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ON EXTREMAL DISTRIBUTIONS AND SHARP L_p-BOUNDS FOR SUMS OF MULTILINEAR FORMS

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In this paper we present a study of the problem of approximating the expectations of functions of statistics in independent and dependent random variables in terms of the expectations of functions of the component random variables. We present results providing sharp analogues of the Burkholder–Rosenthal inequalities and related estimates for the expectations of functions of sums of dependent nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments as well as for sums of multilinear forms. Among others, we obtain the following sharp inequalities: $E(\sum_{k=1}^n X_k)^t \le 2\max(\sum_{k=1}^n EX_k^t, (\sum_{k=1}^n a_k)^t)$ for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $EX_k^t < \infty$, $k = 1, \dots, n$, $1 < t < 2$; $E(\sum_{k=1}^n X_k)^t \le E\theta^t(1)\max(\sum_{k=1}^n b_k, (\sum_{k=1}^n a_k^s)^{t/s})$ for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k^s \mid X_1, \dots, X_{k-1}) \le a_k^s$, $E(X_k^t \mid X_1, \dots, X_{k-1}) \le b_k$, $k = 1, \dots, n$, $1 < t < 2$, $0 < s \le t-1$ or $t \ge 2$, $0 < s \le 1$, where $\theta(1)$ is a Poisson random variable with parameter 1. As applications, new decoupling inequalities for sums of multilinear forms are presented and sharp Khintchine–Marcinkiewicz–Zygmund inequalities for generalized moving averages are obtained. The results can also be used in the study of a wide class of nonlinear statistics connected to problems of long-range dependence and in an econometric setup, in particular, in stabilization policy problems and in the study of properties of moving average and autocorrelation processes. The results are based on the iteration of a series of key lemmas that capture the essential extremal properties of the moments of the statistics involved.

1. Introduction. Let $\{X_k\}$ be a sequence of dependent random variables (r.v.'s). A question of key interest is the approximation of $EH(X_1, \dots, X_n)$, where $H : \mathbf{R}^n \to \mathbf{R}$ is a continuous function. In this paper we present a series of results that provide sharp bounds for the above expectations for a wide class of r.v.'s and functions H, including the cases when $H(x_1, \dots, x_n) = |\sum_{i=1}^n x_i|^t$ and when $H(x_1, \dots, x_n)$ is of the type $|\sum_{i=1}^n x_i|^t \ln|\sum_{i=1}^n x_i|$ (important in the study of entropy conditions) and more general functions when the X_i's have two bounded conditional moments as well as the case of sums of multilinear forms.

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We begin by providing a survey of the known Burkholder–Rosenthal moment inequalities. Let $A(t)$ and $B(t)$ denote constants depending on t only and let L and L_i, $i = 1, 2$, denote absolute constants, not necessarily the same from one place to another. Rosenthal (1970) proved the following inequalities:

$$E\left(\sum_{k=1}^{n} X_k\right)^t \le A(t) \max\left(\sum_{k=1}^{n} EX_k^t, \left(\sum_{k=1}^{n} EX_k\right)^t\right) \tag{1.1}$$

for all independent nonnegative r.v.'s $X_1, \dots, X_n$ with finite tth moment, $t \ge 1$;

$$E\left|\sum_{k=1}^{n} X_k\right|^t \le B(t) \max\left(\sum_{k=1}^{n} E|X_k|^t, \left(\sum_{k=1}^{n} EX_k^2\right)^{t/2}\right) \tag{1.2}$$

for all independent zero-mean r.v.'s $X_1, \dots, X_n$ with finite tth moment, $t \ge 2$. Burkholder (1973) showed that similar inequalities hold for martingales.

Using Sazonov's (1974) results, one can obtain (1.2) with the constant $B(t) = L^t 2^{t^2/4}$, while from the estimates obtained by Nagaev and Pinelis (1977) and Pinelis (1980) it follows that one can take $B(t) = L^t t^t$. Concerning refinements and extensions of relations (1.1) and (1.2) and related inequalities, see also Hitczenko (1990), Nagaev (1990), Wang (1991a, b), Hitczenko (1994a, b, c), Pinelis (1994), Peshkir and Shiryaev (1995) and Nagaev (1998).

Denote by $A^*(t)$ and $B^*(t)$ the best constants in Rosenthal's inequalities for power functions (1.1) and (1.2). Johnson, Schechtman and Zinn (1985) showed that $A^*(t)$ and $B^*(t)$ satisfy the inequalities $L_1^t(t/\ln t)^t \le A^*(t), B^*(t) \le L_2^t(t/\ln t)^t$ [see also Talagrand (1989), Kwapień and Szulga (1991) and Latała (1997)]. Ibragimov and Sharakhmetov (1998) proved that $A^*(t) = 2$, $1 < t < 2$, $A^*(t) = E\theta^t(1)$, $t \ge 2$, and $B^*(2m) = E(\theta(1) - 1)^{2m}$, $m \in \mathbf{N}$, where $\theta(1)$ is a Poisson r.v. with parameter 1 [see also Ibragimov (1997)]. Figiel, Hitczenko, Johnson, Schechtman and Zinn (1997) and Ibragimov and Sharakhmetov (1995, 1997) independently obtained that the best constant $B^*_{\mathrm{sym}}(t)$ in inequality (1.2) in the case of symmetric r.v.'s is given by $B^*_{\mathrm{sym}}(t) = 1 + E|Z|^t$, $2 < t < 4$, $B^*_{\mathrm{sym}}(t) = E|\theta_1(0.5) - \theta_2(0.5)|^t$, $t \ge 4$, where Z is the standard normal r.v. and $\theta_1(0.5)$ and $\theta_2(0.5)$ are independent Poisson r.v.'s with parameter 0.5. In the case of even moments, one can also derive the explicit expression for the constant $B^*_{\mathrm{sym}}(2m)$, $m \in \mathbf{N}$, from the results obtained by Pinelis and Utev (1984). Ibragimov and Sharakhmetov (1995, 1997) found the exact asymptotics of the constant $B^*_{\mathrm{sym}}(t)$ as $t \to \infty$. The proof of the expressions for $B^*_{\mathrm{sym}}(t)$ in Ibragimov and Sharakhmetov (1997) significantly uses ideas and results of Utev (1985), who obtained exact upper and lower bounds for $E|\sum_{k=1}^n X_k|^t$, where $X_1, \dots, X_n$ are independent symmetric r.v.'s with finite tth moment, $t \ge 4$, in terms of $\sum_{k=1}^n E|X_k|^t$ and $(\sum_{k=1}^n EX_k^2)^{t/2}$. In particular, the fact that the maximum of his upper bounds is attained in the case when $\sum_{k=1}^n E|X_k|^t = (\sum_{k=1}^n EX_k^2)^{t/2}$ implies the expression for $B^*_{\mathrm{sym}}(t)$ in the case $t \ge 4$ [see Ibragimov and Sharakhmetov (1995, 1997)].

Recently, Klass and Nowicki (1997), Ibragimov and Sharakhmetov (1998, 1999, 2000) [see also Ibragimov (1997)] and Giné, Latała and Zinn (2000) obtained analogues of Rosenthal's inequalities (1.1) and (1.2) for U-statistics with nonnegative and degenerate kernels. Ibragimov and Sharakhmetov (2000) also showed the significance of each term in the analogues of Rosenthal's bounds for U-statistics of arbitrary order. Ibragimov (1997) proved that the best constants in the analogues of those inequalities grow no slower than $L^t(t/\ln t)^{mt}$, where m is the order of a U-statistic. Giné, Latała and Zinn (2000) proved the analogues of Rosenthal's inequalities for the tth moment of U-statistics of order m with the constants $L_m^t(t/\ln t)^{mt}$, where L_m is a constant depending only on m, and obtained Bernstein-type exponential inequalities for U-statistics. Ibragimov, Cecen and Sharakhmetov (2001) found the best constants in analogues of Rosenthal's ineualities for bilinear forms in the case of the fixed number of r.v.'s.

Let $(\Omega, \Im, P)$ be a probability space with a nondecreasing sequence of σ-algebras $\Im_0 = (\varnothing, \Omega) \subseteq \Im_1 \cdots \subseteq \Im_n \cdots \subseteq \Im$. Pinelis (1980) generalized the results obtained in Nagaev and Pinelis (1977) in the case of martingales having proved the following Burkholder–Rosenthal-type inequality for arbitrary martingale difference (Y_n) with $E|Y_n|^t < \infty$ and $E(Y_n^2|\Im_{n-1}) \le b_n^2 \in \mathbf{R}$ a.s., $n \ge 1$, $t > 2$:

$$E\left|\sum_{k=1}^n Y_k\right|^t \le L^t t^t \max\left(\sum_{k=1}^n E|Y_k|^t, \left(\sum_{k=1}^n b_k^2\right)^{t/2}\right). \tag{1.3}$$

Hitczenko (1990) showed that the following inequalities hold for arbitrary $(\Im_n)$-adapted sequences (X_n) of nonnegative r.v.'s with $EX_n^t < \infty$ and arbitrary martingale differences Y_n with respect to $(\Im_n)$ with $E|Y_n|^t < \infty$:

$$\begin{aligned} &E\left(\sum_{k=1}^n X_k\right)^t \\ &\quad \le (Lt/\ln t)^t \max\left(\sum_{k=1}^n EX_k^t, E\left(\sum_{k=1}^n E(X_k/\Im_{k-1})\right)^t\right), \qquad t > 1, \end{aligned} \tag{1.4}$$

$$\begin{aligned} &E\left|\sum_{k=1}^n Y_k\right|^t \\ &\quad \le (Lt/\ln t)^t \max\left(\sum_{k=1}^n E|Y_k|^t, E\left(\sum_{k=1}^n E(Y_k^2/\Im_{k-1})\right)^{t/2}\right), \qquad t > 2 \end{aligned} \tag{1.5}$$

[see also Hitczenko (1994a, b, c) and Pinelis (1994)]. Several authors [e.g., McConnell and Taqqu (1986), Krakowiak and Szulga (1986), Kwapień and Woyczynski (1992), Szulga (1998) and the references therein] have focused on the study of properties of multilinear forms and their applications. There has also been increasing interest in the study of sums of multilinear forms, partly because these types of r.v.'s represent a special but important case of infinite-degree U-statistics and are related to the study of long-range dependence [cf. Heilig and Nolan (2001)] and moving average processes [e.g., Ho and Hsing (1997)]. In particular, according to Ho and Hsing (1997), for a general class of measurable functions $K:\mathbf{R}\to\mathbf{R}$, stochastic Taylor expansions for functionals $\sum_{n=1}^{N}(K(\sum_{i=1}^{\infty}c_iX_{n-i}) - EK(\sum_{i=1}^{\infty}c_iX_{n-i}))$ of infinite moving averages in independent r.v.'s X_i important in the study of long-range dependence have the form of sums of multilinear forms. We stress here that the increase in technical difficulty in going from problems involving multilinear forms to the case of sums of multilinear forms is justified since, by the use of the above-cited Taylor expansions, results for sums of multilinear forms allow one to study properties of nonlinear statistics.

In the present paper, we determine the exact (sharp) analogues of Burkholder–Rosenthal-type inequalities (1.4) and (1.5) for expectations of functions (generalized moments) of sums of dependent nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments and for sums of multilinear forms. The results are applied to obtain the best constants in Burkholder–Rosenthal inequalities for those objects. The obtained exact inequalities extend the extremal results obtained in Utev (1985), Figiel, Hitczenko, Johnson, Schechtman and Zinn (1997), Ibragimov (1997) and Ibragimov and Sharakhmetov (1995, 1997) and are, to our knowledge, the first attempt to apply methods that were used to investigate extremal problems in moment inequalities for sums of independent r.v.'s, in the case of martingales, sums of dependent nonnegative r.v.'s and sums of multilinear forms.

The paper is organized as follows. Section 2 contains an in-depth study of extremal problems for expectations of statistics $H(X_1,\dots,X_n)$, where $H:\mathbf{R}^n\to\mathbf{R}$ belongs to a class of functions satisfying certain general convexity conditions and the X_i's are independent (dependent) r.v.'s having bounded (conditional) expectations for two different functions, that is, $Eh_k(X_k)\le h_k(a_k)$ and $Ef_k(X_k)\le b_k$ for functions $h_k:\mathbf{R}\to\mathbf{R}$ and $f_k:\mathbf{R}\to\mathbf{R}$, $k=1,\dots,n$. Section 3 applies the results of Section 2 to the special case of sums of r.v.'s and sums of multilinear forms. In particular:

1. Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 provide sharp Burkholder–Rosenthal-type bounds for the case $H(x_1,\dots,x_n)=\phi(\sum_{k=1}^{n}x_i)$.
2. Corollaries 3.3 and 3.4 provide new decoupling inequalities comparing the moments of sums of dependent nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments to the moments of sums of independent r.v.'s.
3. Theorems 3.3 and 3.4 provide the extremal distributions for the moments of sums of multilinear forms in r.v.'s with bounded moments.
4. Theorems 3.5–3.8 provide sharp Burkholder–Rosenthal-type bounds for sums of multilinear forms.

5. Theorem 3.9 provides new decoupling inequalities for sums of multilinear forms.
6. Theorem 3.10 provides exact Khintchine–Marcinkiewicz–Zygmund inequalities for generalized moving averages.

Finally, the Appendix presents the auxiliary results on the extremal properties of moments of sums of independent r.v.'s with fixed sum of tails of distributions used in the proofs.

2. Extrema of some linear functionals on probability distributions of nonnegative and symmetric random variables. This section contains several general results which will be used in Section 3. The reader is advised to first study the statements of the theorems in Section 3 (which are the main results of the paper) and motivate the results of this section.

Let $\mathbf{R}_+ = [0, \infty)$. Denote by J the class of continuous increasing functions $f : \mathbf{R}_+ \to \mathbf{R}_+$, and for $f \in J$ denote by Q_f the class of functions $h \in J$ such that $h(0) = 0$, the function fh^{-1} is convex on $\mathbf{R}_+$ and the function f/h is increasing on $\mathbf{R}_+ \setminus \{0\}$. Examples of functions $f \in J$ and $h \in Q_f$ are given by $f(x) = x^t$ and $h(x) = x^s$, $0 < s < t$.

Let $H : \mathbf{R}_+^n \to \mathbf{R}$ be a continuous function and let $f_i \in J$, $h_i \in Q_{f_i}$, $i = 1, \dots, n$. Set $G_i = f_i/h_i$, $i = 1, \dots, n$. Let $X_1, \dots, X_n$ be independent nonnegative r.v.'s with $Ef_i(X_i) < \infty$, $i = 1, \dots, n$. In what follows, write $(X, n) = (X_1, \dots, X_n)$. Fix values $a_i, b_i > 0$, $f_i(a_i) \le b_i$, $i = 1, \dots, n$. Set

$$M_1^{\text{non}}(n, f, b) = \{(X, n) : Ef_i(X_i) = b_i, i = 1, \dots, n\},$$

$$M_2^{\text{non}}(n, f, b) = \{(X, n) : Ef_i(X_i) \le b_i, i = 1, \dots, n\},$$

$$M_3^{\text{non}}(n, h, f, a, b) = \{(X, n) : Eh_i(X_i) = h_i(a_i), Ef_i(X_i) = b_i, i = 1, \dots, n\},$$

$$M_4^{\text{non}}(n, h, f, a, b) = \{(X, n) : Eh_i(X_i) \le h_i(a_i), Ef_i(X_i) \le b_i, i = 1, \dots, n\}.$$

Let $V_i(h_i, f_i, a_i, b_i)$, $i = 1, \dots, n$, be independent r.v.'s with distributions

$$P\big(V_i(h_i, f_i, a_i, b_i) = G_i^{-1}\big(b_i/h_i(a_i)\big)\big) = \frac{h_i(a_i)}{h_i(G_i^{-1}(b_i/h_i(a_i)))},$$

$$P\big(V_i(h_i, f_i, a_i, b_i) = 0\big) = 1 - \frac{h_i(a_i)}{h_i(G_i^{-1}(b_i/h_i(a_i)))}.$$

LEMMA 2.1. *If the functions $\widetilde{H}_{1k}(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n) = H(z_1, \dots, z_{k-1}, f_k^{-1}(v), z_{k+1}, \dots, z_n)$, $k = 1, \dots, n$, are concave in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$, then*

$$\max_{(X,n) \in M_1^{\text{non}}(n, f, b)} EH(X_1, \dots, X_n) = H\big(f_1^{-1}(b_1), \dots, f_n^{-1}(b_n)\big). \tag{2.1}$$

If, in addition to that, the function $H : \mathbf{R}_+^n \to \mathbf{R}$ *is nondecreasing in each argument, then*

$$\max_{(X,n)\in M_2^{\text{non}}(n,f,b)} EH(X_1,\dots,X_n) = H\big(f_1^{-1}(b_1),\dots,f_n^{-1}(b_n)\big). \tag{2.2}$$

PROOF. Let X_k be a nonnegative r.v. with $Ef_k(X_k) = b_k$ and let $z_1,\dots,z_{k-1}$, $z_{k+1},\dots,z_n \ge 0$. If the function $\tilde{H}_{1k}(z_1,\dots,z_{k-1},v,z_{k+1},\dots,z_n)$ is concave in $v > 0$, then from Jensen's inequality it follows that

$$\begin{aligned}
&EH(z_1,\dots,z_{k-1},X_k,z_{k+1},\dots,z_n)\\
&\quad= E\tilde{H}_{1k}\big(z_1,\dots,z_{k-1},f_k(X_k),z_{k+1},\dots,z_n\big)\\
&\quad\le \tilde{H}_{1k}(z_1,\dots,z_{k-1},b_k,z_{k+1},\dots,z_n)\\
&\quad= H\big(z_1,\dots,z_{k-1},f_k^{-1}(b_k),z_{k+1},\dots,z_n\big).
\end{aligned}$$

This implies that

$$EH(X_1,\dots,X_n) \le EH\big(f_1^{-1}(b_1),\dots,f_n^{-1}(b_n)\big) \tag{2.3}$$

for all $(X,n) \in M_1^{\text{non}}(n,f,b)$. Similarly, if, in addition to concavity, the function $H : \mathbf{R}_+^n \to \mathbf{R}$ is nondecreasing in each argument, then we get in a similar way that (2.3) holds for all $(X,n) \in M_2^{\text{non}}(n,f,b)$. Sharpness of bounds (2.3) follows from the choice $X_i = f_i^{-1}(b_i)$, $i = 1,\dots,n$. Therefore, (2.1) and (2.2) hold. $\square$

LEMMA 2.2. *If* $f_i(0) = 0$, $i = 1,\dots,n$, *and the functions* $\tilde{H}_{2k}(z_1,\dots,z_{k-1}, v, z_{k+1},\dots,z_n) = (H(z_1,\dots,z_{k-1},G_k^{-1}(v),z_{k+1},\dots,z_n) - H(z_1,\dots,z_{k-1},0, z_{k+1},\dots,z_n))/h_k(G_k^{-1}(v))$, $k = 1,\dots,n$, *are concave in* $v > 0$ *for* $z_1,\dots,z_{k-1}$, $z_{k+1},\dots,z_n \ge 0$, *then*

$$\begin{aligned}
&\max_{(X,n)\in M_3^{\text{non}}(n,h,f,a,b)} EH(X_1,\dots,X_n)\\
&\qquad= EH\big(V_1(h_1,f_1,a_1,b_1),\dots,V_n(h_n,f_n,a_n,b_n)\big).
\end{aligned}\tag{2.4}$$

If, in addition to that, the functions $\tilde{H}_{2k}(z_1,\dots,z_{k-1},v,z_{k+1},\dots,z_n)$ *are nonnegative and nondecreasing in* $v > 0$ *for* $z_1,\dots,z_{k-1},z_{k+1},\dots,z_n \ge 0$, *then*

$$\begin{aligned}
&\max_{(X,n)\in M_4^{\text{non}}(n,h,f,a,b)} EH(X_1,\dots,X_n)\\
&\qquad= EH\big(V_1(h_1,f_1,a_1,b_1),\dots,V_n(h_n,f_n,a_n,b_n)\big).
\end{aligned}\tag{2.5}$$

PROOF. Let X_k be a nonnegative r.v. with $Eh_k(X_k) = h_k(a_k)$, $Ef_k(X_k) = b_k$ and let $z_1,\dots,z_{k-1},z_{k+1},\dots,z_n \ge 0$. Suppose that the function $\tilde{H}_{2k}(z_1,\dots,z_{k-1}, v, z_{k+1},\dots,z_n)$ is concave in $v > 0$. Show that

$$\begin{aligned}
&EH(z_1,\dots,z_{k-1},X_k,z_{k+1},\dots,z_n)\\
&\qquad\le EH\big(z_1,\dots,z_{k-1},V_k(h_k,f_k,a_k,b_k),z_{k+1},\dots,z_n\big).
\end{aligned}\tag{2.6}$$

Since the functions h_k, f_k and H are continuous, it suffices to consider only discrete r.v.'s X_k. Let Y_k be the r.v. with distribution $P(Y_k = x) = (h_k(x)/h_k(a_k))P(X_k = x), x \geq 0$. We have that $\sum_x P(Y_k = x) = 1$ and $EG_k(Y_k) = b_k/h_k(a_k)$. Moreover, since $P(X_k = 0) = 1 - h_k(a_k)E(1/h_k(Y_k))$, we have

$$\begin{aligned}
&EH(z_1, \dots, z_{k-1}, X_k, z_{k+1}, \dots, z_n) \\
&\quad = H(z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n) \\
&\qquad + h_k(a_k)E\big((H(z_1, \dots, z_{k-1}, Y_k, z_{k+1}, \dots, z_n) \\
&\qquad\qquad - H(z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n))/h_k(Y_k)\big).
\end{aligned} \tag{2.7}$$

Since the function $\tilde{H}_{2k}$ is concave in $v > 0$, from Jensen's inequality we get

$$\begin{aligned}
&E\big((H(z_1, \dots, z_{k-1}, Y_k, z_{k+1}, \dots, z_n) \\
&\quad - H(z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n))/h_k(Y_k)\big) \\
&\quad = E\tilde{H}_{2k}\big(z_1, \dots, z_{k-1}, G_k(Y_k), z_{k+1}, \dots, z_n\big) \\
&\quad \leq E\tilde{H}_{2k}\big(z_1, \dots, z_{k-1}, b_k/h_k(a_k), z_{k+1}, \dots, z_n\big) \\
&\quad = \big(H\big(z_1, \dots, z_{k-1}, G_k^{-1}\big(b_k/h_k(a_k)\big), z_{k+1}, \dots, z_n\big) \\
&\qquad - H(z_1, \dots, z_{k-1}, 0, z_{k+1}, \dots, z_n)\big)/h_k\big(G_k^{-1}\big(b_k/h_k(a_k)\big)\big).
\end{aligned} \tag{2.8}$$

Equations (2.7) and (2.8) imply (2.6). Using (2.6), we get

$$EH(X_1, \dots, X_n) \leq EH\big(V_1(h_1, f_1, a_1, b_1), \dots, V_n(h_n, f_n, a_n, b_n)\big) \tag{2.9}$$

for all $(X, n) \in M_3^{\text{non}}(h, f, a, b)$. Sharpness of (2.9) follows from the fact that $V_1(h_1, f_1, a_1, b_1), \dots, V_n(h_n, f_n, a_n, b_n) \in M_3^{\text{non}}(h, f, a, b)$. Therefore, (2.4) holds.

Suppose now that the functions $\tilde{H}_{2k}(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$ are concave and nondecreasing in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$ and nonnegative. To prove (2.5), it suffices to show that

$$\begin{aligned}
&EH\big(z_1, \dots, z_{k-1}, V_k(h_k, f_k, a_k, b_k), z_{k+1}, \dots, z_n\big) \\
&\quad \leq EH\big(z_1, \dots, z_{k-1}, V_k(h_k, f_k, a'_k, b'_k), z_{k+1}, \dots, z_n\big)
\end{aligned} \tag{2.10}$$

if $a_k \leq a'_k$, $b_k < b'_k$. Inequality (2.10) is equivalent to the inequality

$$\begin{aligned}
&\tilde{H}_{2k}(z_1, \dots, z_{k-1}, y_k, z_{k+1}, \dots, z_n)r_k \\
&\quad \leq \tilde{H}_{2k}(z_1, \dots, z_{k-1}, y'_k, z_{k+1}, \dots, z_n),
\end{aligned} \tag{2.11}$$

where

$$r_k = h_k(a_k)/h_k(a'_k), \qquad y_k = b_k/h_k(a_k), \qquad y'_k = b'_k/h_k(a'_k).$$

It is evident that $0 < r_k \le 1$, $y_k r_k < y'_k$, $y_k, y'_k > 0$. Let $r_k < 1$. Set $x_k = (y'_k - y_k r_k)/(1 - r_k)$. From the concavity and nonnegativity of $\tilde{H}_{2k}$, it follows that

$$\begin{aligned}
&\tilde{H}_{2k}(z_1, \dots, z_{k-1}, y_k, z_{k+1}, \dots, z_n) r_k \\
&\quad \le \tilde{H}_{2k}(z_1, \dots, z_{k-1}, y_k, z_{k+1}, \dots, z_n) r_k \\
&\qquad + \tilde{H}_{2k}(z_1, \dots, z_{k-1}, x_k, z_{k+1}, \dots, z_n)(1 - r_k) \\
&\quad \le \tilde{H}_{2k}\big(z_1, \dots, z_{k-1}, y_k r_k + x_k(1 - r_k), z_{k+1}, \dots, z_n\big) \\
&\quad = \tilde{H}_{2k}(z_1, \dots, z_{k-1}, y'_k, z_{k+1}, \dots, z_n).
\end{aligned}$$

Since the functions $\tilde{H}_{2k}(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$ are nondecreasing in $v > 0$, we get (2.11) for $r_k = 1$. $\square$

Throughout the paper, $\varepsilon, \varepsilon_1, \dots, \varepsilon_n$ denote independent symmetric Bernoulli r.v.'s.

According to Lemmas 2.3 and 2.4, the class of functions H, f_k and h_k, $k = 1, \dots, n$, such that $\tilde{H}_{2k}$, $k = 1, \dots, n$, satisfy the assumptions of Lemma 2.2, is quite wide and includes, in particular, powers of sums of multilinear forms with nonnegative kernels and moments of sums of symmetrized multilinear forms.

LEMMA 2.3. *Let* $c_{i_1,\dots,i_l} \ge 0$, $1 \le i_1 < \cdots < i_l \le n$, $l = 0, \dots, m$. *If* $H(x_1, \dots, x_n) = (\sum_{l=0}^m \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l} x_{i_1} \cdots x_{i_l})^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $1 < t < 2$, $0 < s_k \le t - 1$ *or* $t \ge 2$, $0 < s_k \le 1$, $k = 1, \dots, n$, *then the functions* $\widetilde{H}_{2k}(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$, $k = 1, \dots, n$, *defined in Lemma* 2.2, *are nonnegative*, *concave and nondecreasing in* $v > 0$ *for* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$. *If* $H(x_1, \dots, x_n) = -(\sum_{l=0}^m \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l} x_{i_1} \cdots x_{i_l})^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $1 < t < 2$, $1 \le s_k < t$ *or* $t \ge 2$, $t - 1 \le s_k < t$, $k = 1, \dots, n$, *then the functions* $\widetilde{H}_{2k}$ *are concave in* $v > 0$ *for* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$.

PROOF. It suffices to show that the function $g(v) = v^{-s/(t-s)}((v^{1/(t-s)} + z)^t - z^t)$ is nondecreasing and concave in $v > 0$ for $z > 0$ if $1 < t < 2, 0 < s \le t-1$ or $t \ge 2, 0 < s \le 1$, and is convex in $v > 0$ for $z > 0$ if $1 < t < 2, 1 \le s < t$ or $t \ge 2$, $t - 1 \le s < t$. It is easy to see that $d^2 g(v)/dv^2 = (t/(t-s)^2) v^{-s/(t-s)-2} z^t (s((1 + u)^{t-1} - 1) - (t-1)u(1+u)^{t-2})$, where $u = v^{1/(t-s)}/z$. Since

$$1 \le (1+u)^{2-t} \le 1 + (2-t)u \tag{2.12}$$

for $1 < t < 2$, $u > 0$, and

$$1 + (2-t)u \le (1+u)^{2-t} \le 1 \tag{2.13}$$

for $t \ge 2$, $u > 0$, we have that $(t-1)((1+u)^{t-1} - 1) \le (t-1)u(1+u)^{t-2} \le (1+u)^{t-1} - 1$ for $1 < t < 2$, $u > 0$, and $(1+u)^{t-1} - 1 \le (t-1)u(1+u)^{t-2} \le$

$(t-1)((1+u)^{t-1}-1)$ for $t \geq 2$, $u > 0$. Therefore, $d^2 g(v)/dv^2 \leq 0$ if $1 < t < 2$, $0 < s \leq t-1$ or $t \geq 2$, $0 < s \leq 1$, and $d^2 g(v)/dv^2 \geq 0$ if $1 < t < 2$, $1 \leq s < t$ or $t \geq 2$, $t-1 \leq s < t$. The fact that $g(v)$ is nondecreasing in $v > 0$ for $z > 0$ if $1 < t < 2$, $0 < s \leq t-1$ or $t \geq 2$, $0 < s \leq 1$ follows from the concavity of $g(v)$ and the evident relation $\lim_{v \to +\infty} g(v) = +\infty$. Indeed, suppose that the function $g(v)$ is not nondecreasing, that is, there exist numbers $v_1 < v_2$ such that $g(v_1) > g(v_2)$. Since $\lim_{v \to +\infty} g(v) = +\infty$, one can find a number $v_3 > v_2$ such that $g(v_3) > g(v_2)$. This implies that

$$\begin{aligned}
\frac{v_3 - v_2}{v_3 - v_1} g(v_1) + \frac{v_2 - v_1}{v_3 - v_1} g(v_3) &> \frac{v_3 - v_2}{v_3 - v_1} g(v_2) + \frac{v_2 - v_1}{v_3 - v_1} g(v_2) \\
&= g(v_2) = g\left(\frac{v_3 - v_2}{v_3 - v_1} v_1 + \frac{v_2 - v_1}{v_3 - v_1} v_3\right),
\end{aligned}$$

which contradicts the fact that $g(v)$ is concave. $\square$

LEMMA 2.4. *Let* $c_{i_1,\dots,i_l} \in \mathbf{R}$, $1 \leq i_1 < \cdots < i_l \leq n$, $l = 0, \dots, m$. *If* $H(x_1, \dots, x_n) = E|\sum_{l=0}^m \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1,\dots,i_l} x_{i_1} \varepsilon_{i_1} \cdots x_{i_l} \varepsilon_{i_l}|^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $2 < t < 4$, $0 < s_k \leq t-2$ *or* $t \geq 4$, $0 < s_k \leq 2$, $k = 1, \dots, n$, *then the functions* $\widetilde{H}_{2k}(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$, $k = 1, \dots, n$, *defined in Lemma* 2.2, *are nonnegative*, *concave and nondecreasing in* $v > 0$ *for* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$. *If* $H(x_1, \dots, x_n) = -E|\sum_{l=0}^m \sum_{1 \leq i_1 < \cdots < i_l \leq n} c_{i_1,\dots,i_l} x_{i_1} \varepsilon_{i_1} \cdots x_{i_l} \varepsilon_{i_l}|^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $3 \leq t < 4$, $2 \leq s_k < t$ *or* $t \geq 4$, $t-2 \leq s_k < t$, $k = 1, \dots, n$, *then the functions* $\widetilde{H}_{2k}$ *are concave in* $v > 0$ *for* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$.

PROOF. To prove the concavity properties, it suffices to show that the function $g(v) = v^{-s/(t-s)}(E|v^{1/(t-s)}\varepsilon + z|^t - z^t)$ is nondecreasing and concave in $v > 0$ for $z > 0$ if $2 < t < 4$, $0 < s_k \leq t-2$ or $t \geq 4$, $0 < s_k \leq 2$, and convex in $v > 0$ for $z > 0$ if $3 \leq t < 4$, $2 \leq s_k < t$ or $t \geq 4$, $t-2 \leq s_k < t$. It is not difficult to check that $d^2 g(v)/dv^2 = (t/(t-s)^2) v^{-s/(t-s)-2} z^t (s(E|1+u\varepsilon|^{t-2}(1+u\varepsilon) - 1) - (t-1)uE|1+u\varepsilon|^{t-2}\varepsilon)$, where, as in the proof of Lemma 2.3, $u = v^{1/(t-s)}/z$. Since [see the proof of Lemmas 1 and 3 in Ibragimov and Sharakhmetov (1997)]

$$g_1(t,u) = (t-2)E|1+u\varepsilon|^{t-2} - uE|1+u\varepsilon|^{t-2}\varepsilon - (t-2) \leq 0 \tag{2.14}$$

for $t \in (2,4)$, $u > 0$ and

$$g_2(t,u) = (t-3)E|1+u\varepsilon|^{t-2}(1+u\varepsilon) - (t-1)E|1+u\varepsilon|^{t-2} + 2 \leq 0 \tag{2.15}$$

for $t \in [3,4)$, $u > 0$, we have that $(t-1)uE|1+u\varepsilon|^{t-2}\varepsilon \geq (t-2)(E|1+u\varepsilon|^{t-2}(1+u\varepsilon) - 1)$ for $t \in (2,4)$, $u > 0$, and $(t-1)uE|1+u\varepsilon|^{t-2}\varepsilon \leq 2(E|1+u\varepsilon|^{t-2}(1+u\varepsilon) - 1)$ for $t \in [3,4)$, $u > 0$, that is, $d^2 g(v)/dv^2 \leq 0$ if $2 < t < 4$, $0 < s \leq t-2$, and $d^2 g(v)/dv^2 \geq 0$ if $3 \leq t < 4$, $2 \leq s < t$. From the proof of Lemmas 3.2 and 3.3 in Utev (1985), it follows that

$$g_1(t,u) \geq 0, \qquad g_2(t,u) \geq 0 \tag{2.16}$$

for $t \geq 4$, $u > 0$. This implies that $d^2g(v)/dv^2 \leq 0$ if $t \geq 4$, $0 < s \leq 2$, and $d^2g(v)/dv^2 \geq 0$ if $t \geq 4$, $t-2 \leq s < t$. The fact that the function $g(v)$ is nondecreasing in $v > 0$ for $2 < t < 4$, $0 < s \leq t-2$ and $t \geq 4$, $0 < s \leq 2$ follows from the relation $\lim_{v\to+\infty} g(v) = +\infty$ and the concavity of $g(v)$. $\square$

LEMMA 2.5. *Let* $l \geq 1$, $H^j : \mathbf{R}_+^n \to \mathbf{R}$, $j = 1, \dots, l$, *be continuous functions*, $\lim_{u\to+\infty} G_k^{-1}(u) = v_k$ (v_k *can be infinite*), $k = 1, \dots, n$, *and let* $\lim_{v\to v_k} H^j(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)/f_k(v) = c_k^j \in \mathbf{R}$, $j = 1, \dots, l, k = 1, \dots, n$, *for all* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$. *If the functions*

$$\begin{aligned}&\widetilde{H}_{3k}^j(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)\\&\qquad = H^j\bigl(z_1, \dots, z_{k-1}, h_k^{-1}(v), z_{k+1}, \dots, z_n\bigr) - c_k^j f_k\bigl(h_k^{-1}(v)\bigr),\end{aligned}$$

$j = 1, \dots, l, k = 1, \dots, n$, *are concave in* $v > 0$ *for* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$, *then*

$$\begin{aligned}&\sup_{(X,n)\in M_3^{\mathrm{non}}(n,h,f,a,b)} E\Biggl(\sum_{j=1}^l H^j(X_1, \dots, X_n)\Biggr)\\&\qquad = \sum_{j=1}^l \sum_{i=1}^n c_i^j\bigl(b_i - f_i(a_i)\bigr) + \sum_{j=1}^l H^j(a_1, \dots, a_n).\end{aligned} \tag{2.17}$$

If, in addition to that, $c_k^j \geq 0$, $j = 1, \dots, l$, $k = 1, \dots, n$, *and the functions* $\widetilde{H}_{3k}^j$ *are nondecreasing in* $v > 0$ *for* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$, $k = 1, \dots, n$, *then*

$$\begin{aligned}&\sup_{(X,n)\in M_4^{\mathrm{non}}(n,h,f,a,b)} E\Biggl(\sum_{j=1}^l H^j(X_1, \dots, X_n)\Biggr)\\&\qquad = \sum_{j=1}^l \sum_{i=1}^n c_i^j\bigl(b_i - f_i(a_i)\bigr) + \sum_{j=1}^l H^j(a_1, \dots, a_n).\end{aligned} \tag{2.18}$$

PROOF. Suppose that the functions $\widetilde{H}_{3k}^j(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$ are concave in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$. Then we have that, if $X_k \geq 0$, $Eh_k(X_k) = h_k(a_k)$, $Ef_k(X_k) = b_k$, then, by Lemma 2.1, $E(\sum_{j=1}^l H^j(z_1, \dots, z_{k-1}, X_k, z_{k+1}, \dots, z_n) - c_k^j f_k(X_k)) \leq \sum_{j=1}^l (H^j(z_1, \dots, z_{k-1}, a_k, z_{k+1}, \dots, z_n) - c_k^j f_k(a_k))$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$. This implies that $E\sum_{j=1}^l H^j(z_1, \dots, z_{k-1}, X_k, z_{k+1}, \dots, z_n) \leq \sum_{j=1}^l c_k^j(b_k - f_k(a_k)) + \sum_{j=1}^l H^j(z_1, \dots, z_{k-1}, a_k, z_{k+1}, \dots, z_n)$ for all $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$, $k = 1, \dots, n$. Using induction, we get, therefore, that

$$E\sum_{j=1}^l H^j(X_1, \dots, X_n) \leq \sum_{j=1}^l \sum_{i=1}^n c_i^j\bigl(b_i - f_i(a_i)\bigr) + \sum_{j=1}^l H^j(a_1, \dots, a_n) \tag{2.19}$$

for all $(X,n) \in M_3^{\mathrm{non}}(n,h,f,a,b)$. Similarly, if the functions $\widetilde{H}_{3k}^j$ are concave and nondecreasing in $v > 0$ for $z_1,\dots,z_{k-1},z_{k+1},\dots,z_n \geq 0$, and $c_k^j \geq 0$, $k = 1,\dots,n$, then (2.19) holds for all $(X,n) \in M_4^{\mathrm{non}}(n,h,f,a,b)$. To finish the proof of (2.17) and (2.18), it suffices to bring an example of a sequence of r.v.'s $X_{km} \geq 0$ with $Eh_k(X_{km}) = h_k(a_k)$, $Ef_k(X_{km}) = b_k$, such that $EH^j(z_1,\dots,z_{k-1},X_{km},z_{k+1},\dots,z_n) \to c_k^j(b_k - f_k(a_k)) + H^j(z_1,\dots,z_{k-1},a_k, z_{k+1},\dots,z_n)$ as $m \to \infty$ for all $j = 1,\dots,l$ and all $z_1,\dots,z_{k-1},z_{k+1},\dots,z_n \geq 0$. If $f_k(a_k) = b_k$, then it suffices to take $X_{km} = a_k$. Let $f_k(a_k) < b_k$. Set $\delta_m = \frac{1}{m}$, $P(X_{km} = a_k) = 1 - \delta_m$, $P(X_{km} = b_{km}) = \delta_m^*$, $P(X_{km} = 0) = \delta_m - \delta_m^*$, where

$$\delta_m^* = \frac{h_k(a_k)\delta_m}{h_k(b_{km})}, \qquad b_{km} = G_k^{-1}\left(\frac{b_k - f_k(a_k)(1-\delta_m)}{h_k(a_k)\delta_m}\right).$$

It is not difficult to see that $b_{km} \geq a_k$, $0 \leq \delta_m^* \leq \delta_m$, $\delta_m \to 0$, $b_{km} \to v_k$, $f_k(b_{km})\delta_m^* = G_k(b_{km})h_k(a_k)\delta_m = b_k - f_k(a_k)(1-\delta_m) \to b_k - f_k(a_k)$ as $m \to \infty$. We have that, for all $j = 1,\dots,l$,

$$\begin{aligned} &EH^j(z_1,\dots,z_{k-1},X_{km},z_{k+1},\dots,z_n) \\ &\quad = H^j(z_1,\dots,z_{k-1},a_k,z_{k+1},\dots,z_n)(1-\delta_m) \\ &\qquad + H^j(z_1,\dots,z_{k-1},0,z_{k+1},\dots,z_n)(\delta_m - \delta_m^*) \\ &\qquad + \bigl(H^j(z_1,\dots,z_{k-1},b_{km},z_{k+1},\dots,z_n) - c_k^j f_k(b_{km})\bigr)\delta_m^* \\ &\qquad + c_k^j f_k(b_{km})\delta_m^*. \end{aligned} \tag{2.20}$$

Since $(H^j(z_1,\dots,z_{k-1},b_{km},z_{k+1},\dots,z_n) - c_k^j f_k(b_{km}))\delta_m^* = (H^j(z_1,\dots,z_{k-1}, b_{km},z_{k+1},\dots,z_n)/f_k(b_{km}) - c_k^j) f_k(b_{km})\delta_m^* \to 0$ as $m \to \infty$, from (2.20) we get that $EH^j(z_1,\dots,z_{k-1},X_{km},z_{k+1},\dots,z_n) \to c_k^j(b_k - f_k(a_k)) + H^j(z_1,\dots,z_{k-1}, a_k,z_{k+1},\dots,z_n)$ as $m \to \infty$ for all $j = 1,\dots,l$ and all $z_1,\dots,z_{k-1},z_{k+1},\dots, z_n \geq 0$. $\square$

REMARK 2.1. The essence of Lemma 2.5 and its proof is that the extrema of the expectations of the statistics $H^j(X_1,\dots,X_n)$ over the classes $M_3^{\mathrm{non}}(n,h,f,a,b)$ and $M_4^{\mathrm{non}}(n,h,f,a,b)$ are attained simultaneously and the sequence of the extremal random vectors is the same for all those statistics. For example, by Lemmas 2.5 and 2.6, if $a,b > 0$, $a^t \leq b$, $z_{1i}, z_{2i} \geq 0$, $i = 1,\dots,l$, $1 < t < 2$, $1 \leq s < t$ or $t \geq 2$, $t-1 \leq s < t$, then the suprema of $E(\sum_{i=l}^{l}(z_{1i}X + z_{2i})^t)$ over all nonnegative r.v.'s X with $EX^s = a^s$, $EX^t = b$ and over all nonnegative r.v.'s X with $EX^s \leq a^s$, $EX^t \leq b$ are given by $\sum_{i=1}^{l} z_{1i}^t(b - a^t) + \sum_{i=1}^{l}(z_{1i}a + z_{2i})^t$. As we will see in the next section, the above fact is important in the problems of determining extrema of expectations of functions of sums of multilinear forms over classes of r.v.'s with fixed moment characteristics.

According to Lemmas 2.6 and 2.7, the assumptions of Lemma 2.5 are satisfied for powers of sums of nonnegative variables and for moments of linear combinations of independent symmetric Bernoulli r.v.'s. The notation in Lemmas 2.6 and 2.7 is the same as that in Lemma 2.5.

LEMMA 2.6. *Let $l = 1$. If $H^1(x_1, \dots, x_n) = -(\sum_{i=1}^n x_i)^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $1 < t < 2$, $0 < s_k \le t - 1$ or $t \ge 2$, $0 < s_k \le 1$, $k = 1, \dots, n$, then $v_k = +\infty$, $c_k^1 = -1$, $k = 1, \dots, n$, and the functions $\widetilde{H}_{3k}^1(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$, $k = 1, \dots, n$, are concave in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$. If $H^1(x_1, \dots, x_n) = (\sum_{i=1}^n x_i)^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $1 < t < 2$, $1 \le s_k < t$ or $t \ge 2$, $t - 1 \le s_k < t$, then $v_k = +\infty$, $c_k^1 = 1$, $k = 1, \dots, n$, and the functions $\widetilde{H}_{3k}^1$ are concave and nondecreasing in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$.*

PROOF. It is evident that $v_k = +\infty$, $k = 1, \dots, n$. The relations for c_k^1 follow from the fact that $\lim_{v \to +\infty} (v + z)^t / v^t = 1$ for $z \ge 0$. To complete the proof, it suffices to show that the function $g(v) = (v^{1/s} + z)^t - v^{t/s}$ is convex in $v > 0$ for $z > 0$ if $1 < t < 2$, $0 < s \le t - 1$ or $t \ge 2$, $0 < s \le 1$, and is concave and nondecreasing in $v > 0$ for $z > 0$ if $1 < t < 2$, $1 \le s < t$ or $t \ge 2$, $t - 1 \le s < t$. It is not difficult to see that $d^2 g(v)/dv^2 = t/s^2 v^{t/s-2}((1 + u)^{t-2}(t + u) - t - s((1 + u)^{t-1} - 1))$, where $u = z/v^{1/s}$. From (2.12) and (2.13), it follows that $(t - 1)((1 + u)^{t-1} - 1) \le (1 + u)^{t-2}(t + u) - t \le (1 + u)^{t-1} - 1$ for $1 < t < 2$, $u > 0$, and $(1 + u)^{t-1} - 1 \le (1 + u)^{t-2}(t + u) - t \le (t - 1)((1 + u)^{t-1} - 1)$ for $t \ge 2$, $u > 0$. The above inequalities imply that $d^2 g(v)/dv^2 \ge 0$ if $1 < t < 2$, $0 < s \le t - 1$ or $t \ge 2$, $0 < s \le 1$, and $d^2 g(v)/dv^2 \le 0$ if $1 < t < 2$, $1 \le s < t$ or $t \ge 2$, $t - 1 \le s < t$. The property that the function $g(v)$ is nondecreasing if $1 < t < 2$, $1 \le s < t$ or $t \ge 2$, $t - 1 \le s < t$ follows from its concavity and the fact that $\lim_{v \to +\infty} g(v) = +\infty$. $\square$

LEMMA 2.7. *Let $l = 1$. If $H^1(x_1, \dots, x_n) = -E|\sum_{i=1}^n x_i \varepsilon_i|^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $3 \le t < 4$, $0 < s_k \le t - 2$ or $t \ge 4$, $0 < s_k \le 2$, $k = 1, \dots, n$, then $v_k = +\infty$, $c_k^1 = -1$, $k = 1, \dots, n$, and the functions $\widetilde{H}_{3k}^1(z_1, \dots, z_{k-1}, v, z_{k+1}, \dots, z_n)$, $k = 1, \dots, n$, are concave in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$. If $H^1(x_1, \dots, x_n) = E|\sum_{i=1}^n x_i \varepsilon_i|^t$, $f_k(x) = x^t$, $h_k(x) = x^{s_k}$, $2 < t < 4$, $2 \le s_k < t$ or $t \ge 4$, $t - 2 \le s_k < t$, then $v_k = +\infty$, $c_k^1 = 1$, $k = 1, \dots, n$, and the functions $\widetilde{H}_{3k}^1$ are concave and nondecreasing in $v > 0$ for $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \ge 0$.*

PROOF. It is evident that $v_k = +\infty$, $k = 1, \dots, n$, and $\lim_{v \to +\infty} E|v\varepsilon + z|^t / v^t = 1$; that is, the relations for c_k^1 hold. To complete the proof, it suffices to show that the function $g(v) = E|v^{1/s}\varepsilon + z|^t - v^{t/s}$ is convex in $v > 0$ for $z > 0$ if $3 \le t < 4$, $0 < s \le t - 2$ or $t \ge 4$, $0 < s \le 2$, and is nondecreasing and concave in $v > 0$ for $z > 0$ if $2 < t < 4$, $2 \le s < t$ or $t \ge 4$, $t - 2 \le s < t$.

It is not difficult to see that $d^2g(v)/dv^2 = t/s^2v^{t/s-2}(E|1+u\varepsilon|^{t-2}(t+u\varepsilon)-t-s(E|1+u\varepsilon|^{t-2}(1+u\varepsilon)-1))$, where $u = z/v^{1/s}$. From (2.14)–(2.16), it follows that $E|1+u\varepsilon|^{t-2}(t+u\varepsilon)-t \le 2(E|1+u\varepsilon|^{t-2}(1+u\varepsilon)-1)$ for $t \in (2,4)$, $u>0$, $E|1+u\varepsilon|^{t-2}(t+u\varepsilon)-t \ge (t-2)(E|1+u\varepsilon|^{t-2}(1+u\varepsilon)-1)$ for $t \in [3,4)$, $u>0$, and $2(E|1+u\varepsilon|^{t-2}(1+u\varepsilon)-1) \le E|1+u\varepsilon|^{t-2}(t+u\varepsilon)-t \le (t-2)(E|1+u\varepsilon|^{t-2}(1+u\varepsilon)-1)$ for $t \ge 4, u>0$. This implies that $d^2g(v)/dv^2 \ge 0$ if $3 \le t < 4, 0 < s \le t-2$ or $t \ge 4$, $0 < s \le 2$, and $d^2g(v)/dv^2 \le 0$ if $2 < t < 4$, $2 \le s < t$ or $t \ge 4$, $t-2 \le s < t$. The property that the function $g(v)$ is nondecreasing in $v > 0$ for $2 < t < 4$, $2 \le s < t$ and $t \ge 4$, $t-2 \le s < t$ follows from the fact that $\lim_{v\to+\infty} g(v) = +\infty$ and the concavity of $g(v)$. $\square$

A sequence of r.v.'s (X_n) on a probability space $(\Omega, \Im, P)$ with a nondecreasing sequence of σ-algebras $\Im_0 = (\varnothing, \Omega) \subseteq \Im_1 \cdots \subseteq \Im_n \cdots \subseteq \Im$ is called $(\Im_n)$-adapted if X_n is $\Im_n$-measurable for $n \ge 1$. An $(\Im_n)$-adapted sequence (S_n) of integrable r.v.'s is called a martingale [with respect to $(\Im_n)$] if $E(S_n \mid \Im_{n-1}) = S_{n-1}$, $n \ge 1$. A sequence (X_n), where $X_n = S_n - S_{n-1}$, is called a martingale difference [of the martingale (S_n)]. A martingale difference (X_n) is conditionally symmetric if X_n and $-X_n$ have the same distribution on the σ-algebra $\Im_{n-1}$.

In what follows, the conditionally symmetric martingale difference properties of a sequence (X_n) are meant to be satisfied with respect to the σ-algebras $\Im_0 = (\varnothing, \Omega)$, $\Im_k = \sigma(X_1, \dots, X_k)$, $k \ge 1$.

Let $m_i \ge 1$, $i = 1, \dots, n$, and let $f_{ij}: \mathbf{R} \to \mathbf{R}$, $j = 1, \dots, m_i$, $i = 1, \dots, n$, $H: \mathbf{R}^n \to \mathbf{R}$, be arbitrary functions. Let $X_1, \dots, X_n$ be r.v.'s with $E|f_{ij}(X_i)| < \infty$, $j = 1, \dots, m_i$, $i = 1, \dots, n$. Fix values $c_{ij} \in \mathbf{R}$, $j = 1, \dots, m_i$, $i = 1, \dots, n$. Set $\overline{M}_1(n, f, c) = \{(X, n): E(f_{ij}(X_i) \mid X_1, \dots, X_{i-1}) = c_{ij}, j = 1, \dots, m_i, i = 1, \dots, n\}$, $\overline{M}_2(n, f, c) = \{(X, n): E(f_{ij}(X_i) \mid X_1, \dots, X_{i-1}) \le c_{ij}, j = 1, \dots, m_i, i = 1, \dots, n\}$. Denote by $\overline{M}_k^{\text{non}}(n, f, c)$, $k = 1, 2$, the subsets of $\overline{M}_k(n, f, c)$, $k = 1, 2$, respectively, consisting of nonnegative r.v.'s $X_1, \dots, X_n$, and by $\overline{M}_k^{\text{sym}}(n, f, c)$, $k = 1, 2$, the subsets of $\overline{M}_k(n, f, c)$, $k = 1, 2$, respectively, consisting of conditionally symmetric martingale differences $X_1, \dots, X_n$. Let $\overline{U}_{1k}(c_{k1}, \dots, c_{km_k})$, $k = 1, \dots, n$, be the sets of r.v.'s X_k such that $Ef_{kj}(X_k) = c_{kj}$, $j = 1, \dots, m_k$, $k = 1, \dots, n$, let $\overline{U}_{2k}(c_{k1}, \dots, c_{km_k})$, $k = 1, \dots, n$, be the sets of r.v.'s X_k such that $Ef_{kj}(X_k) \le c_{kj}$, $j = 1, \dots, m_k$, $k = 1, \dots, n$, and let $\overline{U}_{ik}^{\text{non}}(c_{k1}, \dots, c_{km_k})$ and $\overline{U}_{ik}^{\text{sym}}(c_{k1}, \dots, c_{km_k})$, $i = 1, 2$, $k = 1, \dots, n$, be the subsets of $\overline{U}_{ik}(c_{k1}, \dots, c_{km_k})$, $i = 1, 2$, $k = 1, \dots, n$, consisting of nonnegative and symmetric r.v.'s, respectively.

LEMMA 2.8. *Let $g_k^{\text{non}}, g_k^{\text{sym}}: \mathbf{R}^{m_k} \to \mathbf{R}$, $k = 1, \dots, n$, be some functions, let $Y_k^{\text{non}}(c_{k1}, \dots, c_{km_k})$ and $Y_k^{\text{sym}}(c_{k1}, \dots, c_{km_k})$, $k = 1, \dots, n$, be independent nonnegative and symmetric r.v.'s, respectively, with distributions depending on $c_{k1}, \dots, c_{km_k}$, $k = 1, \dots, n$, and let $i \in \{1, 2\}$. If $EH(z_1, \dots, z_{k-1}, X_k, z_{k+1}, \dots, z_n) \le g_k^{\text{non}}(c_{k1}, \dots, c_{km_k}) + EH(z_1, \dots, z_{k-1}, Y_k^{\text{non}}(c_{k1}, \dots, c_{km_k}), z_{k+1}, \dots, z_n)$*

for all $X_k \in \overline{U}_{ik}^{\text{non}}(c_{k1}, \dots, c_{km_k})$ *and all* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0$, $k = 1, \dots, n$, *then*

$$EH(X_1, \dots, X_n) \leq \sum_{k=1}^{n} g_k^{\text{non}}(c_{k1}, \dots, c_{km_k}) + EH\big(Y_1^{\text{non}}(c_{11}, \dots, c_{1m_1}), \dots, Y_n^{\text{non}}(c_{n1}, \dots, c_{nm_n})\big)$$

for all $(X, n) \in \overline{M}_i^{\text{non}}(n, f, c)$. *If* $EH(z_1, \dots, z_{k-1}, X_k, z_{k+1}, \dots, z_n) \leq g_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}) + EH(z_1, \dots, z_{k-1}, Y_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}), z_{k+1}, \dots, z_n)$ *for all* $X_k \in \overline{U}_{ik}^{\text{sym}}(c_{k1}, \dots, c_{km_k})$ *and all* $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \in \mathbf{R}$, $k = 1, \dots, n$, *then*

$$EH(X_1, \dots, X_n) \leq \sum_{k=1}^{n} g_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}) + EH\big(Y_1^{\text{sym}}(c_{11}, \dots, c_{1m_1}), \dots, Y_n^{\text{sym}}(c_{n1}, \dots, c_{nm_n})\big)$$

for all $(X, n) \in \overline{M}_i^{\text{sym}}(n, f, c)$.

PROOF. Let $i \in \{1, 2\}$. Suppose that $EH(z_1, \dots, z_{k-1}, X_k, z_{k+1}, \dots, z_n) = g_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}) + EH(z_1, \dots, z_{k-1}, Y_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}), z_{k+1}, \dots, z_n)$ for all $X_k \in \overline{U}_{ik}^{\text{sym}}(c_{k1}, \dots, c_{km_k})$ and all $z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n \in \mathbf{R}$, $k = 1, \dots, n$. Let $X_1, \dots, X_{k-1}$ be arbitrary symmetric r.v.'s and let $Y_k^{\text{sym}}(c_{k1}, \dots, c_{km_k})$, $Y_{k+1}^{\text{sym}}(c_{k+1,1}, \dots, c_{k+1,m_{k+1}}), \dots, Y_n^{\text{sym}}(c_{n1}, \dots, c_{nm_n})$ be independent symmetric r.v.'s independent of $X_1, \dots, X_{k-1}$. Then, for $k = n, n-1, \dots, 1$, we have that, for all symmetric r.v.'s X_k independent of $Y_{k+1}^{\text{sym}}, \dots, Y_n^{\text{sym}}$ and such that $E(f_{kj}(X_k) \mid X_1, \dots, X_{k-1}) = c_{kj}$, $j = 1, \dots, m_k$, if $i = 1$, and $E(f_{kj}(X_k) \mid X_1, \dots, X_{k-1}) \leq c_{kj}$, $j = 1, \dots, m_k$, if $i = 2$,

$$\begin{aligned}
&EH\big(X_1, \dots, X_{k-1}, X_k, Y_{k+1}^{\text{sym}}(c_{k+1,1}, \dots, c_{k+1,m_{k+1}}), \dots, Y_n^{\text{sym}}(c_{n1}, \dots, c_{nm_n})\big) \\
&\quad = E\big(E\big(H\big(X_1, \dots, X_{k-1}, X_k, Y_{k+1}^{\text{sym}}(c_{k+1,1}, \dots, c_{k+1,m_{k+1}}), \\
&\qquad\qquad \dots, Y_n^{\text{sym}}(c_{n1}, \dots, c_{nm_n})\big) \big| X_1, \dots, X_{k-1}\big)\big) \\
&\quad \leq g_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}) \\
&\qquad + E\big(E\big(H\big(X_1, \dots, X_{k-1}, Y_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}), \\
&\qquad\qquad Y_{k+1}^{\text{sym}}(c_{k+1,1}, \dots, c_{k+1,m_{k+1}}), \\
&\qquad\qquad \dots, Y_n^{\text{sym}}(c_{n1}, \dots, c_{nm_n})\big) \big| X_1, \dots, X_{k-1}\big)\big) \\
&\quad = g_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}) \\
&\qquad + EH\big(X_1, \dots, X_{k-1}, Y_k^{\text{sym}}(c_{k1}, \dots, c_{km_k}), \\
&\qquad\qquad Y_{k+1}^{\text{sym}}(c_{k+1,1}, \dots, c_{k+1,m_{k+1}}), \dots, Y_n^{\text{sym}}(c_{n1}, \dots, c_{nm_n})\big).
\end{aligned}$$

By induction, we obtain

$$EH(X_1,\dots,X_n) \le \sum_{k=1}^{n} g_k^{\mathrm{sym}}(c_{k1},\dots,c_{km_k}) \\ + EH\big(Y_1^{\mathrm{sym}}(c_{11},\dots,c_{1m_1}),\dots,Y_n^{\mathrm{sym}}(c_{n1},\dots,c_{nm_n})\big)$$

for all $(X,n) \in M_i^{\mathrm{sym}}(n,f,c)$. The rest of the lemma might be proven in a completely similar way. $\square$

3. Sharp moment inequalities for sums of dependent nonnegative random variables, conditionally symmetric martingale differences and multilinear forms. We begin by providing some notation and introducing classes of functions closely linked to the results in Section 2 that will be needed throughout this section. In what follows, Z denotes the standard normal r.v. and, for $d>0$, $\theta(d)$, $\theta_1(d)$ and $\theta_2(d)$ denote independent Poisson r.v.'s with parameter d. Denote by Φ the class of continuous functions $\phi:\mathbf{R}\to\mathbf{R}$ such that there exists a constant $C=C(\phi)$ for which

$$(3.1)\qquad |\phi(a_1+a_2)| \le C\big(1+|\phi(a_1)|\big)\big(1+|\phi(a_2)|\big), \qquad a_1,a_2\in\mathbf{R}.$$

The class Φ includes, for example, all even continuous functions $\phi:\mathbf{R}\to\mathbf{R}$ such that the function $|\phi(x)|$ is nondecreasing on $\mathbf{R}_+$ and the function $\ln|\phi(x)|/x$ is nonincreasing in $x>x_0\in\mathbf{R}_+$ [in other words, Φ includes, basically, all functions growing not faster than an exponent, and, in particular, it includes all powers $\phi(x)=|x|^t$, $t>0$].

Let $f\in J$ and $h\in Q_f$. Denote $G=f/h$. Let $a_i,b_i>0$, $f(a_i)\le b_i$, $i=1,\dots,n$.

Denote by $D^{(1)}$ the class of functions $f\in J$, $h\in Q_f$ and nonnegative nondecreasing convex functions $\phi\in\Phi$ such that $f(0)=0$ and the function $(\phi(G^{-1}(v)+z)-\phi(z))/h(G^{-1}(v))$ is concave and nondecreasing in $v>0$ for $z\ge 0$, and denote by $D^{(2)}$ the class of differentiable convex functions $f\in J$ with $f(0)=0$ and nondecreasing functions $\phi:\mathbf{R}_+\to\mathbf{R}_+$ such that the function $\phi(v+z)-f(v)$ is concave and nondecreasing in $v>0$ for $z\ge 0$. By Lemma 2.3, the class $D^{(1)}$ includes the functions $f(v)=\phi(v)=v^t$, $h(v)=v^s$, where $1<t<2$, $0<s\le t-1$ or $t\ge 2$, $0<s\le 1$. Let $D^{(3)}$ be the class of twice-differentiable functions $f\in J$ with $f(0)=0$ such that the function $f''(v)$ is nonnegative and nonincreasing on $\mathbf{R}_+$, $\lim_{v\to+\infty} f(v+z)/f(v)=1$ for all $z\ge 0$ and $\lim_{v\to+\infty} f(v)/v=+\infty$. It is not difficult to see that if $f\in D^{(3)}$ and $\phi=f$, then $f,\phi\in D^{(2)}$. Indeed, if the function $f''(v)$ is nonincreasing, then $f''(v+z)\le f''(v)$ for all $v,z\ge 0$, and, therefore, $f(v+z)-f(v)$ is concave in $v>0$ for $z\ge 0$. From Proposition 16.B.3 in Marshall and Olkin (1979), it follows that the convexity of f implies that the function $f(v+z)-f(v)$ is nondecreasing in $v>0$

for $z \ge 0$. Indeed, it suffices to consider $z > 0$; according to the proposition, the convexity of f implies that

$$\frac{f((1-\alpha)x+\alpha y)-f(x)}{\alpha} \le \frac{f(y)-f(\beta x+(1-\beta)y)}{\beta}$$

for all $x, y \in \mathbf{R}_+$ and all $\alpha, \beta \in (0,1)$. Taking, for $0 < x_1 < x_2$ and $z > 0$, $x = x_1$, $y = x_2 + z$, $\alpha = \beta = z/(x_2 - x_1 + z)$, we get $f(x_1 + z) - f(x_1) \le f(x_2 + z) - f(x_2)$, $0 < x_1 < x_2$, $z > 0$; that is, the function $f(v+z) - f(v)$ is nondecreasing in $v > 0$ for $z > 0$. The class $D^{(3)}$ includes all functions v^t, $1 < t < 2$. Moreover, it includes all the following modifications of power functions multiplied by the logarithm: $f_1(v) = v^t \ln v - av_0^2 + bv_0 - c$, $v \ge v_0$, $f_1(v) = a(v - v_0)^2 + b(v - v_0) - av_0^2 + bv_0$, $0 \le v < v_0$, where $1 < t < 2$, $v_0 \ge e^{(2t-2)/(2t-t^2)}$, $a = 0.5(t(t-1)v_0^{t-2} \ln v_0 + (2t-1)v_0^{t-2})$, $b = tv_0^{t-1} \ln v_0 + v_0^{t-1}$, $c = v_0^t \ln v_0$ [the function $f_1(v)$ and the function $f_2(v)$ below are defined differently for small and large values of v]. Indeed, we have that, for $v \ge v_0$, $f_1'(v) = tv^{t-1} \ln v + v^{t-1}$, $f_1''(v) = t(t-1)v^{t-2} \ln v + (2t-1)v^{t-2}$, $f_1'''(v) = v^{t-3}(t(t-1)(t-2)\ln v + 3t^2 - 6t + 2)$. Since $v_0 \ge e^{(2t-2)/(2t-t^2)}$, we obtain that $f_1'''(v) \le 0$, $f_1''(v) \ge 0$, $f_1'(v) \ge 0$ for $v \ge v_0$. For $0 < v < v_0$, $f_1'(v) = 2a(v - v_0) + b$, $f_1''(v) = 2a$ and, therefore, $f_1'(v), f_1''(v) \ge 0$, $0 < v < v_0$ (again, since $v_0 \ge e^{(2t-2)/(2t-t^2)}$) and $f_1''(v)$ is nonincreasing on $(0, v_0)$. Moreover, $f_1(0) = 0$ and the definitions of a, b and c assure smoothness of the function f_1: $f_1''(v_0-) = 2a = f_1''(v_0+)$, $f_1'(v_0-) = b = f_1'(v_0+)$, $f_1(v_0-) = bv_0 - av_0^2 = f_1(v_0+)$. Relations $\lim_{v\to+\infty} f_1(v+z)/f_1(v) = 1$ for all $z \ge 0$ and $\lim_{v\to+\infty} f_1(v)/v = +\infty$ are evident. In addition to that, $D^{(3)}$ includes the modification of the first power multiplied by the logarithm $f_2(v) = v \ln v + 0.5$, $v \ge 1$, $f_2(v) = 0.5v^2$, $0 \le v < 1$; and the function $f_3(v) = (v+1)\ln(v+1) - v$, $v \ge 0$. Indeed, we have that $f_2(0) = 0$, and $f_2'(v) = \ln v + 1$, $v \ge 1$, $f_2'(v) = v$, $0 < v \le 1$, $f_2''(v) = 1/v$, $v \ge 1$, $f_2''(v) = 1$, $0 < v \le 1$. Moreover, $f_2(v)$ is smooth in the sense that $f_2(1-) = 0.5 = f_2(1+)$, $f_2'(1-) = 1 = f_2'(1+)$, $f_2''(1-) = 1 = f_2''(1+)$. Therefore, $f_2(v)$ is nondecreasing and $f_2''(v)$ is nonincreasing and nonnegative. Similarly, $f_3(0) = 0$, and $f_3'(v) = \ln(v+1)$, $f_3''(v) = 1/(v+1)$, $v > 0$; that is, $f_3(v)$ is a nondecreasing function such that $f_3''(v)$ is nonincreasing and nonnegative. The relations $\lim_{v\to+\infty} f_i(v+z)/f_i(v) = 1$ for all $z \ge 0$ and $\lim_{v\to+\infty} f_i(v)/v = +\infty$, $i = 2, 3$, are obvious.

In the inequalities throughout the rest of the paper, the extremal cases of the estimates $+\infty \le +\infty$, $-\infty \le +\infty$ and $-\infty \le -\infty$ are considered to be valid inequalities; we, therefore, do not include assumptions on the finiteness of moments of the summand r.v.'s that ensure the finiteness of moments of sums of the r.v.'s into formulations of the results.

The following theorem gives the exact analogues of the Burkholder–Rosenthal inequalities for expectations of functions of sums of dependent nonnegative r.v.'s with bounded conditional moments.

THEOREM 3.1. *If $f, h, \phi \in D^{(1)}$, then the following exact inequality holds*:

$$E\phi\left(\sum_{i=1}^{n} X_i\right) \le E\phi\left[\theta(1)\max\left(f^{-1}\left(\sum_{i=1}^{n} b_i\right), h^{-1}\left(\sum_{i=1}^{n} h(a_i)\right)\right)\right] \tag{3.2}$$

for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(h(X_k) \mid X_1, \dots, X_{k-1}) \le h(a_k)$, $E(f(X_k) \mid X_1, \dots, X_{k-1}) \le b_k$, $k = 1, \dots, n$. If $f, \phi \in D^{(2)}$, then the following exact inequality holds:

$$E\phi\left(\sum_{i=1}^{n} X_i\right) \le \sum_{i=1}^{n} Ef(X_i) + \phi\left(\sum_{i=1}^{n} a_i\right) - f'(0+)\left(\sum_{i=1}^{n} a_i\right) \tag{3.3}$$

for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $k = 1, \dots, n$, where $f'(0+) = \lim_{x \to 0+} f(x)/x$. If $f \in D^{(3)}$ and, in addition to that, $f'(0+) = 0$, then the following exact inequality holds:

$$Ef\left(\sum_{i=1}^{n} X_i\right) \le 2\max\left(\sum_{i=1}^{n} Ef(X_i), f\left(\sum_{i=1}^{n} a_i\right)\right) \tag{3.4}$$

for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $k = 1, \dots, n$.

Theorem 3.1 implies the following corollary. The results in, it in the case of independent r.v.'s and $s = 1$, were obtained by Ibragimov and Sharakhmetov (1998) [see also Ibragimov (1997)].

COROLLARY 3.1. *The constants in the following inequalities are exact*:

$$E\left(\sum_{k=1}^{n} X_k\right)^t \le 2\max\left(\sum_{k=1}^{n} EX_k^t, \left(\sum_{k=1}^{n} a_k\right)^t\right) \tag{3.5}$$

for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $k = 1, \dots, n$, $1 < t < 2$;

$$E\left(\sum_{k=1}^{n} X_k\right)^t \le E\theta^t(1)\max\left(\sum_{k=1}^{n} b_k, \left(\sum_{k=1}^{n} a_k^s\right)^{t/s}\right) \tag{3.6}$$

for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k^s \mid X_1, \dots, X_{k-1}) \le a_k^s$, $E(X_k^t \mid X_1, \dots, X_{k-1}) \le b_k$, $k = 1, \dots, n$, $1 < t < 2$, $0 < s \le t - 1$ or $t \ge 2$, $0 < s \le 1$.

REMARK 3.1. The fact that the functions f_2 and f_3 defined above belong to the class $D^{(3)}$ is important because this fact and Theorem 3.1, together with the property that $f_2'(0+) = f_3'(0+) = 0$, imply that the best constant in Rosenthal's inequality $Ef(\sum_{i=1}^n X_i) \le A(f)\max(\sum_{i=1}^n Ef(X_i), f(\sum_{i=1}^n a_i))$ for nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$ and the entropy-type functions f_2 and f_3 is equal to 2. On the other hand, the best constant in Rosenthal's inequality (1.1) is obviously equal to 1 for $t = 1 : A^*(1) = 1$. This means that even the addition of a logarithm to the first power $f(v) = v$ changes the best constant in Rosenthal's inequality from 1 to 2. One can show, in a similar way, that even the addition of the mth iteration of the logarithm $\ln_m v$, where $\ln_0 v = v$, $\ln_m v = \ln \ln_{m-1} v$, $m = 1, 2, \dots,$ to $f(v) = v$ implies the jump in the best constant in Rosenthal's inequality.

PROOF OF THEOREM 3.1. Let $f, h, \phi \in D^{(1)}$. Fix numbers $D_h, A_f, M_{hf} > 0$. For $f \in J$, $h \in Q_f$, denote by $U_1^{\text{non}}(D_h, A_f)$ the set of independent nonnegative r.v.'s $X_1, \dots, X_n$, $n \ge 1$, such that $\sum_{i=1}^n Eh(X_i) = h(D_h)$, $\sum_{i=1}^n Ef(X_i) = A_f$, denote by $U_2^{\text{non}}(D_h, A_f)$ the set of independent nonnegative r.v.'s $X_1, \dots, X_n$, $n \ge 1$, such that $\sum_{i=1}^n Eh(X_i) \le h(D_h)$, $\sum_{i=1}^n Ef(X_i) \le A_f$, and denote by $U^{\text{non}}(M_{hf})$ the set of independent nonnegative r.v.'s $X_1, \dots, X_n$, $n \ge 1$, such that $\max(f(h^{-1}(\sum_{i=1}^n Eh(X_i))), \sum_{i=1}^n Ef(X_i)) = M_{hf}$. Let $U_3^{\text{non}}(D_h, A_f)$ and $U_4^{\text{non}}(D_h, A_f)$ be the subsets of $U_1^{\text{non}}(D_h, A_f)$ and $U_2^{\text{non}}(D_h, A_f)$, respectively, consisting of identically distributed r.v.'s.

From Lemmas 2.2 and 2.8, it follows that if $f, h, \phi \in D^{(1)}$, then

$$\max_{(X,n)} E\phi\left(\sum_{i=1}^n X_i\right) = E\phi\left(\sum_{i=1}^n V_i(h, f, a_i, b_i)\right), \tag{3.7}$$

where max is taken over all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(h(X_k) \mid X_1, \dots, X_{k-1}) \le h(a_k)$, $E(f(X_k) \mid X_1, \dots, X_{k-1}) \le b_k$, $k = 1, \dots, n$. From Theorem A.1 (see also Remark A.1) and Lemma 2.2, it follows that

$$\begin{aligned} &\sup_{(X,n)\in U_1^{\text{non}}(D_h, A_f)} E\phi\left(\sum_{i=1}^n X_i\right) \\ &\quad = \sup_{(X,n)\in U_3^{\text{non}}(D_h, A_f)} E\phi\left(\sum_{i=1}^n X_i\right) \\ &\quad = \sup_n E\phi\left(\sum_{i=1}^n V_i\big(h, f, h^{-1}\big(h(D_h)n^{-1}\big), A_f n^{-1}\big)\right) \\ &\quad = \sup_n E\phi\left(G^{-1}\big(A_f/h(D_h)\big)\sum_{i=1}^n \overline{X}_i(d/n)\right), \end{aligned} \tag{3.8}$$

where $d = h(D_h)/h(G^{-1}(A_f/h(D_h)))$ and $\overline{X}_i(d/n)$ are defined at the end of the

Appendix and, in addition to that, according to Theorem A.1 and (2.10),

$$\begin{aligned}
&\sup_{(X,n)\in U_2^{\text{non}}(D_h,A_f)} E\phi\left(\sum_{i=1}^n X_i\right)\\
&\quad= \sup_{(X,n)\in U_4^{\text{non}}(D_h,A_f)} E\phi\left(\sum_{i=1}^n X_i\right)\\
&\quad= \sup_{0<A'_f\le A_f,0<D'_h\le D_h} \sup_{(X,n)\in U_3^{\text{non}}(D'_h,A'_f)} E\phi\left(\sum_{i=1}^n X_i\right) \qquad (3.9)\\
&\quad= \sup_n \sup_{0<A'_f\le A_f,0<D'_h\le D_h} E\phi\left(\sum_{i=1}^n V_i\big(h, f, h^{-1}\big(h(D'_h)n^{-1}\big), A'_f n^{-1}\big)\right)\\
&\quad= \sup_n E\phi\left(\sum_{i=1}^n V_i\big(h, f, h^{-1}\big(h(D_h)n^{-1}\big), A_f n^{-1}\big)\right).
\end{aligned}$$

Let $\theta(D_h, A_f) = \theta(h(D_h)/h(G^{-1}(A_f/h(D_h))))$. From (A.6), it follows that

$$\begin{aligned}
&\sup_n E\phi\left(G^{-1}\big(A_f/h(D_h)\big)\sum_{i=1}^n \overline{X}_i(d/n)\right)\\
&\qquad= E\phi\big(G^{-1}\big(A_f/h(D_h)\big)\theta(D_h, A_f)\big).
\end{aligned} \qquad (3.10)$$

Using (3.8)–(3.10), we get that

$$\begin{aligned}
&\sup_{(X,n)\in U_k^{\text{non}}(D_h,A_f)} E\phi\left(\sum_{i=1}^n X_i\right)\\
&\qquad= E\phi\big(G^{-1}\big(A_f/h(D_h)\big)\theta(D_h, A_f)\big), \qquad k = 1, 2.
\end{aligned} \qquad (3.11)$$

Using the evident inequalities

$$\begin{aligned}
&\sup_{(X,n)\in U_1^{\text{non}}(f^{-1}(M_{hf}),M_{hf})} E\phi\left(\sum_{i=1}^n X_i\right)\\
&\qquad\le \sup_{(X,n)\in U^{\text{non}}(M_{hf})} E\phi\left(\sum_{i=1}^n X_i\right)\\
&\qquad\le \sup_{(X,n)\in U_2^{\text{non}}(f^{-1}(M_{hf}),M_{hf})} E\phi\left(\sum_{i=1}^n X_i\right)
\end{aligned} \qquad (3.12)$$

and relations (3.11), we get that

$$\sup_{(X,n)\in U^{\text{non}}(M_{hf})} E\phi\left(\sum_{i=1}^n X_i\right) = E\phi\big(f^{-1}(M_{hf})\theta(1)\big). \qquad (3.13)$$

From (3.7) and (3.13), it follows that (3.2) holds and is exact. Now let $f, \phi \in D^{(2)}$. From Lemmas 2.1 and 2.8, it follows that

$$\max_{(X,n)} E\left(\phi\left(\sum_{i=1}^{n} X_i\right) - \sum_{i=1}^{n} f(X_i)\right) = \phi\left(\sum_{i=1}^{n} a_i\right) - \sum_{i=1}^{n} f(a_i), \tag{3.14}$$

where max is taken over all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $k = 1, \dots, n$. From (3.14) and the inequality $f(x) \ge f'(0+)x$, $x \in \mathbf{R}_+$, implied by the convexity of f, it follows that

$$E\left(\phi\left(\sum_{i=1}^{n} X_i\right) - \sum_{i=1}^{n} f(X_i)\right) \le \phi\left(\sum_{i=1}^{n} a_i\right) - f'(0+)\left(\sum_{i=1}^{n} a_i\right) \tag{3.15}$$

for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $k = 1, \dots, n$. Moreover, (3.15) is sharp as follows from the choice of r.v.'s $X_k = 1/n$ a.s., $k = 1, \dots, n$, and the fact that $\lim_{n\to\infty} nf(1/n) = f'(0+)$. Therefore, (3.3) holds and is exact. From (3.15) and the fact that if $f = \phi \in D^{(3)}$, then $f, \phi \in D^{(2)}$, it follows that if $f \in D^{(3)}$ and $f'(0+) = 0$, then (3.4) holds for all nonnegative r.v.'s $X_1, \dots, X_n$ with $E(X_k \mid X_1, \dots, X_{k-1}) \le a_k$, $k = 1, \dots, n$. From Theorem A.1 and Lemma 2.5, it follows that if $h(x) = x$, $f \in D^{(3)}$ and $f'(0+) = 0$, then (concerning the definitions of classes M_3^{non} and M_4^{non}, see Section 2)

$$\begin{aligned}
&\sup_{(X,n)\in U_1^{\text{non}}(D_h, A_f)} Ef\left(\sum_{i=1}^{n} X_i\right)\\
&\quad= \sup_n \sup_{(X,n)\in M_3^{\text{non}}(n,h,f,D_h/n,A_f/n)} Ef\left(\sum_{i=1}^{n} X_i\right)\\
&\quad= \sup_n \left(A_f + f(D_h) - nf(D_h/n)\right)\\
&\quad= A_f + f(D_h),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&\sup_{(X,n)\in U_2^{\text{non}}(D_h, A_f)} Ef\left(\sum_{i=1}^{n} X_i\right)\\
&\quad= \sup_n \sup_{(X,n)\in M_4^{\text{non}}(n,h,f,D_h/n,A_f/n)} Ef\left(\sum_{i=1}^{n} X_i\right)\\
&\quad= \sup_{0<A'_f\le A_f, 0<D'_h\le D_h} \sup_n \left(A'_f + f(D'_h) - nf(D'_h/n)\right)\\
&\quad= A_f + f(D_h).
\end{aligned} \tag{3.17}$$

From (3.12), (3.16) and (3.17), we get

$$\sup_{(X,n)\in U^{\text{non}}(M_{hf})} Ef\left(\sum_{i=1}^{n} X_i\right) = 2M_{hf}, \tag{3.18}$$

that is, (3.4) is exact. $\square$

Using Lemmas 2.1, 2.2 and 2.8, Theorem A.1 and relation (A.7) similarly to the proof of Theorem 3.1, we get that analogues of relations (3.2) and (3.3) hold in the case of conditionally symmetric martingale differences with bounded conditional moments. In particular, we obtain that the results concerning analogues of (3.3) for independent symmetric r.v.'s obtained in Figiel, Hitczenko, Johnson, Schechtman and Zinn (1997) and Ibragimov and Sharakhmetov (1995, 1997) hold for conditionally symmetric martingale differences with bounded conditional moments as well. Moreover, Theorem 3.2, which generalizes and complements the results obtained by Utev (1985), Figiel, Hitczenko, Johnson, Schechtman and Zinn (1997) and Ibragimov and Sharakhmetov (1995, 1997), holds.

Denote by $D^{(4)}$ the class of functions $f \in J$, $h \in Q_f$ and nonnegative functions $\phi \in \Phi$ such that $f(0) = 0$,

$$E\phi(x\varepsilon) + E\phi(a_1\varepsilon_1 + a_2\varepsilon_2 + x\varepsilon) \geq E\phi(a_1\varepsilon_1 + x\varepsilon_2) + E\phi(a_2\varepsilon_1 + x\varepsilon_2), \tag{3.19}$$

$a_1, a_2, x \in \mathbf{R}$, and the function $(E\phi(G^{-1}(v)\varepsilon + z) - \phi(z))/h(G^{-1}(v))$ is nonnegative, concave and nondecreasing in $v > 0$ for $z \in \mathbf{R}$. Examples of functions $f, h, \phi \in D^{(4)}$ are given by $f(v) = v^t$, $h(v) = v^s$, $\phi(v) = |v|^t$, where $3 \leq t < 4, 0 < s \leq t - 2$ or $t \geq 4$, $0 < s \leq 2$ (see Lemma 2.4 and Remark 3.2).

THEOREM 3.2. *If $f, h, \phi \in D^{(4)}$, then the following exact inequality holds*:

$$E\phi\left(\sum_{i=1}^n X_i\right) \leq E\phi\left[(\theta_1(0.5) - \theta_2(0.5)) \max\left(f^{-1}\left(\sum_{i=1}^n b_i\right), h^{-1}\left(\sum_{i=1}^n h(a_i)\right)\right)\right]$$

for all conditionally symmetric martingale differences $X_1, \dots, X_n$ with $E(h(|X_k|)|X_1, \dots, X_{k-1}) \leq h(a_k)$, $E(f(|X_k|)|X_1, \dots, X_{k-1}) \leq b_k$, $k = 1, \dots, n$.

REMARK 3.2. It is not difficult to show that if a function $\phi: \mathbf{R} \to \mathbf{R}$ is twice differentiable, then (3.19) follows from the condition of convexity of the function $E\phi''(x\varepsilon)$. Indeed, let $E\phi''(x\varepsilon)$ be a convex function. Denote $g(a_1, a_2, x) = E\phi(a_1\varepsilon_1 + a_2\varepsilon_2 + x\varepsilon), a_1, a_2, x \in \mathbf{R}$. Since $(-|a_1| + |a_2| + x, |a_1| - |a_2| + x) \prec (|a_1| + |a_2| + x, -|a_1| - |a_2| + x)$ (see the definition of the majorization relation $\prec$ in the Appendix), from the convexity of $E\phi''(x\varepsilon)$, Proposition 3.C.1 in Marshall and Olkin (1979) and the property that the joint distribution of the r.v.'s $\varepsilon_1, \varepsilon_2$ and ε and the r.v.'s $\varepsilon_1\varepsilon, \varepsilon_2\varepsilon$ and ε is the same (one can show that the latter property holds in a straightforward fashion; it is also implied by the fact that arbitrary r.v.'s assuming two values form a multiplicative system if and only if they are mutually independent; see Remark 3.5), it follows that $\partial^2 g(a_1, a_2, x)/\partial a_1 \partial a_2 = E\phi''(a_1\varepsilon_1 + a_2\varepsilon_2 + x\varepsilon)\varepsilon_1\varepsilon_2 = E\phi''((a_1\varepsilon_1 + a_2\varepsilon_2 + x)\varepsilon)\varepsilon_1\varepsilon_2 \geq 0$. According to Marshall and Olkin (1979), page 150, this inequality means that the function $g(a_1, a_2, x)$ is L-superadditive in a_1, a_2; that is, $g(a_1 + b_1, a_2 + b_2, x) + g(a_1 - b_1, a_2 - b_2, x) \geq g(a_1 + b_1, a_2 - b_2, x) + g(a_1 - b_1, a_2 + b_2, x)$ for all $a_1, a_2, x \in \mathbf{R}$, $b_1, b_2 \geq 0$. Setting in the latter inequality $a_i = b_i = |a_i'|/2, i = 1, 2$, we obtain that $E\phi(x\varepsilon) + E\phi(a_1'\varepsilon_1 + a_2'\varepsilon_2 + x\varepsilon) \geq E\phi(a_1'\varepsilon_1 + x\varepsilon) + E\phi(a_2'\varepsilon_1 + x\varepsilon)$ for all $a_1', a_2', x \in \mathbf{R}$; that is, the function ϕ satisfies condition (3.19).

Furthermore, using, in addition to the above, Lemmas 2.4 and 2.7, taking into account Remark 3.2 and using the exact Khintchine inequality $E|\sum_{i=1}^n a_i\varepsilon_i|^t \leq E|Z|^t(\sum_{i=1}^n a_i^2)^{t/2}$ for all $a_i \in \mathbf{R}$, $i = 1, \dots, n$, $t > 2$ [see Haagerup (1982)], we obtain the following corollary. This corollary in the case of independent r.v.'s and $s = 2$ was obtained independently by Figiel, Hitczenko, Johnson, Schechtman and Zinn (1997) and Ibragimov and Sharakhmetov (1995, 1997).

COROLLARY 3.2. *The constants in the following inequalities are exact*:

$$E\left|\sum_{k=1}^n X_k\right|^t \leq (1 + E|Z|^t) \max\left(\sum_{k=1}^n E|X_k|^t, \left(\sum_{k=1}^n a_k^2\right)^{t/2}\right) \tag{3.20}$$

for all conditionally symmetric martingale differences (X_k) *with* $E(X_k^2|X_1, \dots, X_{k-1}) \leq a_k^2$, $k = 1, \dots, n$, $2 < t < 4$;

$$E\left|\sum_{k=1}^n X_k\right|^t \leq E|\theta_1(0.5) - \theta_2(0.5)|^t \max\left(\sum_{k=1}^n b_k, \left(\sum_{k=1}^n a_k^s\right)^{t/s}\right) \tag{3.21}$$

for all conditionally symmetric martingale differences (X_k) *with* $E(|X_k|^s|X_1, \dots, X_{k-1}) \leq a_k^s$, $E(|X_k|^t|X_1, \dots, X_{k-1}) \leq b_k$, $k = 1, \dots, n$, $3 \leq t < 4$, $0 < s \leq t - 2$ *or* $t \geq 4$, $0 < s \leq 2$.

Suppose that $f \in D^{(3)}$. Then, by the condition that $f''(v)$ is nonnegative, and, therefore, $f(v)$ is convex, we have that $Ef(\sum_{i=1}^n X_i) \geq f(\sum_{i=1}^n EX_i)$ for all nonnegative r.v.'s $X_1, \dots, X_n$, using Jensen's inequality. Moreover, since $f(x)/x$ is nondecreasing on $\mathbf{R}_+$ [it also follows from the fact that $f''(v)$ is nonnegative], we have that $f(\alpha x) \leq \alpha f(x)$ for all $\alpha \in [0, 1]$ and all $x \in \mathbf{R}_+$ and, therefore, $f(x) + f(y) \leq f(x + y)$ for all $x, y \in \mathbf{R}_+$; indeed, it suffices to consider $x, y > 0$:

$$\begin{aligned} f(x) + f(y) &= f\left(\frac{x}{x+y}(x+y)\right) + f\left(\frac{y}{x+y}(x+y)\right) \\ &\leq \frac{x}{x+y} f(x+y) + \frac{y}{x+y} f(x+y) = f(x+y). \end{aligned}$$

The latter inequality implies that $Ef(\sum_{i=1}^n X_i) \geq \sum_{i=1}^n Ef(X_i)$ for all nonnegative r.v.'s $X_1, \dots, X_n$. Combining the above with relation (3.4), we obtain the following result.

COROLLARY 3.3. *The following decoupling inequality holds*: $Ef(\sum_{i=1}^n X_i) \leq 2Ef(\sum_{i=1}^n \tilde{X}_i)$ *for all functions* $f \in D^{(3)}$ *and all nonnegative r.v.'s* $X_1, \dots, X_n$, $\tilde{X}_1, \dots, \tilde{X}_n$ *with* $E(X_i \mid X_1, \dots, X_{i-1}) \leq E\tilde{X}_i$, $Ef(X_i) \leq Ef(\tilde{X}_i)$, $i = 1, \dots, n$.

Similarly, from Corollaries 3.1 and 3.2 and the lower Rosenthal bounds $E(\sum_{i=1}^n X_i)^t \geq \max(\sum_{i=1}^n EX_i^t, (\sum_{i=1}^n EX_i)^t)$ for all nonnegative r.v.'s $X_1, \dots, X_n$ with finite tth moment, $t \geq 1$ (note that independence of the r.v.'s is not necessary here), and $E|\sum_{i=1}^n X_i|^t \geq \max(\sum_{i=1}^n E|X_i|^t, (\sum_{i=1}^n EX_i^2)^{t/2})$ for all independent symmetric r.v.'s $X_1, \dots, X_n$ with finite tth moment, $t \geq 2$, it follows that the best constants $A^*(t)$ and $B^*_{\mathrm{sym}}(t)$ from the Introduction dominate the best constants in decoupling inequalities for dependent nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments. More precisely, the following corollary holds.

COROLLARY 3.4. *The inequality* $E|\sum_{i=1}^n X_i|^t \leq C(t)E|\sum_{i=1}^n \tilde{X}_i|^t$ *holds with the constant* $C(t) = 2$ *for all nonnegative r.v.'s* $X_1, \dots, X_n, \tilde{X}_1, \dots, \tilde{X}_n$ *with* $E(X_i \mid X_1, \dots, X_{i-1}) \leq E\tilde{X}_i$, $EX_i^t \leq E\tilde{X}_i^t$, $i = 1, \dots, n$, $1 < t < 2$; *with the constant* $C(t) = E\theta^t(1)$ *for all nonnegative r.v.'s* $X_1, \dots, X_n, \tilde{X}_1, \dots, \tilde{X}_n$ *with* $E(X_i \mid X_1, \dots, X_{i-1}) \leq E\tilde{X}_i$, $E(X_i^t \mid X_1, \dots, X_{i-1}) \leq E\tilde{X}_i^t$, $i = 1, \dots, n$, $t \geq 2$; *with the constant* $C(t) = 1 + E|Z|^t$ *for a conditionally symmetric martingale difference* $X_1, \dots, X_n$ *and a sequence of independent symmetric r.v.'s* $\tilde{X}_1, \dots, \tilde{X}_n$ *with* $E(X_i^2 \mid X_1, \dots, X_{i-1}) \leq E\tilde{X}_i^2$, $E|X_i|^t \leq E|\tilde{X}_i|^t$, $i = 1, \dots, n$, $2 < t < 4$; *with the constant* $C(t) = E|\theta_1(0.5) - \theta_2(0.5)|^t$ *for a conditionally symmetric martingale difference* $X_1, \dots, X_n$ *and a sequence of independent symmetric r.v.'s* $\tilde{X}_1, \dots, \tilde{X}_n$ *with* $E(X_i^2 \mid X_1, \dots, X_{i-1}) \leq E\tilde{X}_i^2$, $E(|X_i|^t | X_1, \dots, X_{i-1}) \leq E|\tilde{X}_i|^t$, $i = 1, \dots, n$, $t \geq 4$.

REMARK 3.3. The classes of nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments are quite wide. For example, if X_k, $k = 1, \dots, n$, is a sequence of independent nonnegative r.v.'s on a probability space $(\Omega, \Im, P)$ with $EX_k^t < \infty$, and τ is a stopping time with respect to $\sigma(X_1, \dots, X_k)$, $k = 0, 1, \dots, n$ [we assume that $\sigma(X_1, \dots, X_k) = (\varnothing, \Omega)$ for $k = 0$], then, for the r.v.'s $\tilde{X}_k = X_k I(\tau \geq k)$, $k = 1, \dots, n$ [$I(\cdot)$ is the indicator function], $E(\tilde{X}_k \mid X_1, \dots, X_{k-1}) \leq EX_k$, $E(\tilde{X}_k^t \mid X_1, \dots, X_{k-1}) \leq EX_k^t$. Similarly, if X_k, $k = 1, \dots, n$, is a sequence of independent symmetric r.v.'s on $(\Omega, \Im, P)$ with $E|X_k|^t < \infty$, and τ is a stopping time with respect to $\sigma(X_1, \dots, X_k)$, $k = 0, 1, \dots, n$, then the sequence $\tilde{X}_k = X_k I(\tau \geq k)$, $k = 1, \dots, n$, is a conditionally symmetric martingale difference with respect to $\sigma(X_1, \dots, X_k)$, $k = 0, 1, \dots, n$, and $E(\tilde{X}_k^2 \mid X_1, \dots, X_{k-1}) \leq EX_k^2$, $E(|\tilde{X}_k|^t \mid X_1, \dots, X_{k-1}) \leq E|X_k|^t$. Moreover, if X_k, $k = 1, \dots, n$, is a sequence of independent symmetric r.v.'s with $E|X_k|^t < \infty$, and v_{k-1}, $k = 1, \dots, n$, are $\sigma(X_1, \dots, X_{k-1})$-measurable r.v.'s such that $|v_{k-1}| \leq 1$, then the sequence $v_{k-1}X_k$ is a conditionally symmetric martingale difference with respect to $\sigma(X_1, \dots, X_k)$, $k = 0, 1, \dots, n$, and $E(v_{k-1}^2 X_k^2 \mid X_1, \dots, X_{k-1}) \leq EX_k^2$, $E(|v_{k-1}X_k|^t \mid X_1, \dots, X_{k-1}) \leq E|X_k|^t$. Therefore, the results in Corollaries 3.1–3.4 hold for the randomly stopped sums $\sum_{k=1}^{\tau \wedge n} X_k$ and the martingale transforms $\sum_{k=1}^n v_{k-1}X_k$.

REMARK 3.4. Let $\Im_0 = (\varnothing, \Omega) \subseteq \Im_1 \cdots \subseteq \Im_n \cdots \subseteq \Im$ and let (X_n) be a sequence of $(\Im_n)$-adapted r.v.'s on a probability space $(\Omega, \Im, P)$. According to Kwapień and Woyczynski [(1992), pages 104–105] there exists (maybe on a different probability space) a sub-σ-field $\overline{\Im}$ of $\Im$ and a sequence $\overline{X}_n$ of $(\Im_n)$-adapted r.v.'s such that, for each n, $\mathcal{L}(X_n \mid \Im_{n-1}) = \mathcal{L}(\overline{X}_n \mid \Im_{n-1}) = \mathcal{L}(\overline{X}_n \mid \overline{\Im})$. Hitczenko (1994c) showed that for any sequence of nonnegative $(\Im_n)$-adapted r.v.'s (X_n) the following inequality holds and the constant 2^{t-1} in it is exact: $E(\sum_{i=1}^n X_i)^t \leq 2^{t-1} E(\sum_{i=1}^n \overline{X}_i)^t$, $t \geq 1$. Moreover, according to Hitczenko (1994a), the following more general inequality is valid: $E|\sum_{i=1}^n X_i|^t \leq L^t E|\sum_{i=1}^n \overline{X}_i|^t$, $t \geq 1$, for all $(\Im_n)$-adapted r.v.'s (X_n), where L is an absolute constant. Using the above domination inequalities and the sharp moment inequalities for sums of independent r.v.'s that follow from the results presented in this section, one can easily obtain, similarly to de la Peña and Zamfirescu (2002), moment estimates for sums of adapted r.v.'s and martingales. For example, from the former domination inequality and Corollary 3.1, it follows that inequality (1.4) holds with the constant 2^t if $1 < t < 2$, and $2^{t-1} E\theta^t(1)$ if $t \geq 2$. Similarly, the latter domination inequality and Corollary 3.2 imply that inequality (1.5) holds with the constant $L^t(1 + E|Z|^t)$ if $2 < t < 4$, and $L^t E|\theta_1(0.5) - \theta_2(0.5)|^t$ if $t \geq 4$. Using the fact that the actual rate of growth of $\|\theta(1)\|_t$ and $\|\theta_1(0.5) - \theta_2(0.5)\|_t$ is $t/\ln t$ as $t \to \infty$ [see the calculation of the asymptotics of Bell numbers in Sachkov (1996) and the derivation of the asymptotics of the best constant in Rosenthal's inequality for independent symmetric r.v.'s in Ibragimov and Sharakhmetov (1997)], from the above we obtain a new proof of the property that [e.g., Hitczenko (1990)] the actual rate of growth of the best constants in Burkholder inequalities for L_t-norms of sums of adapted nonnegative r.v.'s and martingales is $t/\ln t$.

Let $1 \leq s < t$ and let $X_1, \dots, X_n$ be independent r.v.'s with finite tth moment. Fix values $a_i, b_i > 0$, $a_i^t \leq b_i$, $i = 1, \dots, n$. Set

$$M_1^{\text{ind}}(n, s, t, a, b) = \{(X, n) : E|X_i|^s = a_i^s,\ E|X_i|^t = b_i,\ i = 1, \dots, n\},$$
$$M_2^{\text{ind}}(n, s, t, a, b) = \{(X, n) : E|X_i|^s \leq a_i^s,\ E|X_i|^t \leq b_i,\ i = 1, \dots, n\}.$$

Let $M_k^{\text{non,ind}}(n, s, t, a, b)$, $k = 1, 2$, be the subsets of $M_k^{\text{ind}}(n, s, t, a, b)$, $k = 1, 2$, respectively, consisting of nonnegative r.v.'s, and let $M_k^{\text{sym,ind}}(n, s, t, a, b)$, $k = 1, 2$, be the subsets of $M_k^{\text{ind}}(n, s, t, a, b)$, $k = 1, 2$, respectively, consisting of symmetric r.v.'s. Let $V_1(s, t, a_1, b_1), \dots, V_n(s, t, a_n, b_n)$ be independent r.v.'s with distributions $P(V_k(s, t, a_k, b_k) = 0) = 1 - (a_k^t/b_k)^{s/(t-s)}$, $P(V_k(s, t, a_k, b_k) = (b_k/a_k^s)^{1/(t-s)}) = (a_k^t/b_k)^{s/(t-s)}$, $k = 1, \dots, n$, and let $W_1(s, t, a_1, b_1), \dots, W_n(s, t, a_n, b_n)$ be independent r.v.'s with distributions $P(W_k(s, t, a_k, b_k) = 0) = 1 - (a_k^t/b_k)^{s/(t-s)}$, $P(W_k(s, t, a_k, b_k) = (b_k/a_k^s)^{1/(t-s)}) = P(W_k(s, t, a_k, b_k) = -(b_k/a_k^s)^{1/(t-s)}) = \frac{1}{2}(a_k^t/b_k)^{s/(t-s)}$, $k = 1, \dots, n$.

Let $0 \le m \le n$. Denote (we assume below that $c_{i_1,\dots,i_l} = \text{const}$ for $l = 0$)

$$F_1(m,n,s,t,a,b,c) = E\left(\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} V_{i_1}(s,t,a_{i_1},b_{i_1}) \cdots V_{i_l}(s,t,a_{i_l},b_{i_l})\right)^t,$$

$$G_1(m,n,s,t,a,b,c) = \sum_{q=0}^{m} \sum_{1\le j_1<\cdots<j_q\le n} \prod_{r=1}^{q} \left(b_{j_r} - a_{j_r}^t\right) \times \left(\sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}} a_{i_1}\cdots a_{i_{l-q}}\right)^t,$$

$c_{i_1,\dots,i_l} \ge 0,\ 1 \le i_1 < \cdots < i_l \le n,\ l = 0,\dots,m,\ c_{i_{\pi(1)},\dots,i_{\pi(l)}} = c_{i_1,\dots,i_l},\ 1 \le i_1 < \cdots < i_l \le n$, for all permutations $\pi : \{1,\dots,l\} \to \{1,\dots,l\}$, $l = 2,\dots,m$;

$$F_2(m,n,s,t,a,b,c) = E\left|\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} W_{i_1}(s,t,a_{i_1},b_{i_1}) \cdots W_{i_l}(s,t,a_{i_l},b_{i_l})\right|^t,$$

$$G_2(m,n,s,t,a,b,c) = \sum_{q=0}^{m} \sum_{1\le j_1<\cdots<j_q\le n} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \times E\left|\sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}} a_{i_1}\cdots a_{i_{l-q}} \varepsilon_{i_1}\cdots\varepsilon_{i_{l-q}}\right|^t,$$

$c_{i_1,\dots,i_l} \in \mathbf{R},\ 1 \le i_1 < \cdots < i_l \le n,\ l = 0,\dots,m,\ c_{i_{\pi(1)},\dots,i_{\pi(l)}} = c_{i_1,\dots,i_l},\ 1 \le i_1 < \cdots < i_l \le n$, for all permutations $\pi : \{1,\dots,l\} \to \{1,\dots,l\}$, $l = 2,\dots,m$.

THEOREM 3.3. *Let* $c_{i_1,\dots,i_l} \ge 0,\ 1 \le i_1 < \cdots < i_l \le n,\ l = 0,\dots,m$. *If* $1 < t < 2$, $0 < s \le t-1$ *or* $t \ge 2$, $0 < s \le 1$, *then*

$$\sup_{(X,n)\in M_k^{\text{non,ind}}(n,s,t,a,b)} E\left(\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right)^t = F_1(m,n,s,t,a,b,c), \qquad k = 1,2, \tag{3.22}$$

$$\inf_{(X,n)\in M_1^{\text{non,ind}}(n,s,t,a,b)} E\left(\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right)^t = G_1(m,n,s,t,a,b,c). \tag{3.23}$$

If $1 < t < 2$, $1 \le s < t$ *or* $t \ge 2$, $t-1 \le s < t$, *then*

$$\sup_{(X,n)\in M_k^{\text{non,ind}}(n,s,t,a,b)} E\left(\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right)^t = G_1(m,n,s,t,a,b,c), \qquad k=1,2, \tag{3.24}$$

$$\inf_{(X,n)\in M_1^{\text{non,ind}}(n,s,t,a,b)} E\left(\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right)^t = F_1(m,n,s,t,a,b,c). \tag{3.25}$$

THEOREM 3.4. *Let* $c_{i_1,\dots,i_l} \in \mathbf{R}$, $1 \le i_1 < \cdots < i_l \le n$, $l = 0,\dots,m$. *If* $2 < t < 4$, $0 < s \le t-2$ *or* $t \ge 4$, $0 < s \le 2$, *then*

$$\sup_{(X,n)\in M_k^{\text{sym,ind}}(n,s,t,a,b)} E\left|\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right|^t = F_2(m,n,s,t,a,b,c), \qquad k=1,2. \tag{3.26}$$

If $3 \le t < 4$, $0 < s \le t-2$ *or* $t \ge 4$, $0 < s \le 2$, *then*

$$\inf_{(X,n)\in M_1^{\text{sym,ind}}(n,s,t,a,b)} E\left|\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right|^t = G_2(m,n,s,t,a,b,c). \tag{3.27}$$

If $2 < t < 4$, $2 \le s < t$ *or* $t \ge 4$, $t-2 \le s < t$, *then*

$$\sup_{(X,n)\in M_k^{\text{sym,ind}}(n,s,t,a,b)} E\left|\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right|^t = G_2(m,n,s,t,a,b,c), \qquad k=1,2. \tag{3.28}$$

If $3 \le t < 4$, $2 \le s < t$ *or* $t \ge 4$, $t-2 \le s < t$, *then*

$$\inf_{(X,n)\in M_1^{\text{sym,ind}}(n,s,t,a,b)} E\left|\sum_{l=0}^{m} \sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right|^t = F_2(m,n,s,t,a,b,c). \tag{3.29}$$

REMARK 3.5. It is important that the quantity G_2 has a simple structure in a particular case of sums of multilinear forms, namely, in the case of generalized moving averages $\sum_{i=1}^{n-h_m} c_i X_{i+h_1} X_{i+h_2} \cdots X_{i+h_m}$, $0 \le h_1 < \cdots < h_m \le n-1$, $c_i \in \mathbf{R}$, $i = 1,\dots,n-h_m$. Let $\varepsilon_i' = \varepsilon_{i+h_1}\varepsilon_{i+h_2}\cdots\varepsilon_{i+h_m}$, $i = 1, \dots, n-h_m$. It is not difficult to see that the r.v.'s ε_i', $i = 1,\dots, n-h_m$, are mutually independent. Indeed, we have that the r.v.'s ε_i' satisfy the conditions $P(\varepsilon'_i = 1) = P(\varepsilon'_i = -1) = \frac{1}{2}$, $i = 1, \dots, n - h_m$. In addition to that, for arbitrary $1 \le j_1 < \cdots < j_c \le n - h_m$, $c = 1, \dots, n - h_m$, the r.v. $\varepsilon_{j_1+h_1}$ is independent of the r.v.'s $\varepsilon_{j_1+h_2}, \dots, \varepsilon_{j_1+h_m}, \varepsilon_{j_2+h_1}, \varepsilon_{j_2+h_2}, \dots, \varepsilon_{j_2+h_m}, \dots, \varepsilon_{j_c+h_1}, \varepsilon_{j_c+h_2}, \dots, \varepsilon_{j_c+h_m}$, and, therefore, $E\varepsilon'_{j_1} \cdots \varepsilon'_{j_c} = E\prod_{k=1}^{c}\prod_{l=1}^{m}\varepsilon_{j_k+h_l} = E\varepsilon_{j_1+h_1}E\prod_{k=2}^{c}\varepsilon_{j_k+h_1}\prod_{k=1}^{c}\prod_{l=2}^{m}\varepsilon_{j_k+h_l} = 0 = E\varepsilon'_{j_1} \cdots E\varepsilon'_{j_c}$ (i.e., ε'_i, $i = 1, \dots, n - h_m$, is a multiplicative system of order 1). Since $I(\varepsilon'_i = t_i) = (1 + t_i\varepsilon'_i)/2$, $t_i \in \{-1, 1\}$, $i = 1, \dots, n - h_m$ a.s. [$I(\cdot)$ is the indicator function], the latter relation implies that

$$\begin{aligned} P(\varepsilon'_{j_1} = t_{j_1}, \dots, \varepsilon'_{j_c} = t_{j_c}) &= EI(\varepsilon'_{j_1} = t_{j_1}) \cdots I(\varepsilon'_{j_c} = t_{j_c}) \\ &= EI(\varepsilon'_{j_1} = t_{j_1}) \cdots EI(\varepsilon'_{j_c} = t_{j_c}) \\ &= P(\varepsilon'_{j_1} = t_{j_1}) \cdots P(\varepsilon'_{j_c} = t_{j_c}), \end{aligned}$$

$t_{j_1}, \dots, t_{j_c} \in \{-1, 1\}$, $1 \le j_1 < \cdots < j_c \le n - h_m$, $c = 1, \dots, n - h_m$; that is, the r.v.'s ε'_i, $i = 1, \dots, n - h_m$, are mutually independent [Sharakhmetov (1997) proved a more general fact, namely, that arbitrary r.v.'s assuming $\alpha + 1$ values form a multiplicative system of order α if and only if they are mutually independent]. The above means, in particular, that in the case of generalized moving averages $\sum_{i=1}^{n-h_m} c_i X_{i+h_1}X_{i+h_2} \cdots X_{i+h_m}$, $0 \le h_1 < \cdots < h_m \le n - 1$, $c_i \in \mathbf{R}$, $i = 1, \dots, n - h_m$; that is, for sums of multilinear forms $\sum_{l=0}^{m}\sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l}X_{i_1} \cdots X_{i_l}$ with $c_{i_1,\dots,i_l} = 0$, $1 \le i_1 < \cdots < i_l \le n$, $l = 0, \dots, m-1$; $c_{i_1,\dots,i_m} = 0$, $(i_1, \dots, i_m) \ne (j + h_1, \dots, j + h_m)$, $j = 1, \dots, n - h_m$; $c_{i+h_1,\dots,i+h_m} = c_i$, $i = 1, \dots, n - h_m$, the quantity G_2 is just a sum of moments of linear combinations of independent symmetric Bernoulli r.v.'s. Moreover, independence of ε'_i, $i = 1, \dots, n - h_m$, implies that all moment and probability inequalities and limit theorems for linear combinations of independent Bernoulli r.v.'s hold for generalized moving averages in independent Bernoulli r.v.'s. An application of the above facts is given in Theorem 3.10.

The following theorems give exact analogues of Rosenthal's inequalities for sums of multilinear forms in nonnegative r.v.'s.

THEOREM 3.5. *Let* $c_{i_1,\dots,i_l} \ge 0$, $1 \le i_1 < \cdots < i_l \le n$, $l = 0, \dots, m$, $c_{i_{\pi(1)},\dots,i_{\pi(l)}} = c_{i_1,\dots,i_l}$, $1 \le i_1 < \cdots < i_l \le n$, *for all permutations* $\pi : \{1, \dots, l\} \to \{1, \dots, l\}$, $l = 2, \dots, m$. *The constants in the following inequality are exact*:

$$\begin{aligned} &E\left(\sum_{l=0}^{m}\sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l}X_{i_1} \cdots X_{i_l}\right)^t \\ &\quad \le \sum_{q=0}^{m}\sum_{1 \le j_1 < \cdots < j_q \le n}\prod_{r=1}^{q} EX_{j_r}^t \\ &\qquad \times \left(\sum_{l=q}^{m}\sum_{i_1 < \cdots < i_{l-q} \in \{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}}EX_{i_1} \cdots EX_{i_{l-q}}\right)^t \end{aligned} \tag{3.30}$$

for all independent nonnegative r.v.'s $X_1, \dots, X_n$ *with finite* t*th moment*, $1 < t < 2$.

THEOREM 3.6. *Let* $c_{i_1,\dots,i_l} \ge 0$, $1 \le i_1 < \dots < i_l \le n$, $l = 0, \dots, m$, $c_{i_{\pi(1)},\dots,i_{\pi(l)}} = c_{i_1,\dots,i_l}$, $1 \le i_1 < \dots < i_l \le n$, *for all permutations* $\pi : \{1, \dots, l\} \to \{1, \dots, l\}$, $l = 2, \dots, m$. *The following inequality holds*:

$$\begin{aligned}
&\max_{q=0,\dots,m} \sum_{1 \le j_1 < \dots < j_q \le n} \prod_{r=1}^{q} EX_{j_r}^t \\
&\qquad \times \left(\sum_{l=q}^{m} \sum_{i_1 < \dots < i_{l-q} \in \{1,\dots,n\} \setminus \{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}} EX_{i_1} \cdots EX_{i_{l-q}} \right)^t \\
&\quad \le E\left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \dots < i_l \le n} c_{i_1,\dots,i_l} X_{i_1} \cdots X_{i_l} \right)^t \\
&\quad \le (m+1) \max_{q=0,\dots,m} \sum_{1 \le j_1 < \dots < j_q \le n} \prod_{r=1}^{q} EX_{j_r}^t \\
&\qquad\qquad \times \left(\sum_{l=q}^{m} \sum_{i_1 < \dots < i_{l-q} \in \{1,\dots,n\} \setminus \{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}} \right. \\
&\qquad\qquad\qquad \left. \times EX_{i_1} \cdots EX_{i_{l-q}} \right)^t
\end{aligned} \tag{3.31}$$

for all independent nonnegative r.v.'s $X_1, \dots, X_n$ *with finite* t*th moment*, $1 < t < 2$.

Theorems 3.7 and 3.8 provide a link between Rosenthal's and Khintchine's inequalities for sums of multilinear forms in independent symmetric r.v.'s and give analogues of Rosenthal's bounds for those objects.

Let $\varepsilon_{p1}, \dots, \varepsilon_{pn}$, $p = 1, \dots, m$, be independent symmetric Bernoulli r.v.'s and let $Kh^{*\mathrm{reg}}(m, t)$ and $Kh^{*\mathrm{dec}}(m, t)$, $t > 0$, denote the best upper constants in Khintchine's inequalities for sums of regular and decoupled multilinear forms in independent symmetric Bernoulli r.v.'s, respectively:

$$E\left| \sum_{l=0}^{m} \sum_{1 \le i_1 < \dots < i_l \le n} c_{i_1,\dots,i_l} \varepsilon_{i_1} \cdots \varepsilon_{i_l} \right|^t \le
Kh^{*\mathrm{reg}}(m, t) E\left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \dots < i_l \le n} c_{i_1,\dots,i_l}^2 \right)^{t/2},$$

$$E\left|\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\ldots,i_l}\varepsilon_{1,i_1}\cdots\varepsilon_{l,i_l}\right|^t \le Kh^{*\mathrm{dec}}(m,t)E\left(\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c^2_{i_1,\ldots,i_l}\right)^{t/2}$$

for all $c_{i_1,\ldots,i_l}\in\mathbf{R}$. The existence of such constants follows from Khintchine's inequalities for multilinear forms [e.g., McConnell and Taqqu (1986), Krakowiak and Szulga (1986), de la Peña (1992) and Ibragimov and Sharakhmetov(1998, 1999, 2000)].

THEOREM 3.7. *Let* $c_{i_1,\ldots,i_l}\in\mathbf{R}$, $1\le i_1<\cdots<i_l\le n$, $l=0,\ldots,m$, $c_{i_{\pi(1)},\ldots,i_{\pi(l)}}=c_{i_1,\ldots,i_l}$, $1\le i_1<\cdots i_l\le n$, *for all permutations* $\pi:\{1,\ldots,l\}\to\{1,\ldots,l\}$, $l=2,\ldots,m$. *The following inequality holds*:

$$\begin{aligned}E\left|\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\ldots,i_l}X_{i_1}\cdots X_{i_l}\right|^t &\le \sum_{q=0}^{m-1}\sum_{1\le j_1<\cdots<j_q\le n} Kh^{*\mathrm{reg}}(m-q,t)\prod_{r=1}^{q}E|X_{j_r}|^t\\ &\quad\times\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\backslash\{j_1,\ldots,j_q\}} c^2_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}EX^2_{i_1}\cdots EX^2_{i_{l-q}}\right)^{t/2}\\ &\quad+\sum_{1\le i_1<\cdots<i_m\le n}\prod_{r=1}^{m}E|X_{i_r}|^t\end{aligned}\tag{3.32}$$

for all independent symmetric r.v.'s $X_1,\ldots,X_n$ *with finite* t*th moment*, $2<t<4$.

THEOREM 3.8. *Let* $c_{i_1,\ldots,i_l}\in\mathbf{R}$, $1\le i_1<\cdots<i_l\le n$, $l=0,\ldots,m$, $c_{i_{\pi(1)},\ldots,i_{\pi(l)}}=c_{i_1,\ldots,i_l}$, $1\le i_1<\cdots<i_l\le n$, *for all permutations* $\pi:\{1,\ldots,l\}\to\{1,\ldots,l\}$, $l=2,\ldots,m$. *The following inequalities hold*:

$$\begin{aligned}&\max_{q=0,\ldots,m}\sum_{1\le j_1<\cdots<j_q\le n}\prod_{r=1}^{q}E|X_{j_r}|^t\\ &\qquad\times\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\backslash\{j_1,\ldots,j_q\}} c^2_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}}EX^2_{i_1}\cdots EX^2_{i_{l-q}}\right)^{t/2}\\ &\le E\left|\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\ldots,i_l}X_{i_1}\cdots X_{i_l}\right|^t\end{aligned}\tag{3.33}$$

$$
\begin{aligned}
&\le \left(1+\sum_{q=1}^{m} K h^{*\mathrm{reg}}(q,t)\right)\\
&\quad\times \max_{q=0,\dots,m} \sum_{1\le j_1<\cdots<j_q\le n} \prod_{r=1}^{q} E|X_{j_r}|^t\\
&\qquad\times \left(\sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c^2_{j_1,\dots,j_q,i_1,\dots,i_{l-q}}\right.\\
&\qquad\qquad\left.\times EX^2_{i_1}\cdots EX^2_{i_{l-q}}\right)^{t/2}
\end{aligned}
$$

for all independent symmetric r.v.'s $X_1,\dots,X_n$ *with finite* t*th moment*, $2<t<4$.

REMARK 3.6. Using the fact that sums of decoupled multilinear forms $\sum_{l=0}^m \sum_{i_1=1}^n \cdots \sum_{i_l=1}^n c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l}$, where $X_{p1},\dots,X_{pn}$, $p=1,\dots,m$, are independent r.v.'s, can be represented as sums of regular multilinear forms with many zero coefficients, we obtain that analogues of Theorems 3.3–3.8 hold for sums of decoupled multilinear forms as well. Using Lemma 2.8, we also obtain, as in the case of sums of r.v.'s, that analogues of the above theorems hold for sums of multilinear forms in nonnegative r.v.'s and conditionally symmetric martingale differences with bounded conditional moments and nonrandom conditional moments.

Theorem 3.9 gives new decoupling inequalities for sums of multilinear forms that complement the results obtained in McConnell and Taqqu (1986), de la Peña (1992) and de la Peña and Montgomery-Smith (1995) [here and in what follows, $C_n^k = n!/(k!(n-k)!)$, $0\le k\le n$].

THEOREM 3.9. *Let* $c_{i_1,\dots,i_l}\in\mathbf{R}$, $1\le i_1<\cdots<i_l\le n$, $l=0,\dots,m$. *The following decoupling inequalities hold*:

$$
\begin{aligned}
&(m+1)^{-1} E\left(\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l}\right)^t\\
&\quad\le E\left(\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l}\right)^t \qquad (3.34)\\
&\quad\le \left(\sum_{q=0}^{m}(C_m^q)^t\right) E\left(\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l}\right)^t
\end{aligned}
$$

for all independent nonnegative r.v.'s $X_1,\dots,X_n$, $X_{p1},\dots,X_{pn}$, $p=1,\dots,m$, *with finite* t*th moment*, $1<t<2$, *such that, for* $i=1,\dots,n$, *the r.v.'s* X_{pi},

$p = 1, \dots, m$, *and* X_i *have the same distribution*,

$$\begin{aligned}
&\left(1 + \sum_{q=1}^{m} Kh^{*\mathrm{dec}}(q,t)\right)^{-1} E\left|\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l} X_{1,i_1} \cdots X_{l,i_l}\right|^t \\
&\quad \le E\left|\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l} X_{i_1} \cdots X_{i_l}\right|^t \qquad (3.35)\\
&\quad \le \left(1 + \sum_{q=1}^{m} (C_m^q)^t Kh^{*\mathrm{reg}}(q,t)\right) E\left|\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l} X_{1,i_1} \cdots X_{l,i_l}\right|^t
\end{aligned}$$

for all independent symmetric r.v.'s $X_1, \dots, X_n, X_{p1}, \dots, X_{pn}$, $p = 1, \dots, m$, *with finite* t*th moment*, $2 < t < 4$, *such that*, *for* $i = 1, \dots, n$, *the r.v.'s* X_{pi}, $p = 1, \dots, m$, *and* X_i *have the same distribution*.

According to the following theorem, the best constants in Khintchine–Marcinkiewicz–Zygmund inequalities for generalized moving averages in independent symmetric r.v.'s do not depend on the order m and are the same as in the independent case.

THEOREM 3.10. *Let* $0 \le h_1 < \cdots < h_m$. *The best constants* $\overline{K}h_1^*(t,m)$ *and* $\overline{K}h_2^*(t,m)$ *in the following Khintchine–Marcinkiewicz–Zygmund inequalities*

$$\begin{aligned}
&\overline{K}h_1(t,m) E\left(\sum_{i=1}^{n-h_m} c_i^2 X_{i+h_1}^2 X_{i+h_2}^2 \cdots X_{i+h_m}^2\right)^{t/2} \\
&\quad \le E\left|\sum_{i=1}^{n-h_m} c_i X_{i+h_1} X_{i+h_2} \cdots X_{i+h_m}\right|^t \\
&\quad \le \overline{K}h_2(t,m) E\left(\sum_{i=1}^{n-h_m} c_i^2 X_{i+h_1}^2 X_{i+h_2}^2 \cdots X_{i+h_m}^2\right)^{t/2}
\end{aligned}$$

for all $n > h_m$, $c_i \in \mathbf{R}$, $i = 1, \dots, n - h_m$, *and all independent symmetric r.v.'s* $X_1, \dots, X_n$ *with finite* t*th moment*, $t > 0$, *are given by* $Kh_1^*(t,m) = Kh_1^*(t) = 2^{t/2-1}$, $0 < t \le t_0$, $Kh_1^*(t,m) = Kh_1^*(t) = E|Z|^t$, $t_0 \le t \le 2$, $Kh_1^*(t,m) = Kh_1^*(t) = 1$, $t \ge 2$, $Kh_2^*(t,m) = Kh_2^*(t) = 1$, $0 < t \le 2$, $Kh_2^*(t,m) = Kh_2^*(t) = E|Z|^t$, $t \ge 2$, *where* t_0 *is the nontrivial solution of the equation* $\Gamma((t_0+1)/2) = \Gamma(3/2)$, $\Gamma(x)$ *is the Gamma-function*, $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\,dt$ *and* Z *is the standard normal r.v.*

REMARK 3.7. From Theorem 3.10, it follows that in the case of generalized moving averages, one can set $Kh^{*\mathrm{reg}}(m-q,t) = E|Z|^t$, $q = 0, \dots, m-1$, $Kh^{*\mathrm{reg}}(q,t) = E|Z|^t$, $q = 1, \dots, m$, in inequalities (3.32) and (3.33).

PROOF OF THEOREMS 3.3 AND 3.4. Relations (3.22) and (3.25) follow from Lemmas 2.2 and 2.3, relations (3.26) and (3.29) follow from Lemmas 2.2 and 2.4. Let us prove (3.24). We first consider the case $m = 2$ in order to illustrate the general argument. Suppose that $1 < t < 2$, $1 \le s < t$ or $t \ge 2$, $t - 1 \le s < t$. Let $a_i, b_i > 0$, $a_i^t \le b_i$, $i = 1, \dots, n$, and let $c_k, c_{ij} = c_{ji} \ge 0$, $k = 0, 1, \dots, n$, $1 \le i < j \le n$. Determine the extrema of $ET^t(X_1, \dots, X_n)$, where $T(X_1, \dots, X_n) = c_0 + \sum_{i=1}^n c_i X_i + \sum_{1 \le i < j \le n} c_{ij} X_i X_j$, over the sets $M_k^{\text{non,ind}}(n, s, t, a, b)$, $k = 1, 2$. We have

$$T(X_1, \dots, X_n) = X_n\left(c_n + \sum_{i=1}^{n-1} c_{in} X_i\right) + c_0 + \sum_{i=1}^{n-1} c_i X_i + \sum_{1 \le i < j \le n-1} c_{ij} X_i X_j. \tag{3.36}$$

Let $\overline{a}^l = (a_1, \dots, a_l)$, $\overline{b}^l = (b_1, \dots, b_l)$, $l = 1, \dots, n-1$. For $l = 1, \dots, n$ and r.v.'s $X_1, \dots, X_{l-1} \in M_k^{\text{non,ind}}(l-1, s, t, \overline{a}^{l-1}, \overline{b}^{l-1})$, denote by $H_l^k(a_l, b_l)$ the class of nonnegative r.v.'s X_l independent of $X_1, \dots, X_{l-1}$ and such that $EX_l^s = a_l^s$, $EX_l^t = b_l$ if $k = 1$, and $EX_l^s \le a_l^s$, $EX_l^t \le b_l$ if $k = 2$. Using (3.36) and Lemmas 2.5 and 2.6, we get that, for all r.v.'s $X_1, \dots, X_{n-1} \in M_k^{\text{non,ind}}(n-1, s, t, \overline{a}^{n-1}, \overline{b}^{n-1})$,

$$\begin{aligned}
&\sup_{X_n \in H_n^k(a_n, b_n)} ET^t(X_1, \dots, X_n) \\
&\quad = (b_n - a_n^t) E\left(c_n + \sum_{i=1}^{n-1} c_{in} X_i\right)^t \\
&\qquad + E\left(a_n\left(c_n + \sum_{i=1}^{n-1} c_{in} X_i\right) + c_0 + \sum_{i=1}^{n-1} c_i X_i + \sum_{1 \le i < j \le n-1} c_{ij} X_i X_j\right)^t \\
&\quad = (b_n - a_n^t) E\left(c_n + \sum_{i=1}^{n-1} c_{in} X_i\right)^t + ET^t(X_1, \dots, X_{n-1}, a_n), \qquad k = 1, 2.
\end{aligned}$$

From the same lemmas (see also Remark 2.1), it follows therefore that, for all r.v.'s $X_1, \dots, X_{n-2} \in M_k^{\text{non,ind}}(n-2, s, t, \overline{a}^{n-2}, \overline{b}^{n-2})$,

$$\begin{aligned}
&\sup_{X_{n-1} \in H_{n-1}^k(a_{n-1}, b_{n-1})} \sup_{X_n \in H_n^k(a_n, b_n)} ET^t(X_1, \dots, X_n) \\
&\quad = c_{n-1,n}^t (b_{n-1} - a_{n-1}^t)(b_n - a_n^t) \\
&\qquad + (b_n - a_n^t) E\left(c_n + c_{n-1,n} a_{n-1} + \sum_{i=1}^{n-2} c_{in} X_i\right)^t
\end{aligned}$$

$$+ (b_{n-1} - a_{n-1}^t) E \left(c_{n-1} + c_{n,n-1} a_n + \sum_{i=1}^{n-2} c_{i,n-1} X_i \right)^t$$
$$+ ET^t(X_1, \ldots, X_{n-2}, a_{n-1}, a_n).$$

Continuing in the same way, we get that, for all r.v.'s $X_1, \ldots, X_{l-1} \in M_k^{\text{non,ind}}(l-1, s, t, \overline{a}^{l-1}, \overline{b}^{l-1})$,

$$\sup_{X_l \in H_l^k(a_l, b_l)} \cdots \sup_{X_{n-1} \in H_{n-1}^k(a_{n-1}, b_{n-1})} \sup_{X_n \in H_n^k(a_n, b_n)} ET^t(X_1, \ldots, X_n)$$
$$= \sum_{l \le i < j \le n} c_{ij}^t (b_i - a_i^t)(b_j - a_j^t)$$
$$+ \sum_{j=l}^{n} (b_j - a_j^t) E \left(c_j + \sum_{i=l, i \ne j}^{n} c_{ij} a_i + \sum_{i=1}^{l-1} c_{ij} X_i \right)^t$$
$$+ ET^t(X_1, \ldots, X_{l-1}, a_l, \ldots, a_n), \qquad k = 1, 2.$$

Therefore, $\sup_{(X,n) \in M_k^{\text{non,ind}}(n,s,t,a,b)} ET^t(X_1, \ldots, X_n) = \sum_{1 \le i < j \le n} c_{ij}^t (b_i - a_i^t) \times (b_j - a_j^t) + \sum_{i=1}^{n} (b_i - a_i^t)(c_i + \sum_{j=1, j \ne i}^{n} c_{ij} a_j)^t + T^t(a_1, a_2, \ldots, a_n), k = 1, 2,$ and we get that (3.24) holds in the case $m = 2$.

Let us now turn to the case of arbitrary m. Let us use induction on the number of r.v.'s $X_1, \ldots, X_n$. Suppose we have already proven relation (3.24) for all sums of multilinear forms of order not greater than m, $1 \le m \le n-1$, in the case of $n-1$ r.v.'s; that is, suppose that the relation

$$\sup_{(X,n-1) \in M_k^{\text{non,ind}}(n-1,s,t,\overline{a}^{n-1},\overline{b}^{n-1})} E \left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n-1} c_{i_1,\ldots,i_l} X_{i_1} \cdots X_{i_l} \right)^t$$
$$= \sum_{q=0}^{m} \sum_{1 \le j_1 < \cdots < j_q \le n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t)$$
$$\times \left(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n-1\} \setminus \{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1} \cdots a_{i_{l-q}} \right)^t,$$
$$k = 1, 2,$$

is valid. From Lemmas 2.5 and 2.6, we get that, for all r.v.'s $X_1, \ldots, X_{n-1}$ from the class $M_k^{\text{non,ind}}(n-1, s, t, \overline{a}^{n-1}, \overline{b}^{n-1})$,

$$\sup_{X_n \in H_n^k(a_n, b_n)} E \left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\ldots,i_l} X_{i_1} \cdots X_{i_l} \right)^t \tag{3.37}$$

$$= (b_n - a_n^t) E\left(\sum_{l=0}^{m-1} \sum_{1 \le i_1 < \cdots < i_l \le n-1} c_{i_1,\ldots,i_l,n} X_{i_1} \cdots X_{i_l}\right)^t$$
$$+ E\left(E\left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\ldots,i_l} X_{i_1} \cdots X_{i_l} \,\Big|\, X_n = a_n\right)\right)^t$$

[note that $E(f(X_1, \ldots, X_n) \mid X_n = a_n) = f(X_1, \ldots, X_{n-1}, a_n)$]. From the induction hypothesis, it follows that (we assume $c_{i_1,\ldots,i_m,n} = 0$)

$$\sup_{(X,n-1) \in M_k^{\text{non,ind}}(n-1,s,t,\overline{a}^{n-1},\overline{b}^{n-1})} E\left(E\left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\ldots,i_l} X_{i_1} \cdots X_{i_l} \,\Big|\, X_n = a_n\right)\right)^t$$
$$= \sup_{(X,n-1) \in M_k^{\text{non,ind}}(n-1,s,t,\overline{a}^{n-1},\overline{b}^{n-1})} E\left(\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n-1} (c_{i_1,\ldots,i_l} + c_{i_1,\ldots,i_l,n} a_n) X_{i_1} \cdots X_{i_l}\right)^t$$
$$(3.38) \qquad = \sum_{q=0}^{m} \sum_{1 \le j_1 < \cdots < j_q \le n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \times \left(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n-1\} \setminus \{j_1,\ldots,j_q\}} (c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} + c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n} a_n) a_{i_1} \cdots a_{i_{l-q}}\right)^t$$
$$= \sum_{q=0}^{m} \sum_{1 \le j_1 < \cdots < j_q \le n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \times \left(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n\} \setminus \{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} a_{i_1} \cdots a_{i_{l-q}}\right)^t.$$

Moreover,

$$\sup_{(X,n-1) \in M_k^{\text{non,ind}}(n-1,s,t,\overline{a}^{n-1},\overline{b}^{n-1})} E\left(\sum_{l=0}^{m-1} \sum_{1 \le i_1 < \cdots < i_l \le n-1} c_{i_1,\ldots,i_l,n} X_{i_1} \cdots X_{i_l}\right)^t$$
$$(3.39) \qquad = \sum_{q=0}^{m-1} \sum_{1 \le j_1 < \cdots < j_q \le n-1} \prod_{r=1}^{q} (b_{j_r} - a_{j_r}^t) \times \left(\sum_{l=q}^{m-1} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\ldots,n-1\} \setminus \{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q},n} a_{i_1} \cdots a_{i_{l-q}}\right)^t.$$

From (3.37)–(3.39) and Lemmas 2.5 and 2.6 (see also Remark 2.1), it follows that

$$\sup_{(X,n)\in M_k^{\mathrm{non,ind}}(n,s,t,a,b)} E\left(\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l}X_{i_1}\cdots X_{i_l}\right)^t$$

$$= \sum_{q=0}^{m}\sum_{1\le j_1<\cdots<j_q\le n-1}\prod_{r=1}^{q}(b_{j_r}-a_{j_r}^t)$$

$$\times\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}}a_{i_1}\cdots a_{i_{l-q}}\right)^t$$

$$+ (b_n-a_n^t)\sum_{q=0}^{m-1}\sum_{1\le j_1<\cdots<j_q\le n-1}\prod_{r=1}^{q}(b_{j_r}-a_{j_r}^t)$$

$$\times\left(\sum_{l=q}^{m-1}\sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n-1\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q},n}a_{i_1}\cdots a_{i_{l-q}}\right)^t$$

$$= \sum_{q=0}^{m}\sum_{1\le j_1<\cdots<j_q\le n}\prod_{r=1}^{q}(b_{j_r}-a_{j_r}^t)$$

$$\times\left(\sum_{l=q}^{m}\sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}}a_{i_1}\cdots a_{i_{l-q}}\right)^t,$$

where the next to the last term is the case when $j_{q+1}=n$, $1\le j_1<\cdots<j_q\le n-1$, $q=0,1,\dots,m-1$, missing to complete the sum. The fact that relation (3.24) is obviously valid for sums of multilinear forms in one r.v. completes the proof by induction. Relations (3.23), (3.27) and (3.28) might be proven in a similar way [to prove (3.27) and (3.28), one uses Lemma 2.7 instead of Lemma 2.6]. $\square$

PROOF OF THEOREMS 3.5–3.10. Inequality (3.30) immediately follows from relation (3.24). Let $c_{i_1,\dots,i_l}=0$, $1\le i_1<\cdots<i_l\le n$, $l=0,\dots,m-1$, $c_{i_1,\dots,i_m}=(\sum_{q=0}^{m}(1/q!)(1/(m-q)!)^t)^{-t}$, $1\le i_1<\cdots<i_m\le n$, and let $a_i=b_i=1/n$, $i=1,\dots,n$. From relation (3.24) it follows that

$$\sup_{(X,n)\in M_1^{\mathrm{non,ind}}(n,1,t,a,b)} E\left(\sum_{l=0}^{m}\sum_{1\le i_1<\cdots<i_l\le n} c_{i_1,\dots,i_l}X_{i_1}\cdots X_{i_l}\right)^t$$

$$= \sum_{q=0}^{m} C_n^q(n^{-1}-n^{-t})^q\left(\sum_{l=q}^{m} C_{n-q}^{l-q}n^{-(l-q)}c_{i_1,\dots,i_l}\right)^t$$

$$
\begin{aligned}
&= \sum_{q=0}^{m} C_n^q (n^{-1} - n^{-t})^q (C_{n-q}^{m-q} n^{-(m-q)} c_{1,\dots,m})^t \\
&\sim \sum_{q=0}^{m} \frac{1}{q!} (\frac{1}{(m-q)!})^t c_{1,\dots,m}^t = 1.
\end{aligned}
$$

Moreover, for all r.v.'s $(X, n) \in M_1^{\text{non,ind}}(n, 1, t, a, b)$,

$$
\begin{aligned}
&\sum_{q=0}^{m} \sum_{1 \le j_1 < \cdots < j_q \le n} \prod_{r=1}^{q} E X_{j_r}^t \\
&\quad \times \left(\sum_{l=q}^{m} \sum_{i_1 < \cdots < i_{l-q} \in \{1,\dots,n\} \setminus \{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}} E X_{i_1} \cdots E X_{i_{l-q}} \right)^t \\
&\quad = \sum_{q=0}^{m} C_n^q n^{-q} (C_{n-q}^{m-q} n^{-(m-q)} c_{1,\dots,m})^t \sim 1.
\end{aligned}
$$

This proves exactness of the constants in inequality (3.30). The right-hand side of inequality (3.31) is an evident consequence of (3.30). The left-hand side of inequality (3.31) easily follows from the nonnegativity of the r.v.'s X_1, $\dots, X_n$ and Jensen's inequality. Inequality (3.32) follows from relation (3.28). The upper bound in (3.33) is an immediate consequence of (3.32). The lower estimate is an evident consequence of the lower Khintchine bound $E(\sum_{l=0}^{m} \times \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l}^2 X_{i_1}^2 \cdots X_{i_l}^2)^{t/2} \le E|\sum_{l=0}^{m} \sum_{1 \le i_1 < \cdots < i_l \le n} c_{i_1,\dots,i_l} X_{i_1} \cdots X_{i_l}|^t$, $t \ge 2$, implied by Jensen's inequality. Let us prove (3.34). Let us again consider first the case $m = 2$. Let $c_k, c_{ij} = c_{ji} \ge 0$, $k = 0, 1, \dots, n$, $1 \le i < j \le n$. From (3.30) (see Remark 3.6) and the left-hand side of inequality in (3.31), it follows that

$$
\begin{aligned}
&E\left(c_0 + \sum_{i=1}^{n} c_i X_{1i} + \sum_{1 \le i < j \le n} c_{ij} X_{1i} X_{2j} \right)^t \\
&\quad \le \sum_{1 \le i < j \le n} c_{ij}^t E X_{1i}^t X_{2j}^t \\
&\qquad + \sum_{i=1}^{n} E X_{1i}^t \left(c_i + \sum_{j=i+1}^{n} c_{ij} E X_{2j} \right)^t \\
&\qquad + \sum_{j=1}^{n} E X_{2j}^t \left(\sum_{i=1}^{j-1} c_{ij} E X_{1i} \right)^t + \left(\sum_{1 \le i < j \le n} E X_{1i} E X_{2j} \right)^t
\end{aligned}
$$

$$\leq \sum_{1\leq i<j\leq n} c_{ij}^t EX_i^t X_j^t + \sum_{i=1}^n EX_i^t \left(c_i + \sum_{j\neq i}^n c_{ij} EX_j\right)^t$$
$$+ \left(\sum_{1\leq i<j\leq n} EX_i EX_j\right)^t$$
$$\leq 3E\left(c_0 + \sum_{i=1}^n c_i X_i + \sum_{1\leq i<j\leq n} c_{ij} X_i X_j\right)^t.$$

From (3.30) and the nonnegativity of $X_{11}, \ldots, X_{1n}, X_{21}, \ldots, X_{2n}$, it also follows that

$$E\left(c_0 + \sum_{i=1}^n c_i X_i + \sum_{1\leq i<j\leq n} c_{ij} X_i X_j\right)^t$$
$$\leq \sum_{1\leq i<j\leq n} c_{ij}^t EX_i^t X_j^t + \sum_{i=1}^n EX_i^t \left(c_i + \sum_{j\neq i}^n c_{ij} EX_j\right)^t$$
$$+ \left(\sum_{1\leq i<j\leq n} EX_i EX_j\right)^t$$
$$\leq \sum_{1\leq i<j\leq n} c_{ij}^t EX_{1i}^t X_{2j}^t + 2^{t-1} \sum_{i=1}^n EX_{1i}^t \left(c_i + \sum_{j=i+1}^n c_{ij} EX_{2j}\right)^t$$
$$+ 2^{t-1} \sum_{j=1}^n EX_{2j}^t \left(\sum_{i=1}^{j-1} c_{ij} EX_{1i}\right)^t + \left(\sum_{1\leq i<j\leq n} EX_{1i} EX_{2j}\right)^t$$
$$\leq (2+2^t) E\left(\sum_{1\leq i<j\leq n} c_{ij} X_{1i} X_{2j}\right)^t.$$

Therefore, (3.34) holds for $m = 2$. Let us now turn to the case of arbitrary m. Let $c_{i_1,\ldots,i_l} \geq 0$, $1 \leq i_1 < \cdots < i_l \leq n$, $l = 0, \ldots, m$, $c_{i_{\pi(1)},\ldots,i_{\pi(l)}} = c_{i_1,\ldots,i_l}$, $1 \leq i_1 < \cdots i_l \leq n$, for all permutations $\pi : \{1, \ldots, l\} \to \{1, \ldots, l\}$, $l = 2, \ldots, m$. It is obvious that

$$\sum_{1\leq j_1<\cdots<j_q\leq n} \prod_{r=1}^q EX_{j_r}^t$$
$$\times \left(\sum_{l=q}^m \sum_{i_1<\cdots<i_{l-q}\in\{1,\ldots,n\}\setminus\{j_1,\ldots,j_q\}} c_{j_1,\ldots,j_q,i_1,\ldots,i_{l-q}} EX_{i_1} \cdots EX_{i_{l-q}}\right)^t$$

$$
\geq \sum_{1\leq j_1<\cdots<j_q\leq m} \sum_{1\leq i_{j_1}<\cdots<i_{j_q}\leq n} \prod_{r=1}^{q} EX^t_{j_r,i_{j_r}}
$$

$$
\times \left(\sum_{l=j_q}^{m} \sum_{\substack{1\leq i_p\leq n, i_{p_1}<i_{p_2}, p_1<p_2,\\ p,p_1,p_2=1,\dots,l, p\neq j_1,\dots,j_q}} c_{i_1,\dots,i_l} \prod_{k=1,\dots,l, k\neq j_1,\dots,j_q} EX_{k,i_k} \right)^t
$$

for all independent nonnegative r.v.'s $X_1,\dots,X_n,\ X_{p1},\dots,X_{pn},\ p=1,\dots,m$, with finite tth moment, $1<t<2$, such that, for $i=1,\dots,n,\ X_{pi},\ p=1,\dots,m$, and X_i have the same distribution. This inequality, the estimate

$$
E\left(\sum_{l=0}^{m} \sum_{1\leq i_1<\cdots<i_l\leq n} c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l} \right)^t
$$

$$
\leq \sum_{q=0}^{m} \sum_{1\leq j_1<\cdots<j_q\leq m} \sum_{1\leq i_{j_1}<\cdots<i_{j_q}\leq n} \prod_{r=1}^{q} EX^t_{j_r,i_{j_r}}
$$

$$
\times \left(\sum_{l=j_q}^{m} \sum_{\substack{1\leq i_p\leq n, i_{p_1}<i_{p_2}, p_1<p_2,\\ p,p_1,p_2=1,\dots,l, p\neq j_1,\dots,j_q}} c_{i_1,\dots,i_l} \prod_{k=1,\dots,l, k\neq j_1,\dots,j_q} EX_{k,i_k} \right)^t
$$

for all independent nonnegative r.v.'s $X_{p1},\dots,X_{pn},\ p=1,\dots,m$, with finite tth moment, $1<t<2$, implied by (3.30) (see Remark 3.6) and the left-hand side inequality in (3.31) imply that

$$
E\left(\sum_{l=0}^{m} \sum_{1\leq i_1<\cdots<i_l\leq n} c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l} \right)^t
$$

$$
\leq (m+1) E\left(\sum_{l=0}^{m} \sum_{1\leq i_1<\cdots<i_l\leq n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l} \right)^t.
$$

Similarly, from the inequality $(\sum_{k=1}^{N} z_k)^t \leq N^{t-1}\sum_{k=1}^{N} z_k^t$ for $z_k\geq 0,\ k=1,\dots,N,\ t>1$, it follows that

$$
\sum_{1\leq j_1<\cdots<j_q\leq n} \prod_{r=1}^{q} EX^t_{j_r}
$$

$$
\times \left(\sum_{l=q}^{m} \sum_{i_1<\cdots<i_{l-q}\in\{1,\dots,n\}\setminus\{j_1,\dots,j_q\}} c_{j_1,\dots,j_q,i_1,\dots,i_{l-q}} EX_{i_1}\cdots EX_{i_{l-q}} \right)^t
$$

$$\leq (C_m^q)^{t-1} \sum_{1\leq j_1<\cdots<j_q\leq m} \sum_{1\leq i_{j_1}<\cdots<i_{j_q}\leq n} \prod_{r=1}^{q} EX^t_{j_r,i_{j_r}}$$

$$\times \left(\sum_{l=j_q}^{m} \sum_{\substack{1\leq i_p\leq n, i_{p_1}<i_{p_2}, p_1<p_2,\\ p,p_1,p_2=1,\dots,l, p\neq j_1,\dots,j_q}} c_{i_1,\dots,i_l} \prod_{k=1,\dots,l,k\neq j_1,\dots,j_q} EX_{k,i_k} \right)^t.$$

This, together with inequality (3.30) and the inequality

$$\max_{q=0,\dots,m} \max_{1\leq j_1<\cdots<j_q\leq m} \sum_{1\leq i_{j_1}<\cdots<i_{j_q}\leq n} \prod_{r=1}^{q} EX^t_{j_r,i_{j_r}}$$

$$\times \left(\sum_{l=j_q}^{m} \sum_{\substack{1\leq i_p\leq n, i_{p_1}<i_{p_2}, p_1<p_2,\\ p,p_1,p_2=1,\dots,l, p\neq j_1,\dots,j_q}} c_{i_1,\dots,i_l} \prod_{k=1,\dots,l,k\neq j_1,\dots,j_q} EX_{k,i_k} \right)^t$$

$$\leq E\left(\sum_{l=0}^{m} \sum_{1\leq i_1<\cdots<i_l\leq n} c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l} \right)^t,$$

which follows from the nonnegativity of $X_{p1},\dots,X_{pn}$, $p=1,\dots,m$, and Jensen's inequality, imply that

$$E\left(\sum_{l=0}^{m} \sum_{1\leq i_1<\cdots<i_l\leq n} c_{i_1,\dots,i_l} X_{i_1}\cdots X_{i_l} \right)^t$$

$$\leq \left(\sum_{q=0}^{m} (C_m^q)^t \right) E\left(\sum_{l=0}^{m} \sum_{1\leq i_1<\cdots<i_l\leq n} c_{i_1,\dots,i_l} X_{1,i_1}\cdots X_{l,i_l} \right)^t;$$

that is, inequality (3.34) holds. Decoupling inequalities (3.35) might be proven in a similar way, with the help of the inequalities $\sum_{k=1}^N |z_k|^t \leq |\sum_{k=1}^N z_k|^t \leq N^{t-1}\sum_{k=1}^N |z_k|^t$ for $z_k \in \mathbf{R}$, $k=1,\dots,N$, $t>1$, estimate (3.32), left-hand side inequality (3.33) and their implications for sums of decoupled mulitilinear forms. Theorem 3.10 follows from the results of Haagerup (1982) and independence of the r.v.'s $\varepsilon_i' = \varepsilon_{i+h_1}\cdots\varepsilon_{i+h_m}$, $i=1,\dots,n-h_m$ (Remark 3.5). $\square$

APPENDIX

Auxiliary results on extremal properties of sums of independent random variables with fixed sum of tails of distributions. Let $\Im$ be the σ-algebra of Borel subsets of $\mathbf{R}$ and let Λ be the class of finite positive σ-additive measures λ

on $\Im$ such that $\lambda(\{0\}) = 0$. Set $\Lambda_1 = \{\lambda \in \Lambda : \lambda(B) = \lambda(B \cap \mathbf{R}_+), B \in \Im\}$, $\Lambda_2 = \{\lambda \in \Lambda : \lambda(B) = \lambda(-B), B \in \Im\}$. For a measure $\lambda \in \Lambda$ denote by $T(\lambda)$ the r.v. with characteristic function $Ee^{itT(\lambda)} = \exp(\int_{-\infty}^{+\infty}(e^{itx} - 1)\, d\lambda(x))$. Let $\lambda_1 \in \Lambda_1$, $\lambda_2 \in \Lambda_2$. Set [here, as in Section 3, (X, n) denotes a set of independent r.v.'s $(X_1, \dots, X_n)$]

$$W_1(\lambda_1) = \Biggl\{(X, n) : n \ge 1, X_i \text{ is nonnegative}, \\ \sum_{i=1}^n P(X_i \in B \setminus \{0\}) = \lambda_1(B), B \in \Im\Biggr\},$$

$$W_1(\lambda_2) = \Biggl\{(X, n) : n \ge 1, X_i \text{ is symmetric}, \\ \sum_{i=1}^n P(X_i \in B \setminus \{0\}) = \lambda_2(B), B \in \Im\Biggr\}.$$

Denote by $W_2(\lambda_j)$, $j = 1, 2$, the subsets of $W_1(\lambda_j)$ consisting of identically distributed r.v.'s.

The following theorem refines and complements the results obtained in Utev (1985).

As in the beginning of Section 3, Φ is the class of continuous functions $\phi : \mathbf{R} \to \mathbf{R}$ satisfying condition (3.1). Let $f \in J$, $h \in Q_f$ (concerning the definitions of the classes J and Q_f, see the beginning of Section 2), $f(0) = 0$, $D_h, A_f > 0$, and let, similarly to the proof of Theorem 3.1, $U_1^{(1)}(D_h, A_f)$ and $U_2^{(1)}(D_h, A_f)$ be the sets of independent nonnegative r.v.'s $X_1, \dots, X_n$, $n \ge 1$, satisfying the conditions

$$\text{(A.1)} \qquad \sum_{i=1}^n Eh(|X_i|) = h(D_h), \qquad \sum_{i=1}^n Ef(|X_i|) = A_f$$

and

$$\text{(A.2)} \qquad \sum_{i=1}^n Eh(|X_i|) \le h(D_h), \qquad \sum_{i=1}^n Ef(|X_i|) \le A_f,$$

respectively. Denote by $U_1^{(2)}(D_h, A_f)$ and $U_2^{(2)}(D_h, A_f)$ the sets of independent symmetric r.v.'s $X_1, \dots, X_n$, $n \ge 1$, satisfying inequalities (A.1) and (A.2), and denote by $U_3^{(j)}(D_h, A_f)$ and $U_4^{(j)}(D_h, A_f)$ the subsets of $U_1^{(j)}(D_h, A_f)$ and $U_2^{(j)}(D_h, A_f)$, $j = 1, 2$, respectively, consisting of identically distributed r.v.'s. Let

$$\Lambda_1^{(j)}(D_h, A_f) = \biggl\{\lambda \in \Lambda_j : \int_{-\infty}^{\infty} h(|x|)\, d\lambda(x) = h(D_h),$$

$$\int_{-\infty}^{\infty} f(|x|)\,d\lambda(x) = A_f\Big\},$$

$$\Lambda_2^{(j)}(D_h, A_f) = \Big\{\lambda \in \Lambda_j : \int_{-\infty}^{\infty} h(|x|)\,d\lambda(x) \le h(D_h),$$

$$\int_{-\infty}^{\infty} f(|x|)\,d\lambda(x) \le A_f\Big\}, \qquad j = 1, 2.$$

THEOREM A.1. *If a nonnegative function $\phi_1 \in \Phi$ is convex, and a nonnegative function $\phi_2 \in \Phi$ satisfies condition* (3.19) *then*

$$\sup_{(X,n)\in U_k^{(j)}(D_h,A_f)} E\phi_j\left(\sum_{i=1}^{n} X_i\right) = \sup_{(X,n)\in U_{k+2}^{(j)}(D_h,A_f)} E\phi_j\left(\sum_{i=1}^{n} X_i\right)$$
$$= \sup_{\lambda\in\Lambda_k^{(j)}(D_h,A_f)} E\phi_j\big(T(\lambda)\big), \qquad k, j = 1, 2.$$

REMARK A.1. Theorem A.1 means that the following important fact holds: in problems of determining extrema of expectations of functions ϕ_1 of sums of independent nonnegative r.v.'s with fixed sums of generalized moments and expectations of functions ϕ_2 of sums of independent symmetric r.v.'s with fixed sums of generalized moments, it suffices to consider only identically distributed r.v.'s.

Theorem A.1 immediately follows from the evident relations $U_k^{(j)}(D_h, A_f) = \bigcup_{\lambda\in\Lambda_k^{(j)}(D_h,A_f)} W_1(\lambda)$, $U_{k+2}^{(j)}(D_h, A_f) = \bigcup_{\lambda\in\Lambda_k^{(j)}(D_h,A_f)} W_2(\lambda)$, $k = 1, 2$, $j = 1, 2$, and the following lemma.

LEMMA A.1. *Let $\lambda_j \in \Lambda_j$, $j = 1, 2$, let a function $\phi_1 \in \Phi$ be convex and let a function $\phi_2 \in \Phi$ satisfy condition* (3.19). *If $\int_{-\infty}^{+\infty} |\phi_j(x)|\,d\lambda_j(x) < \infty$, then*

$$E\big|\phi_j\big(T(\lambda_j)\big)\big| < \infty,$$

$$\sup_{(X,n)\in W_1(\lambda_j)} E\phi_j\left(\sum_{i=1}^{n} X_i\right) \le E\phi_j\big(T(\lambda_j)\big), \qquad j = 1, 2. \tag{A.3}$$

If, in addition to that, the functions ϕ_j, $j = 1, 2$, are nonnegative, then

$$\sup_{(X,n)\in W_k(\lambda_j)} E\phi_j\left(\sum_{i=1}^{n} X_i\right) = E\phi_j(T(\lambda_j)), \qquad j, k = 1, 2. \tag{A.4}$$

Let us formulate some auxiliary results needed for the proof of Lemma A.1. For a vector $a \in \mathbf{R}^n$, denote by $a_{[1]} \ge \cdots \ge a_{[n]}$ its components arranged in descending order.

DEFINITION A.1 [Marshall and Olkin (1979)]. Let $x, y \in \mathbf{R}^n$. The vector x is said to be majorized by the vector y ($x \prec y$) if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$, $k = 1, \dots, n-1$, $\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$.

DEFINITION A.2 [Marshall and Olkin (1979)]. Let $A \subseteq \mathbf{R}^n$. A function $\phi : A \to \mathbf{R}$ is said to be S-convex (resp. S-concave) on A if $(x \prec y) \Rightarrow (\phi(x) \le \phi(y))$ [resp. $(x \prec y) \Rightarrow (\phi(x) \ge \phi(y))$] for all $x, y \in A$.

LEMMA A.2. *A continuous function $\phi_1 : \mathbf{R}_+ \to \mathbf{R}$ is convex on $\mathbf{R}_+$ if and only if*

$$(n-1)\phi_1(x) + \phi_1\left(\sum_{i=1}^{n} a_i + x\right) \ge \sum_{i=1}^{n} \phi_1(a_i + x), \qquad a_1, \dots, a_n, x \in \mathbf{R}_+, \ n \ge 1. \tag{A.5}$$

PROOF. Let $\phi_1 : \mathbf{R}_+ \to \mathbf{R}$ be a continuous and convex function on $\mathbf{R}_+$. Then from Proposition 3.C.1 in Marshall and Olkin (1979), it follows that $\sum_{i=1}^{n} \phi_1(x_i)$ is an S-convex function on $\mathbf{R}_+^n$. Since $(a_1 + x, \dots, a_n + x) \prec (x, \dots, x, \sum_{i=1}^{n} a_i + x)$, this implies inequality (A.5). Let now a continuous function $\phi_1 : \mathbf{R}_+ \to \mathbf{R}$ satisfy inequality (A.5) and let $0 \le y \le z$. Setting in (A.5) $n = 2, x = y, a_1 = a_2 = (z - y)/2$, we obtain that $(\phi_1(y) + \phi_1(z))/2 \ge \phi_1((y + z)/2)$; that is, the function ϕ_1 is convex. □

LEMMA A.3. *A function $\phi_2 : \mathbf{R} \to \mathbf{R}$ satisfies condition* (3.19) *if and only if*

$$(n-1)E\phi_2(x\varepsilon) + E\phi_2\left(\sum_{i=1}^{n} a_i \varepsilon_i + x\varepsilon\right) \ge \sum_{i=1}^{n} E\phi_2(a_i \varepsilon_1 + x\varepsilon_2), \qquad a_1, \dots, a_n, x \in \mathbf{R}, \ n \ge 1.$$

The proof can be easily obtained by induction.

LEMMA A.4. *Let $X^{(1)}, Y^{(1)}$ be nonnegative r.v.'s, let $X^{(2)}, Y^{(2)}$ be symmetric r.v.'s, let $\phi_1 \in \Phi$ be a convex function and let $\phi_2 \in \Phi$ be a function satisfying condition* (3.19). *Suppose that for $j = 1, 2$ the r.v. $X^{(j)}$ has a distribution $\lambda_j \in \Lambda_j$, the r.v.'s $X^{(j)}, Y^{(j)}, T(\lambda_j)$ are independent, and $E|\phi_j(X^{(j)})| < \infty$, $E|\phi_j(Y^{(j)})| < \infty$. Then $E|\phi_j(T(\lambda_j) + Y^{(j)})| < \infty$ and $E\phi_j(X^{(j)} + Y^{(j)}) \le E\phi_j(T(\lambda_j) + Y^{(j)})$, $j = 1, 2$.*

PROOF. The distributions of the r.v.'s $T(\lambda_j)$, $j = 1, 2$, are the same as the distributions of the r.v.'s $\sum_{i=1}^{\theta(1)} X_i^{(j)}$, respectively, where the r.v. $\theta(1)$ is independent of $Y^{(j)}$ and $X_1^{(j)}, X_2^{(j)}, \dots$ are sequences of independent r.v.'s with distributions λ_j independent of $Y^{(j)}$ and $\theta(1)$. According to Lemma 3.4 in Utev (1985), from the condition that $\phi_j \in \Phi$ and, therefore, ϕ_j satisfy (3.1), it follows that, for all $a_1, \dots, a_n \in \mathbf{R}$, $|\phi_j(\sum_{i=1}^n a_i)| \le q_j^{n-1} \prod_{i=1}^n (1 + |\phi_j(a_i)|)$, $j = 1, 2$, where $q_j = \max(1, 2C(\phi_j))$, $C(\phi_j)$ are the constants in (3.1). Consequently,

$$
\begin{aligned}
&E|\phi_j(T(\lambda_j) + Y^{(j)})| \\
&\quad = e^{-1} \sum_{k=0}^{\infty} E\left|\phi_j\left(\sum_{i=1}^{k} X_i^{(j)} + Y^{(j)}\right)\right| \Big/ k! \\
&\quad \le e^{-1}(1 + E|\phi_j(Y^{(j)})|) \sum_{k=0}^{\infty} \left(q_j(1 + E|\phi_j(X_i^{(j)})|)\right)^k \Big/ k! \\
&\quad = (1 + E|\phi_j(Y^{(j)})|) \exp\left(q_j(1 + E|\phi_j(X_i^{(j)})|) - 1\right) < \infty, \qquad j = 1, 2.
\end{aligned}
$$

From Lemmas A.2 and A.3, it follows that

$$
\begin{aligned}
&e^{-1} \sum_{k=1}^{\infty} E\phi_j\left(\sum_{i=1}^{k} X_i^{(j)} + Y^{(j)}\right) \Big/ k! \\
&\qquad \ge e^{-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} E\phi_j(X_i^{(j)} + Y^{(j)}) - (k-1)E\phi_j(Y^{(j)})\right) \Big/ k! \\
&\qquad = E\phi_j(X^{(j)} + Y^{(j)}) - e^{-1} E\phi_j(Y^{(j)}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E\phi_j(T(\lambda_j) + Y^{(j)}) &= e^{-1} \sum_{k=0}^{\infty} E\phi_j\left(\sum_{i=1}^{k} X_i^{(j)} + Y^{(j)}\right) \Big/ k! \\
&\ge E\phi_j(X^{(j)} + Y^{(j)}), \qquad j = 1, 2. \qquad \square
\end{aligned}
$$

PROOF OF LEMMA A.1. The inequalities in (A.3) are evident consequences of Lemma A.4. Let us prove the relations in (A.4). Let $\phi_1 \in \Phi$ be a nonnegative convex function, let $\phi_2 \in \Phi$ be a nonnegative function satisfying condition (3.19) and let $\lambda_j \in \Lambda_j$, $j = 1, 2$. It suffices to prove the exactness of upper bounds in (A.4). Take $n \ge \max_{j=1,2} \lambda_j(\mathbf{R})$. Let $X_{1n}^{(1)}, \dots, X_{nn}^{(1)}$ be independent nonnegative r.v.'s and let $X_{1n}^{(2)}, \dots, X_{nn}^{(2)}$ be independent symmetric r.v.'s such that $P(X_{in}^{(j)} \in B \setminus \{0\}) = n^{-1}\lambda_j(B)$ for $B \in \Im$, $i = 1, \dots, n$, $j = 1, 2$. Then $\sum_{i=1}^n P(X_{in}^{(j)} \in B \setminus \{0\}) = \lambda_j(B)$, and the characteristic function of the r.v. $\sum_{i=1}^n X_{in}^{(j)}$ is given by $(1+n^{-1}\int_{-\infty}^{+\infty}(e^{itx}-1)\,d\lambda_j(x))^n \to \exp(\int_{-\infty}^{+\infty}(e^{itx}-1)\,d\lambda_j(x))$, $j=1,2$, as $n\to\infty$. Since the functions ϕ_j are continuous, this implies that $\phi_j(\sum_{i=1}^n X_{in}^{(j)}) \to \phi_j(T(\lambda_j))$, $j=1,2$ (in distribution), as $n\to\infty$. Therefore [e.g., Billingsley (1999), Theorem 3.4], $\liminf_{n\to\infty} E\phi_j(\sum_{i=1}^n X_{in}^{(j)}) \geq E\phi_j(T(\lambda_j))$, $j=1,2$. $\square$

For $n \geq d > 0$, set $H(d) = \{(p_1,\dots,p_n): 0 \leq p_i \leq 1, i=1,\dots,n, \sum_{i=1}^n p_i = d\}$. Let $\overline{X}_1(p_1),\dots,\overline{X}_n(p_n)$ be independent r.v.'s with distributions $P(\overline{X}_i(p_i)=1)=p_i$, $P(\overline{X}_i(p_i)=0)=1-p_i$, $i=1,\dots,n$, and let $\overline{Y}_1(p_1),\dots,\overline{Y}_n(p_n)$ be independent r.v.'s with distributions $P(\overline{Y}_i(p_i)=1)=P(\overline{Y}_i(p_i)=-1)=p_i/2$, $P(\overline{Y}_i(p_i)=0)=1-p_i$, $i=1,\dots,n$. Then, for all $(p_1,\dots,p_n)\in H(d)$ and $B\in\Im$, we have $\sum_{i=1}^n P(\overline{X}_i(p_i)\in B\setminus\{0\})=\lambda_1(B)$, $\sum_{i=1}^n P(\overline{Y}_i(p_i)\in B\setminus\{0\})=\lambda_2(B)$, where $\lambda_1(\{1\})=d$, $\lambda_2(\{1\})=\lambda_2(\{-1\})=d/2$, $\lambda_1(\mathbf{R})=\lambda_2(\mathbf{R})=d$. The distributions of r.v.'s $T(\lambda_1)$ and $T(\lambda_2)$ are the same as distributions of r.v.'s $\theta(d)$ and $\theta_1(d/2)-\theta_2(d/2)$, respectively. Using Lemma A.1, we obtain that if $\phi_1\in\Phi$ is a nonnegative convex function and $\phi_2\in\Phi$ is a nonnegative function satisfying condition (3.19), then

$$\begin{aligned}\sup_{(p_1,\dots,p_n)\in H(d)} & E\phi_1\left(\sum_{i=1}^n \overline{X}_i(p_i)\right)\\ &= \sup_n E\phi_1\left(\sum_{i=1}^n \overline{X}_i(d/n)\right) = E\phi_1\big(\theta(d)\big) < \infty,\end{aligned} \tag{A.6}$$

$$\begin{aligned}\sup_{(p_1,\dots,p_n)\in H(d)} & E\phi_2\left(\sum_{i=1}^n \overline{Y}_i(p_i)\right)\\ &= \sup_n E\phi_2\left(\sum_{i=1}^n \overline{Y}_i(d/n)\right) = E\phi_2\big(\theta_1(d/2)-\theta_2(d/2)\big) < \infty.\end{aligned} \tag{A.7}$$

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