}_\infty(x', y) = \partial_y \bar{p}_\infty(x', y)$$

and

$$\bar{q}_\infty(x', y) = \lim_{t\to\infty} \bar{q}_t(x, x'; y),$$

and for any function $g : \mathbb{R}^d \to \mathbb{R}$ which satisfies a bound of the type

$$|g(x)| \le C(1 + |x|^n),$$

the following integral converges:

$$\int_0^\infty dt \int_{\mathbb{R}^d} [\bar{q}_t(x, x'; y) - \bar{q}_\infty(x', y)]g(x')\, dx'.$$

More precisely, we now show the following theorem.

THEOREM 6. *Assume that conditions* (H_b), (H_a) *and* $(H^{1+\alpha,1})$ *and the above mentioned assumptions on* $\bar{p}$ *hold. Then for each* $k, m' > 0$, *there exists* $C, m \in \mathbb{R}$ *such that for all* $y \in \mathbb{R}^\ell$, $x, x' \in \mathbb{R}^d$, $t \ge 1$,

$$|\bar{q}_t(x, x'; y) - \bar{q}_\infty(x'; y)| \le C \frac{1 + |x|^m}{(1 + |x'|^{m'})(1 + t)^k}, \tag{29}$$

$$|\bar{q}_\infty(x, y)| \le \frac{C}{1 + |x|^m} \tag{30}$$

and moreover,

$$\bar{q}_\infty(x, y) = \partial_y \bar{p}_\infty(x, y).$$

Note that $\bar{p}$, $\bar{q}$ stand for p and q correspondingly.

PROOF OF THEOREM 6. Estimate (30) follows clearly from (29) and (24), considered at $t = 1$, $x = 0$. The last statement follows from taking the limit as $t \to \infty$ in the identity

$$\bar{p}_t(x, x'; y + he_i) - \bar{p}_t(x, x'; y) = \int_0^h \bar{q}_t^i(x, x'; y + \alpha e_i)\, d\alpha$$

since the function $\bar{q}_t^i(x, x', \cdot)$ is continuous, which is shown exactly as for q_t.

Let us establish (29). We have

$$\begin{aligned}
&\bar{q}_t(x, x'; y) - \bar{q}_\infty(x, x'; y) \\
&\quad = \int_0^t ds \int p_{t-s}(x, x''; y) f_s^1(x'', x'; y)\, dx'' \\
&\qquad - \int_0^\infty ds \int p_\infty(x'', y) f_s^1(x'', x'; y)\, dx'' \\
&\quad = \int_0^{t/2} ds \int \big(p_{t-s}(x, x''; y) - p_\infty(x'', y)\big) f_s^1(x'', x'; y)\, dx'' \\
&\qquad + \int_{t/2}^t ds \int p_{t-s}(x, x''; y) f_s^1(x'', x'; y)\, dx'' \\
&\qquad - \int_{t/2}^\infty ds \int p_\infty(x'', y) f_s^1(x'', x'; y)\, dx''.
\end{aligned}$$

We now successively estimate each term of the above right-hand side.

Consider the first term. It follows from (21) that

$$|p_{t-s}(x, x''; y) - p_\infty(x'', y)| \le C \frac{1 + |x|^m}{(1 + t - s)^k (1 + |x''|)^j}.$$

So, estimating f_s^1 with the help of Propostion 2 between 0 and 1 and of the estimate (20) between 1 and $t/2$, we get

$$\begin{aligned}
&\int_0^{t/2} ds \int dx'' |p_{t-s}(x, x''; y) - p_\infty(x'', y)| |f_s^1(x'', x'; y)| \\
&\quad \le C \int_0^1 ds \int dx'' \frac{1 + |x|^m}{(1 + t - s)^k (1 + |x''|^j)} s^{-d/2} \exp\left[-c \frac{|x'' - x'|^2}{s}\right] \\
&\qquad + C \int_1^{t/2} ds \int dx'' \frac{1 + |x|^m}{(1 + t - s)^k (1 + |x''|^j)} \frac{1 + |x''|^{j'}}{1 + |x'|^{m'}} \\
&\quad \le C \frac{(1 + |x|^m)}{(1 + |x'|^{m'})(1 + t)^{k-1}},
\end{aligned}$$

provided $j > j' + d$, and using Lemma 2.

Consider the second integral using (22):

$$\begin{aligned}
&\int_0^1 ds \int p_s(x, x''; y) \big|\big(\partial_y \bar{L}(x'')\big) \bar{p}_{t-s}(x'', x'; y)\big|\, dx'' \\
&\quad \le C \int_0^1 \int s^{-d/2} \exp\big(-(x - x'')^2/(2cs)\big) \frac{(1 + |x''|^m)}{(1 + t)^k (1 + |x'|^{m'})}\, ds\, dx'' \\
&\quad \le C \int_0^1 \frac{(1 + |x''|^m)}{t^k (1 + |x'|^{m'})}\, ds = C \frac{(1 + |x''|^m)}{t^k (1 + |x'|^{m'})},
\end{aligned}$$

where we have used Lemma 1.

Another part of this term is estimated by

$$\int_1^{t/2} ds \int p_s(x, x''; y) \big| \big(\partial_y \bar{L}(x'')\big) \bar{p}_{t-s}(x'', x'; y) \big| \, dx''$$

$$\le C \int_1^{t/2} \int \frac{(1+|x|^m)}{(1+s)^k(1+|x''|^{m''})} \frac{(1+|x''|^m)}{(1+t/2)^k(1+|x'|^{m'})} \, ds \, dx''$$

$$\le C \int_1^{t/2} \frac{(1+|x|^m)}{(1+s)^k(1+|x'|^{m'})(1+t/2)^k} \, ds \le C \frac{(1+|x|^m)}{(1+t/2)^k(1+|x'|^{m'})}.$$

Finally, we use (19) and (22), yielding

$$\int_{t/2}^{\infty} ds \int p_\infty(x'', y) \big| \big(\partial_y \bar{L}(x'')\big) \bar{p}_s(x'', x'; y) \big| \, dx''$$

$$\le \int_{t/2}^{\infty} ds \int \frac{C}{1+|x''|^{m'}} \frac{C(1+|x''|^m)}{(1+s)^k(1+|x'|^j)} \, dx''$$

$$\le \frac{C}{(1+t)^{k-1}(1+|x'|^j)},$$

provided $m' > m + d$. $\square$

Note that f_t^1 could be replaced by a finite sum $\sum_{k=1}^m \frac{\partial \bar{L}^k}{\partial y} \bar{p}^k(x, x', y)$, where each pair $(\bar{L}^k, \bar{p}^k)$ satisfies the assumptions stated for $(\bar{L}, \bar{p})$. Let us describe the notation for the induction step j:

(IS_j) $\bar{p}^i := p^{(i)}, i = 0, 1, \dots, j-1$; $\bar{L}^i = C_j^i \partial^{j-i} L / \partial y^{j-i}$ (tensor), and $\bar{p}^j = p^{(j)}$ where $p^{(j)}$ is defined by formula (34) below; assumption $(H^{2+\alpha, j})$ is required for jth step.

4.4. *Derivatives with respect to* $x, x' : \partial_x, \partial_{x'} q_t$. Again, during the first reading, $\bar{p} = p$ and $\bar{q} = q$. We first study the derivatives $\partial_{x'} \bar{q}_t$ and $\partial_{x'} \bar{q}_\infty$ under assumptions $(H^{1+\alpha,1})$ and $(H^{2+\alpha,1})$ (the latter is used only once). The existence of both derivatives under $(H^{1+\alpha,1})$ follow from the explicit expressions for $\bar{q}_t$ and $\bar{q}_\infty$. Indeed, we clearly have that

$$\frac{\partial \bar{q}_t(x, x', y)}{\partial x'} = \int_0^t ds \int dx'' p_{t-s}(x, x'', y) \frac{\partial f_s^1(x'', x', y)}{\partial x'}.$$

We will show that this function converges, as $t \to \infty$, to

$$\int_0^{\infty} ds \int dx'' p_\infty(x'', y) \frac{\partial f_s^1(x'', x', y)}{\partial x'}.$$

Moreover, the last quantity is the derivative of q_∞ with respect to x', and the same is true for second order derivatives with respect to x'.

THEOREM 7. *Assume that conditions* (H_b), (H_a) *and* $(H^{1+\alpha,1})$ *hold. Then for all* $k, m' \geq 0$, *there exist real numbers* C, m *such that for all* y, x, x' *and all* $t \geq 1$,

$$\left|\partial_{x'}\bar{q}_t(x, x'; y) - \partial_{x'}\bar{q}_\infty(x', y)\right| \leq C\frac{1+|x|^m}{(1+|x'|^{m'})(1+t)^k}$$

and

$$|\partial_{x'}\bar{q}_\infty(x', y)| \leq \frac{C}{(1+|x'|^{m'})}.$$

PROOF. All terms can be estimated exactly as in the proof of Theorem 6, except

$$\int_0^1 ds \int \left(p_{t-s}(x, x''; y) - p_\infty(x'', y)\right) f_s^1(x'', x'; y)\, dx'',$$

which now becomes

$$\int_0^1 ds \int \left(p_{t-s}(x, x''; y) - p_\infty(x'', y)\right)\partial_{x'} f_s^1(x'', x'; y)\, dx''.$$

We have (see Proposition 3)

$$\begin{aligned}
&\int_0^1 ds \int |p_{t-s}(x, x''; y) - p_\infty(x'', y)||\partial_{x'} f_s^1(x'', x'; y)|\, dx'' \\
&\quad \leq C\int_0^1 s^{-1/2}\, ds \int \frac{1+|x|^m}{(1+t)^k(1+|x''|^j)} s^{-d/2} \exp\left(-\frac{c|x'-x''|^2}{s}\right) dx'' \\
&\quad \leq C\frac{1+|x|^m}{(1+t)^k(1+|x'|^j)}.
\end{aligned}$$

The second inequality follows from the estimates

$$\begin{aligned}
|\partial_x\bar{q}_\infty(x, y)| &= \left|\partial_x \int_0^\infty ds \int dx'\, p_\infty(x', y) f_s^1(x', x; y)\right| \\
&= \left|\partial_x \int_0^\infty ds \int dx'\, L_y^* p_\infty(x', y)\bar{p}_s(x', x; y)\right| \\
&\leq \left|\partial_x \int_0^1 ds \int dx'\, L_y^* p_\infty(x', y)\bar{p}_s(x', x; y)\right| \\
&\quad + \left|\partial_x \int_1^\infty ds \int dx'\, L_y^* p_\infty(x', y)[\bar{p}_s(x', x; y) - \bar{p}_\infty(x, y)]\right| \\
&\leq C\int_0^1 ds \int dx' \frac{1}{1+|x'|^j}\frac{1}{s^{d/2}} \exp\left(-c\frac{|x'-x|^2}{s}\right)
\end{aligned}$$

$$+C\int_1^\infty ds\int dx'\frac{1}{1+|x'|^j}\frac{1+|x'|^m}{(1+|x|^{m'})(1+s)^k}$$

$$\le \frac{C}{1+|x|^{m'}}$$

since j can be chosen arbitrarily large, in particular, greater than $m'+d$ and, as usual, $k>1$. $\square$

We now study the derivative $\partial_x\bar{q}_t$. Arguing as above, we can show that for fixed t and any $\delta\le t/2$,

$$\begin{aligned}\partial_x\bar{q}_t(x,x';y) &= \int_0^\delta ds\int_{\mathbb{R}^d}\frac{\partial p_s}{\partial x}(x,x'';y)\frac{\partial\bar{L}}{\partial y}(x'',y)\bar{p}_{t-s}(x'',x';y)\,dx'' \\ &\quad+\int_\delta^{t-\delta} ds\int_{\mathbb{R}^d}\frac{\partial p_s}{\partial x}(x,x'';y)\frac{\partial\bar{L}}{\partial y}(x'',y)\bar{p}_{t-s}(x'',x';y)\,dx'' \\ &\quad+\int_{t-\delta}^t ds\int_{\mathbb{R}^d}\frac{\partial\bar{L}^*}{\partial y}(x'',y)\frac{\partial p_s}{\partial x}(x,x'';y)\bar{p}_{t-s}(x'',x';y)\,dx''.\end{aligned}\tag{31}$$

We can prove the following theorem.

THEOREM 8. *Assume that conditions* (H_b), (H_a) *and* $(H^{1+\alpha,1})$ *hold. Then for all* $k,m'\ge 0$, *there exist real numbers* C,m *such that for all* $y\in\mathbb{R}^\ell$, $x,x'\in\mathbb{R}^d$ *and all* $t\ge 1$,

$$|\partial_x\bar{q}_t(x,x';y)|\le C\frac{1+|x|^m}{(1+|x'|^{m'})(1+t)^k}.$$

PROOF. We choose $\delta=1/2$ in formula (31). We can estimate the right-hand side of (31), using Propositions 2 and 3. We obtain that the absolute value of $\partial_x\bar{q}_t(x,x';y)$ is bounded by a constant times

$$\begin{aligned}&\int_0^{1/2} s^{-(d+1)/2}\,ds\int_{\mathbb{R}^d}\exp\left[-c\frac{|x-x''|^2}{s}\right]\frac{1+|x''|^m}{(1+t-s)^k(1+|x'|^j)}\,dx'' \\ &\quad+\int_{1/2}^{t-1/2} ds\int_{\mathbb{R}^d}\frac{1+|x|^m}{(1+s)^k(1+|x''|^j)}\frac{1+|x''|^{j'}}{(1+t-s)^{k'}(1+|x'|^{m'})}\,dx'' \\ &\quad+\int_{t-1/2}^t ds\int_{\mathbb{R}^d}\frac{1+|x|^m}{(1+s)^k(1+|x''|^j)}(t-s)^{-d/2}\exp\left[-c\frac{|x'-x''|^2}{t-s}\right]dx'',\end{aligned}$$

from which the result follows, using the assumption $j>j'+d$, Lemma 2 and the freedom of choice of the various parameters. $\square$

For small t we can establish the following estimate. It is the only place where we will use $(H^{2+\alpha,1})$ in this section.

PROPOSITION 4. *Under condition* $(H^{1+\alpha,1})$, *for any* $0 < t \le 1$,

$$|\partial_x \bar{q}_t(x, x'; y)| \le \frac{C}{t^{(d+1)/2}} \exp\left(-c\frac{|x-x'|^2}{t}\right). \tag{32}$$

If $(H^{2+\alpha,1})$ *is valid then also*

$$|\partial_{x'} \bar{q}_t(x, x'; y)| \le \frac{C}{t^{(d+1)/2}} \exp\left(-c\frac{|x-x'|^2}{t}\right). \tag{33}$$

PROOF. We have, with some bounded coefficients $\alpha(\cdot)$ and $\beta(\cdot)$,

$$\begin{aligned}
&\partial_{x'}\bar{q}_t(x, x'; y)\\
&\quad = \int_0^t ds \int dx''\, p_s(x, x''; y)\partial_{x'} f^1_{t-s}(x'', x'; y)\\
&\quad = \int_0^{t/2} ds \int dx''\, \big[\alpha_1(x'', y) p_s(x, x''; y)\\
&\qquad\qquad + \beta_1(x'', y)\partial_{x''} p_s(x, x''; y)\big]\partial^2_{x'x''}\bar{p}_{t-s}(x'', x'; y)\\
&\qquad + \int_{t/2}^t ds \int dx''\, \big[\alpha_2(x'', y) p_s(x, x''; y)\\
&\qquad\qquad + \beta_2(x'', y)\partial_{x''} p_s(x, x''; y)\\
&\qquad\qquad + \gamma(x'', y)\partial^2_{x''} p_s(x, x''; y)\big]\partial_{x'}\bar{p}_{t-s}(x'', x'; y)
\end{aligned}$$

with some $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$. It follows from the above estimates that

$$\begin{aligned}
&\left|\int_0^t ds \int dx''\, p_s(x, x''; y)\partial_{x'} f^1_{t-s}(x'', x'; y)\right|\\
&\quad \le C\int_0^{t/2} ds \int dx''\, \frac{1}{(t-s)^{(d+2)/2}} \exp\left(-c\frac{|x-x''|^2}{t-s}\right)\\
&\qquad\qquad \times \frac{1}{s^{(d+1)/2}} \exp\left(-c\frac{|x'-x''|^2}{s}\right)\\
&\qquad + C\int_{t/2}^t ds \int dx''\, \frac{1}{(t-s)^{(d+1)/2}} \exp\left(-c\frac{|x-x''|^2}{t-s}\right)\\
&\qquad\qquad \times \frac{1}{s^{(d+2)/2}} \exp\left(-c\frac{|x'-x''|^2}{s}\right)\\
&\quad \le \frac{C}{t^{d/2}} \exp\left(-c\frac{|x-x'|^2}{t}\right)\left[\int_0^{t/2} \frac{ds}{(t-s)\sqrt{s}} + \int_{t/2}^t \frac{ds}{\sqrt{t-s}s}\right]\\
&\quad \le \frac{C}{t^{(d+1)/2}} \exp\left(-c\frac{|x-x'|^2}{t}\right)
\end{aligned}$$

which gives assertion (33). Inequality (32) follows similarly under $(H^{1+\alpha,1})$ from the representation

$$\begin{aligned}
\partial_x \bar{q}_t(x,x',y) &= \int_0^t ds \int dx'' \big(\partial_x p_s(x,x'',y)\big) f^1_{t-s}(x'',x',y) \\
&= \int_0^{t/2} ds \int dx'' \big(\partial_x p_s(x,x'',y)\big)\big(\partial_y \bar{L}(x'')\big) \bar{p}_{t-s}(x'',x',y) \\
&\quad - \int_{t/2}^t ds \int dx'' \big[\partial^2_{xx''} p_{t-s}(x,x'',y)\big]\big[\beta(x'',y)\partial_{x''}\bar{p}_s(x,x'',y)\big] \\
&\quad + \text{lower order terms},
\end{aligned}$$

which again leads to estimations

$$\begin{aligned}
&\int_0^{t/2} ds \int dx'' \frac{1}{(t-s)^{(d+2)/2}} \exp\biggl(-c\frac{|x-x''|^2}{t-s}\biggr) \\
&\qquad\qquad \times \frac{1}{s^{(d+1)/2}} \exp\biggl(-c\frac{|x'-x''|^2}{s}\biggr) \\
&\quad + C\int_{t/2}^t ds \int dx'' \frac{1}{(t-s)^{(d+1)/2}} \exp\biggl(-c\frac{|x-x''|^2}{t-s}\biggr) \\
&\qquad\qquad \times \frac{1}{s^{(d+2)/2}} \exp\biggl(-c\frac{|x'-x''|^2}{s}\biggr) \\
&\quad \le \frac{C}{t^{(d+1)/2}} \exp\biggl(-c\frac{|x-x'|^2}{t}\biggr).
\end{aligned}$$

Inequality (32) and Proposition 4 are thus proved. $\square$

The same note as at the end of Section 4.3 applies here.

4.5. *Higher order derivatives* $\partial_y^j p_t$, $j \ge 2$. We now want to study the tensor valued function of t, x, x', y:

$$p_t^{(j)}(x,x';y) := \partial_y^{j-1} q_t(x,x';y) = \partial_y^j p_t(x,x';y)$$

for $j \ge 2$ by induction.

THEOREM 9. *Assume* $(H^{2+\alpha,j})$, *the existence of* (*matrix or tensor*)*-functions*

$$p_t^{(i)}(x,x';y) := \partial_y^i p_t(x,x';y)$$

for any $0 \le i \le j-1$, *the existence of the limits* $p_\infty^{(i)}(x',y)$ *as* $t \to \infty$ *and the* *following inequalities: for any m', k there exist C, m such that*

$$\left|\partial_{x'} p_t^{(i)}(x, x'; y)\right| \le C t^{-(d+1)/2} \exp\bigl(-c|x - x'|^2/t\bigr), \qquad 0 < t \le 1,$$

$$\left|\partial_{x'} p_t^{(i)}(x, x'; y) - \partial_{x'} p_\infty^{(i)}(x', y)\right| \le C \frac{1 + |x|^m}{(1 + |x'|^{m'})(1 + t)^k}, \qquad t > 1,$$

$$\left|\partial_{x'} p_\infty^{(i)}(x', y)\right| \le \frac{C}{(1 + |x'|^{m'})}.$$

Then there exists $\partial_y^j p_t(x, x'; y) =: p_t^{(j)}(x, x'; y)$, *there exists a limit*

$$\lim_{t\to\infty} p_t^{(j)}(x, x'; y) =: p_\infty^{(j)}(x', y)$$

and

$$\left|p_t^{(j)}(x, x'; y) - p_\infty^{(j)}(x', y)\right| \le C \frac{1 + |x|^m}{(1 + |x'|^{m'})(1 + t)^k}, \qquad t > 1,$$

$$\left|\partial_{x'} p_t^{(j)}(x, x'; y)\right| \le C t^{-(d+1)/2} \exp(-c|x - x'|^2/t), \qquad 0 < t \le 1,$$

$$\left|\partial_{x'} p_t^{(j)}(x, x'; y) - \partial_{x'} p_\infty^{(j)}(x', y)\right| \le C \frac{1 + |x|^m}{(1 + |x'|^{m'})(1 + t)^k}, \qquad t > 1,$$

$$\left|\partial_{x'} p_\infty^{(j)}(x')\right| \le \frac{C}{(1 + |x'|^{m'})(1 + t)^k}.$$

If, moreover, for all $i \le j - 1$,

$$\left|\partial_x p_t^{(i)}(x, x'; y)\right| \le C t^{-(d+1)/2} \exp\bigl(-c|x - x'|^2/t\bigr), \qquad 0 < t \le 1,$$

and

$$\left|\partial_x p_t^{(i)}(x, x'; y)\right| \le \frac{C}{(1 + |x'|^{m'})(1 + t)^k}, \qquad t \ge 1,$$

then the following inequalities hold true:

$$\left|\partial_x p_t^{(j)}(x, x'; y)\right| \le C t^{-(d+1)/2} \exp\bigl(-c|x - x'|^2/t\bigr), \qquad 0 < t \le 1,$$

and

$$\left|\partial_x p_t^{(j)}(x, x'; y)\right| \le \frac{C}{(1 + |x'|^{m'})(1 + t)^k}.$$

PROOF. We define $p_t^{(j)}$ by the formula [cf. with (27)]

$$
\begin{aligned}
&p_t^{(j)}(x, x'; y) \\
(34) \qquad &:= \int_0^t ds \int p_{t-s}(x'', x'; y) \sum_{0 \le i < j} p^{(i)}(x, x''; y) L_{i,j}(x'', y)\, dx''
\end{aligned}
$$

with

$$L_{i,j} = C_j^i \frac{\partial^{j-i} L}{\partial y^{j-i}}$$

and repeat the arguments from Section 4.3—the first induction step—to show that indeed

$$p_t^{(j)}(x, x'; y) = \partial_y p_t^{(j-1)}(x, x'; y).$$

The other assertions of Theorem 9 follow immediately from results proved above for $\bar{p}_t$ and $\bar{q}_t$ with $\bar{p}_t = p_t^{(j)}$, $\bar{L}^i = L_{i,j}$ [cf. induction assumptions (IS_j)]. □

Note that some assertions of the last theorem hold true under a weaker assumption $(H^{1+\alpha,j})$, similar to the previous section.

5. PDE solution bounds.

5.1. *First derivative $q_t^f = \partial_y p_t^f$.* We now establish the differentiability of p_t^f with respect to y under less restrictive assumptions than needed for the transition density p_t. Firstly, we study the case $f(x,t) \equiv g(x)$, that is, when the function f does not depend on t. Let $g(x)$, $x \in \mathbb{R}^d$, $g \in H^\alpha$,

$$|g(x)| \le C(1 + |x|^k).$$

The result of this subsection is the following theorem.

THEOREM 10. *Assume that the conditions (H_b), (H_a) and either $((H^{2+\alpha,1}) + (g \in C^\alpha))$ or $((H^{\alpha,1}) + (g \in C^{2+\alpha}))$ hold. Then for each $t > 0$, $x, x' \in \mathbb{R}^d$, $y \in \mathbb{R}^\ell$, $p_t(x, g; y)$ is differentiable in the variable y, and the gradient $\partial_y p_t(x, g; y)$ is given by the formula*

$$
\begin{aligned}
&\partial_y p_t(x, g, y) = q_t(x, g; y) \\
(35) \qquad &:= \int_0^t ds \int_{\mathbb{R}^d} p_s(x, x''; y) \frac{\partial L}{\partial y}(x'', y) p_{t-s}(x'', g; y)\, dx''.
\end{aligned}
$$

The function q_t is bounded and continuous in y for any $t > 0$, x, x'.

PROOF. Under assumption $(\mathrm{H}^{2+\alpha,1}) + (g \in C^{\alpha})$ all assertions follow from considerations in Theorem 5 with additional integration with respect to x' variable. Under assumption $(H^{\alpha,1}) + (g \in C^{2+\alpha})$ the same calculus becomes even easier due to the bound $\|p_t(\cdot, g, y)\|_{2+\alpha} \le C\|g\|_{2+\alpha}$ where C does not depends on t ([8], combination of Theorems 3.6 and 3.5). $\square$

5.2. *Behavior of the first derivative q_t^f as $t \to \infty$.* Recall the notation

$$f_t^1(x, g; y) = \frac{\partial L}{\partial y} p_t(x, g; y).$$

We need to show that the following integral converges:

$$\int_0^\infty dt \int_{\mathbb{R}^d} [q_t(x, g; y) - q_\infty(g, y)]\, dx',$$

where

$$q_\infty(g, y) = \lim_{t\to\infty} q_t(x, g; y) = \partial_y p_\infty(g, y),$$

and we have the representation

$$q_\infty(g, y) := \int_0^t ds \int dx''\, p_\infty(x, x''; y) f_s^1(x'', g; y).$$

More precisely, we now show the following theorem.

THEOREM 11. *Assume that conditions $(H_b), (H_a)$ and either $((H^{2+\alpha,1}) + (g \in H^{\alpha}))$ or $((H^{\alpha,1}) + (g \in C^{2+\alpha}))$ hold. Then for each k, there exists $C, m \in \mathbb{R}$ such that for all $y \in \mathbb{R}^\ell$, $x \in \mathbb{R}^d$, $t \ge 1$,*

$$|q_t(x, g; y) - q_\infty(g; y)| \le C \frac{1 + |x|^m}{(1+t)^k},$$
$$|q_\infty(g, y)| \le C,$$

and moreover,

$$q_\infty(g, y) = \partial_y p_\infty(g, y).$$

The same assertion is valid for the second derivative, $\partial_y^2 p_t(x, g, y)$ under assumption either $((H^{1+\alpha,2}) + (g \in H^{\alpha}))$ or $((H^{\alpha,2}) + (g \in C^2))$. Moreover, for any k, there exists $C, m \in \mathbb{R}$ such that for all $y \in \mathbb{R}^\ell$, $x \in \mathbb{R}^d$, $t \ge 1$,

$$|\partial_x q_t(x, g; y)| \le C \frac{1 + |x|^m}{(1+t)^k}.$$

Finally, for any $t > 0$ and y, $p_t(\cdot, g, y) \in C^{2+\alpha}$.

PROOF. Similarly to the previous theorem, again under assumption $((H^{2+\alpha,1}) + (g \in C^{\alpha}))$, the proof follows from considerations in Theorem 5 with additional integration with respect to x' variable, while under assumption $((H^{\alpha,1}) + (g \in C^{2+\alpha}))$—from the same calculus due to the bound $\|p_t(\cdot, g, y)\|_{2+\alpha} \le C\|g\|_{2+\alpha}$. The assertion concerning $\partial_x p_t^{(j)}(x, g, y)$ follows from the inequalities of Theorem 8 after integration with respect to variable x'. The last assertion, $p_t(\cdot, g, y) \in C^{2+\alpha}$, is classical; see [8], Theorem 3.5 or [12], Chapter 4. $\square$

PROOF OF THEOREM 2. For the case $j > 0$, the proof follows from Theorems 10 and 11 and the linearity of the operator $g \to p_t(x, g, y)$ which implies the formula

$$\partial_y^j p_t(x, f, y) = \sum_{\ell=0}^{j} C_j^\ell (\partial_y^\ell p_t)(x, g, y)\big|_{g=\partial_y^{j-\ell} f}.$$

The case $j = 0$ is implied by Theorem 1, also for $j = 0$, after integration with respect to the variable x'. $\square$

REFERENCES

[1] BAKHVALOV, N. S. (1974). Averaged characteristics of bodies with periodic structure. *Soviet Phys. Dokl.* **19** 650–651. [Translated from *Dokl. Akad. Nauk SSSR* **218** (1974) 1046–1048.]

[2] BAKHVALOV, N. S. (1975). Averaging of partial differential equations with rapidly oscillating coefficients. *Sov. Math. Dokl.* **16** 351–355. [Translated from *Dokl. Akad. Nauk SSSR* **221** (1975) 516–519.]

[3] BENVENISTE, A., METIVIER, M. and PRIOURET, P. (1987). *Algorithmes adaptifs et approximations stochastiques. Théorie et applications à l'identification, au traitment du signal et à reconnaissance des formes.* Masson, Paris. [English translation (1990) *Adaptive Algorithms and Stochastic Approximations.* Springer, Berlin.]

[4] EIDELMAN, S. D. (1956). On the fundamental solutions of parabolic systems. *Soviet Math. Sbornik* **38** 51–92. [English translation *Amer. Math. Soc. Transl.* **41** (1964) 1–48.]

[5] EIDELMAN, S. D. (1964). *Parabolic Systems*. Nauka, Moscow (in Russian).

[6] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes. Characterization and Convergence.* Wiley, New York.

[7] FREIDLIN, M. I. and WENTZELL, S. D. (1984). *Random Perturbations of Dynamical Systems.* Springer, New York.

[8] FRIEDMAN, A. (1964) *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, NJ.

[9] GLYNN, P. W. and MEYN, S. P. (1996). A Liapounov bound for solutions of the Poisson equation. *Ann. Probab.* **24** 916–931.

[10] KIFER, YU. (2001). Averaging and climate models. In *Stochastic Climate Models* (P. Imkeller and J. von Stroch, eds.). Birkhäuser, Basel.

[11] KRYLOV, N. V. (1980). *Controlled Diffusion Processes*. Springer, Berlin.

[12] LADYZENSKAJA, O., SOLONNIKOV, V. and URAL'CEVA, N. (1968). *Linear and Quasilinear Equations of Parabolic Type.* Amer. Math. Soc., Providence, RI.

[13] PAPANICOLAOU, G. C., STROOCK, D. W. and VARADHAN, S. R. S (1976). Martingale approach to some limit theorems. In *Statistical Mechanics and Dynamical Systems* (D. Ruelle ed.) 1–12. Duke Univ. Math. Series **3**.

[14] PARDOUX, É. and VERETENNIKOV, A. YU. (2000). On regularity of an invariant density of a Markov chain in a parameter. *Russian Math. Dokl.* **370** 158–160.

[15] PARDOUX, É. and VERETENNIKOV, A. YU. (2001). On Poisson equation and diffusion approximation 1. *Ann. Probab.* **29** 1061–1085.

[16] VERETENNIKOV, A. YU. (1997). On polynomial mixing bounds for stochastic differential equations. *Stoch. Processes Appl.* **70** 115–127.

[17] VERETENNIKOV, A. YU. (1999). On polynomial mixing and convergence rate for stochastic difference and differential equations. *Teor. Veroyatnost. i Primenen.* **44** 312–327 (in Russian). [English translation *Theory Probab. Appl.* **44** (2000) 361–374.]

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